Linear Compounds & Linear Combination:

Linear Compound:

$$Y_{1\times 1} = \underset{1\times p}{\underline{a}'}\ \underset{p\times 1}{\underline{x}} = a_1X_1 + a_2X_2 + \cdots + a_pX_p$$

is a linear compound.

Mean:

$$E(Y) = E[\underline{a}'\underline{x}] = \underline{a}'E(\underline{x}) = \underline{a}'\underline{\mu}$$

Variance:

$$Var(Y) = Var[\underline{a}'\underline{x}]$$

$$Var(Y) = E\{(\underline{a}'\underline{x} - E(\underline{a}'\underline{x}))^2\}.$$

$$Var(Y) = E[\underline{a}'\underline{x} - \underline{a}'\underline{\mu}]^2$$

$$Var(Y) = E[\underline{a}'(\underline{x} - \underline{\mu})]^2.$$

$$Var(Y) = E\left[\{\underset{\sim}{a}'(\underset{\sim}{x} - \underset{\sim}{\mu})\}\{\underset{\sim}{a}'(\underset{\sim}{x} - \underset{\sim}{\mu})\}\right]$$

$$Var(Y) = E\left\{[\underset{\sim}{a}'(\underset{\sim}{x} - \underset{\sim}{\mu})][\underset{\sim}{a}'(\underset{\sim}{x} - \underset{\sim}{\mu})]'\right\}$$

$\because$ $\underset{\sim}{a}'(\underset{\sim}{x} - \underset{\sim}{\mu})$ is a scaler so it is equivalent to its transpose.

$$Var(Y) = E\left\{[\underset{\sim}{a}'(\underset{\sim}{x} - \underset{\sim}{\mu})(\underset{\sim}{x} - \underset{\sim}{\mu})'\underset{\sim}{a}]\right\}$$

$$= \underset{\sim}{a}' E(\underset{\sim}{x} - \underset{\sim}{\mu})(\underset{\sim}{x} - \underset{\sim}{\mu})'\underset{\sim}{a}$$

$$Var(Y) = \underset{\sim}{a}' \, Var(\underset{\sim}{x}) \, \underset{\sim}{a}$$

$$Var(Y) = \underset{\sim}{a}' \underset{\sim}{\Sigma} \underset{\sim}{a}.$$

Linear Combination:

let $\underset{q \times 1}{\underset{\sim}{z}} = \underset{q \times p}{\underset{\sim}{c}} \; \underset{p \times 1}{\underset{\sim}{x}}$

Mean:

$$E(\underset{\sim}{z}) = E(\underset{\sim}{c}\underset{\sim}{x}) = \underset{\sim}{c} E(\underset{\sim}{x}) = \underset{\sim}{c}\underset{\sim}{\mu}$$

Variance:

$$Var(\underset{\sim}{z}) = E\left\{[\underset{\sim}{z} - E(\underset{\sim}{z})][\underset{\sim}{z} - E(\underset{\sim}{z})]'\right\}$$

$$= E\left\{(\underset{\sim}{c}\underset{\sim}{x} - \underset{\sim}{c}\underset{\sim}{\mu})(\underset{\sim}{c}\underset{\sim}{x} - \underset{\sim}{c}\underset{\sim}{\mu})'\right\}$$

$$= E\{[\underset{\sim}{c}(\underline{x}-\underline{\mu})][\underset{\sim}{c}(\underline{x}-\underline{\mu})]'\}$$

$$= E\{\underset{\sim}{c}(\underline{x}-\underline{\mu})(\underline{x}-\underline{\mu})'\underset{\sim}{c}'\}$$

$$= \underset{\sim}{c}\,E(\underline{x}-\underline{\mu})(\underline{x}-\underline{\mu})'\underset{\sim}{c}'$$

$$= \underset{\sim}{c}\,\mathrm{Var}(\underline{x})\underset{\sim}{c}'$$

$$= \underset{\sim}{c}\,\underset{\sim}{\Sigma}\,\underset{\sim}{c}'$$

Question:

Consider the covariance matrix

$$\underset{\sim}{\Sigma} = \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}$$ of three

random variables. X_1, X_2, X_3.

a) Identify ρ_{23}.

b) Obtain the corelation b/w X_1 and $\frac{X_2+X_3}{2}$

Solution:

$$\because\ \underset{\sim}{\rho} = \underset{\sim}{D}^{-1/2}\,\underset{\sim}{\Sigma}\,\underset{\sim}{D}^{-1/2}$$

where $\underset{\sim}{D} = \text{Diagonal}(\underset{\sim}{\Sigma})$.

