CHAPTER 25

ELECTRIC CHARGE AND COULOMB'S LAW

We begin here a detailed study of electromagnetism, *which will extend throughout most of the remainder of this text. Electromagnetic forces are responsible for the structure of atoms and for the binding of atoms in molecules and solids. Many properties of materials that we have studied so far are electromagnetic in their nature, such as the elasticity of solids and the surface tension of liquids. The spring force, friction, and the normal force all originate with the electromagnetic force between atoms.*

Among the examples of electromagnetism that we shall study are the force between electric charges, such as occurs between an electron and the nucleus in an atom; the motion of a charged body subject to an external electric force, such as an electron in an oscilloscope beam; the flow of electric charges through circuits and the behavior of circuit elements; the force between permanent magnets and the properties of magnetic materials; and electromagnetic radiation, which ultimately leads to the study of optics, *the nature and propagation of light.*

In this chapter we begin with a discussion of electric charge, some properties of charged bodies, and the fundamental electric force between two charged bodies.

25-1 ELECTROMAGNETISM: A PREVIEW

What do the following have in common?

1. You turn on the light switch in your room. The consumption of fuel at a power plant produces electromagnetic energy by causing a loop of electrically conducting wire to rotate in the vicinity of a magnet. Ultimately some of this energy is transferred to the electrons in the filament of your lightbulb, which can transform the electrical energy into visible light.

2. You enter a command on your computer keyboard. A stream of electrons is formed to transmit your instructions. There are many thousands of possible pathways for the electrons through the computer circuitry, but most are blocked by electronic gates. Electrons can move only through the gates that have been opened by your command so that the stream of electrons reaches its destination and your command is executed.

3. You push the channel select button on the remote control of your TV set. Electromagnetic waves travel from the remote control unit to a receptor on the set, which then tunes the set to accept another electromagnetic wave that originates from a satellite orbiting high above the Earth. The waves from the satellite provide instructions for your set to use electric and magnetic forces to focus and direct a beam of electrons that strikes the surface of the picture tube and produces a visible image.

The common factor in these diverse phenomena is that they all depend on forces that we describe as *electric* or *magnetic* to control and direct the flow of energy or particles. These forces form the basis of our study of *electromagnetism.* We will find in our study that all electromagnetic effects can be explained by a set of four basic equations, called *Maxwell's equations.* These equations represent individual laws of electromagnetism, just as we have previously discussed equations that represent Newtons' laws of mechanics or the laws of thermodynamics.

Our study will first consider electric phenomena and then magnetic phenomena. Later we will show that the two cannot be separated; certain electric phenomena produce magnetic effects, and certain magnetic phenomena produce electric effects. This leads us to unify electric and magnetic phenomena under the common name of electromagnetism. The development of the laws of electromagnetism and their unification was a great triumph of 19th-century physics. Their application has led directly to a great range of devices of practical use, such as motors, radios and televisions, radar, microwave ovens, and cellular phones.

The development of electromagnetic theory continued in the 20th century with three very significant advancements. In 1905, Albert Einstein showed that, to a moving observer, electric effects could appear as magnetic effects, and thus observers in relative motion could disagree in assigning their measurements to electric or magnetic causes. This conclusion formed the basis of the special theory of relativity, which ultimately was to revolutionize our concepts of space and time. The second development was the introduction of a quantum theory of electromagnetism, called *quantum electrodynamics*, which reached its fruition around 1949 and enabled properties of the atom to be calculated with incredible precision, currently about 11 significant figures. The third development of the 20th century was the unification of electromagnetism with another force, called the "weak" force, which is responsible for certain radioactive decay processes and other interactions between particles. Just as electric and magnetic effects were unified into the electromagnetic interaction, so electromagnetic and weak effects were shown in the 1960s to be unified under the *electroweak* interaction. For our study of electric and magnetic forces, however, the electroweak interaction does not yield anything new, and it is more convenient to consider the separate electromagnetic interaction.

Figure 25-1 is a time line of some of the major events in the development of our understanding of electromagnetism.

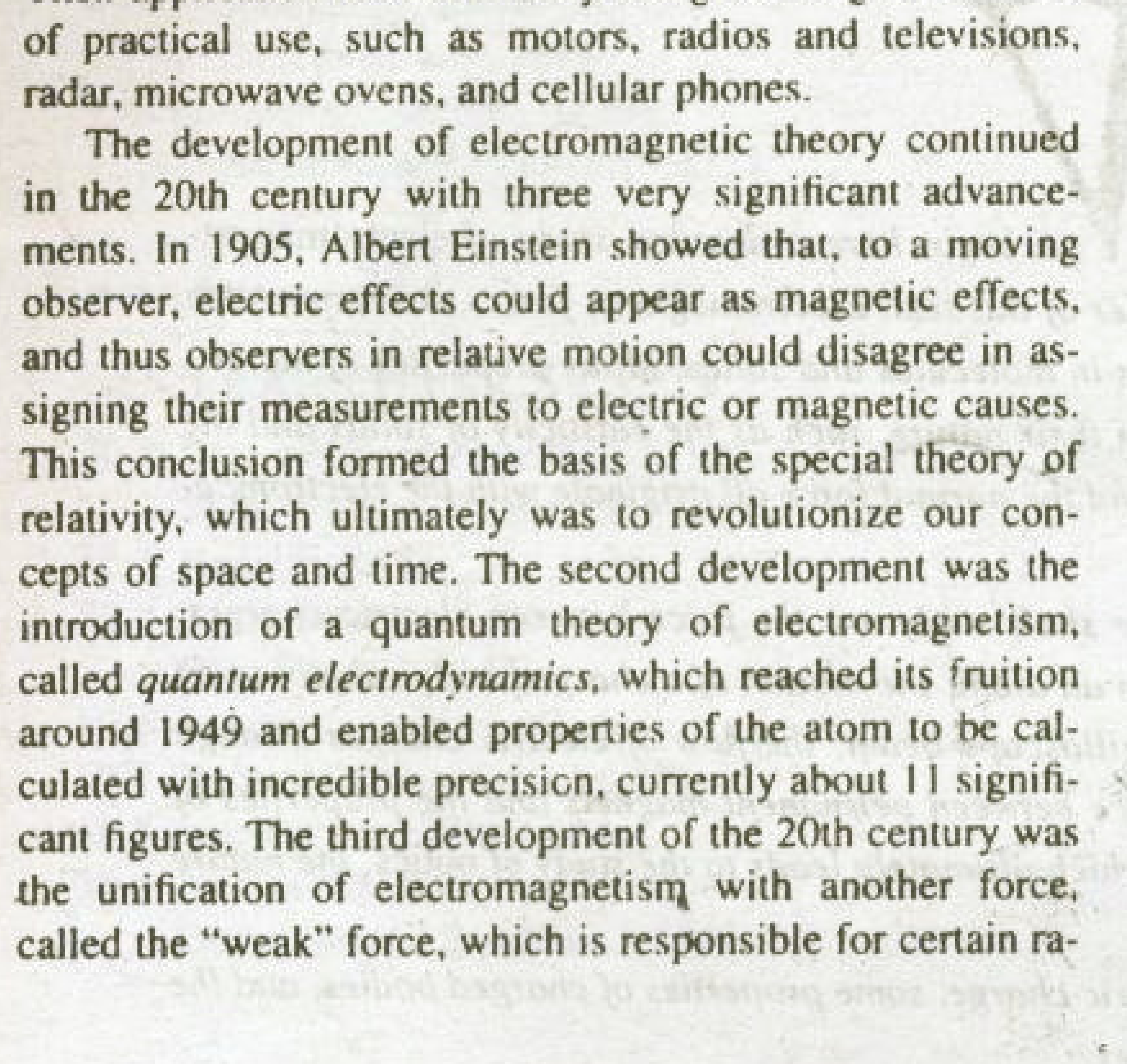

FIGURE 25-1. Time line of major developments in electromagnetism.

25-2 ELECTRIC CHARGE

After you pass a plastic comb through your hair a few times, you will find that the comb can exert a force on individual strands of your hair. You may also observe that, once the strands of hair are attracted to the comb and come into contact with it, they may no longer be attracted to it.

It seems reasonable to conclude that the attraction between the comb and the hair is a result of some physical entity being transferred from one to the other when they rub together, with the same physical entity being transferred back again to neutralize the attraction when they come into contact. This physical entity is called *electric charge*, and today we understand this transfer on the basis of electrons that can be removed from the atoms of one object and attached to the atoms of the other object.

The transfer of electric charge by means of friction is a commonly observed phenomenon. It was known to the ancient Greeks, who observed that pieces of amber rubbed with fur could attract bits of straw. When you walk across a carpet and are shocked by touching a metal door knob, or when a lightning flash stretches between a cloud and the ground, you are observing the effects of this transfer of charge.

When we "charge" an object (that is, when we transfer charge to it), we find that it can exert a force on another charged object. Early observations that this force can be either attractive or repulsive led to the conclusion that there are two kinds of electrical charge, which are called positive and negative.*

Although effects resulting from the transfer of charge can be powerful, it is remarkable that they originate from the transfer of only a tiny fraction of the electric charge that is contained in objects. Ordinary matter is made of electrically neutral atoms or molecules that contain equal amounts of positive charge (the nucleus) and negative charge (the electrons). When two objects rub together, relatively few electrons from the atoms of one object are

* The positive and negative labels for electric charge were chosen arbitrarily by Benjamin Franklin (1706–1790) who, among his other accomplishments, was a scientist of international reputation. In fact, his scientific reputation may have enabled his diplomatic triumphs in France during the American War of Independence.

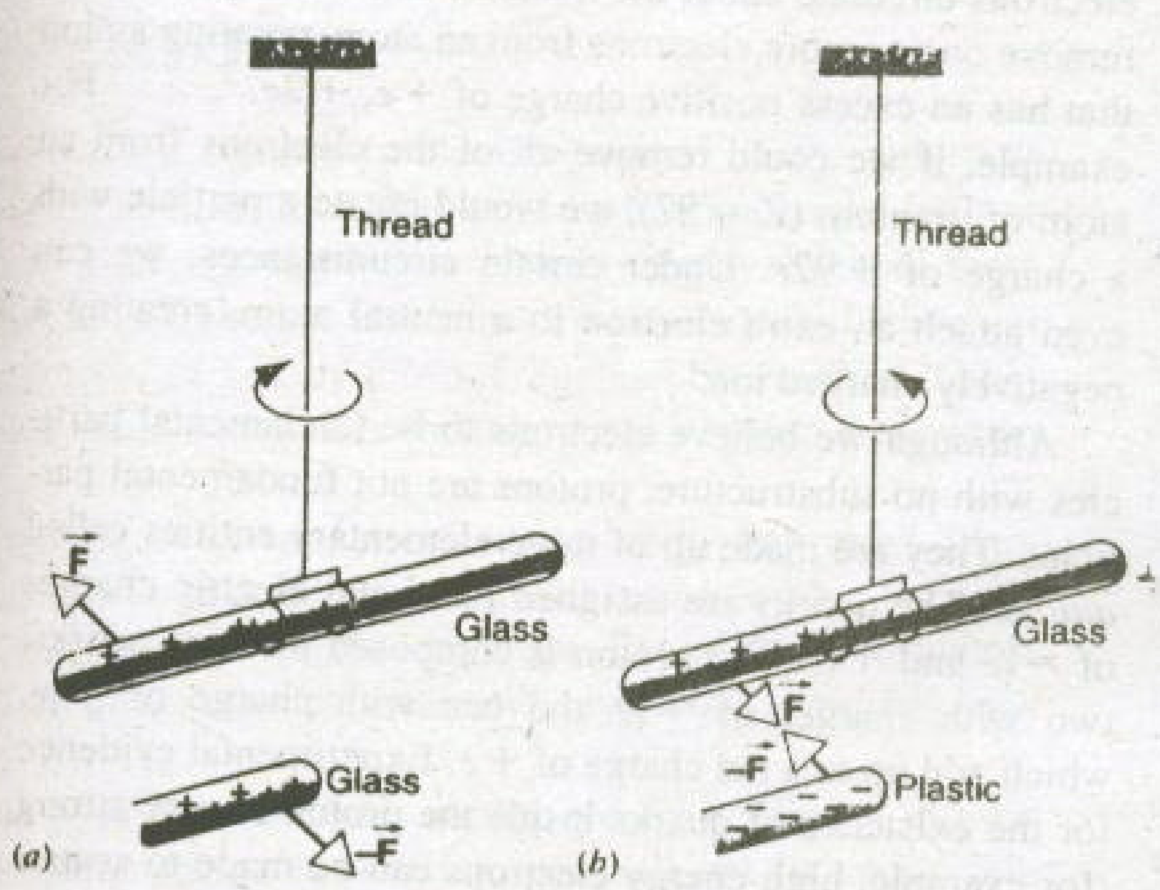

FIGURE 25-2. (*a*) Two similarly charged rods repel each other. (*b*) Two oppositely charged rods attract each other.

transferred to the other; most of the electrons remain undisturbed. It is this slight upset of the balance between the enormous but equal amounts of positive and negative charge in an object that is responsible for most commonly observed electrical effects.

When we rub a plastic rod with fur, electrons are transferred to the rod; because it has an excess of electrons (which carry a negative charge), the rod becomes negatively charged. The fur now has a deficiency of electrons and so it is positively charged. We can see the attraction of the rod for individual strands of the fur, which results from the charge on each. In a similar way, we can rub a glass rod with silk and observe that both become charged and can attract one another. In each case, we have transferred a relatively small number of electrons and upset the electrical neutrality of these objects.

Let us charge a glass rod by rubbing one end of it with silk and then suspend it from a thread, as in Fig. 25-2. If we place a similarly charged glass rod nearby, we find that the two rods repel one another, as in Fig. 25-2*a*. However, if we place a charged plastic rod (charged by rubbing with fur) nearby, the two rods attract one another, as in Fig. 25-2*b*.

We account for the existence of these two kinds of forces in terms of two kinds of charge. When plastic is rubbed with fur, electrons are transferred to the plastic and it becomes negatively charged. When glass is rubbed with silk, electrons are transferred to the silk, leaving the glass with a deficiency of electrons and therefore a net positive charge. The forces observed in Fig. 25-2 can be summarized by the following rule:

Charges of the same sign repel one another, and charges of the opposite sign attract one another.

In Section 25-4, we put this rule into quantitative form, as Coulomb's law of force. We consider only charges that are either at rest with respect to each other or moving very slowly, a restriction that defines the subject of *electrostatics*.

FIGURE 25-3. A carrier bead from a Xerox photocopier, covered with toner particles that stick to it by electrostatic attraction. The diameter of the bead is about 0.3 mm.

Electrical forces between charged bodies have many industrial applications, including electrostatic paint spraying and powder coating, fly-ash precipitation, nonimpact ink-jet printing, and photocopying. Figure 25-3 for example, shows a tiny carrier bead in a photocopying machine, covered with particles of black powder called *toner*, that stick to the carrier bead by electrostatic forces. These negatively charged toner particles are eventually attracted from their carrier beads to a positively charged latent image of the document to be copied, which is formed on a rotating drum. A charged sheet of paper then attracts the toner particles from the drum to itself, after which they are heat-fused in place to make the final copy.

The net electric charge of an object is usually represented by the symbol q. The charge is a scalar quantity. It can be positive or negative, depending on whether the object has a net positive or negative charge. Electric charge is measured in units of coulombs (C). The coulomb is a very large unit of charge; it takes about 6×10^{18} electrons to make one coulomb of charge.

The coulomb cannot be derived from previously defined units. Because electric charge is a new quantity, we are free to define its basic unit in any convenient way. One possible way would be in terms of the force exerted between two standard charges at a given separation, such as the quantity of charge that exerts a force of one newton on a similar charge a distance of one meter away. However, the force between static charges is difficult to measure, and so in practice it is more useful to define the coulomb in terms of the magnetic force between current-carrying wires (which is discussed in Chapter 33). This force can be measured more precisely than the electric force between static charges. It is therefore more convenient to define an SI base unit in terms of current (rate of flow of electric charge per unit time). The coulomb as a unit for electric charge is then a derived unit, obtained from the fundamental units of current and time (see Appendix A).

Electric Charge Is Quantized

When we transfer electric charge from one object to another, the transfer cannot be done in arbitrarily small units. That is, the flow of charge as a current is not a continuous flow, but is made up of discrete elements.* Experiments show that the electric charge always exists only in quantities that are integer multiples of a certain elementary quantity of charge e. That is,

$$q = ne \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots, \qquad (25\text{-}1)$$

where (to four significant figures)

$$e = 1.602 \times 10^{-19}\ \text{C}.$$

The *elementary charge e* is one of the fundamental constants of nature whose experimental value has been determined to an uncertainty of about 4 parts in 10^8.

The electron and the proton are examples of commonly occurring particles that each carry one fundamental unit of charge. The electron has a charge of $-e$ and the proton has a charge of $+e$. Some particles, such as the neutron, carry no net electric charge. Other elementary particles are known that carry charges that are small multiples of e, usually ± 1, ± 2, or ± 3. Each particle has a corresponding *antiparticle,* which has the same mass but the opposite electric charge; the antielectron, which is known as the *positron,* has a charge of $+e$. Antiparticles do not commonly exist in nature, but can be created in decays and reactions of nuclei and elementary particles.

Equation 25-1 tells us that it is possible to have a net charge on an object of $+10e$ or $-6e$ but never $3.57e$. When the values of a property are restricted to discrete multiples of a basic quantity, we say that the property is *quantized.*

Because the elementary charge is small, under ordinary circumstances we are not aware of the discrete nature of the flow of charge. For example, in a wire of an electronic circuit in which small currents of one milliampere are typical, 6×10^{15} electrons pass through any cross section of the wire every second!

Ordinary atoms are electrically neutral, which means that they contain equal quantities of positive and negative charge. The nucleus of the atom contains Z protons (where Z is called the *atomic number* of the atom) and thus a charge of $+Ze$. In a neutral atom, Z negatively charged electrons circulate about the nucleus. It is often possible to remove one or more electrons from an atom, creating an ion that has an excess positive charge of $+e$, $+2e$, ... For example, if we could remove all of the electrons from an atom of uranium ($Z = 92$), we would create a particle with a charge of $+92e$. Under certain circumstances, we can even attach an extra electron to a neutral atom, creating a negatively charged ion.

Although we believe electrons to be fundamental particles with no substructure, protons are not fundamental particles. They are made up of more elementary entities called *quarks.* The quarks are assigned fractional electric charges of $-\frac{1}{3}e$ and $+\frac{2}{3}e$. The proton is composed of three quarks, two with charges of $+\frac{2}{3}e$ and one with charge of $-\frac{1}{3}e$, which add up to a net charge of $+e$. Experimental evidence for the existence of quarks inside the proton is very strong (for example, high-energy electrons can be made to scatter from the fractionally charged quarks inside the proton), but, no matter how violently the protons are made to collide, no free quark has been released. As a result, no free particle with a fractional charge has ever been observed. This fact can be understood if the attractive force that one quark exerts on another *increases* with their separation. This is in contrast to the electromagnetic and gravitational forces, both of which *decrease* as the distance between a pair of interacting bodies increases.

SAMPLE PROBLEM 25-1. A penny, being electrically neutral, contains equal amounts of positive and negative charge. What is the magnitude of these equal charges?

Solution The charge q is given by NZe, in which N is the number of atoms in a penny and Ze is the magnitude of the positive and the negative charges carried by each atom.

The number N of atoms in a penny, assumed for simplicity to be made of copper, is $N_A m/M$, in which N_A is the Avogadro constant. The mass m of the coin is 3.11 g, and the mass M of 1 mol of copper (called its *molar mass*) is 63.5 g. We find

$$N = \frac{N_A m}{M} = \frac{(6.02 \times 10^{23}\ \text{atoms/mol})(3.11\ \text{g})}{63.5\ \text{g/mol}}$$
$$= 2.95 \times 10^{22}\ \text{atoms}.$$

Every neutral atom has a negative charge of magnitude Ze associated with its electrons and a positive charge of the same magnitude associated with its nucleus. Here e is the elementary charge, 1.60×10^{-19} C, and Z is the atomic number of the element in question. For copper, Z is 29. The magnitude of the total negative or positive charge in a penny is then

$$q = NZe = (2.95 \times 10^{22})(29)(1.60 \times 10^{-19}\ \text{C})$$
$$= 1.37 \times 10^5\ \text{C}.$$

This is an enormous charge. By comparison, the charge that you might get by rubbing a plastic rod is perhaps 10^{-9} C, smaller by a factor of about 10^{14}. For another comparison, it would take 1–2 days for a charge of 1.37×10^5 C to flow through the filament of a typical lightbulb. There is a lot of electric charge in ordinary matter.

* In Franklin's day, electric charge was thought to be a substance and to flow as a continuous fluid. Today we know that fluids are made up of individual atoms and molecules—matter is discrete. Similarly, the "electric fluid" is not continuous but discrete.

25-3 CONDUCTORS AND INSULATORS

Materials are commonly classified based on the ability of electrons to flow through them. In some materials, such as metals, electrons can move relatively freely. We call these materials *conductors.* Electrons deposited at one location in the material can easily move throughout the material. Other examples of conductors include tap water and the human body.

In other materials, electrons can hardly flow at all. Electrons deposited at one location will remain at that location. We call these materials *insulators.* Examples of insulators include glass, plastics, and many crystalline materials such as NaCl.

If you try to charge a copper rod by holding it in your hand and rubbing it with fur, you will not be successful. Electrons may be transferred from the rod to the fur as a result of friction, but additional electrons will easily flow from your body through your hand into the rod to replace those that were removed. As a result, no net charge builds up on the rod due to rubbing. We can consider the Earth to possess an infinite supply of electrons, some of which can flow into your body to replace those that have been lost to the rod. When we have a pathway through which electrons can flow between an object and the Earth, the object is said to be electrically *grounded.*

If instead we attach a plastic handle to the copper rod, we find that we can build up a charge by rubbing it. The insulating handle blocks the flow of electrons between the rod and your body.

Isolated atoms of a conducting material such as copper generally contain loosely bound electrons that can easily be detached, leaving a positively charged ion. When copper atoms come together to form solid copper, these loosely bound electrons do not remain attached to individual atoms but become free to wander through the material. These free electrons are called *conduction electrons;* in copper, a typical conductor, there are about 10^{23} conduction electrons per cm^3. The positively charged ions are not free to move and remain fixed within the solid lattice structure of the copper.

