Partitioning Of A Random Vector "$\underline{X}$":

let

$$\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_q \\ \hdashline X_{q+1} \\ X_{q+2} \\ \vdots \\ X_p \end{bmatrix} = \begin{bmatrix} \underline{x}_1 \\ \hdashline \underline{x}_2 \end{bmatrix} \begin{matrix} q\times 1 \\ \\ (p-q)\times 1 \end{matrix} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix},$$

Mean Vector:

$$E(\underline{X}) = E\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_q \\ \hline X_{q+1} \\ \vdots \\ X_p \end{bmatrix} = \begin{bmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_q) \\ \hline E(X_{q+1}) \\ \vdots \\ E(X_p) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_q \\ \hline \mu_{q+1} \\ \vdots \\ \mu_p \end{bmatrix} = \begin{bmatrix} \underline{\mu}_1 \\ \hline \underline{\mu}_2 \end{bmatrix}$$

Covariance Matrix:

$$\underset{\sim}{\Sigma} = E\left\{(\underline{X} - \underline{\mu})(\underline{X} - \underline{\mu})'\right\}.$$

$$\underset{\sim}{\Sigma} = E\left\{\begin{bmatrix} \underline{X}_1 - \underline{\mu}_1 \\ \underline{X}_2 - \underline{\mu}_2 \end{bmatrix} \begin{bmatrix} \underline{X}_1 - \underline{\mu}_1 \\ \underline{X}_2 - \underline{\mu}_2 \end{bmatrix}'\right\}.$$

$$\underset{\sim}{\Sigma} = E\left\{\begin{bmatrix} \underline{X}_1 - \underline{\mu}_1 \\ \underline{X}_2 - \underline{\mu}_2 \end{bmatrix} \left[(\underline{X}_1 - \underline{\mu}_1)' \quad (\underline{X}_2 - \underline{\mu}_2)'\right]\right\},$$

$$\underset{\sim}{\Sigma} = \begin{bmatrix} (\underline{X}_1 - \underline{\mu}_1)(\underline{X}_1 - \underline{\mu}_1)' & (\underline{X}_1 - \underline{\mu}_1)(\underline{X}_2 - \underline{\mu}_2)' \\ (\underline{X}_2 - \underline{\mu}_2)(\underline{X}_1 - \underline{\mu}_1)' & (\underline{X}_2 - \underline{\mu}_2)(\underline{X}_2 - \underline{\mu}_2)' \end{bmatrix}$$

applying expectation: ~~on both sides.~~

$$\underset{\sim}{\Sigma} = \begin{bmatrix} E(\underline{X}_1 - \underline{\mu}_1)(\underline{X}_1 - \underline{\mu}_1)' & E(\underline{X}_1 - \underline{\mu}_1)(\underline{X}_2 - \underline{\mu}_2)' \\ E(\underline{X}_2 - \underline{\mu}_2)(\underline{X}_1 - \underline{\mu}_1)' & E(\underline{X}_2 - \underline{\mu}_2)(\underline{X}_2 - \underline{\mu}_2)' \end{bmatrix}$$

$$\underset{\sim}{\Sigma} = \begin{bmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22} \end{bmatrix}$$

where;

$$\underset{\sim}{\Sigma}_{11} = E(\underline{X}_1 - \underline{\mu}_1)(\underline{X}_1 - \underline{\mu}_1)'$$

$$= E\begin{bmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 \\ \vdots \\ X_{q_1} - \mu_{q_1} \end{bmatrix} \begin{bmatrix} X_1 - \mu_1 & X_2 - \mu_2 & \cdots & X_{q_1} - \mu_{q_1} \end{bmatrix}$$

$$= E\begin{bmatrix} (X_1 - \mu_1)^2 & (X_1 - \mu_1)(X_2 - \mu_2) & \cdots & (X_1 - \mu_1)(X_{q_1} - \mu_{q_1}) \\ (X_2 - \mu_2)(X_1 - \mu_1) & (X_2 - \mu_2)^2 & \cdots & (X_2 - \mu_2)(X_{q_1} - \mu_{q_1}) \\ \vdots & \vdots & & \vdots \\ (X_{q_1} - \mu_{q_1})(X_1 - \mu_1) & (X_{q_1} - \mu_{q_1})(X_2 - \mu_2) & \cdots & (X_{q_1} - \mu_{q_1})^2 \end{bmatrix}$$