$$\underset{\sim}{D} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$D^{-1/2} = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.3333 \end{bmatrix}$$

$$D^{-1/2} \Sigma D^{-1/2} = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.3333 \end{bmatrix} \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix} \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.3333 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.3333 \end{bmatrix} \begin{bmatrix} 5 & -1 & 1.3332 \\ -0.4 & 2 & 0.333 \\ 0.8 & 0.5 & 2.9997 \end{bmatrix}$$

$$\rho = \begin{bmatrix} 1 & -0.2 & 0.26664 \\ -0.2 & 1 & 0.1665 \\ 0.26664 & 0.1665 & 1 \end{bmatrix}$$

$$\Rightarrow \boxed{\rho_{23} = 0.1665}$$

let

$$Z_1 = X_1 = X_1 + 0X_2 + 0X_3$$

$$Z_2 = \frac{X_2 + X_3}{2} = 0X_1 + \frac{1}{2}X_2 + \frac{1}{2}X_3$$

$$\Rightarrow \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} X_1 + 0X_2 + 0X_3 \\ 0X_1 + \frac{1}{2}X_2 + \frac{1}{2}X_3 \end{bmatrix}$$

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$\underset{\sim}{Z} = \underset{\sim}{C}\,\underset{\sim}{x}$$

Variance :

$$V(\underset{\sim}{Z}) = E\left\{ \left(\underset{\sim}{C}\underset{\sim}{x} - \underset{\sim}{C}\underset{\sim}{\mu} \right) \left(\underset{\sim}{C}\underset{\sim}{x} - \underset{\sim}{C}\underset{\sim}{\mu} \right)' \right\}$$

$$V(\underset{\sim}{Z}) = E\left\{ \underset{\sim}{C} \left(\underset{\sim}{x} - \underset{\sim}{\mu} \right) \left(\underset{\sim}{x} - \underset{\sim}{\mu} \right)' \underset{\sim}{C}' \right\}$$

$$\underset{\sim}{\Sigma}_z = V(\underset{\sim}{Z}) = \underset{\sim}{C}\,\underset{\sim}{\Sigma}\,\underset{\sim}{C}'$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \\ 0 & 1/2 \end{bmatrix}$$

$$\underset{\sim}{\Sigma}_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 25 & 1 \\ -2 & 2.5 \\ 4 & 5 \end{bmatrix}$$

$$\underset{\sim}{\Sigma}_z = \begin{bmatrix} 25 & 1 \\ 1 & 3.75 \end{bmatrix}$$

$$\underset{\sim}{\rho}_z = \underset{\sim}{D}^{-1/2}\,\underset{\sim}{\Sigma}_z\,\underset{\sim}{D}^{-1/2}$$

$$\underset{\sim}{\rho}_z = \begin{bmatrix} (25)^{-1/2} & 0 \\ 0 & (3.75)^{-1/2} \end{bmatrix} \begin{bmatrix} 25 & 1 \\ 1 & 3.75 \end{bmatrix} \begin{bmatrix} (25)^{-1/2} & 0 \\ 0 & (3.75)^{-1/2} \end{bmatrix}$$

$$\underset{\sim}{\rho}_z = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.5164 \end{bmatrix} \begin{bmatrix} 5 & 0.5164 \\ 0.2 & 1.9365 \end{bmatrix}.$$

$$\rho_{z} = \begin{bmatrix} 1 & 0.10328 \\ 0.10328 & 1 \end{bmatrix}$$

$$\mathrm{cov}(Z_1, Z_2) = \mathrm{cos}\left(X_1, \frac{X_2+X_3}{2}\right) = 0.10328.$$

Question:

Obtain the mean and covariance matrix of the given linear combination

$X_1 + X_2$, $X_1 - X_2$, $X_1 + X_2 + X_3$

$$\Sigma = \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}, \quad \mu = \begin{bmatrix} 11 \\ 9 \\ 2 \end{bmatrix}.$$

Solution:

let

$$Z_1 = X_1 + X_2 + 0X_3$$
$$Z_2 = X_1 - X_2 + 0X_3$$
$$Z_3 = X_1 + X_2 + X_3$$

$$\Rightarrow \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} X_1 + X_2 + 0X_3 \\ X_1 - X_2 + 0X_3 \\ X_1 + X_2 + X_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$\Rightarrow Z = C X$$

Mean:

$$E(\underset{\sim}{z}) = E(\underset{\sim}{C}\underset{\sim}{x})$$

$$E(\underset{\sim}{z}) = \underset{\sim}{C}\,E(\underset{\sim}{x}) = \underset{\sim}{C}\,\underset{\sim}{\mu}$$

$$E(\underset{\sim}{z}) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 11 \\ 9 \\ 2 \end{bmatrix}$$

$$E(\underset{\sim}{z}) = \begin{bmatrix} 20 \\ 2 \\ 22 \end{bmatrix}$$ Answer.

Variance:

$$V(\underset{\sim}{z}) = \underset{\sim}{C}\,\underset{\sim}{\Sigma}\,\underset{\sim}{C}'$$

$$V(\underset{\sim}{z}) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$V(\underset{\sim}{z}) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 23 & 27 & 27 \\ 2 & -6 & 3 \\ 5 & 3 & 14 \end{bmatrix}$$

$$V(\underset{\sim}{z}) = \begin{bmatrix} 25 & 21 & 30 \\ 21 & 33 & 24 \\ 30 & 24 & 44 \end{bmatrix}$$ Answer.