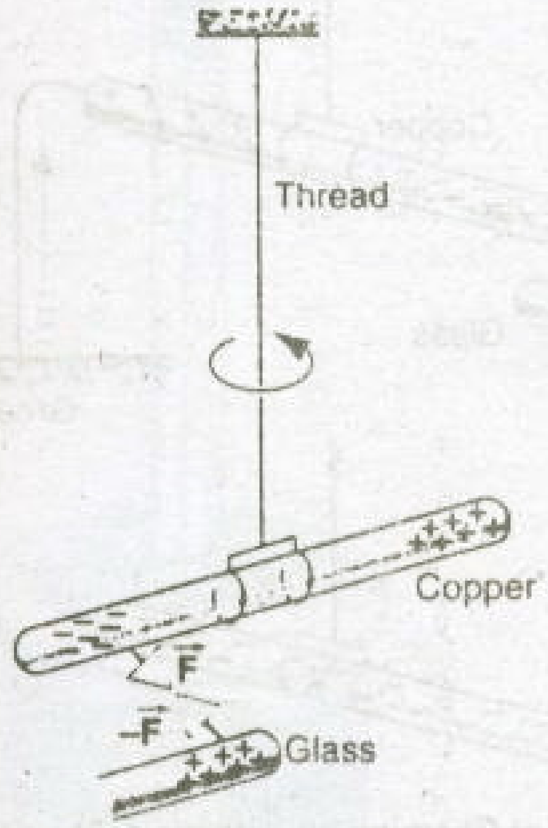

FIGURE 25-4. Either end of an isolated uncharged copper rod is attracted by a charged rod of either sign. In this case, conduction electrons in the copper rod are attracted to the near end of the copper rod, leaving the far end with a net positive charge.

The experiment of Fig. 25-4 demonstrates the mobility of charge in a conductor. An uncharged copper rod is suspended by an insulating thread. When a positively charged glass rod is brought near one end of the copper, the mobile conduction electrons in the copper are attracted by the positive charges on the glass. The flow of electrons to the near end of the copper leaves the far end with a deficiency of electrons and a net positive charge. The negatively charged end of the copper rod and the positively charged glass exert attractive forces on one another. Note that this situation is very different from that of Fig. 25-2; in Fig. 25-4, the glass attracts a copper rod that carries no net charge. (As we will discuss in the next section, the electrical force depends inversely on the separation between the charges; therefore the attractive force between the glass and the negative end of the copper rod is much stronger than the repulsive force between the glass and the positive end of the copper rod.)

If instead of a positively charged glass rod in Fig. 25-4 we used a negatively charged plastic rod, the effect would be the same: an attractive force between the plastic and the copper. In this case the negatively charged plastic would repel the conduction electrons of the copper, leaving the near end of the copper with a positive charge. There would be an attractive force between the negatively charged plastic and the positively charged end of the copper rod.

It is also possible to have an attractive force between a charged body and an uncharged insulator. Figure 25-5*a* shows a charged comb attracting uncharged bits of paper. The explanation for this attraction is different from that of the attraction between the glass rod and the copper rod. In this case, the paper is an insulator, and it is not possible for electrons to collect on one end of the paper (as was the case for the conductor in Fig. 25-4). Instead, electrons in individual molecules in the bits of paper are repelled by the negatively charged comb, and so the electrons are preferentially located on the side of each molecule that is away from the comb. In each molecule, the positive end (the end with a lack of electrons) is closest to the comb and feels a greater force of attraction to the comb. This is responsible for the net attractive force between the comb and the paper (Fig. 25-5*b*). The same attractive force would occur if the comb were positively charged.

The separation of positive and negative charge in an isolated object under the influence of a nearby charged object is known as *polarization.* Polarization can occur at the macroscopic level, such as in the copper rod of Fig. 25-4, or at the molecular level, as in Fig. 25-5.

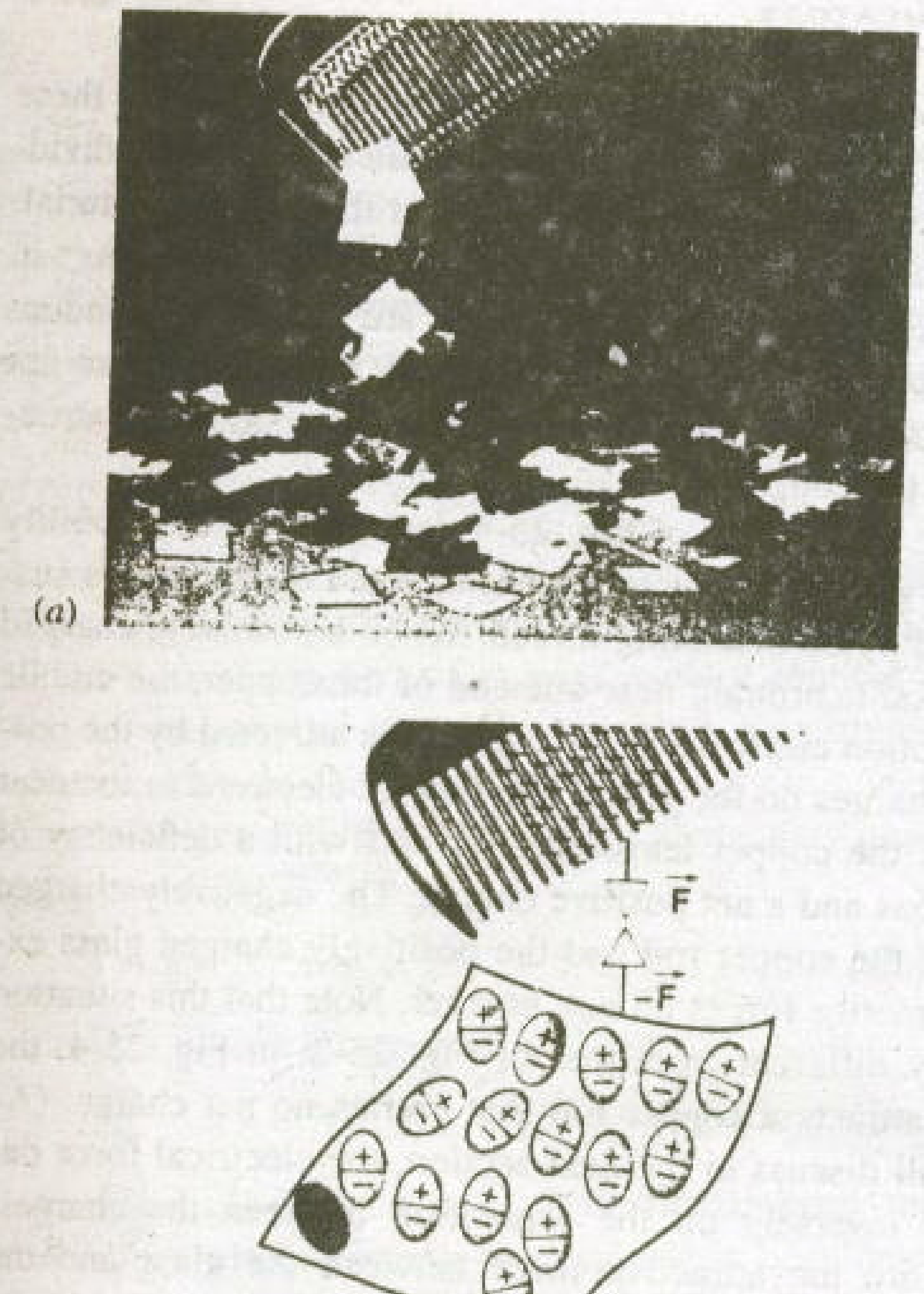

FIGURE 25-5. (*a*) A charged comb attracts uncharged bits of paper. (*b*) The negatively charged comb polarizes the charges in the molecules, resulting in an attractive force between the comb and the paper.

Charging by Contact and by Induction

Suppose now we touch a positively charged glass rod to an uncharged copper rod (as in Fig. 25-6). Electrons will flow from the copper to neutralize the positive charges on the glass. However, because electrons do not flow through the glass, they can neutralize only those positive charges at the point of contact with the copper. To remove additional electrons from the copper, we can "wipe" the glass rod along the copper rod, thereby transferring electrons to fresh, unneutralized areas of the glass that come into contact with the copper (Fig. 25-6*a*). If we then remove the glass rod, the copper remains with a deficiency of electrons and therefore a net positive charge. Electrons will flow through the copper so that the positive charges (the ion cores) are evenly distributed along the surface of the copper. This direct transfer of charge from one object to another is called *charging by contact*. Even though negative electrons were actually transferred, it is often convenient to regard the experiment shown in Fig. 25-6 as if positive charges were transferred from the glass rod to the copper.

Let us return to the situation of Fig. 25-4. If we attach a wire between the positive end of the copper and ground (as

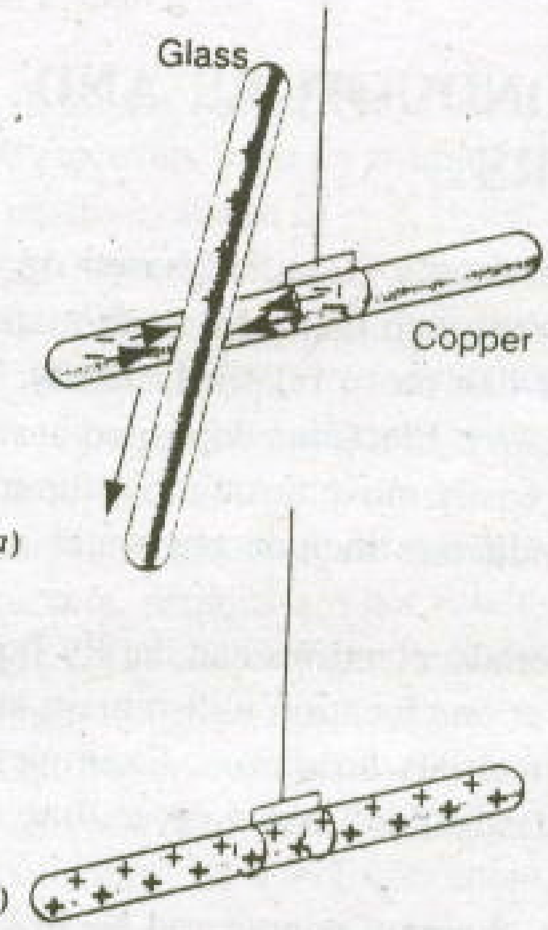

FIGURE 25-6. (*a*) Charging by contact. Electrons flow from the copper to neutralize positive charges at the point of contact with the glass. (*b*) The resulting charge on the copper when the glass is removed.

in Fig. 25-7*a*), electrons will flow from ground to neutralize the positive charges in the copper. Leaving the glass rod in place, if we then remove the connection to ground, the copper rod retains a net negative charge. If we then remove the glass rod the negative charges will distribute themselves over the surface of the copper (Fig. 25-7*b*) to be as far apart from one another as possible. This method of charging an object is called *charging by induction*. Note that we have been able to use the positively charged glass rod to transfer either positive charge to the copper by contact or negative charge (from ground) by induction.

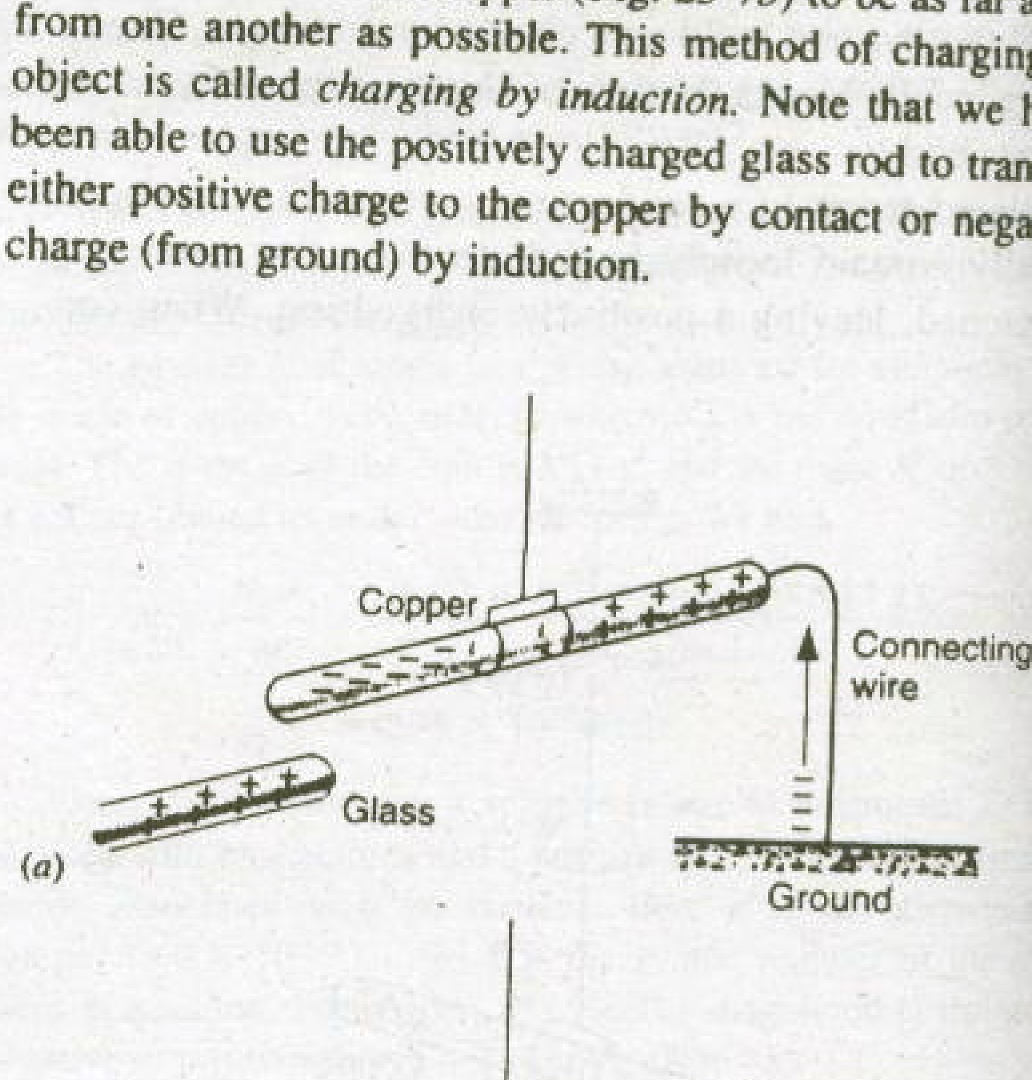

FIGURE 25-7. (*a*) Charging by induction. Electrons flow from ground to neutralize the positive charges on the far end of the copper rod. (*b*) The resulting charge on the copper when the glass is removed.

25-4 COULOMB'S LAW

So far in this chapter we have established that there are two kinds of electric charge and that charges exert forces on each other. It is now our goal to understand the nature of this force.

The first successful quantitative experiments to study the force between electric charges were done by Charles Augustin Coulomb (1736–1806), who measured electrical attractions and repulsions and deduced the law that governs them. In principle, Coulomb's apparatus is similar to Fig. 25-2, except that he used small charged spheres indicated by *a* and *b* in Fig. 25-8.

If *a* and *b* are charged, the electric force on *a* tends to twist the suspension fiber. Coulomb cancelled out this twisting effect by turning the suspension head through the angle θ needed to keep the two charges at a particular separation. The angle θ is then a relative measure of the electric force acting on charge *a*. The device of Fig. 25-8 is a *torsion balance;* a similar arrangement was used later by Cavendish to measure gravitational attractions (Section 14-3).

Experiments by Coulomb and his contemporaries showed that the electrical force exerted by one charged body on another depends directly on the product of the magnitudes of the two charges and inversely on the square of their separation. That is,

$$F \propto \frac{|q_1||q_2|}{r^2}.$$

Here F is the magnitude of the mutual force that acts on each of the two charges q_1 and q_2, and r is the distance between their centers. The force on each charge due to the other acts along the line connecting the charges. As required by Newton's third law, the force exerted by q_1 on q_2 is equal in magnitude but opposite in direction to the force exerted by q_2 on q_1, even though the magnitudes of the charges may be different.

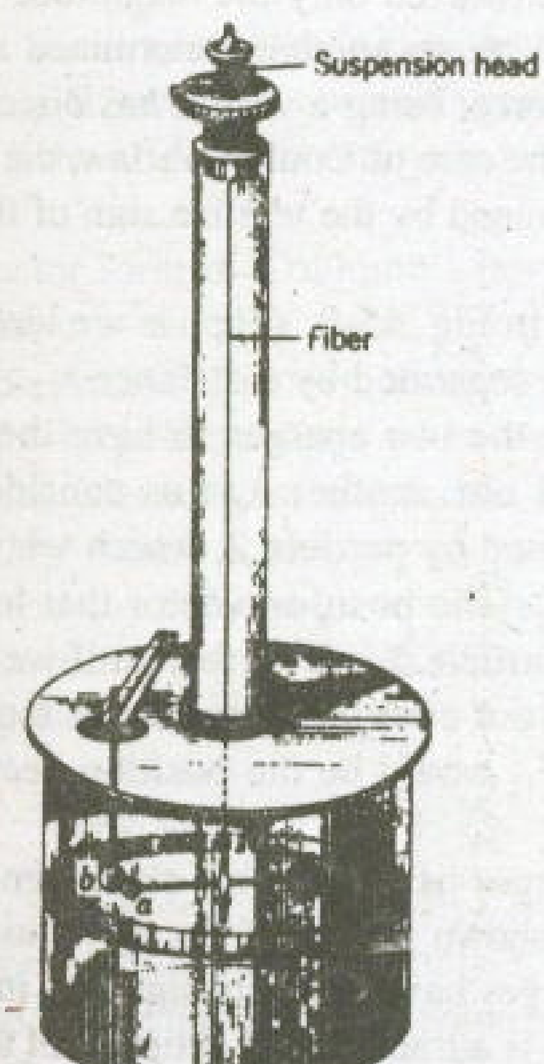

FIGURE 25-8. Coulomb's torsion balance, from his 1785 memoir to the Paris Academy of Sciences.

To turn the above proportionality into an equation, we introduce a constant of proportionality K, which we can call the Coulomb constant. We thus obtain, for the force between the charges,

$$F = K\frac{|q_1||q_2|}{r^2}. \tag{25-2}$$

Equation 25-2, which is called *Coulomb's law,* generally holds only for charged objects whose sizes are much smaller than the distance between them. We often say that it holds only for *point charges.*

Our belief in Coulomb's law does not rest quantitatively on Coulomb's experiments. Such measurements could not, for example, convince us that the exponent of r in Eq. 25-2 is exactly 2 and not, say 2.0001. In Section 27-7 we show that Coulomb's law can also be deduced from an indirect experiment, which shows that, if the exponent in Eq. 25-2 is not exactly 2, it differs from 2 by at most 1×10^{-16}.

Coulomb's law resembles Newton's inverse square law of gravitation, $F = Gm_1m_2/r^2$, which was already 100 years old at the time of Coulomb's experiments. Both are inverse square laws, and the charge q plays the same role in Coulomb's law that the mass m plays in Newton's law of gravitation. One difference between the two laws is that gravitational forces are always attractive, whereas electrostatic forces can be repulsive or attractive, depending on whether the two charges have the same or opposite signs.

In the SI system, the constant K is expressed in the following form:

$$K = \frac{1}{4\pi\epsilon_0}. \tag{25-3}$$

Although the choice of this form for the constant K appears to make Coulomb's law needlessly complex, it ultimately results in a simplification of formulas of electromagnetism that are used more often than Coulomb's law.

The constant ϵ_0, which is called the *electric constant* (also known as the *permittivity*), has a value that is determined by the value of the speed of light, as we discuss in Chapter 39. Its exact value is

$$\epsilon_0 = 8.85418781762 \times 10^{-12}\ \mathrm{C^2/N\cdot m^2}.$$

The Coulomb constant K has the corresponding value (to three significant figures)

$$K = \frac{1}{4\pi\epsilon_0} = 8.99 \times 10^9\ \mathrm{N\cdot m^2/C^2}.$$

With this choice of the constant K, Coulomb's law can be written

$$F = \frac{1}{4\pi\epsilon_0}\frac{|q_1||q_2|}{r^2}. \tag{25-4}$$

When K has the above value, expressing q in coulombs and r in meters gives the force in newtons.

The significance of Coulomb's law goes far beyond the description of the forces exerted by charged spheres on each other. This law, when incorporated into the structure of quantum physics, correctly describes (1) the electrical forces that bind the electrons of an atom to its nucleus, (2) the forces that bind atoms together to form molecules, and (3) the forces that bind atoms and molecules together to form solids or liquids. Thus most of the forces of our daily experience that are not gravitational in nature are electrical.

SAMPLE PROBLEM 25-2. In Sample Problem 25-1 we saw that a copper penny contains both positive and negative charges, each of a magnitude 1.37×10^5 C. Suppose that these charges could be concentrated into two separate bundles, held 100 m apart. What attractive force would act on each bundle?

Solution From Eq. 25-4 we have

$$F = \frac{1}{4\pi\epsilon_0}\frac{|q|^2}{r^2} = \frac{(8.99 \times 10^9\ \mathrm{N\cdot m^2/C^2})(1.37 \times 10^5\ \mathrm{C})^2}{(100\ \mathrm{m})^2}$$

$$= 1.69 \times 10^{16}\ \mathrm{N}.$$

This is about 2×10^{12} tons of force! Even if the charges were separated by one Earth diameter, the attractive force would still be about 120 tons. In all of this, we have sidestepped the problem of forming each of the separated charges into a "bundle" whose dimensions are small compared to their separation. Such bundles, if they could ever be formed, would be blasted apart by mutual Coulomb repulsion forces.

The lesson of this sample problem is that you cannot disturb the electrical neutrality of ordinary matter very much. If you try to pull out any sizable fraction of the charge contained in a body, a large Coulomb force appears automatically, tending to pull it back.

SAMPLE PROBLEM 25-3. The average distance r between the electron and the proton in the hydrogen atom is 5.3×10^{-11} m. (*a*) What is the magnitude of the average electrostatic force that acts between these two particles? (*b*) What is the magnitude of the average gravitational force that acts between these particles?