By applying expectation:

$$\underset{\sim}{\Sigma}_{11} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1q} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2q} \\ \vdots & & & \\ \sigma_{q1} & \sigma_{q2} & \cdots & \sigma_{qq} \end{bmatrix}$$

Similarly;

$$\underset{\sim}{\Sigma}_{12} = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ q \end{array} \begin{array}{c} \begin{array}{cccc} q+1 & q+2 & \cdots & p \end{array} \\ \begin{bmatrix} \sigma_{1,q+1} & \sigma_{1,q+2} & \cdots & \sigma_{1p} \\ \sigma_{2,q+1} & \sigma_{2,q+2} & \cdots & \sigma_{2p} \\ \vdots & \vdots & & \vdots \\ \sigma_{q,q+1} & \sigma_{q,q+2} & \cdots & \sigma_{qp} \end{bmatrix} \end{array}$$

$$\underset{\sim}{\Sigma}_{21} = \begin{array}{c} \\ q+1 \\ q+2 \\ \vdots \\ p \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & q \end{array} \\ \begin{bmatrix} \sigma_{q+1,1} & \sigma_{q+1,2} & \cdots & \sigma_{q+1,q} \\ \sigma_{q+2,1} & \sigma_{q+2,2} & \cdots & \sigma_{q+2,q} \\ \vdots & & & \\ \sigma_{p,1} & \sigma_{p2} & \cdots & \sigma_{pq} \end{bmatrix} \end{array}$$

$$\underset{\sim}{\Sigma}_{22} = \begin{array}{c} \\ q+1 \\ q+2 \\ \vdots \\ p \end{array} \begin{array}{c} \begin{array}{cccc} q+1 & q+2 & \cdots & p \end{array} \\ \begin{bmatrix} \sigma_{q+1,q+1} & \sigma_{q+1,q+2} & \cdots & \sigma_{q+1,p} \\ \sigma_{q+2,q+1} & \sigma_{q+2,q+2} & \cdots & \sigma_{q+2,p} \\ \vdots & \vdots & & \vdots \\ \sigma_{p,q+1} & \sigma_{p,q+2} & \cdots & \sigma_{pp} \end{bmatrix} \end{array}$$

Results:

$$\Rightarrow \underset{\sim}{\Sigma}_{12} = \underset{\sim}{\Sigma}_{21}'$$

and

$$\underset{\sim}{\Sigma} = \begin{array}{c|cc} & q & p-q \\ \hline q & \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ p-q & \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22} \end{array} = \begin{bmatrix} \underset{\sim}{\Sigma}_{11}(q \times q) & \underset{\sim}{\Sigma}_{12}(q \times (p-q)) \\ \underset{\sim}{\Sigma}_{21}((p-q) \times q) & \underset{\sim}{\Sigma}_{22}(p-q) \times (p-q) \end{bmatrix}$$

$$\underset{\sim}{\Sigma} = \begin{bmatrix} Var(\underline{X}_1) & Cov(\underline{X}_1, \underline{X}_2) \\ Cov(\underline{X}_2, \underline{X}_1) & Var(\underline{X}_2) \end{bmatrix}$$

Question:

Consider a random vector $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix}$ with mean vector $\underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix}$

Carry out the covariance matrix of $\underline{x}_1$ and $\underline{x}_2$. Where

$$\underline{x}_1 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \text{ and } \underline{x}_2 = \begin{bmatrix} X_3 \\ X_4 \\ X_5 \end{bmatrix}$$

Solution:

Given that $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix}$ with $\underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_5 \end{bmatrix}$.