Solution (*a*) From Eq. 25-4 we have, for the electrostatic force,

$$F_e = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} = \frac{(8.99 \times 10^9\ \mathrm{N\cdot m^2/C^2})(1.60 \times 10^{-19}\ \mathrm{C})^2}{(5.3 \times 10^{-11}\ \mathrm{m})^2}$$

$$= 8.2 \times 10^{-8}\ \mathrm{N}.$$

Although this force may seem small (it is about equal to the weight of a speck of dust), it produces an enormous acceleration of the electron within the atom, about 10^{23} m/s².
(*b*) For the gravitational force, we have

$$F_g = G\frac{m_e m_p}{r^2}$$

$$= \frac{(6.67 \times 10^{-11}\ \mathrm{N\cdot m^2/kg^2})(9.11 \times 10^{-31}\ \mathrm{kg})(1.67 \times 10^{-27}\ \mathrm{kg})}{(5.3 \times 10^{-11}\ \mathrm{m})^2}$$

$$= 3.6 \times 10^{-47}\ \mathrm{N}.$$

We see that the gravitational force is weaker than the electrostatic force by a factor of about 10^{39}. Although the gravitational force is weak, it is always attractive. Thus it can act to build up very large masses, as in the formation of stars and galaxies, so that large gravitational forces can develop. The electrostatic force, on the other hand, is repulsive for charges of the same sign, so that it is not possible to accumulate large concentrations of either positive or negative charge. We must always have the two types of charge together, so that they largely compensate for each other. The charges that we are accustomed to in our daily experiences are slight disturbances of this overriding balance.

SAMPLE PROBLEM 25-4. The nucleus of an iron atom has a radius of about 4×10^{-15} m and contains 26 protons. What repulsive electrostatic force acts between two protons in such a nucleus if they are separated by a distance of one radius?

Solution From Eq. 25-4 we have

$$F = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} = \frac{(8.99 \times 10^9\ \mathrm{N\cdot m^2/C^2})(1.60 \times 10^{-19}\ \mathrm{C})^2}{(4 \times 10^{-15}\ \mathrm{m})^2}$$

$$= 14\ \mathrm{N}.$$

The large repulsive electrostatic force, more than 3 lb and acting on a single proton, must be balanced by the attractive nuclear force that binds the nucleus together. This force, whose range is so short that its effects cannot be felt very far outside the nucleus, is known as the "strong nuclear force" and is very well named.

Coulomb's Law: Vector Form

So far we have considered only the magnitude of the force exerted by one charge on another, determined according to Coulomb's law. Force, being a vector, has directional properties as well. In the case of Coulomb's law, the direction of the force is determined by the relative sign of the two electric charges.

As illustrated in Fig. 25-9, suppose we have two point charges q_1 and q_2 separated by a distance r_{12}. For the moment, we assume the two charges to have the same sign, so that they repel one another. Let us consider the force *on* particle 1 exerted *by* particle 2, which we write in our usual form as $\vec{\mathbf{F}}_{12}$. The position vector that locates particle 1 relative to particle 2 is $\vec{\mathbf{r}}_{12}$; that is, if we were to define the origin of our coordinate system at the location of particle 2, then $\vec{\mathbf{r}}_{12}$ would be the position vector of particle 1.

If the two charges have the same sign, then the force is repulsive and, as shown in Fig. 25-9*a*, $\vec{\mathbf{F}}_{12}$ must be parallel to $\vec{\mathbf{r}}_{12}$. If the charges have opposite signs, as in Fig. 25-9*b*, then the force $\vec{\mathbf{F}}_{12}$ is attractive and antiparallel to $\vec{\mathbf{r}}_{12}$. In either case, we can represent the force as

$$\vec{\mathbf{F}}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{\mathbf{r}}_{12}. \qquad (25\text{-}5)$$

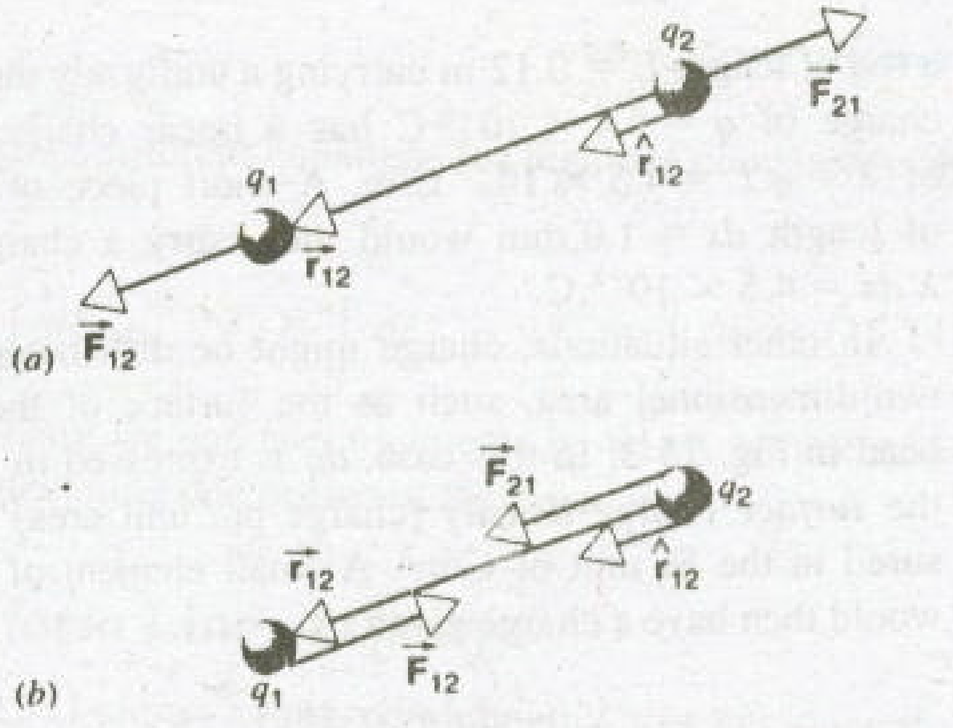

FIGURE 25-9. (*a*) Two point charges q_1 and q_2 of the same sign exert equal and opposite repulsive forces on one another. The vector $\vec{r}_{12}$ locates q_1 relative to q_2, and the unit vector $\hat{r}_{12}$ points in the direction of $\vec{r}_{12}$. Note that $\vec{F}_{12}$ is parallel to $\vec{r}_{12}$. (*b*) The two charges now have opposite signs, and the force is attractive. Note that $\vec{F}_{12}$ is antiparallel to $\vec{r}_{12}$.

Here r_{12} represents the magnitude of the vector $\vec{r}_{12}$, and $\hat{r}_{12}$ indicates the *unit vector* in the direction of $\vec{r}_{12}$. That is,

$$\hat{r}_{12} = \frac{\vec{r}_{12}}{r_{12}}. \tag{25-6}$$

We used a form similar to Eq. 25-5 to express the gravitational force (see Eqs. 14-2 and 14-3).

One other feature is apparent from Fig. 25-9. According to Newton's third law, the force exerted *on* particle 2 *by* particle 1, $\vec{F}_{21}$, is opposite to $\vec{F}_{12}$. This force can then be expressed in eactly the same form:

$$\vec{F}_{21} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{21}^2}\hat{r}_{21}. \tag{25-7}$$

Here $\hat{r}_{21}$ is a unit vector that points from particle 1 to particle 2; that is, it would be the unit vector in the direction of particle 2 if the origin of coordinates were at the location of particle 1.

The vector form of Coulomb's law is useful because it carries within it the directional information about $\vec{F}$ and whether the force is attractive or repulsive. Using the vector form is of critical importance when we consider the forces acting on an assembly of more than two charges. In this case, Eq. 25-5 would hold for every pair of charges, and the total force on any one charge would be found by taking the *vector* sum of the forces due to each of the other charges. For example, the force on particle 1 in an assembly would be

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \cdots, \tag{25-8}$$

where $\vec{F}_{12}$ is the force on particle 1 from particle 2, $\vec{F}_{13}$ is the force on particle 1 from particle 3, and so on. Equation 25-8 is the mathematical representation of the *principle of superposition* applied to electric forces. It asserts that the force acting on one charge due to another is independent of whether or not other charges are present, and therefore we can calculate the force separately for each pair of charges and then take their vector sum to find the net force on any charge. For instance, the force $\vec{F}_{13}$ that particle 3 exerts on particle 1 is completely unaffected by the presence of particle 2. The principle of superposition is not at all obvious and may fail in the case of very strong electric forces. Only through experiment can its applicability be verified. For all situations we meet in this text, however, the principle of superposition is valid.

SAMPLE PROBLEM 25-5. Figure 25-10 shows three charged particles, held in place by forces not shown. What electrostatic force, due to the other two charges, acts on q_1? Take $q_1 = -1.2\ \mu C$, $q_2 = +3.7\ \mu C$, $q_3 = -2.3\ \mu C$, $r_{12} = 15$ cm, $r_{13} = 10$ cm, and $\theta = 32°$.

Solution This problem calls for the use of the superposition principle. We start by computing the magnitudes of the forces that q_2 and q_3 exert on q_1. We substitute the magnitudes of the charges into Eq. 25-5:

$$F_{12} = \frac{1}{4\pi\epsilon_0}\frac{|q_1||q_2|}{r_{12}^2}$$

$$= \frac{(8.99 \times 10^9\ N\cdot m^2/C^2)(1.2 \times 10^{-6}\ C)(3.7 \times 10^{-6}\ C)}{(0.15\ m)^2}$$

$$= 1.77\ N.$$

The charges q_1 and q_2 have opposite signs so that the force exerted by q_2 on q_1 is attractive. Hence $\vec{F}_{12}$ points to the right in Fig. 25-10.

We also have

$$F_{13} = \frac{(8.99 \times 10^9\ N\cdot m^2/C^2)(1.2 \times 10^{-6}\ C)(2.3 \times 10^{-6}\ C)}{(0.10\ m)^2}$$

$$= 2.48\ N.$$

These two charges have the same (negative) sign so that the force exerted by q_3 on q_1 is repulsive. Thus $\vec{F}_{13}$ points as shown in Fig. 25-10.

The components of the resultant force $\vec{F}_1$ acting on q_1 are determined by the corresponding components of Eq. 25-8, or

$$F_{1x} = F_{12x} + F_{13x} = F_{12} + F_{13}\sin\theta$$

$$= 1.77\ N + (2.48\ N)(\sin 32°) = 3.08\ N$$

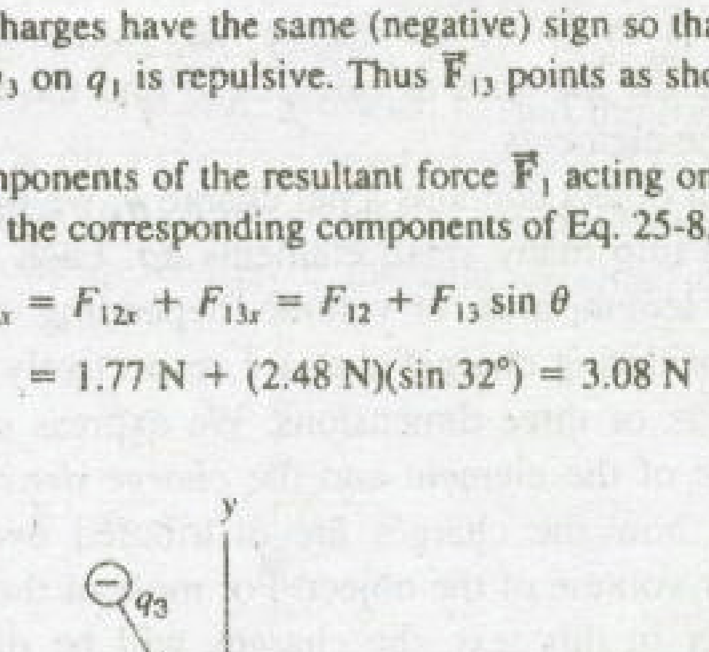

FIGURE 25-10. Sample Problem 25-5. The three charges exert three pairs of action–reaction forces on each other. Only the two forces acting on q_1 are shown here.

and

$$F_{1y} = F_{12y} + F_{13y} = 0 - F_{13}\cos\theta$$
$$= -(2.48\ \text{N})(\cos 32°) = -2.10\ \text{N}.$$

From these components, you can show that the magnitude of $\vec{F}_1$ is 3.73 N and that this vector makes an angle of −34° with the x axis.

25-5 CONTINUOUS CHARGE DISTRIBUTIONS

So far we have seen how to calculate the forces due to point charges. In many applications, however, electric forces are exerted by charged objects in the form of rods, plates, or solids. For simplicity we assume that the objects are insulators and that charge is spread throughout the surface or volume of the object, forming a *continuous charge distribution.*

Figure 25-2 showed forces exerted by one charged rod on another. Coulomb's law applies only to point charges, and we therefore cannot use Coulomb's law in its point charge form to calculate the force exerted by one charged rod on the other. It is possible to imagine the rods to be covered with point charges and to use Coulomb's law to calculate the force exerted by each point charge of one rod on each point charge of the other, but such an approach would be hopelessly complicated—if the rods carry the small charge of only 1 nC, it would be necessary to consider 10^{10} point charges on each rod!

Instead, we return to an idea from Franklin's time and regard the charge as a continuous property. The basic procedure is to divide the charge into infinitesimal elements and use the methods of calculus to find the total force due to all the elements.

If an object contains a net charge q, we imagine it to be divided into many small elements dq. Each element has a certain length, area, or volume, depending on whether we are considering charges that are respectively distributed in one, two, or three dimensions. We express dq in terms of the size of the element and the *charge density,* which describes how the charges are distributed over the length, area, or volume of the object. For most of the problems we consider in this text, the charges will be distributed uniformly over the object, which means that the charge density has the same value everywhere on the object.

In some situations, charges are distributed in one dimension, such as the thin charged rods of Fig. 25-2. In this case, we express dq in terms of the *linear charge density* (charge per unit length) λ, whose basic unit is C/m. A small element of the rod of length dx has a charge dq given by

$$dq = \lambda\, dx. \tag{25-9}$$

If the rod is uniformly charged, so that a total charge q is spread evenly over its length L, then $\lambda = q/L$. For example, a rod of length $L = 0.12$ m carrying a uniformly distributed charge of $q = 5.4 \times 10^{-6}$ C has a linear charge density of $\lambda = q/L = 4.5 \times 10^{-5}$ C/m. A short piece of the rod of length $dx = 1.0$ mm would then carry a charge $dq = \lambda\, dx = 4.5 \times 10^{-8}$ C.

In other situations, charge might be distributed over a two-dimensional area, such as the surface of the carrier bead in Fig. 25-3. In this case, dq is expressed in terms of the *surface charge density* (charge per unit area) σ, measured in the SI unit of C/m². A small element of area dA would then have a charge given by

$$dq = \sigma\, dA. \tag{25-10}$$

If a charge q is spread uniformly over a surface of area A, then $\sigma = q/A$.

The charge also might be spread throughout the volume of a three-dimensional object. In this case, we use the *volume charge density* (charge per unit volume) ρ, whose SI unit is C/m³. The charge dq in an element of volume dV would then be

$$dq = \rho\, dV. \tag{25-11}$$

If the charge q is distributed uniformly throughout the volume V, then $\rho = q/V$.

To illustrate these concepts, we will calculate expressions for the force exerted by a continuous charge distribution on a point charge q_0. By extending these methods, it is possible to calculate the force exerted by one continuous charge distribution on another.

The procedure for finding the force exerted by a continuous charge distribution on a point charge is as follows:

1. Imagine the continuous charge distribution to be divided into a large number of small charge elements.

2. Pick an arbitrary charge element and express its charge dq in terms of Eq. 25-9, 25-10, or 25-11, depending respectively on whether the charge is distributed over a line, an area, or a volume.

3. Because dq is infinitesimally small, we can treat it as a point charge. Express the magnitude of the force element dF exerted by the charge dq on the charge q_0 in terms of Coulomb's law, Eq. 25-4:

$$dF = \frac{1}{4\pi\epsilon_0}\frac{|dq||q_0|}{r^2}, \tag{25-12}$$

where r is the distance between dq and q_0.

4. By taking into account the signs and locations of dq and q_0, determine the direction of the force element $d\vec{F}$.

5. The total force is then found by adding all the infinitesimal force elements, which involves the integral

$$\vec{F} = \int d\vec{F}. \tag{25-13}$$

In performing this integral, we usually need to take into account that different charge elements dq may give force elements $d\vec{F}$ in different directions. Equation 25-13 really means three different equations for the three components of $\vec{F}$:

$$F_x = \int dF_x, \qquad F_y = \int dF_y, \qquad F_z = \int dF_z. \quad (25\text{-}14)$$

Occasionally we can use arguments based on symmetry to avoid calculating one or two of these integrals.

A Uniform Line of Charge

Figure 25-11 shows a thin rod of length L that lies along the z axis and carries a uniformly distributed positive charge q, so that its linear charge density is $\lambda = q/L$. We want to find the force exerted by the rod on a positive point charge q_0 located on the perpendicular bisector of the rod (the positive y axis) a distance y from its center.

The figure shows the results of performing steps 1, 2, and 3 of our procedure. We imagine the rod divided into small elements of length dz. An arbitrary element of charge $dq = \lambda\, dz$ is located a distance z from its center and exerts a force dF on q_0, where

$$dF = \frac{1}{4\pi\epsilon_0}\frac{q_0 dq}{r^2}.$$

The direction of the force $d\vec{F}$ is shown in the figure. There is no component of $d\vec{F}$ in the x direction (perpendicular to the page), so $F_x = 0$. We can also use a symmetry argument to show that $F_z = 0$. For every charge element dq located at position $+z$, there is another charge element dq located at $-z$. When we add the forces due to the charge elements at $+z$ and $-z$, we find that the z components have equal magnitudes but point in opposite directions, so their sum is zero. Because the charge q_0 is located in the median plane of the rod, this cancellation will occur for every such pair of charge elements along the entire length of the rod. We therefore conclude that $F_z = 0$.

Only F_y remains to be calculated. The element $dF_y = dF \cos\theta$ is shown in Fig. 25-11. With $dq = \lambda\, dz$, $r^2 = y^2 + z^2$, and $\cos\theta = y/r$, we have

$$dF_y = dF\cos\theta = \frac{1}{4\pi\epsilon_0}\frac{q_0\lambda\, dz}{(y^2+z^2)}\frac{y}{\sqrt{y^2+z^2}}$$

$$F_y = \int dF_y = \frac{1}{4\pi\epsilon_0} q_0\lambda y \int_{-L/2}^{L/2}\frac{dz}{(y^2+z^2)^{3/2}}.$$

Evaluating the integral (see Appendix I and note that y is a constant), we obtain

$$F_y = \frac{1}{4\pi\epsilon_0}\frac{q_0 q}{y\sqrt{y^2+L^2/4}}. \quad (25\text{-}15)$$

This force is in the positive y direction when q_0 and q are positive. If the charge q_0 is moved to other locations in the xy plane, the expression for the force may change (see Exercise 14).

It is often instructive to evaluate expressions such as this one in various limiting cases. Let us consider the result when $y \gg L$, in which case the force becomes

$$F_y \approx \frac{1}{4\pi\epsilon_0}\frac{q_0 q}{y^2},$$

which is just the expression for the force of one point charge on another. When we are very far from the charged rod, or when the rod is very small, it looks like a point charge.

A Ring of Charge

Figure 25-12 shows a thin ring of radius R carrying a uniformly distributed positive charge q, so that its linear charge density is $\lambda = q/2\pi R$. We wish to find the force exerted by the ring on a positive point charge q_0 located on the axis of the ring (which we choose as the positive z axis) a distance z from the center of the ring. A small charge

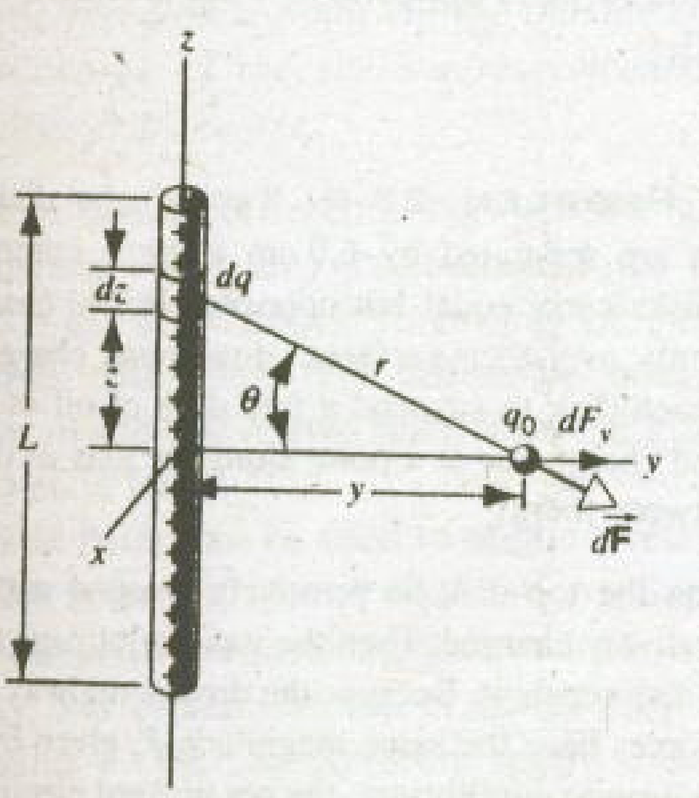

FIGURE 25-11. A uniformly charged rod. To find the force on the point charge q_0, we consider the rod to consist of many individual charge elements such as dq.

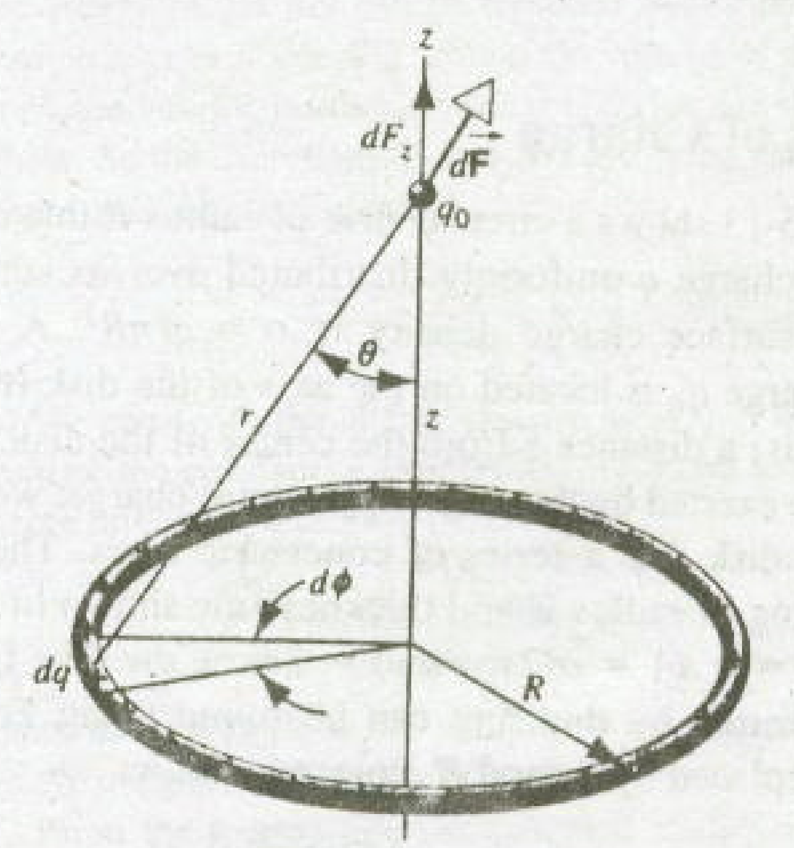

FIGURE 25-12. A uniformly charged ring. To find the force on a point charge q_0, we consider the ring to consist of many individual charge elements such as dq.

element of the ring has a length $R\,d\phi$ and thus carries a charge $dq = \lambda R\,d\phi$. The force dF exerted on q_0 by dq is

$$dF = \frac{1}{4\pi\epsilon_0}\frac{q_0 dq}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{q_0\lambda R\,d\phi}{(z^2+R^2)}.$$

We can use a symmetry argument to establish that the only nonzero component of $\vec{\mathbf{F}}$ is its z component. For every element dq of the ring, there will be another element of equal charge dq at the opposite end of a diameter through the center of the ring; when the force elements on q_0 due to these two charge elements are added, all force components other than F_z will cancel. With $\cos\theta = z/r$, we find

$$F_z = \int dF_z = \int dF\cos\theta$$

$$= \int \frac{1}{4\pi\epsilon_0}\frac{q_0\lambda R\,d\phi}{(z^2+R^2)}\frac{z}{\sqrt{z^2+R^2}}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q_0\lambda R z}{(z^2+R^2)^{3/2}}\int_0^{2\pi} d\phi.$$

The integral evaluated around the ring gives 2π, and so the final result for the force is

$$F_z = \frac{1}{4\pi\epsilon_0}\frac{q_0 q z}{(z^2+R^2)^{3/2}}. \qquad (25\text{-}16)$$

Is this result also valid if q_0 is located on the negative z axis? (See Exercise 15.)

We can examine this result in the limiting case as $z \to \infty$. For $z >> R$, we obtain

$$F_z \approx \frac{1}{4\pi\epsilon_0}\frac{q_0 q}{z^2},$$

which again gives the result for point charges. When we are very far from the ring, it appears to be a point charge.

Note also that $F_z = 0$ for $z = 0$. This is reasonable, because at the center of the ring the charge q_0 would be pushed equally in all directions by the charge elements that make up the ring.

A Disk of Charge

Figure 25-13 shows a circular disk of radius R that carries a positive charge q uniformly distributed over its surface, so that its surface charge density is $\sigma = q/\pi R^2$. A positive point charge q_0 is located on the axis of the disk (the positive z axis) a distance z from the center of the disk. To find the force exerted by the disk on the point charge, we can divide the disk into a series of concentric rings. The charge on the ring of radius w and thickness dw shown in Fig. 25-13 is $dq = \sigma\,dA = \sigma(2\pi w\,dw) = 2\pi\sigma w\,dw$. The force dF_z on q_0 exerted by this ring can be found using Eq. 25-16 with q replaced by dq and R replaced with w:

$$dF_z = \frac{1}{4\pi\epsilon_0}\frac{q_0(2\pi\sigma w\,dw)z}{(z^2+w^2)^{3/2}}.$$

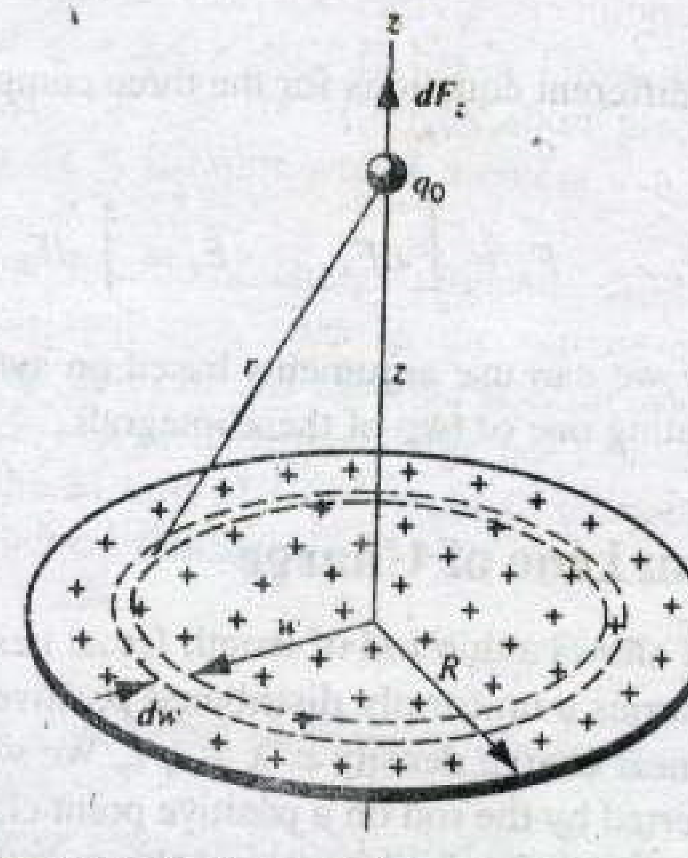

FIGURE 25-13. A circular disk carrying a uniform surface charge density. The force on a point charge q_0 is determined by dividing the disk into thin circular rings.

To add the force elements due to all of the rings, we integrate as w ranges from 0 to R:

$$F_z = \frac{1}{4\pi\epsilon_0} q_0 2\pi\sigma z \int_0^R \frac{w\,dw}{(z^2+w^2)^{3/2}}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{2q_0 q}{R^2}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right). \qquad (25\text{-}17)$$

Note that the integral is of the form $\int u^{-3/2}du$, which can be evaluated directly. How would this equation differ for $z < 0$? (See Exercise 15.) As $z \to \infty$, you can use the binomial expansion (see Appendix I) to show that this expression reduces to Coulomb's law for point charges.

In these three examples, we have assumed that all charges are positive. If either the point charge or the extended object (but not both!) carry negative charge, the direction of the force is opposite to what is shown in Figs. 25-11 to 25-13.

SAMPLE PROBLEM 25-6. Two circular disks of radius $R = 5.0$ cm are separated by 6.0 cm along a common vertical axis. The disks carry equal but opposite electric charges distributed uniformly over their surfaces. How much charge q must be placed on each disk to suspend a tiny drop of oil of mass 4.0×10^{-15} kg and charge $-e$ at a point along the axis of the disks and midway between them?

Solution Let the top disk be positively charged and the bottom disk be negatively charged. Then the top disk attracts the drop and the bottom disk repels it. Because the drop is midway between the disks, the forces have the same magnitude F_z given by Eq. 25-17. To hold the drop in equilibrium, the net upward electrostatic force $2F_z$ must equal the weight mg of the drop. Setting these two forces equal and using Eq. 25-17, we obtain

$$q = \frac{mg}{\dfrac{4e}{4\pi\epsilon_0 R^2}\left(1 - \dfrac{z}{\sqrt{z^2 + R^2}}\right)}.$$

Solving, we find $q = 35$ nC.

This method was used by Robert A. Millikan in a series of experiments begun in 1906 to measure the charge of the electron. (See Section 26-6.)

A Special Case

There is one special case in which a continuous charge distribution can be treated as a point charge, which allows us to use Coulomb's law in its point-charge form. This occurs when the charge is distributed with spherical symmetry. That is, the volume charge density may vary with radius, but the density is uniform in a thin shell at any radius.

First we consider a thin spherical shell. In Section 14-5 we established two properties of the gravitational force exerted by a uniformly dense spherical shell of matter on a point mass. These properties are: (1) the force on a particle inside the shell is zero, and (2) the force on an external particle is the same as if the entire mass of the shell were concentrated at its center.

The symmetry between the gravitational and electrostatic force laws (both depend on $1/r^2$) will allow us to make many analogies between gravitation and electrostatics. We can often apply results from gravitation directly to electrostatics without additional calculation or proof. This is true for the properties of uniform shells. The proofs of these two important results for electrostatics follow exactly the corresponding proofs in Section 14-5 for the gravitational force.

> *A uniformly charged spherical shell exerts no electrostatic force on a point charge located anywhere inside the shell.*

> *A uniformly charged spherical shell exerts an electrostatic force on a point charge outside the shell as if all the charge of the shell were concentrated in a point charge at its center.*

There is one difference between the gravitational and electrostatic cases: the gravitational force is always attractive, but the electrostatic force may be either attractive or repulsive. However, this difference does not affect the transfer of the above two rules from the gravitational to the electrostatic force.

These rules can be used to obtain a related result that is valid for spherical charge distributions. Suppose we have a spherical distribution of charge in which the volume charge density ρ is either constant or varies only as a function of the radius r. We can therefore regard the sphere as being composed of a set of thin spherical shells. Each shell is uniformly charged; that is, the charge density of one shell may differ from the charge density of another shell, but on each individual shell the charge is uniformly distributed. Then we can apply each of the two rules to every shell of the sphere. If our test charge is somewhere inside the sphere, the shells outside of the charge exert no force on it according to the first rule. (This result was used for the gravitational force in Sample Problem 14-4.) If the test charge is outside the sphere, then every shell can be replaced by a point charge at the center, and so the entire sphere can be replaced by a point charge equal to the total charge of the sphere.

For this reason the force exerted by the nucleus of an atom on its electrons generally cannot give us any information about the distribution of positive charge within the nucleus. For spherical nuclei in which the charge density depends only on r, *all* charge distributions give identical forces on an electron outside the nucleus. However, occasionally an electron may wander *inside* the nucleus and give us information about its distribution of positive charge.

SAMPLE PROBLEM 25-7. The spherical nucleus of a certain atom contains a positive charge Ze in a volume of radius R. Compare the force exerted on an electron inside the nucleus at radius $0.5R$ with the force at radius R for a nucleus in which (*a*) the charge density is constant throughout its volume, and (*b*) the charge density increases in direct proportion to the radius r.

Solution (*a*) For the uniformly charged nucleus (which is a good approximation to the behavior of many nuclei), the volume charge density is

$$\rho = \frac{q}{V} = \frac{Ze}{\frac{4}{3}\pi R^3}.$$

When the electron is at $r = R$, the entire nucleus can be replaced by a point charge $q = Ze$ located at its center, so the force $F(r)$ at $r = R$ is

$$F(R) = \frac{1}{4\pi\epsilon_0}\frac{Ze^2}{R^2}.$$

When the electron is at $r = R/2$, the charge at all larger radii exerts no force on the electron (by the first rule for spherical shells). The charge inside $R/2$ can be replaced with a point charge. How much charge is inside $R/2$? Since the volume of a sphere depends on r^3, the volume inside $r = R/2$ is $(1/2)^3 = 1/8$ the volume of the sphere. So the charge inside $r = R/2$ is 1/8 the charge of the entire sphere, and we conclude that

$$\frac{F(R/2)}{F(R)} = \frac{1}{8}.$$

(*b*) We can write the charge density as $\rho(r) = br$. First we must evaluate the constant of proportionality b. We know that the total charge on the nucleus must be Ze, so

$$\int_0^R \rho\, dV = \int_0^R (br)4\pi r^2\, dr = Ze,$$

where $dV = 4\pi r^2 dr$ is the volume of a spherical shell. Carrying out the integral, we find $b = Ze/\pi R^4$.

From the second rule for spherical shells, we know that the force $F(R)$ is the same for both charge distributions. However,

$F(R/2)$ will be different for the two distributions. To find $F(R/2)$, we need to know how much charge q' is contained within the sphere of radius $R/2$, since the charge outside of radius $R/2$ exerts no force on the electron. This charge is

$$q' = \int_0^{R/2} \rho\, dV = \int_0^{R/2} \frac{Zer}{\pi R^4} 4\pi r^2\, dr = \frac{Ze}{16}.$$

The force on the electron at $r = R/2$ can be found by replacing the sphere within $R/2$ by a point charge q' at its center, which gives a force 1/16 the force at the surface:

$$\frac{F(R/2)}{F(R)} = \frac{1}{16}.$$

This result is very different from that of the uniform sphere in part (*a*), demonstrating that, although the electron outside the nucleus cannot distinguish between the two distributions, the electron inside clearly can.

Atomic electrons can occasionally penetrate the nucleus, and electrons can be fired into the nucleus by an accelerator. These two methods can give us information about the distribution of charge within the nucleus. One result of these experiments is that the charge density is found to be very nearly uniform for most nuclei. Despite the Coulomb repulsion of the protons (which is expected to drive them toward the nuclear surface) and despite the strong nuclear force between the protons (which is expected to make them congregate near the center of the nucleus), the protons in the nucleus are distributed with a roughly uniform density. Furthermore, this density is approximately the same for light nuclei as it is for heavy nuclei. These surprising results offer insight into important properties of the nuclear force.

25-6 CONSERVATION OF CHARGE

When a glass rod is rubbed with silk, a positive charge appears on the rod. Measurement shows that a corresponding negative charge appears on the silk. This suggests that rubbing does not create charge but merely transfers it from one object to another, disturbing slightly the electrical neutrality of each. This hypothesis of the *conservation of charge* has stood up under careful experimental tests both for large-scale objects and for atoms, nuclei, and elementary particles. No exceptions have ever been found.

In analogy with other conservation laws, such as conservation of momentum or conservation of energy, we can express conservation of electric charge as

$$\sum q = \text{constant} \quad \text{or} \quad q_i = q_f. \qquad (25\text{-}18)$$

In any process occurring in an isolated system the net initial charge must equal the net final charge. In finding the net charge, it is important to take into account the signs of the individual charges.

An interesting example of charge conservation comes about when an electron (charge $= -e$) and an antielectron or positron (charge $= +e$) are brought close to each other. The two particles may annihilate one another, converting all their rest energy into radiant energy. The radiant energy may appear in the form of two gamma rays (high-energy packets of electromagnetic radiation, which are chargeless):

$$e^- + e^+ \rightarrow \gamma + \gamma.$$

The net charge is zero both before and after the event, and charge is conserved.

Certain uncharged particles, such as the neutral π meson, sometimes decay into two gamma rays:

$$\pi^0 \rightarrow \gamma + \gamma.$$

This decay conserves charge, the total charge again being 0 before and after the decay. For another example, a neutron ($q = 0$) decays into a proton ($q = +e$) and an electron ($q = -e$) plus another neutral particle, a neutrino ($q = 0$):

$$n \rightarrow p + e^- + \nu.$$

The total charge is zero, both before and after the decay, and charge is conserved. Experiments have been done to search for decays of the neutron into a proton with no electron emitted, which would violate charge conservation. No such events have been found.

The decay of an electron ($q = -e$) into neutral particles, such as gamma rays (γ) or neutrinos (ν), is forbidden; for example

$$e^- \nrightarrow \gamma + \nu,$$

because that decay would violate charge conservation. Attempts to observe this decay have likewise been unsuccessful, indicating that, if the decay does occur, the electron must have a lifetime of at least 10^{23} years!

Another example of charge conservation is found in the fusion of two deuterium nuclei ^{2}H (called "heavy hydrogen") to make helium. Among the possible reactions are

$$^2\text{H} + {}^2\text{H} \rightarrow {}^3\text{H} + \text{p},$$
$$^2\text{H} + {}^2\text{H} \rightarrow {}^3\text{He} + \text{n}.$$

The deuterium nucleus contains one proton and one neutron and therefore has a charge of $+e$. The nucleus of the isotope of hydrogen with mass 3, written ^{3}H and known as *tritium*, contains one proton and two neutrons, and thus also has a charge of $+e$. The first reaction therefore has a net charge of $+2e$ on each side and conserves charge. In the second reaction, the neutron is uncharged, while the nucleus of the isotope of helium with mass 3 contains two protons and one neutron and therefore has a charge of $+2e$. The second reaction thus also conserves charge. Conservation of charge explains why we never see a proton emitted along with ^{3}He or a neutron along with ^{3}H

To sum up, charge is conserved in *all* known interactions among particles; no exceptions have ever been observed.

MULTIPLE CHOICE

25-1 Electromagnetism: A Preview

25-2 Electric Charge

1. Electric charges A and B are attracted to each other. Electric charges B and C repel each other. If A and C are held close together they will
   (A) attract. (B) repel.
   (C) not affect each other.
   (D) More information is needed to answer.
2. Electric charges A and B are attracted to each other. Electric charges B and C are also attracted to each other. If A and C are held close together they will
   (A) attract. (B) repel.
   (C) not affect each other.
   (D) More information is needed to answer.
3. Electric charges A and B repel each other. Electric charges B and C also repel each other. If A and C are held close together they will
   (A) attract. (B) repel.
   (C) not affect each other.
   (D) More information is needed to answer.

25-3 Conductors and Insulators

4. If an object made of substance A rubs an object made of substance B, then A becomes positively charged and B becomes negatively charged. If, however, an object made of substance A is rubbed against an object made of substance C, then A becomes negatively charged. What will happen if an object made of substance B is rubbed against an object made of substance C?
   (A) B becomes positively charged and C becomes positively charged.
   (B) B becomes positively charged and C becomes negatively charged.
   (C) B becomes negatively charged and C becomes positively charged.
   (D) B becomes negatively charged and C becomes negatively charged.
5. A positively charged rod is held near a ball suspended by an insulating thread. The ball is seen to swing toward the charged rod. What can be concluded?
   (A) The ball must have had a charge opposite to that of the rod.
   (B) The ball may have been neutral originally, but it became charged when the rod was held near it.
   (C) The ball must be a conductor.
   (D) The ball is not positively charged, but it could be neutral.
6. A spherical conducting ball is suspended by a grounded conducting thread. A positive point charge is moved near the ball. The ball will
   (A) be attracted to the point charge and swing toward it.
   (B) be repelled from the point charge and swing away from it.
   (C) not be affected by the point charge.
7. A spherical conducting ball is suspended by an insulating thread. A positive point charge is moved near the ball. The ball will
   (A) be attracted to the point charge and swing toward it.
   (B) be repelled from the point charge and swing away from it.
   (C) not be affected by the point charge.

25-4 Coulomb's Law

8. A 3-μC point charge q_1 is located a distance d away from a -6-μC point charge q_2. What is the ratio $|\vec{F}_{12}|/|\vec{F}_{21}|$?
   (A) 1/2 (B) 1 (C) 2 (D) 18
9. Two 200-pound lead balls are separated by a distance of 1 m. Both balls have the same positive charge q. What charge will produce an electrostatic force between the balls that is of the same order of magnitude as the weight of one ball?
   (A) 1×10^{-14} C (B) 1×10^{-7} C
   (C) 3×10^{-4} C (D) 2×10^{-2} C
10. Two identical, small, conducting spheres are separated by a distance of 1 m. The spheres originally have the same positive charge, and the force between them is F_0. Half of the charge on one sphere is then moved to the other sphere. The force between the spheres is now
    (A) $F_0/4$. (B) $F_0/2$. (C) $3F_0/4$.
    (D) $3F_0/2$. (E) $3F_0$.
11. Two identical, small, conducting spheres are separated by a distance of 1 m. The spheres originally have equal but opposite charges, and the force between them is F_0. Half of the charge on one sphere is then moved to the other sphere. The force between the spheres is now
    (A) $F_0/4$. (B) $F_0/2$. (C) $3F_0/4$.
    (D) $3F_0/2$. (E) $3F_0$.

25-5 Continuous Charge Distributions

12. A point charge q is located a distance a from the surface of a sphere of radius $2a$. A charge Q is distributed uniformly throughout the volume of the sphere. The magnitude of the electrostatic force between the point charge q and the sphere is F, where
    (A) $F = |qQ|/4\pi\epsilon_0 a^2$.
    (B) $|qQ|/4\pi\epsilon_0 a^2 > F > |qQ|/12\pi\epsilon_0 a^2$.
    (C) $|qQ|/12\pi\epsilon_0 a^2 > F > |qQ|/20\pi\epsilon_0 a^2$.
    (D) $|qQ|/20\pi\epsilon_0 a^2 > F > |qQ|/36\pi\epsilon_0 a^2$.
    (E) $F = |qQ|/36\pi\epsilon_0 a^2$.

25-6 Conservation of Change

13. A positively charged rod is held near a neutral conducting sphere suspended by an insulating thread. The sphere will
    (A) be unaffected, because it is neutral.
    (B) remain neutral, but be repelled from the rod anyway.
    (C) remain neutral, but be attracted to the rod anyway.
    (D) acquire a negative charge and be repelled from the rod.
    (E) acquire a negative charge and be attracted to the rod.
14. Objects A, B, and C are three identical, insulated, spherical conductors. Originally A and B both have charges of $+3$ mC, while C has a charge of -6 mC. Objects A and C are allowed to touch, then they are moved apart. Then objects B and C are allowed to touch, and they are moved apart.
    (*a*) If objects A and B are now held near each other, they will
    (A) attract. (B) repel.
    (C) have no effect on each other.
    (*b*) If instead objects A and C are held near each other, they will
    (A) attract. (B) repel.
    (C) have no effect on each other.

QUESTIONS

1. You are given two metal spheres mounted on portable insulating supports. Find a way to give them equal and opposite charges. You may use a glass rod rubbed with silk but may not touch it to the spheres. Do the spheres have to be of equal size for your method to work?

2. In Question 1, find a way to give the spheres equal charges of the same sign. Again, do the spheres need to be of equal size for your method to work?

3. A charged rod attracts bits of dry cork dust, which, after touching the rod, often jump violently away from it. Explain.