We have to find the covariance matrix of $\underline{x}_1$ and $\underline{x}_2$ where

$$\underline{x}_1 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_{2\times1}, \quad \underline{x}_2 = \begin{bmatrix} X_3 \\ X_4 \\ X_5 \end{bmatrix}_{3\times1}$$

Similarly;

$$\underline{\mu}_1 = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \underline{\mu}_2 = \begin{bmatrix} \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix}$$

Covariance Matrix:

$$\underset{\sim}{\Sigma} = E\{(\underline{X}-\underline{\mu})(\underline{X}-\underline{\mu})'\}.$$

$$\underset{\sim}{\Sigma} = E\left\{\begin{bmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 \end{bmatrix}\begin{bmatrix} \underline{X}_1 - \underline{\mu}_1 \\ X_2 - \underline{\mu}_2 \end{bmatrix}'\right\}.$$

$$\underset{\sim}{\Sigma} = E\left\{\begin{bmatrix} \underline{X}_1 - \underline{\mu}_1 \\ X_2 - \mu_2 \end{bmatrix}\begin{bmatrix}(\underline{X}_1 - \mu_1)' & (\underline{X}_2 - \underline{\mu}_2)'\end{bmatrix}\right\}.$$

$$\underset{\sim}{\Sigma} = E\begin{bmatrix} (\underline{X}_1 - \underline{\mu}_1)(\underline{X}_1 - \underline{\mu}_1)' & (\underline{X}_1 - \underline{\mu}_1)(\underline{X}_2 - \underline{\mu}_2)' \\ (\underline{X}_2 - \underline{\mu}_2)(\underline{X}_1 - \underline{\mu}_1)' & (\underline{X}_2 - \underline{\mu}_2)(\underline{X}_2 - \underline{\mu}_2)' \end{bmatrix}.$$

applying expectation.

$$\underset{\sim}{\Sigma} = \begin{bmatrix} E(\underline{X}_1 - \underline{\mu}_1)(\underline{X}_1 - \underline{\mu}_1)' & E(\underline{X}_1 - \underline{\mu}_1)(\underline{X}_2 - \underline{\mu}_2)' \\ E(\underline{X}_2 - \underline{\mu}_2)(\underline{X}_1 - \underline{\mu}_1)' & E(\underline{X}_2 - \underline{\mu}_2)(\underline{X}_2 - \underline{\mu}_2) \end{bmatrix}.$$

$$\underset{\sim}{\Sigma} = \begin{bmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22} \end{bmatrix}.$$

where,

$$\underset{\sim}{\Sigma}_{11} = E\{(\underset{\sim}{X_1} - \underset{\sim}{\mu_1})(\underset{\sim}{X_1} - \underset{\sim}{\mu_1})'\}$$

$$\underset{\sim}{\Sigma}_{11} = E\left\{\begin{bmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 \end{bmatrix}\begin{bmatrix} X_1 - \mu_1 & X_2 - \mu_2 \end{bmatrix}\right\}.$$

$$\underset{\sim}{\Sigma}_{11} = E\begin{bmatrix} (X_1 - \mu_1)^2 & (X_1 - \mu_1)(X_2 - \mu_2) \\ (X_2 - \mu_2)(X_1 - \mu_1) & (X_2 - \mu_2)^2 \end{bmatrix}$$

$$\underset{\sim}{\Sigma}_{11} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}_{2\times 2}$$

$\because \; E(X_1 - \mu_1)^2 = Var(X_1) = \sigma_{11}.$

Similarly;

$$\underset{\sim}{\Sigma}_{21} = \begin{bmatrix} \sigma_{31} & \sigma_{32} \\ \sigma_{41} & \sigma_{42} \\ \sigma_{51} & \sigma_{52} \end{bmatrix}_{3\times 2}$$

$$\underset{\sim}{\Sigma}_{12} = \begin{bmatrix} \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \sigma_{23} & \sigma_{24} & \sigma_{25} \end{bmatrix}_{2\times 3}$$

and

$$\underset{\sim}{\Sigma}_{22} = \begin{bmatrix} \sigma_{33} & \sigma_{34} & \sigma_{35} \\ \sigma_{43} & \sigma_{44} & \sigma_{45} \\ \sigma_{53} & \sigma_{54} & \sigma_{55} \end{bmatrix}_{3\times 3}$$

$\because \; \underset{\sim}{\Sigma}_{21} = \underset{\sim}{\Sigma}_{12}'.$