4. How would your answers to Multiple-choice question 1, Multiple-choice question 2, and Multiple-choice question 3 change if any of the objects *A*, *B*, or *C* could be uncharged?

5. The experiments described in Section 25-2 could be explained by postulating four kinds of charge—that is, on glass, silk, plastic, and fur. What is the argument against this?

6. A positive charge is brought very near to an uncharged insulated conductor. The conductor is grounded while the charge is kept near. Is the conductor charged positively, negatively, or not at all if (*a*) the charge is taken away and then the ground connection is removed and (*b*) the ground connection is removed and then the charge is taken away?

7. A charged insulator can be discharged by passing it just above a flame. Explain how.

8. If you rub a coin briskly between your fingers, it will not seem to become charged by friction. Why?

9. If you walk briskly across a carpet, you often experience a spark on touching a door knob. (*a*) What causes this? (*b*) How might it be prevented?

10. Why do electrostatic experiments not work well on humid days?

11. Why is it recommended that you touch the metal frame of your personal computer before installing any internal accessories?

12. An insulated rod is said to carry an electric charge. How could you verify this and determine the sign of the charge?

13. If a charged glass rod is held near one end of an insulated uncharged metal rod as in Fig. 25-14, electrons are drawn to one end, as shown. Why does the flow of electrons cease? After all, there is an almost inexhaustible supply of them in the metal rod.

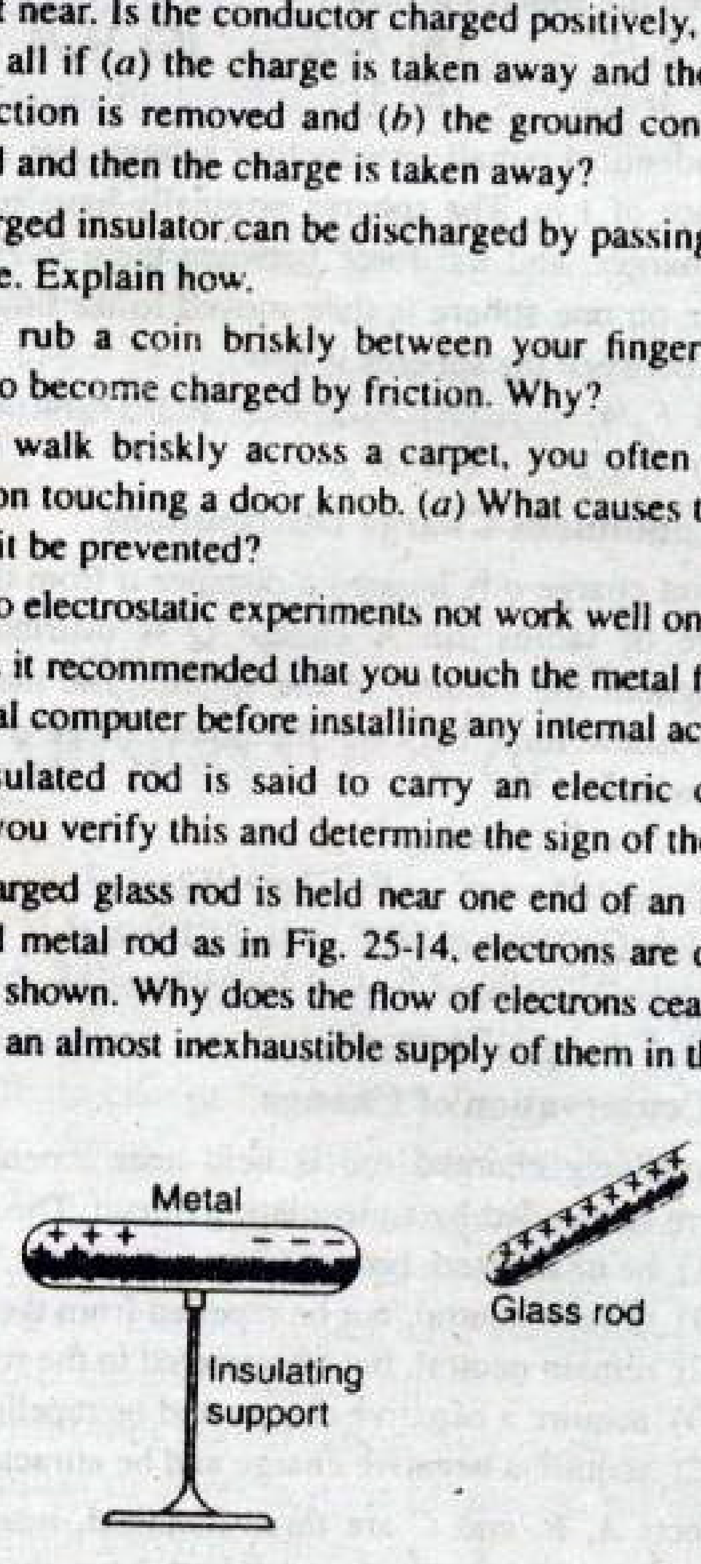

FIGURE 25-14. Questions 13 and 14

14. In Fig. 25-14, does any resultant electric force act on the metal rod? Explain.

15. A person standing on an insulating stool touches a charged, insulated conductor. Is the conductor discharged completely?

16. (*a*) A positively charged glass rod attracts a suspended object. Can we conclude that the object is negatively charged? (*b*) A positively charged glass rod repels a suspended object. Can we conclude that the object is positively charged?

17. Explain what is meant by the statement that electrostatic forces obey the principle of superposition.

18. Is the electric force that one charge exerts on another changed if other charges are brought nearby?

19. A solution of copper sulfate is a conductor. What particles serve as the charge carriers in this case?

20. If the electrons in a metal such as copper are free to move about, they must often find themselves headed toward the metal surface. Why do they not keep on going and leave the metal?

21. Would it have made any important difference if Benjamin Franklin had chosen, in effect, to call electrons positive and protons negative?

22. Coulomb's law predicts that the force exerted by one point charge on another is proportional to the product of the two charges. How might you go about testing this aspect of the law in the laboratory?

23. Explain how an atomic nucleus can be stable if it is composed of particles that are either neutral (neutrons) or carry like charges (protons).

24. An electron (charge $= -e$) circulates around a helium nucleus (charge $= +2e$) in a helium atom. Which particle exerts the larger force on the other?

25. The charge of a particle is a true characteristic of the particle, independent of its state of motion. Explain how you can test this statement by making a vigorous experimental check of whether the hydrogen atom is truly electrically neutral.

26. Suppose the charge in Fig. 25-11 were not distributed uniformly along the length of the rod but instead were concentrated at its center and tapered off at the same rate toward either end. Will the force now have a z component? If this rod had the same total charge q as the uniformly charged rod, how will the magnitude of F_y compare with Eq. 25-15? Repeat both questions if the charge is distributed along the rod so that there is a deficiency near the center and the charge density increases at the same rate toward either end.

27. Earnshaw's theorem says that no particle can be in stable equilibrium under the action of electrostatic forces alone. Consider, however, point P at the center of a square of four equal positive charges, as in Fig. 25-15. If you put a positive test charge there it might seem to be in stable equilibrium. Every one of the four external charges pushes it toward P, yet Earnshaw's theorem holds. Can you explain how?

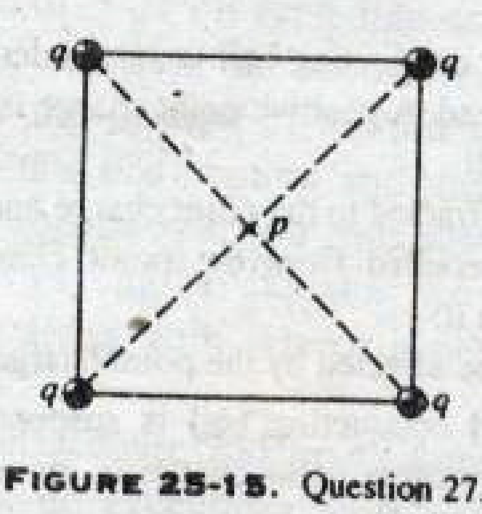

FIGURE 25-15. Question 27.

28. The quantum of charge is 1.60×10^{-19} C. Is there a corresponding quantum of mass?

29. What does it mean to say that a physical quantity is (*a*) quantized or (*b*) conserved? Give some examples.

30. In Sample Problem 25-3 we show that the electrical force is about 10^{39} times stronger than the gravitational force. Can you conclude from this that a galaxy, a star, or a planet must be essentially neutral electrically?

31. How do we know that electrostatic forces are not the cause of gravitational attraction—between the Earth and Moon, for example?

EXERCISES

25-1 Electromagnetism: A Preview

25-2 Electric Charge

1. In the return stroke of a typical lightning bolt (see Fig. 25-16), a current of 2.5×10^4 C/s flows for 20 μs. How much charge is transferred in this event?

FIGURE 25-16. Exercise 1.

25-3 Conductors and Insulators

25-4 Coulomb's Law

2. What must be the distance between point charge $q_1 = 26.3\ \mu$C and point charge $q_2 = -47.1\ \mu$C for the attractive electrical force between them to have a magnitude of 5.66 N?

3. A point charge of $+3.12 \times 10^{-6}$ C is 12.3 cm distant from a second point charge of -1.48×10^{-6} C. Calculate the magnitude of the force on each charge.

4. Two equally charged particles, held 3.20 mm apart, are released from rest. The initial acceleration of the first particle is observed to be 7.22 m/s^2 and that of the second to be 9.16 m/s^2. The mass of the first particle is 6.31×10^{-7} kg. Find (*a*) the mass of the second particle and (*b*) the magnitude of the common charge.

5. Figure 25-17*a* shows two charges, q_1 and q_2, held a fixed distance d apart. (*a*) Find the strength of the electric force that acts on q_1. Assume that $q_1 = q_2 = 21.3\ \mu$C and $d = 1.52$ m. (*b*) A third charge $q_3 = 21.3\ \mu$C is brought in and placed as shown in Fig. 25-17*b*. Find the strength of the electric force on q_1 now.

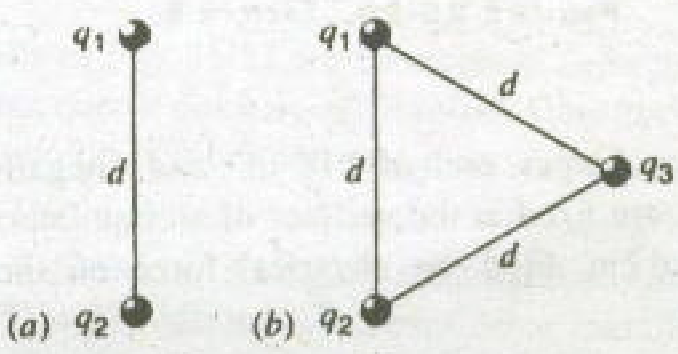

FIGURE 25-17. Exercise 5.

6. Two identical conducting spheres, 1 and 2, carry equal amounts of charge and are fixed a distance apart that is large compared with their diameters. The spheres repel each other with an electrical force of 88 mN. Suppose now that a third identical sphere 3, having an insulating handle and initially uncharged, is touched first to sphere 1, then to sphere 2, and finally removed. Find the force between spheres 1 and 2 now. See Fig. 25-18.

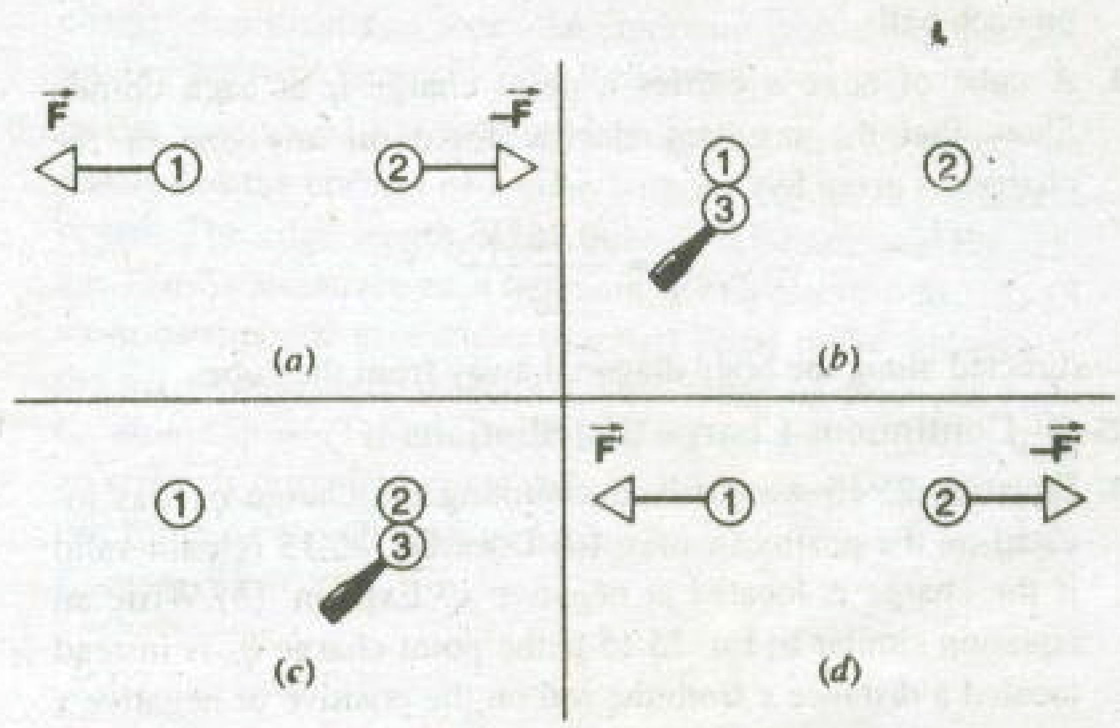

FIGURE 25-18. Exercise 6.

7. Three charged particles lie on a straight line and are separated by a distance d as shown in Fig. 25-19. Charges q_1 and q_2 are held fixed. Charge q_3, which is free to move, is found to be in equilibrium under the action of the electric forces. Find q_1 in terms of q_2.

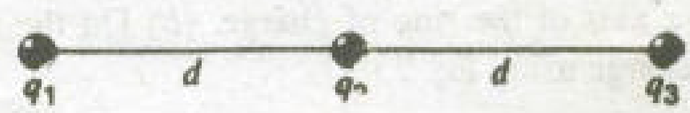

FIGURE 25-19. Exercise 7.

8. In Fig. 25-20, find (*a*) the horizontal components and (*b*) the vertical components of the resultant electric force on the charge in the lower left corner of the square. Assume that $q = 1.13\ \mu C$ and $a = 15.2$ cm. The charges are at rest.

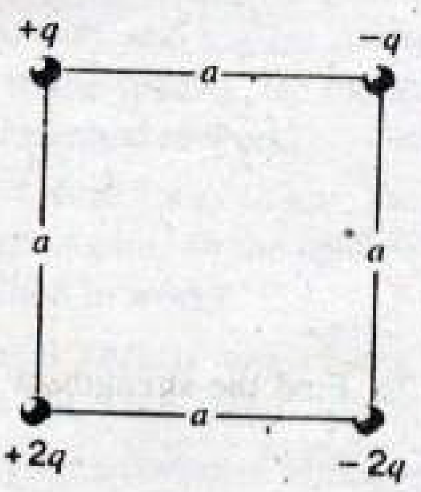

FIGURE 25-20. Exercise 8.

9. Two positive charges, each of 4.18 μC, and a negative charge, −6.36 μC, are fixed at the vertices of an equilateral triangle of side 13.0 cm. Find the electrical force on the negative charge.

10. Each of two small spheres is charged positively, the total charge being 52.6 μC. Each sphere is repelled from the other with a force of 1.19 N when the spheres are 1.94 m apart. Calculate the charge on each sphere.

11. Two fixed charges, +1.07 μC and −3.28 μC, are 61.8 cm apart. Where may a third charge be located so that no net force acts on it?

12. Three small balls, each of mass 13.3 g, are suspended separately from a common point by silk threads, each 1.17 m long. The balls are identically charged and hang at the corners of an equilateral triangle 15.3 cm on a side. Find the charge on each ball.

13. A cube of edge a carries a point charge q at each corner. Show that the resultant electric force on any one of the charges is given by

$$F = \frac{0.262q^2}{\epsilon_0 a^2},$$

directed along the body diagonal away from the cube.

25-5 Continuous Charge Distributions

14. Equation 25-15 was derived assuming the charge q_0 was located on the positive y axis. (*a*) Does Eq. 25-15 remain valid if the charge is located at negative y? Explain. (*b*) Write an equation similar to Eq. 25-15 if the point charge q_0 is instead located a distance x from the rod on the positive or negative x axis. (*c*) Write an equation in vector component form for the force when q_0 is located a distance d from the rod on the 45° line that bisects the positive x and y axes. (*d*) Write an equation in vector component form that gives the force when q_0 is located at an arbitrary point x,y anywhere in the xy plane. Check that the components have the correct signs when the point x,y is located in each of the four quadrants.

15. (*a*) Starting with Eq. 25-16, write an equation in vector form that gives the force when q_0 is located either on the positive or negative z axis of the ring of charge. (*b*) Do the same for the disk of charge using Eq. 25-17.

16. Find the force on a positive point charge q located a distance x from the end of a rod of length L with uniformly distributed positive charge Q. (See Fig. 25-21.)

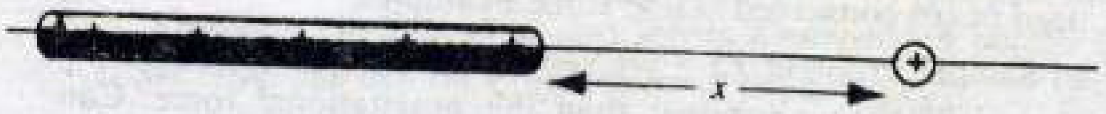

FIGURE 25-21. Exercise 16.

17. Consider the rod and charge q_0 in Fig. 25-11. Where can you place a second point charge q (equal to the charge on the rod) so that q_0 is in equilibrium (ignore gravity)? Solve this problem assuming that (*a*) q is positive and (*b*) q is negative.

18. Show that the equilibrium of q_0 in Exercise 17 is unstable. (Hint: This problem can be solved by symmetry arguments, and actually requires very little math!)

19. Assume that the rod in Fig. 25-11 has a uniform positive charge density λ on the top half of the rod and a uniform charge density $-\lambda$ on the bottom half of the rod. Find the net force on the point charge q_0.

20. Four charged rods form the sides of a square in the horizontal (xy) plane. Each rod has a length $L = 25.0$ cm and each carries a uniformly distributed positive charge Q. A small sphere, which can be considered to be a point charge of mass 3.46×10^{-4} g and charge $q = +2.45 \times 10^{-12}$ C, is in equilibrium a distance $z = 21.4$ cm above the center of the square. Find the value of Q.

25-6 Conservation of Charge

21. Identify the element X in the following nuclear reactions:
(*a*) $^1H + {^9Be} \rightarrow X + n$;
(*b*) $^{12}C + {^1H} \rightarrow X$;
(*c*) $^{15}N + {^1H} \rightarrow {^4He} + X$.
(Hint: See Appendix E.)

22. In the radioactive decay of ^{238}U ($^{238}U \rightarrow {^4He} + {^{234}Th}$), the center of the emerging ^4He particle is, at a certain instant, 12×10^{-15} m from the center of the residual ^{234}Th nucleus. At this instant, (*a*) what is the force on the ^4He particle, and (*b*) what is its acceleration?

23. In a crystal of salt, an atom of sodium transfers one of its electrons to a neighboring atom of chlorine, forming an ionic bond. The resulting positive sodium ion and negative chlorine ion attract each other by the electrostatic force. Calculate the force of attraction if the ions are 282 pm apart.

24. The electrostatic force between two identical ions that are separated by a distance of 5.0×10^{-10} m is 3.7×10^{-9} N. (*a*) Find the charge on each ion. (*b*) How many electrons are missing from each ion?

25. A neutron is thought to be composed of one "up" quark of charge $+\frac{2}{3}e$ and two "down" quarks each having charge $-\frac{1}{3}e$. If the down quarks are 2.6×10^{-15} m apart inside the neutron, what is the repulsive electrical force between them?

26. (*a*) How many electrons would have to be removed from a penny to leave it with a charge of $+1.15 \times 10^{-7}$ C? (*b*) To what fraction of the electrons in the penny does this correspond? See Sample Problem 25-1.

27. An electron is in a vacuum near the surface of the Earth. Where should a second electron be placed so that the net force on the first electron, due to the other electron and to gravity, is zero?

28. Find the total charge in coulombs of 75.0 kg of electrons.

29. Calculate the number of coulombs of positive charge in a glass of water. Assume the volume of the water to be 250 cm^3.

30. Two physics students (Mary at 52.0 kg and John at 90.7 kg) are 28.0 m apart. Let each have a 0.01% imbalance in their amounts of positive and negative charge, one student being positive and the other negative. Estimate the electrostatic force of attraction between them. (Hint: Replace the students by spheres of water and use the result of Exercise 29.)

31. (*a*) What equal amounts of positive charge would have to be placed on the Earth and on the Moon to neutralize their gravitational attraction? Do you need to know the Moon's distance to solve this problem? Why or why not? (*b*) How many metric tons of hydrogen would be needed to provide the positive charge calculated in part (*a*)? The molar mass of hydrogen is 1.008 g/mol.

PROBLEMS

1. Two identical conducting spheres, having charges of opposite sign, attract each other with a force of 0.108 N when separated by 50.0 cm. The spheres are suddenly connected by a thin conducting wire, which is then removed, and thereafter the spheres repel each other with a force of 0.0360 N. What were the initial charges on the spheres?

2. A charge Q is fixed at each of two opposite corners of a square. A charge q is placed at each of the other two corners. (*a*) If the resultant electrical force on Q is zero, how are Q and q related? (*b*) Could q be chosen to make the resultant electrical force on every charge zero? Explain your answer.

3. Two free point charges $+q$ and $+4q$ are a distance L apart. A third charge is placed so that the entire system is in equilibrium. (*a*) Find the sign, magnitude, and location of the third charge. (*b*) Show that the equilibrium is unstable.

4. Two similar tiny balls of mass m are hung from silk threads of length L and carry equal charges q as in Fig. 25-22. Assume that θ is so small that $\tan\theta$ can be replaced by its approximate equal, $\sin\theta$. (*a*) To this approximation show that, for equilibrium,

$$x = \left(\frac{q^2L}{2\pi\epsilon_0 mg}\right)^{1/3},$$

where x is the separation between the balls. (*b*) If $L =$ 122 cm, $m =$ 11.2 g, and $x =$ 4.70 cm, what is the value of q?

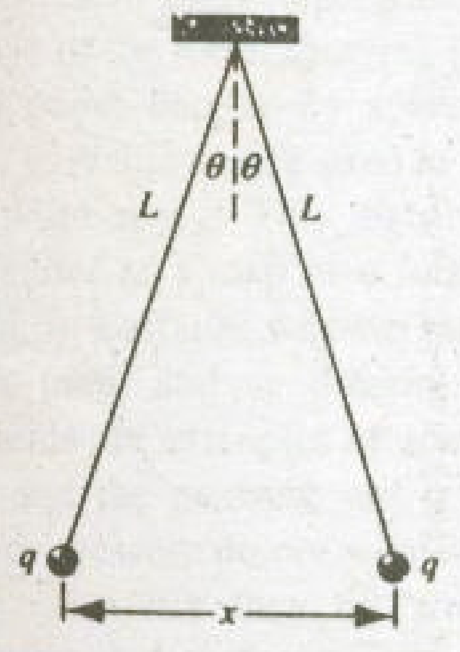

FIGURE 25-22. Problems 4, 5, and 6.

5. If the balls of Fig. 25-22 are conducting, (*a*) what happens to them after one is discharged? Explain your answer. (*b*) Find the new equilibrium separation.

6. Assume that each ball in Problem 4 is losing charge at the rate of 1.20 nC/s. At what instantaneous relative speed ($= dx/dt$) do the balls approach each other initially?

7. A certain charge Q is to be divided into two parts, $Q - q$ and q. What is the relation of Q to q if the two parts, placed a given distance apart, are to have a maximum Coulomb repulsion?

8. Two positive charges $+Q$ are held fixed a distance d apart. A particle of negative charge $-q$ and mass m is placed midway between them, then is given a small displacement perpendicular to the line joining them and released. Show that the particle describes simple harmonic motion of period $(\epsilon_0 m\pi^3 d^3/qQ)^{1/2}$.

9. Calculate the period of oscillation for a particle of positive charge $+q$ displaced from the midpoint and along the line joining the two charges $+Q$ in Problem 8.

10. In the compound CsCl (cesium chloride), the Cs atoms are situated at the corners of a cube with a Cl atom at the cube's center. The edge length of the cube is 0.40 nm; see Fig. 25-23. The Cs atoms are each deficient in one electron and the Cl atom carries one excess electron. (*a*) What is the strength of the net electric force on the Cl atom resulting from the eight Cs atoms shown? (*b*) Suppose that the Cs atom marked with an arrow is missing (crystal defect). What now is the net electric force on the Cl atom resulting from the seven remaining Cs atoms?

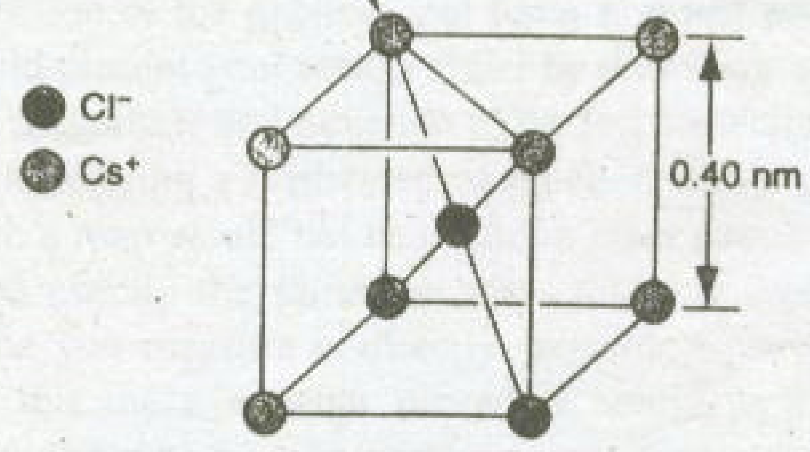

FIGURE 25-23. Problem 10.

11. Two equal positive point charges q are held a fixed distance $2a$ apart. A point test charge is located in a plane that is normal to the line joining these charges and midway between them. Find the radius R of the circle in this plane for which the force on the test particle has a maximum value. See Fig. 25-24.

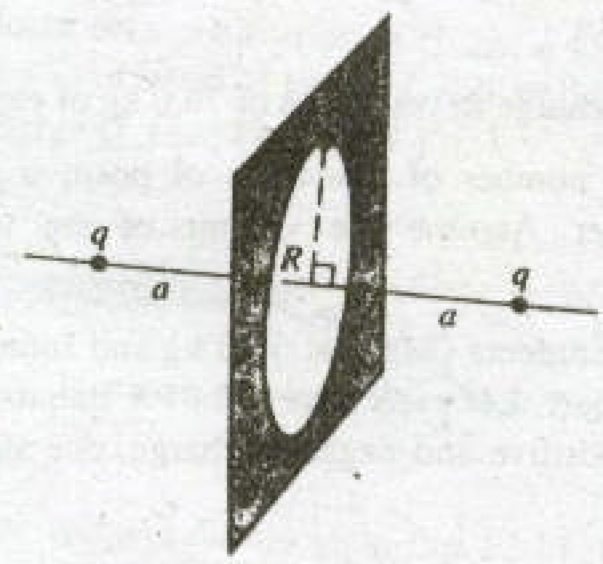

FIGURE 25-24. Problem 11.

COMPUTER PROBLEMS

1. Calculate the force of attraction between two rings with uniformly distributed charges $+q$ and $-q$. The axis of the rings is the x axis, each has a radius R, and the rings are separated by a distance of $2R$. Your final answer should be of the form $F = C_r q^2/4\pi\epsilon_0 R^2$, where C_r is a dimensionless constant that you will find.
2. Repeat Computer Problem 1 for the case of two disks with uniformly distributed charges $+q$ and $-q$. Your final answer will still be of the form $F = C_d q^2/4\pi\epsilon_0 R^2$, where C_d is a dimensionless constant that you will find; C_d for the disk *is different* from C_r for the ring.
3. Calculate the force of attraction between two solid spheres with uniformly distributed charges $+q$ and $-q$. The spheres are centered on the x axis, each has a radius R, and the centers of the spheres are separated by a distance of $d > 2R$. Your final answer should be of the form $F = C_s q^2/4\pi\epsilon_0 d^2$, where C_s is a dimensionless constant that you will find.
4. A uniform ring of charge Q has a radius $R = 1.00$ cm. An electron is constrained to move in the plane of the ring. (*a*) Assuming that $Q = -100\ \mu$C, find the speed of an electron that would move in a circular orbit of radius $r = 0.50$ cm, concentric with the ring. (*b*) Assuming that $Q = +100\ \mu$C, find the speed of an electron that would move in a circular orbit of radius $r = 1.50$ cm, concentric with the ring. (*c*) Show, by numerically integrating the motion, that neither orbit is stable.

CHAPTER 26

THE ELECTRIC FIELD

Electric charges can interact with one another over vast distances. Electrons or ionized atoms at the furthest reaches of the known universe can exert forces that cause electrons to move on Earth.

How can we explain these interactions? We do so in terms of the electric field—the distant charges set up an electric field, which exists throughout the space between Earth and the origin of the field. The motion of the charges causes disturbances in the field, which travel through space with the speed of light and are detected eons later (as radiation) when they cause motion of electrons in circuits on Earth.

In this chapter we consider only the static electric field due to charges at rest. Later in this book we expand our discussion to show how moving charges are responsible for the fields associated with electromagnetic radiation, such as radio or light.

26-1 WHAT IS A FIELD?

The temperature has a definite value at every point in the room in which you are sitting. You can measure the temperature at each point by putting a thermometer there. You could then represent that temperature distribution either by drawing a map of the room showing the measured temperature at each point, or else by specifying a mathematical function $T(x, y, z)$ that can be used to calculate the temperature at any point x, y, z. This distribution of temperature, represented either as a map or a function, is called a *temperature field.* In a similar way we might measure the pressure at each point and so determine the *pressure field.* These two fields are examples of *scalar fields,* because the temperature and the pressure are scalar quantities. If the temperature or pressure do not vary with time, they are also *static fields;* otherwise they are *time-varying fields* that might be represented mathematically by a time-dependent function such as $T(x, y, z, t)$.

If, on the other hand, you wanted to measure the velocity at every point in a flowing fluid, you would need to specify the value of the velocity *vector* at each point. Once again, you could draw a map showing the magnitude and direction of the velocity at any point, or you could specify a mathematical function $\vec{v}(x, y, z)$ that would allow the flow velocity to be calculated at any point. This is an example of a *vector field.*

The Earth's gravitational field is another example of a vector field. You might measure the value of the gravitational force at any point by attaching a test mass m_0 to a spring scale. You could then determine the magnitude and direction of the gravitational force $\vec{F}$ at any point, and you could present your results either by drawing a map showing the magnitude and direction of the force at various points or by specifying a mathematical function $\vec{F}(x, y, z)$. However, such a map would not be useful to other people unless they used exactly the same test mass that you used. Since the force you measure is directly proportional to the value of the test mass, a better procedure would be to produce a map showing not the force on your test mass but instead the force per unit test mass, or $\vec{F}/m_0$. This quantity, which would have units of N/kg, would be independent of the value of the test mass m_0. Choosing a test mass of a different size would give exactly the same map with the same values of the force per unit mass at every point.* We call the quantity $\vec{F}/m_0$ the *gravitational field.* You will recognize it as also being equal to the free-fall acceleration $\vec{g}$ at any point:

$$\vec{g} = \frac{\vec{F}}{m_0}. \qquad (26\text{-}1)$$

The field $\vec{g}$ is a vector whose direction gives the direction of the gravitational force at that point and whose magnitude indicates the "strength" of the gravitational effect at that point. We could find the force on a mass m at any point by multiplying $\vec{g}$ at that point by the value of the mass:

$$\vec{F} = m\vec{g}. \qquad (26\text{-}2)$$

In this chapter, we will develop the useful concept of an *electric field* based on a similar procedure that involves determining the electrical force per unit charge (rather than the gravitational force per unit mass). Because a force is involved, the electric field is a vector field. For now we deal only with static fields, but later when we discuss electromagnetic radiation we will consider time-varying electric fields.

Before the concept of fields became widely accepted, the force exerted by one gravitating body on another was thought of as a direct and instantaneous interaction. This view, called *action at a distance,* was also used for electromagnetic forces. In the case of gravitation, it can be represented schematically as

mass ⇄ mass,

indicating that the two masses interact directly with one another. According to this view, the effect of a movement of one body is instantaneously transmitted to the other body. This view violates the special theory of relativity, which limits the speed at which such information can be transmitted to the speed of light c, at most. The present interpretation, based on the field concept, can be represented as

mass ⇄ field ⇄ mass,

in which each mass interacts not directly with the other but instead with the gravitational field established by the other. That is, the first mass sets up a field that has a certain value at every point in space; the second mass then interacts with the field at its particular location. The field plays the role of an intermediary between the two bodies. The force exerted on the second mass can be calculated from Eq. 26-2, given the value of the field $\vec{g}$ due to the first mass. The situation is completely symmetrical from the point of view of the first mass, which interacts with the gravitational field established by the second mass. Changes in the location of one mass cause variations in its gravitational field; these variations travel at the speed of light, so the field concept is consistent with the restrictions imposed by special relativity.

26-2 THE ELECTRIC FIELD

The previous description of the gravitational field can be carried directly over to electrostatics. Coulomb's law for the force of one electric charge on another encourages us to think in terms of action at a distance, represented as

charge ⇄ charge.

Again introducing the field as an intermediary between the charges, we can represent the interaction as

charge ⇄ field ⇄ charge.

That is, the first charge sets up an *electric field,* and the second charge interacts with the electric field of the first charge. Our problem of determining the interaction between the charges is therefore reduced to two separate problems: (1) determine, by measurement or calculation, the electric field established by the first charge at every point in space, and (2) calculate the force that the field exerts on the second charge placed at a particular point in space.

In analogy with Eq. 26-1 for the gravitational field, we define the electric field $\vec{E}$ associated with a certain collection of charges in terms of the force exerted on a positive test charge q_0 at a particular point, or

$$\vec{E} = \frac{\vec{F}}{q_0}. \qquad (26\text{-}3)$$

The direction of the vector $\vec{E}$ is the same as the direction of $\vec{F}$, because q_0 is a positive scalar. Defined in this way, the electric field is independent of the magnitude of the test charge q_0.

Figure 26-1 suggests how we use this definition to determine the electric field at a particular point P. We place a positive test charge q_0 at P, and we determine the electrostatic force on q_0 due to objects in the surrounding area, which are not shown in the figure. Equation 26-3 then gives the electric field shown in Fig. 26-1b. Note that $\vec{E}$ and $\vec{F}$ are parallel, as they must be from the definition of Eq. 26-3.

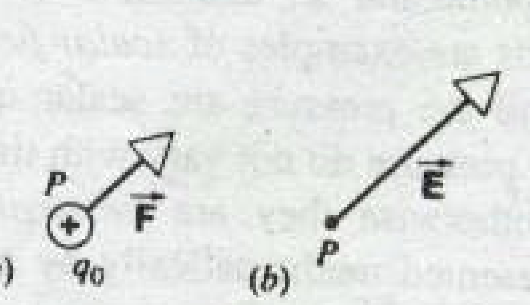

FIGURE 26-1. (*a*) Charged objects in the environment exert a force $\vec{F}$ on a positive test charge q_0 at point P. (*b*) The electric field at point P due to the charged objects in the environment.

* We usually specify that the test mass m_0 must be small. That is, we do not want it to change the Earth's gravitational field. If, for instance, we used a test mass of the size of the Moon, its gravitational force on the Earth would cause tidal effects that would change the distribution of mass on the Earth and thus change the gravitational force at various locations. To prevent this from happening, we keep m_0 much smaller than the mass of the Earth.

TABLE 26-1 Some Electric Fields

Location	*Electric Field* (N/C)
At the surface of a uranium nucleus	3×10^{21}
In a hydrogen atom, at the electron's average radius	5×10^{11}
Electric breakdown occurs in air	3×10^{6}
At the charged drum of a photocopier	10^5
The electron beam accelerator in a TV set	10^5
Near a charged plastic comb	10^3
In the lower atmosphere	10^2
Inside the copper wire of household circuits	10^{-2}

Dimensionally, the electric field is the force per unit charge, and its SI unit is the newton/coulomb (N/C), although it is more often given, as we discuss in Chapter 28, in the equivalent unit of volt/meter (V/m). Note the similarity with the gravitational field, in which g (which is usually expressed in units of m/s^2) can also be expressed as the force per unit mass in units of newton/kilogram (N/kg). Both the gravitational and electric fields can be expressed as a force divided by a property (mass or charge) of the test body. Table 26-1 shows some electric fields that occur in a few situations.

As we did in Eq. 26-2 for the gravitational force on a body, we can use the electric field to calculate the force on any charged body. Once we have found the electric field at a point (using our test body, for instance), we can find the electric force exerted on any object of charge q at that location as

$$\vec{\mathbf{F}} = q\vec{\mathbf{E}}. \tag{26-4}$$

Here the electric field $\vec{\mathbf{E}}$ is caused by other charges that may be present, not by the charge q. Equation 26-4 is simply a way of specifying the force that these other charges exert on q.

We can now understand how the electric field acts as an intermediary in the interaction between two charges q_1 and q_2. As illustrated in Fig. 26-2*a*, charge q_1 located at A sets up an electric field at all surrounding points. Let $\vec{\mathbf{E}}_1$ be its value at location B. We can find the value of the field by placing our test charge at B and measuring the force exerted on it by q_1. If we place a different charge q_2 at B, it will experience an electric force $\vec{\mathbf{F}}_{21}$, which we can calculate using Eq. 26-4: $\vec{\mathbf{F}}_{21} = q_2\vec{\mathbf{E}}_1$ (Fig. 26-2*b*). The situation is completely symmetric: we could instead first use the test charge to determine the field $\vec{\mathbf{E}}_2$ at A due to q_2 (Fig. 26-2*c*). Then placing q_1 at A, we find the force exerted on q_1 by q_2: $\vec{\mathbf{F}}_{12} = q_1\vec{\mathbf{E}}_2$ (Fig. 26-2*d*). From Newton's third law the forces are equal and opposite ($\vec{\mathbf{F}}_{21} = -\vec{\mathbf{F}}_{12}$), even though the electric fields set up by the two charges are different.

To use Eq. 26-3 as an operational procedure for measuring the electric field, we must apply the same caution we did in using a test mass to measure the gravitational field: the test charge should be sufficiently small so that it does not disturb the distribution of charges whose electric field we are trying to measure. That is, we should more properly write Eq. 26-3 as

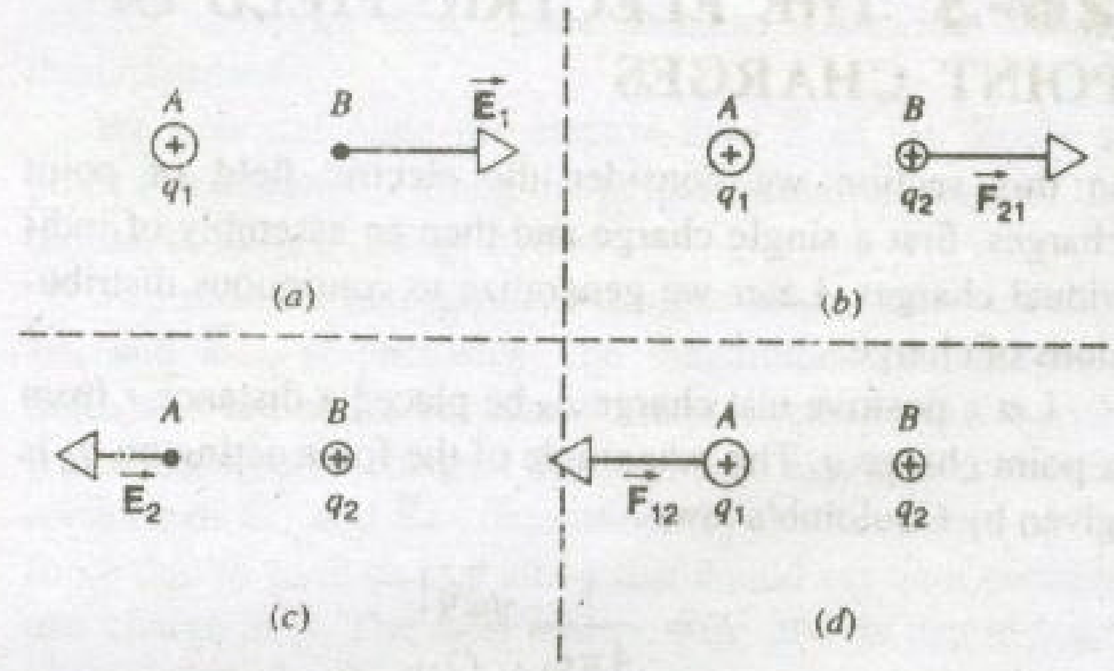

FIGURE 26-2. (*a*) q_1 at A sets up an electric field at B. (*b*) The electric field at B exerts a force on q_2. (*c*) q_2 at B sets up an electric field at A. (*d*) The electric field at A exerts a force on q_1. Note that $\vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21}$.

$$\vec{\mathbf{E}} = \lim_{q_0 \to 0} \frac{\vec{\mathbf{F}}}{q_0}, \tag{26-5}$$

even though we know from Chapter 25 that this limit in actuality cannot be taken to zero because the test charge can never be smaller than the elementary charge e. Of course, if we are *calculating* (rather than measuring) the electric field due to a specified collection of charges at fixed positions, neither the magnitude nor the sign of q_0 affects the result. As we show later in this chapter, electric fields of collections of charges can be calculated using Eq. 26-3 and Coulomb's law without direct reference to Eq. 26-5.

SAMPLE PROBLEM 26-1. An electron ($q = -e$) placed near a charged body experiences a force in the $+y$ direction of magnitude 3.60×10^{-8} N. (*a*) What is the electric field at that location? (*b*) What would be the force exerted by the same charged body on an alpha particle ($q = +2e$) placed at the location formerly occupied by the electron?

Solution (*a*) Using Eq. 26-4, we have

$$E_y = \frac{F_y}{q} = \frac{3.60 \times 10^{-8}\ \text{N}}{-1.60 \times 10^{-19}\ \text{C}} = -2.25 \times 10^{11}\ \text{N/C}.$$

The electric field is in the negative y direction.

(*b*) The force on the alpha particle follows from Eq. 26-4:

$$F_y = qE_y = 2(+1.60 \times 10^{-19}\ \text{C})(-2.25 \times 10^{11}\ \text{N/C}) = -7.20 \times 10^{-8}\ \text{N}.$$

The force is in the negative y direction, the same direction as the electric field but opposite to the direction of the force on the electron. In the same electric field the force on the alpha particle is twice as large as the force on the electron, because the charge of the alpha particle is twice the magnitude of the electron's charge.

26-3 THE ELECTRIC FIELD OF POINT CHARGES

In this section we consider the electric field of point charges, first a single charge and then an assembly of individual charges. Later we generalize to continuous distributions of charge.

Let a positive test charge q_0 be placed a distance r from a point charge q. The magnitude of the force acting on q_0 is given by Coulomb's law,

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_0|q|}{r^2}.$$

The magnitude of the electric field at the site of the test charge is, from Eq. 26-3,

$$E = \frac{F}{q_0} = \frac{1}{4\pi\epsilon_0}\frac{|q|}{r^2}. \qquad (26\text{-}6)$$

The direction of $\vec{\mathbf{E}}$ is the same as the direction of $\vec{\mathbf{F}}$, along a radial line from q, pointing outward if q is positive and inward if q is negative. Figure 26-3 shows the magnitude and direction of the electric field $\vec{\mathbf{E}}$ at various points near a positive point charge. How would this figure be drawn if the charge were negative?

To find $\vec{\mathbf{E}}$ for a group of N point charges, the procedure is as follows: (1) Calculate $\vec{\mathbf{E}}_n$ due to each charge n at the given point *as if it were the only charge present.* (2) Add these separately calculated fields vectorially to find the resultant field $\vec{\mathbf{E}}$ at the point. In equation form,

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 + \vec{\mathbf{E}}_3 + \cdots$$
$$= \sum \vec{\mathbf{E}}_n \qquad (n = 1, 2, 3, \ldots, N). \qquad (26\text{-}7)$$

The sum is a vector sum, taken over all the charges. Equation 26-7 (like Eq. 25-8) is an example of the application of the *principle of superposition,* which states, in this context, that at a given point the electric fields due to separate charge distributions simply add up (vectorially) or superimpose independently. This principle may fail when the magnitudes of the fields are extremely large, but it will be valid in all situations we discuss in this text.

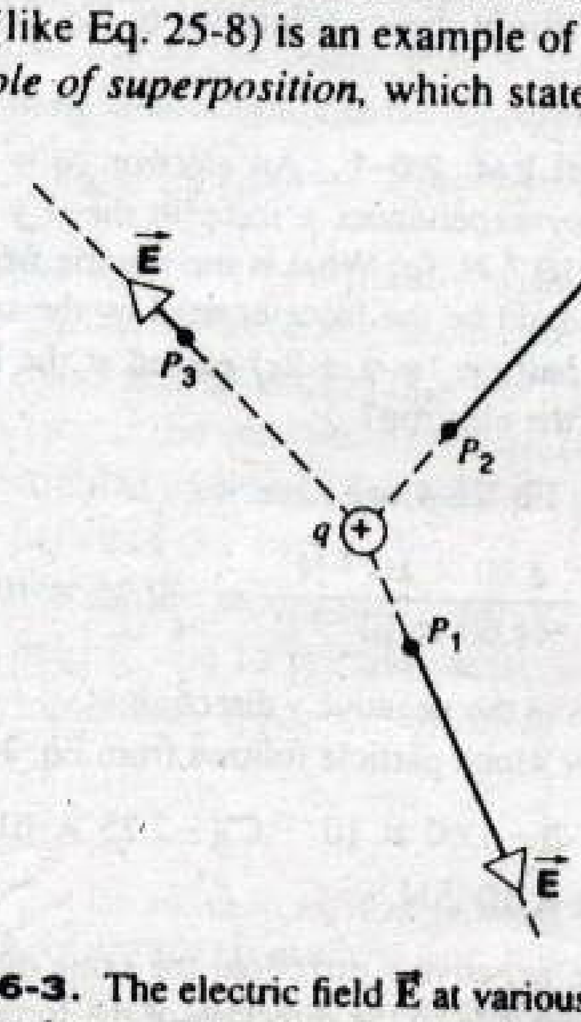

FIGURE 26-3. The electric field $\vec{\mathbf{E}}$ at various points near a positive point charge q. Note that the direction of $\vec{\mathbf{E}}$ is everywhere radially outward from q. The fields at P_1 and P_2, which are the same distance from q, are equal in magnitude. The field at P_3, which is twice as far from q as P_1 or P_2, has one-quarter the magnitude of the field at P_1 or P_2.

SAMPLE PROBLEM 26-2. In an ionized helium atom (a helium atom in which one of the two electrons has been removed), the electron and the nucleus are separated by a distance of 26.5 pm. What is the magnitude of the electric field due to the nucleus at the location of the electron?

Solution We use Eq. 26-6, with q (the charge of the nucleus) equal to $+2e$:

$$E = \frac{1}{4\pi\epsilon_0}\frac{|q|}{r^2} = \frac{(8.99 \times 10^9\ \mathrm{N\cdot m^2/C^2})[2(1.60 \times 10^{-19}\ \mathrm{C})]}{(26.5 \times 10^{-12}\ \mathrm{m})^2}$$
$$= 4.10 \times 10^{12}\ \mathrm{N/C}.$$

This value is 8 times the electric field that acts on an electron in hydrogen (see Table 26-1). The increase comes about because (1) the nuclear charge in helium is twice that in hydrogen, and (2) the electron in ionized helium is closer to its nucleus (by a factor of two) than is the case for an electron in the hydrogen atom.

SAMPLE PROBLEM 26-3. Figure 26-4 shows a charge q_1 of $+1.5\ \mu\mathrm{C}$ and a charge q_2 of $+2.3\ \mu\mathrm{C}$. The first charge is at the origin of an x axis, and the second is at a position $x = L$, where $L = 13$ cm. At what point P along the x axis is the electric field zero?

Solution The point must lie between the charges because only in this region do the forces exerted by q_1 and q_2 on a test charge oppose each other. If $\vec{\mathbf{E}}_1$ is the electric field due to q_1 and $\vec{\mathbf{E}}_2$ is that due to q_2, the magnitudes of these vectors must be equal, or

$$E_1 = E_2.$$

From Eq. 26-6 we then have

$$\frac{1}{4\pi\epsilon_0}\frac{q_1}{x^2} = \frac{1}{4\pi\epsilon_0}\frac{q_2}{(L-x)^2},$$

where x is the coordinate of point P. Taking the square root of each side and solving for x, we obtain

$$x = \frac{L}{1 \pm \sqrt{q_2/q_1}},$$

where we have taken into account that the square root may have either a positive or a negative value. Substituting numerical values for L, q_1, and q_2, we obtain

$$x = 5.8\ \mathrm{cm} \quad \text{and} \quad x = -54.6\ \mathrm{cm}.$$

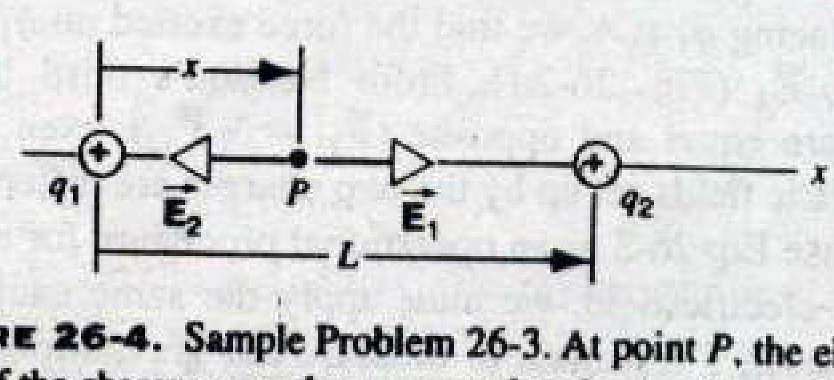

FIGURE 26-4. Sample Problem 26-3. At point P, the electric fields of the charges q_1 and q_2 are equal and opposite, so the net field at P is zero.

The first solution, which defines a point between the charges, is the solution we seek. The second solution defines a point to the left of the two charges. At this point it is true that $E_1 = E_2$, but the fields point in the same direction so their vector sum cannot be zero. Thus we are justified in discarding the second solution.

The Electric Dipole

Many objects found in nature can be successfully analyzed as isolated bodies having a net charge, as we have done so far in this chapter. Others show different kinds of behavior. One type of behavior is characteristic of an object that has no *net* charge but instead consists of equal positive and negative charges $+q$ and $-q$ separated by a fixed distance d. For example, an ionic molecule such as NaCl (in the high-temperature vapor state, not the familiar crystalline form) is electrically neutral but can be considered as a Na^+ ion joined to a Cl^- ion. For another example, a similar type of behavior of the water molecule is in part responsible for the large solubilities of many substances in water.

The configuration of two equal and opposite charges separated by a distance is called an *electric dipole*. In equations that describe electric dipoles, we find that the magnitude of the charge q on each of the components and their separation d often occur together as the product qd. It is convenient to define this quantity as the *electric dipole moment p:*

$$p = qd. \tag{26-8}$$

It turns out that this quantity behaves like a vector. We define the vector electric dipole moment to have a magnitude $p = qd$ and a direction pointing from the negative charge to the positive charge along the line joining the two charges. Figure 26-5a shows an electric dipole and its vector dipole moment. For example, in NaCl the magnitude of the charge q on each ion is e and the measured separation distance is 0.236 nm, so we expect the dipole moment of the molecule to be

$$p = ed = (1.60 \times 10^{-19}\ \text{C})(0.236 \times 10^{-9}\ \text{m}) = 3.78 \times 10^{-29}\ \text{C}\cdot\text{m}.$$

The measured value is 3.00×10^{-29} C·m, indicating that the electron is not entirely removed from Na and attached

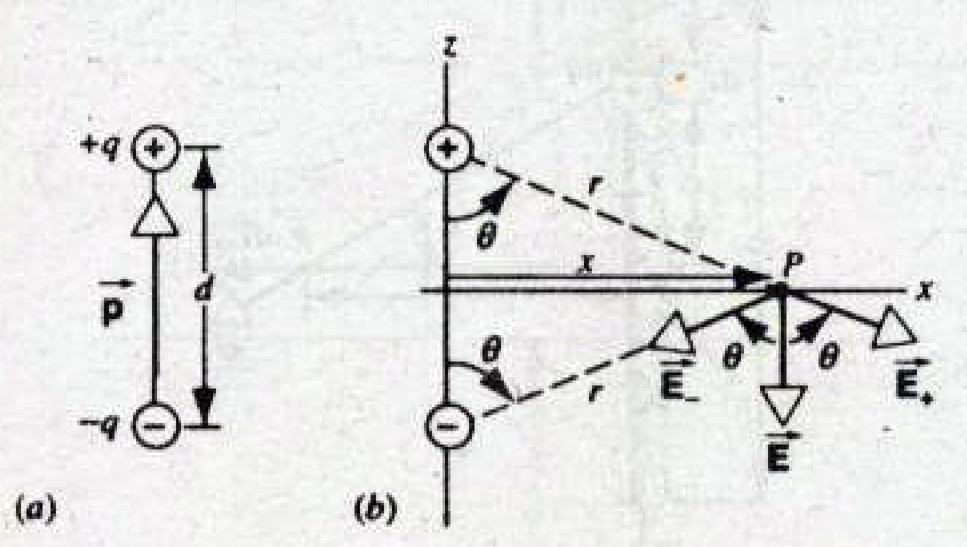

FIGURE 26-5. (a) Positive and negative charges of equal magnitude form an electric dipole. (b) The electric field $\vec{E}$ at any point is the vector sum of the fields due to the individual charges. At point P on the x axis, the field has only a z component.

to Cl. To a certain extent, the electron is shared between Na and Cl, resulting in a dipole moment somewhat smaller than expected.

We now calculate the electric field $\vec{E}$ of the dipole at point P a distance x along the perpendicular bisector of the dipole, as shown in Fig. 26-5b.

The positive and negative charges set up electric fields $\vec{E}_+$ and $\vec{E}_-$, respectively. The magnitudes of these two fields at P are equal, because P is equidistant from the positive and negative charges. Figure 26-5b also shows the directions of $\vec{E}_+$ and $\vec{E}_-$, determined by the directions of the force due to each charge alone that would act on a positive test charge at P. The total electric field at P is determined, according to Eq. 26-7, by the vector sum

$$\vec{E} = \vec{E}_+ + \vec{E}_-.$$

From Eq. 26-6, the magnitudes of the fields from each charge are given by

$$E_+ = E_- = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{q}{x^2 + (d/2)^2}. \tag{26-9}$$

Because the fields $\vec{E}_+$ and $\vec{E}_-$ have equal magnitudes and lie at equal angles θ with respect to the z direction as shown, the x component of the total field is $E_+ \sin\theta - E_- \sin\theta = 0$. The total field $\vec{E}$ therefore has only a z component, of magnitude

$$E = E_+ \cos\theta + E_- \cos\theta = 2E_+ \cos\theta. \tag{26-10}$$

From the figure we see that the cosine of the angle θ is determined according to

$$\cos\theta = \frac{d/2}{\sqrt{x^2 + (d/2)^2}}.$$

Substituting this result and Eq. 26-9 into Eq. 26-10, we obtain

$$E = (2)\frac{1}{4\pi\epsilon_0}\frac{q}{x^2 + (d/2)^2}\frac{d/2}{\sqrt{x^2 + (d/2)^2}}$$

or

$$E = \frac{1}{4\pi\epsilon_0}\frac{p}{[x^2 + (d/2)^2]^{3/2}}, \tag{26-11}$$

using Eq. 26-8 ($p = qd$) for the dipole moment.

Equation 26-11 gives the magnitude of the electric field at P due to the dipole. Note that the problem has cylindrical symmetry about the z axis; that is, we could have chosen the x axis to have any direction perpendicular to the dipole axis, and the field would be given by Eq. 26-11.

Often we observe the field of an electric dipole at points P whose distance x from the dipole is very large compared with the separation d. In this case we can simplify the dipole field somewhat by making use of the binomial expansion,

$$(1 + y)^n = 1 + ny + \frac{n(n-1)}{2!}y^2 + \cdots.$$

Let us rewrite Eq. 26-11 as

$$E = \frac{1}{4\pi\epsilon_0}\frac{p}{x^3}\frac{1}{[1 + (d/2x)^2]^{3/2}}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{p}{x^3}\left[1 + \left(\frac{d}{2x}\right)^2\right]^{-3/2}$$

and apply the binomial expansion to the factor in brackets, which gives

$$E = \frac{1}{4\pi\epsilon_0}\frac{p}{x^3}\left[1 + \left(-\frac{3}{2}\right)\left(\frac{d}{2x}\right)^2 + \cdots\right]$$

For $x >> d$ it is sufficient to keep only the first term in the brackets (the 1), and so we find an expression for the magnitude of the electric field due to a dipole at distant points in its median plane:

$$E = \frac{1}{4\pi\epsilon_0}\frac{p}{x^3}. \qquad (26\text{-}12)$$

An expression of a similar form is obtained for the field along the dipole axis (the z axis of Fig. 26-5*b*); see Problem 1. A more general result for the field at any point in the xz plane can also be calculated; see Problem 2. In all cases, the field at distant points varies with the distance r from the dipole as $1/r^3$. This is a characteristic result for the electric dipole field. The field varies more rapidly with distance than the $1/r^2$ dependence characteristic of a point charge.

There are also more complicated charge distributions that give electric fields that vary as higher inverse powers of r. See Exercise 11 and Problem 4 for examples of the $1/r^4$ variation of the field of an electric *quadrupole*.

26-4 ELECTRIC FIELD OF CONTINUOUS CHARGE DISTRIBUTIONS

In Section 25-5 we discussed the force exerted on a point charge by various continuous charge distributions. We analyzed those continuous charge distributions by considering them to be collections of infinitesimal charge elements, which we treated as point charges, and then integrating over the distribution to find the force. We use a similar method here to calculate the electric field due to continuous charge distributions. In fact, as we shall see, we can use the results of Section 25-5 to obtain the electric field due to the distributions considered in that section.

We first discuss the general method for finding the electric field of a continuous charge distribution. We divide the charge distribution into infinitesimal elements dq, expressing the charge element dq as $\lambda\, ds$, $\sigma\, dA$, or $\rho\, dV$, depending on whether the charge is distributed over a line (λ = linear charge density or charge per unit length), surface (σ = surface charge density or charge per unit area), or volume (ρ = volume charge density or charge per unit volume). Choosing an arbitrary charge element, we write the magnitude of the contribution to the electric field at the observation point P as if dq were a point charge:

$$dE = \frac{1}{4\pi\epsilon_0}\frac{|dq|}{r^2}, \qquad (26\text{-}13)$$

using Eq. 26-6. The direction of the vector $d\vec{E}$ is determined by the sign of dq according to the direction of the force that dq would exert on a positive test charge at P. The total resultant field at P for the entire distribution is obtained by adding the contributions from all the charge elements of the object, taking into account the different directions that all the $d\vec{E}$ might have:

$$\vec{E} = \int d\vec{E}. \qquad (26\text{-}14)$$

In Cartesian coordinates, we can regard Eq. 26-14 as a shorthand representation of the three component equations:

$$E_x = \int dE_x, \quad E_y = \int dE_y, \quad E_z = \int dE_z. \quad (26\text{-}15)$$

As we discuss below, we can often simplify the calculation by arguing on the basis of symmetry that one or two of these integrals vanish or that two of them have identical values.

A Uniform Line of Charge

As an example of the application of Eqs. 26-13 to 26-15, we consider the electric field due to a line of charge (a thin charged rod, for example) of length L having a uniform positive linear charge density $\lambda = q/L$, where q is the total charge carried by the rod. Figure 26-6 shows the geometry for the calculation. We wish to find the field at point P a distance y from the rod along its perpendicular bisector (the positive y axis). The magnitude of the electric field $d\vec{E}$ at point P due to the charge element dq is given by Eq. 26-13. We can conclude that $E_x = 0$, because none of the charge

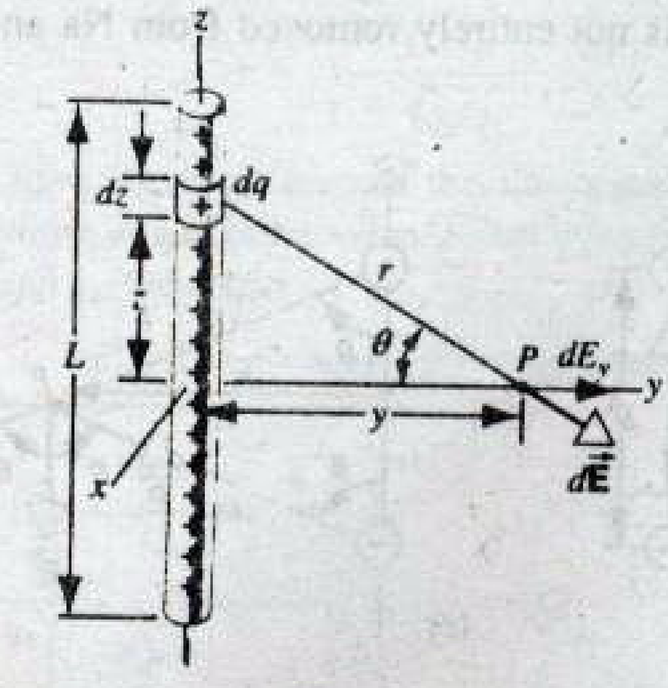

FIGURE 26-6. A uniformly charged rod. The electric field at point P is due to the total effect of all charge elements such as dq.

elements dq anywhere on the rod produces a $d\vec{\mathbf{E}}$ with an x component. We can also conclude from symmetry that $E_z = 0$, because for every dq at positive z there is a corresponding dq at negative z such that, when we add the $d\vec{\mathbf{E}}$ vectors from the two charge elements, the z components cancel. The only nonzero component of the electric field at P is E_y. We therefore have

$$dE_y = dE\cos\theta = \frac{1}{4\pi\epsilon_0}\frac{\lambda\,dz}{y^2+z^2}\frac{y}{\sqrt{y^2+z^2}},$$

where we have used Eq. 26-13 for dE with $dq = \lambda\,dz$, $\cos\theta = y/r$, and $r^2 = y^2 + z^2$. The total field at P is

$$E_y = \int dE_y = \int_{-L/2}^{+L/2}\frac{1}{4\pi\epsilon_0}\frac{\lambda y\,dz}{(y^2+z^2)^{3/2}}.$$

Carrying out the integration over z, with y held constant, we obtain (see integral 18 in Appendix I)

$$E_y = \frac{1}{4\pi\epsilon_0}\frac{\lambda L}{y\sqrt{y^2+L^2/4}}. \qquad (26\text{-}16)$$

This equation gives the electric field at point P on the positive y axis due to the line of charge. Note that we could have obtained this result directly from Eq. 25-15 for the force between the line of charge and the point charge q_0 by substituting λL for q and using Eq. 26-3, $E_y = F_y/q_0$.

As was the case with the electric dipole, this problem also has cylindrical symmetry about the z axis, and we could have chosen the y axis to point in any direction perpendicular to the axis of the rod and through its midpoint. Figure 26-7 shows a representation of the field in the xy plane due to a uniform positively charged rod.

As we did for the force calculated in Chapter 25, it is important to check our electric field calculations to verify that they have the correct limits. In the limit $y \to \infty$, Eq. 26-16 approaches the expression for the electric field of a point charge,

$$E_y = \frac{1}{4\pi\epsilon_0}\frac{q}{y^2},$$

where we have used $q = \lambda L$.

Often in situations involving line charges, our observation point is very close to the line, such that y is small in comparison with L. Taking the limit of Eq. 26-16 when $L \gg y$ with λ remaining constant, we have the electric field due to an infinitely long line of charge:

$$E_y = \frac{\lambda}{2\pi\epsilon_0 y}. \qquad (26\text{-}17)$$

The field is directed radially outward from the rod and depends inversely on the distance from the rod.

You may wonder about the usefulness of calculating the field due to an infinite line of charge when any real line of charge must have a finite length. However, for points close to the line and far from either end, Eq. 26-17 gives a very good and useful approximation to the electric field. The difference between the approximate result, Eq. 26-17, and the exact result, Eq. 26-16, is often negligible. The approximate result in this case may give more physical insight, because the variation of E with distance from the rod is more immediately apparent.

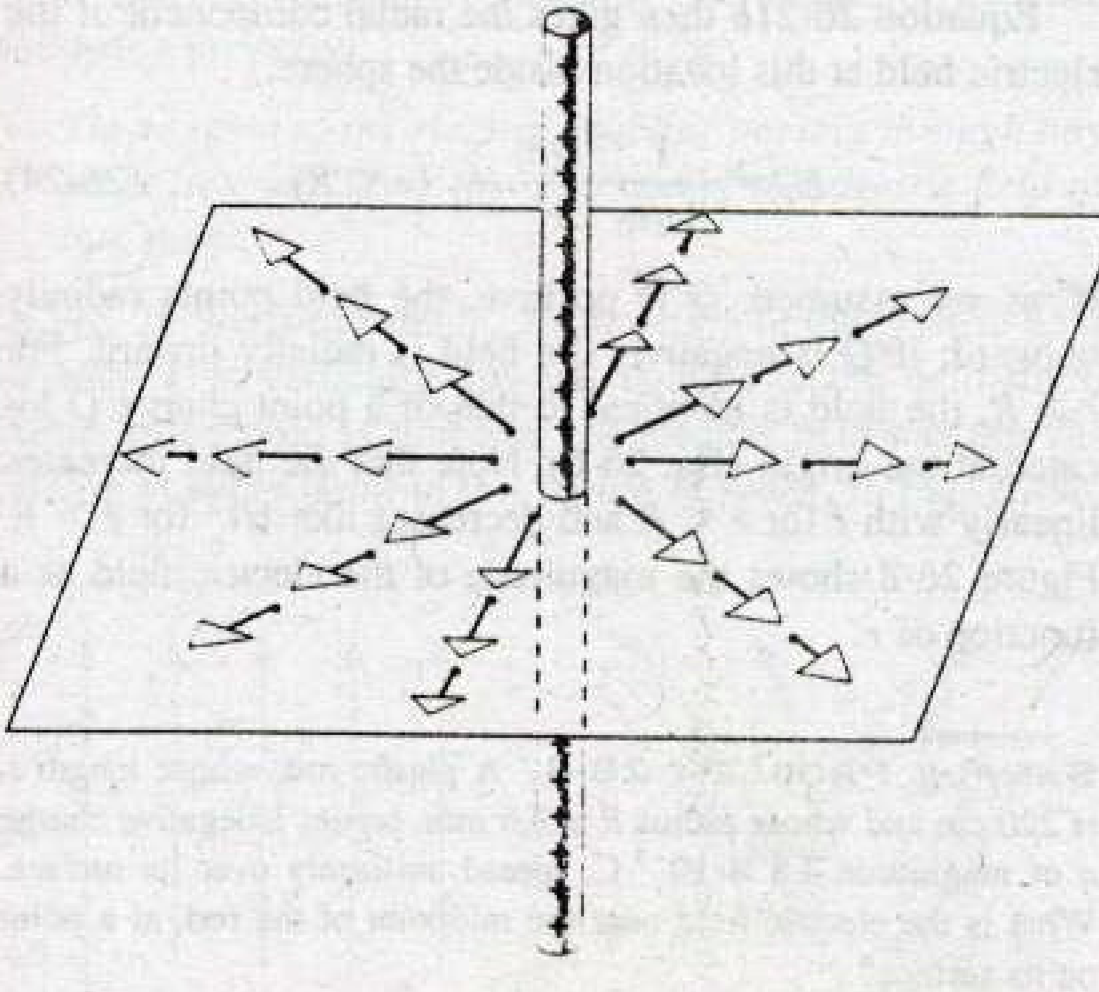

FIGURE 26-7. Electric field due to a positively charged rod. The field has cylindrical symmetry about the axis of the rod.

A Uniform Ring or Disk of Charge

To discuss the electric field due to a ring or disk of radius R carrying a uniform charge density, it is not necessary to do the complete calculation starting with Eq. 26-13. We have already calculated the force exerted on a point charge q_0 by a ring of charge or a disk of charge. The force exerted by a ring of charge on a point charge q_0 on the axis of the ring was given by Eq. 25-16. Using Eq. 26-3, $E_z = F_z/q_0$, we can find the electric field at a point on the positive z axis due to a ring of charge directly from Eq. 25-16:

$$E_z = \frac{\lambda}{2\epsilon_0}\frac{Rz}{(z^2+R^2)^{3/2}}, \qquad (26\text{-}18)$$

where we have used $q = \lambda(2\pi R)$. The electric field is directed along the axis of the ring (the z axis) and away from the ring. Equation 26-18 is valid for positive as well as negative z. If the ring is negatively charged, the field points along the axis in the opposite direction (toward the ring).

In a similar fashion, we can find the electric field due to a disk of charge from Eq. 25-17:

$$E_z = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right). \qquad (26\text{-}19)$$

Here we have expressed the electric field in terms of the surface charge density of the disk using $q = \sigma A = \sigma(\pi R^2)$. This expression gives the field at a point on the positive z

axis a distance z from the disk. The field points away from the disk if the disk is positively charged. Equation 26-19 is valid only for $z > 0$. How should it be modified if P is located on the negative z axis?

An Infinite Sheet of Charge

Let us now consider the limiting case of Eq. 26-19 as $R \to \infty$, so that the charged disk becomes an infinite sheet of charge. We assume that, as R increases, we add charge to the disk so that the surface charge density σ remains constant. Under these conditions, we can approximate Eq. 26-19 as

$$E_z = \frac{\sigma}{2\epsilon_0}. \qquad (26\text{-}20)$$

This turns out to be a very useful result, which is approximately valid for a disk of uniform charge density when we are close to the disk and far from any of its edges. In fact, if we are far from its edges, we cannot tell whether the charge distribution is spread over a circular area or over one that is square, rectangular, or irregularly shaped. As we will derive in the next chapter, this result is valid for any large uniformly charged sheet, no matter what its shape. The field has a uniform magnitude and (for a positively charged sheet) is directed away from the sheet of charge.

A Uniform Spherical Shell of Charge

In Section 25-5, we established two properties of a uniformly charged spherical shell: it exerts no force on a test charge in its interior, and at exterior points the force that it exerts on a test charge is the same as if all the charge of the shell were concentrated in a point at its center. We can use these properties to deduce the electric field due to a thin uniformly charged shell. Let the shell have radius R and charge q, which we assume for now to be positive. We then have the following results for the electric field at various distances from the center of the shell:

$$E = 0 \qquad (r < R) \qquad (26\text{-}21a)$$

$$E_r = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \qquad (r \geq R). \qquad (26\text{-}21b)$$

The subscript r on the electric field reminds us that the field points in the radial direction. These results follow directly from the force on a test charge at the different locations. Inside the shell, the electric field is zero. At exterior points, the electric field is radial and identical to that of a point charge, so it would look just like the field displayed in Fig. 26-3.

We can use the properties of shells of charge to deduce the electric field due to a spherically symmetric charge distribution in a sphere of radius R. For simplicity we assume the charge to be distributed uniformly throughout the sphere, so that its volume charge density is a constant. If Q is the total charge on the sphere, then the volume charge density is

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3}. \qquad (26\text{-}22)$$

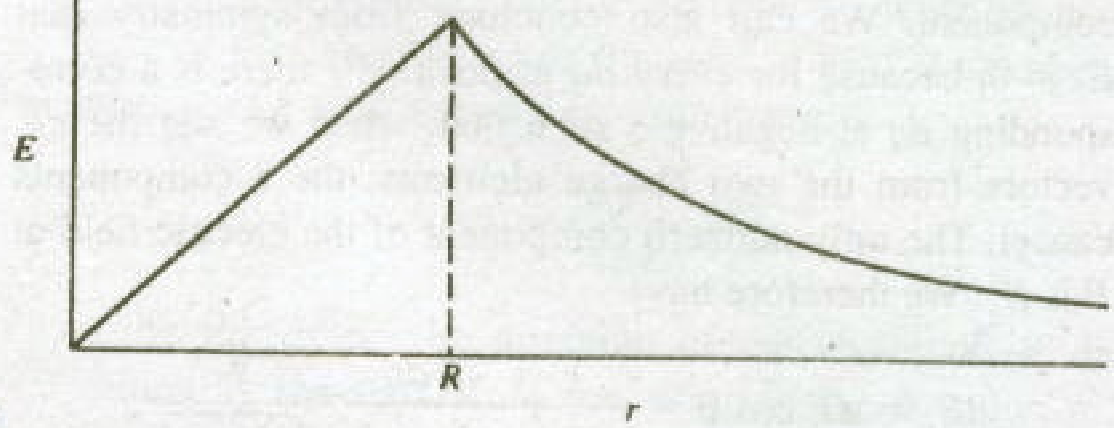

FIGURE 26-8. The magnitude of the electric field due to a uniformly charged sphere of radius R.

Imagine the sphere to be divided into many thin shells of radius r and thickness dr. If we place a test charge at a distance r from the origin and inside the shell ($r < R$), the electric field at the location of the test charge is due only to the shells at smaller radii; we know from Eq. 26-21a that $E = 0$ for all the shells of larger radii. Furthermore, we know from Eq. 26-21b that the field due to all the shells of smaller radii is the same as that of a point charge at the origin. The magnitude of that point charge is the same as the total charge of all the shells with radii smaller than r, or equivalently the total charge q inside the sphere of radius r, which is given by the volume charge density times the volume of the sphere of radius r:

$$q = \rho(\tfrac{4}{3}\pi r^3) = Q\frac{r^3}{R^3}, \qquad (26\text{-}23)$$

using the charge density from Eq. 26-22.

Equation 26-21b then gives the radial component of the electric field at this location inside the sphere:

$$E_r = \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3} \qquad (r < R). \qquad (26\text{-}24)$$

If, as we assumed, Q is positive, the field points radially outward; if Q is negative, the field is radially inward. For $r > R$, the field is identical to that of a point charge Q located at the origin (Eq. 26-6). Note that the field increases linearly with r for $r < R$ and decreases like $1/r^2$ for $r > R$. Figure 26-8 shows the magnitude of the electric field as a function of r.

SAMPLE PROBLEM 26-4. A plastic rod, whose length L is 220 cm and whose radius R is 3.6 mm, carries a negative charge q of magnitude 3.8×10^{-7} C, spread uniformly over its surface. What is the electric field near the midpoint of the rod, at a point on its surface?

Solution Although the rod is not infinitely long, for a point on its surface and near its midpoint it is effectively very long, so that we are justified in using Eq. 26-17. The linear charge density for the rod is

$$\lambda = \frac{q}{L} = \frac{-3.8 \times 10^{-7}\ \mathrm{C}}{2.2\ \mathrm{m}} = -1.73 \times 10^{-7}\ \mathrm{C/m}.$$

From Eq. 26-17 we then have, for $y = 0.0036$ m,

$$\begin{aligned} E_y &= \frac{\lambda}{2\pi\epsilon_0 y} \\ &= \frac{-1.73 \times 10^{-7}\ \mathrm{C/m}}{(2\pi)(8.85 \times 10^{-12}\ \mathrm{C^2/N\cdot m^2})(0.0036\ \mathrm{m})} \\ &= -8.6 \times 10^5\ \mathrm{N/C}. \end{aligned}$$

The negative sign tells us that, because the rod is negatively charged, the direction of the electric field is radially inward, toward the axis of the rod. Sparking occurs in dry air at atmospheric pressure at an electric field strength of about 3×10^6 N/C. The field strength we calculated is lower than this by a factor of about 3.4, so that sparking should not occur.

26-5 ELECTRIC FIELD LINES

The concept of the electric field was introduced in the early 19th century by Michael Faraday. Faraday did not develop the mathematical representation of the electric field; instead, he developed a graphical representation, in which he imagined the space around an electric charge to be filled with *lines of force*. Today we no longer attach the same reality to the lines of force that Faraday did, but we retain them as a convenient way to visualize the electric field. We refer to these lines as *electric field lines*.

Figure 26-9*a* shows the electric field lines representing a uniform field. Note that the lines are parallel and equally spaced. Figure 26-9*b* shows lines representing a nonuniform field. By convention, we draw the field lines with the following property:

> *The tangent to the electric field line passing through any point in space gives the direction of the electric field at that point.*

In Fig. 26-9*a*, for example, the direction of the electric field at point P is vertically upward, tangent to the field lines. Because the field is uniform, the electric field has this direction at every point in this region of space. In Fig. 26-9*b*, which shows a nonuniform field, the electric field has different directions at points P_1 and P_2, in each case tangent to the electric field line passing through that point.

For the electric field lines to have this property, they must also be drawn so that

> *The electric field lines start on positive charges and end on negative charges.*

For example, Fig. 26-10 represents the field lines for an isolated positive point charge (or a small sphere of positive charge). The lines point radially outward, so that at any point P the field is radial. The field lines begin on the positive charge and extend to infinity, since there are no negative charges in this region. If the charge were negative, the field lines would point in the opposite direction (radially inward).

One final property of electric field lines is that

> *The magnitude of the electric field at any point is proportional to the number of field lines per unit cross-sectional area perpendicular to the lines.*

In other words, the more densely packed the field lines are near any point, the stronger is the field at that point. Figure 26-9*b*, for example, suggests that the magnitude of the field is larger at the bottom of the drawing (near point P_1) than it is at the top of the drawing (near point P_2). In Fig. 26-9*a*, on the other hand, the spacing of the field lines is the same at all points, suggesting that the field has the same magnitude everywhere. For a point charge (Fig. 26-10), the field lines are close together near the charge and further apart away from the charge, which indicates that the field grows weaker as the distance from the charge increases.

The uniform field near a large sheet of positive charge is shown in Fig. 26-11. The direction of the field is perpendicular to the sheet. Near the edges of the sheet, the field becomes nonuniform and is no longer directed perpendicular to the sheet, but as long as we stay close to the center of the sheet and far away from any edge, the field is very nearly uniform. Once again, the field lines extend to infinity.

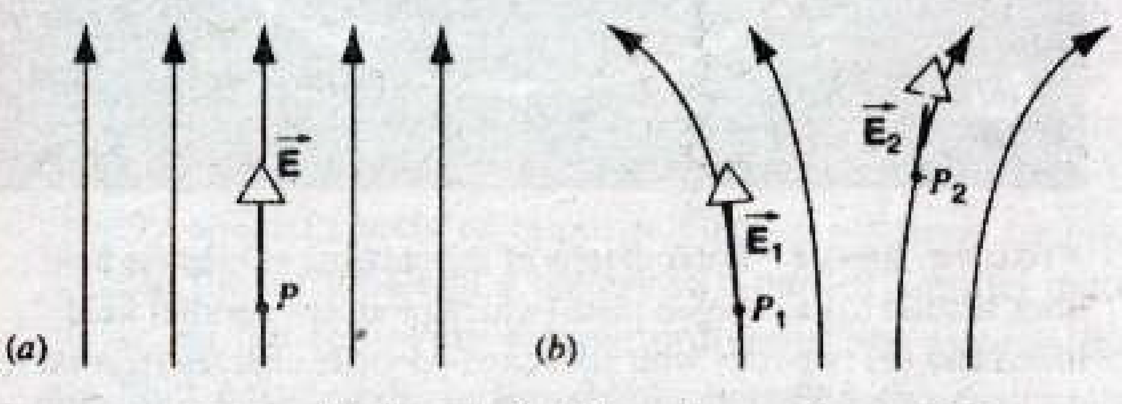

FIGURE 26-9. (*a*) Electric field lines for a uniform field. (*b*) Electric field lines for a nonuniform field.

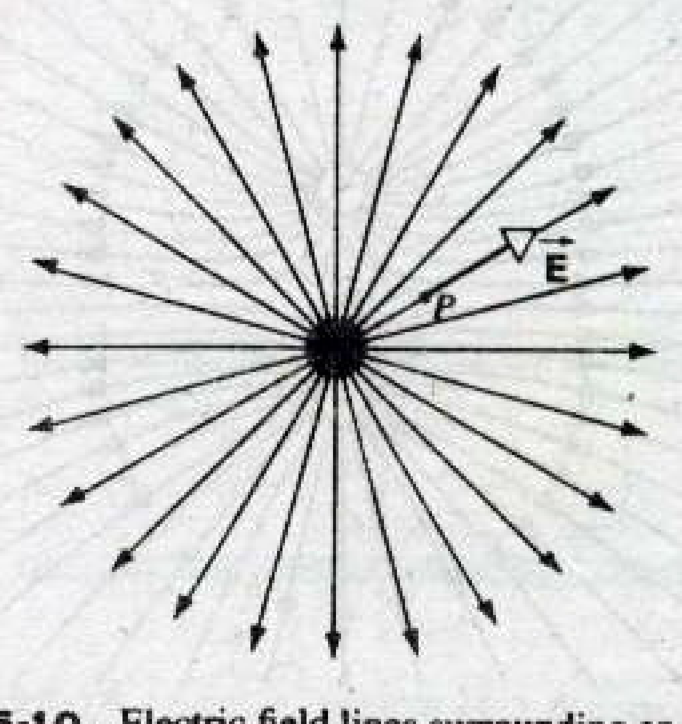

FIGURE 26-10. Electric field lines surrounding an isolated positive point charge or positively charged uniform sphere. The field at an arbitrary point P is shown.

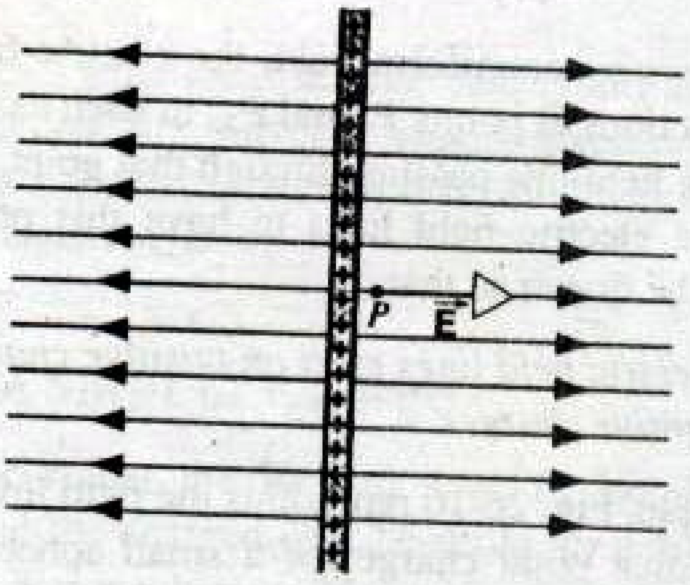

FIGURE 26-11. Electric field lines near a thin uniform sheet of charge. Here we are looking at the edge of the sheet, which is oriented perpendicular to the page.

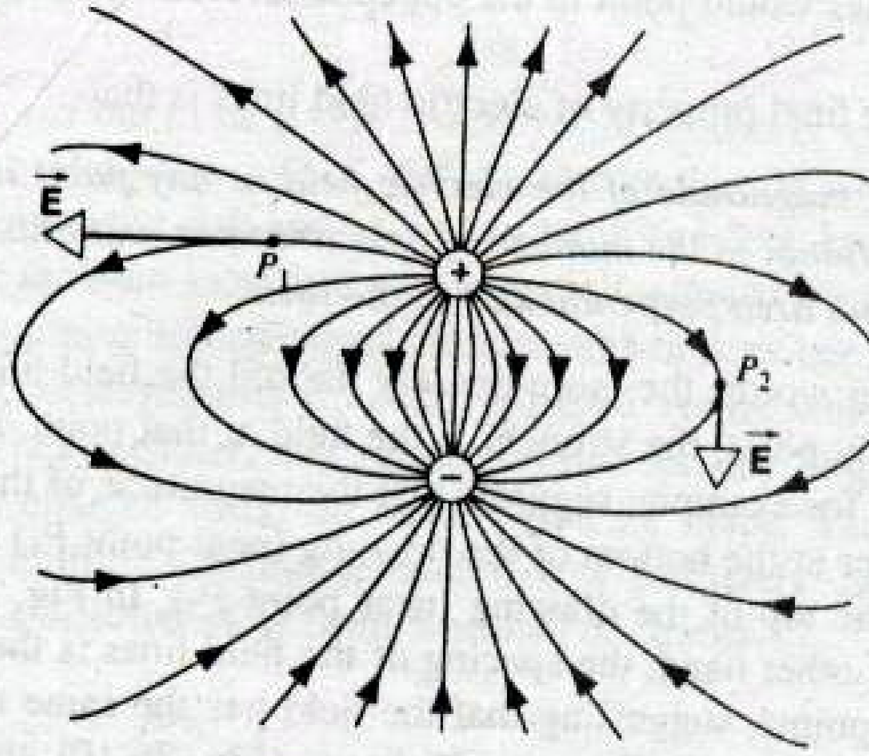

FIGURE 26-12. Electric field lines for an electric dipole.

Figure 26-12 shows the field near a dipole (which illustrates how the field lines begin on positive charges and end on negative charges), and Fig. 26-13 shows the field near two equal positive charges. Note the differences between the two patterns. In the region directly between the charges, the density of field lines is larger in Fig. 26-12 than in Fig.

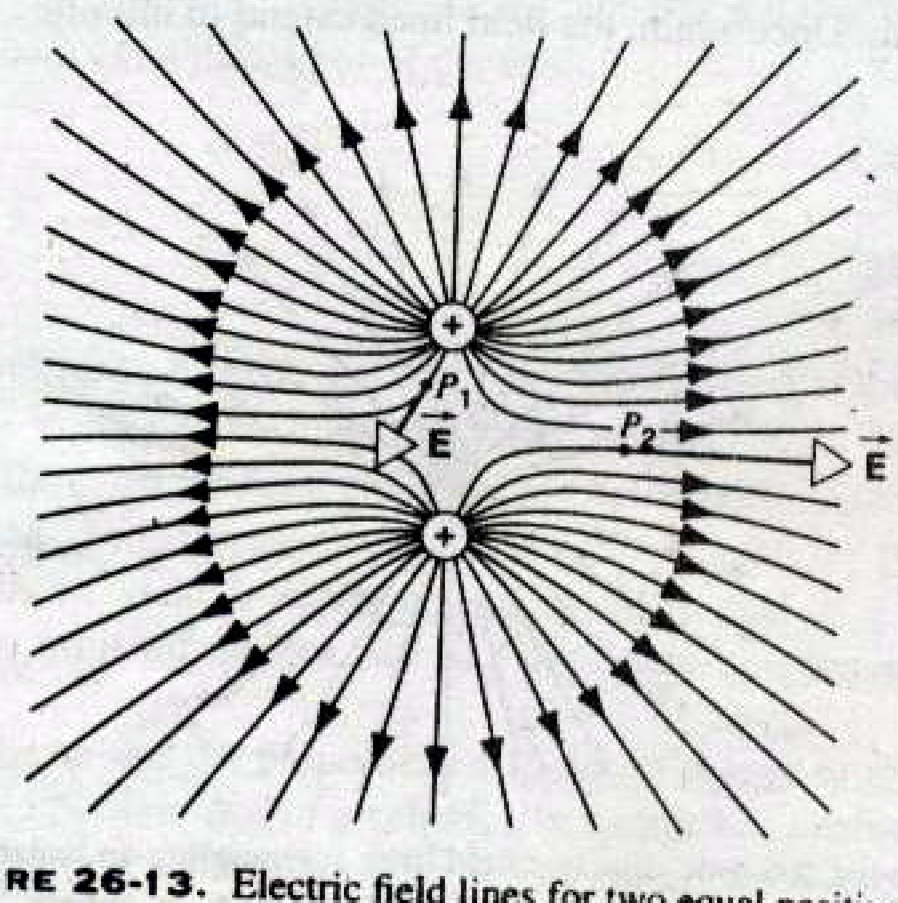

FIGURE 26-13. Electric field lines for two equal positive charges.

26-13, suggesting that the dipole gives a larger field in that region than the two positive charges. Because the electric field is zero halfway between the charges in Fig. 26-13, no electric field lines can be drawn through that point. As we travel far from the charges in Fig. 26-13, the pattern begins

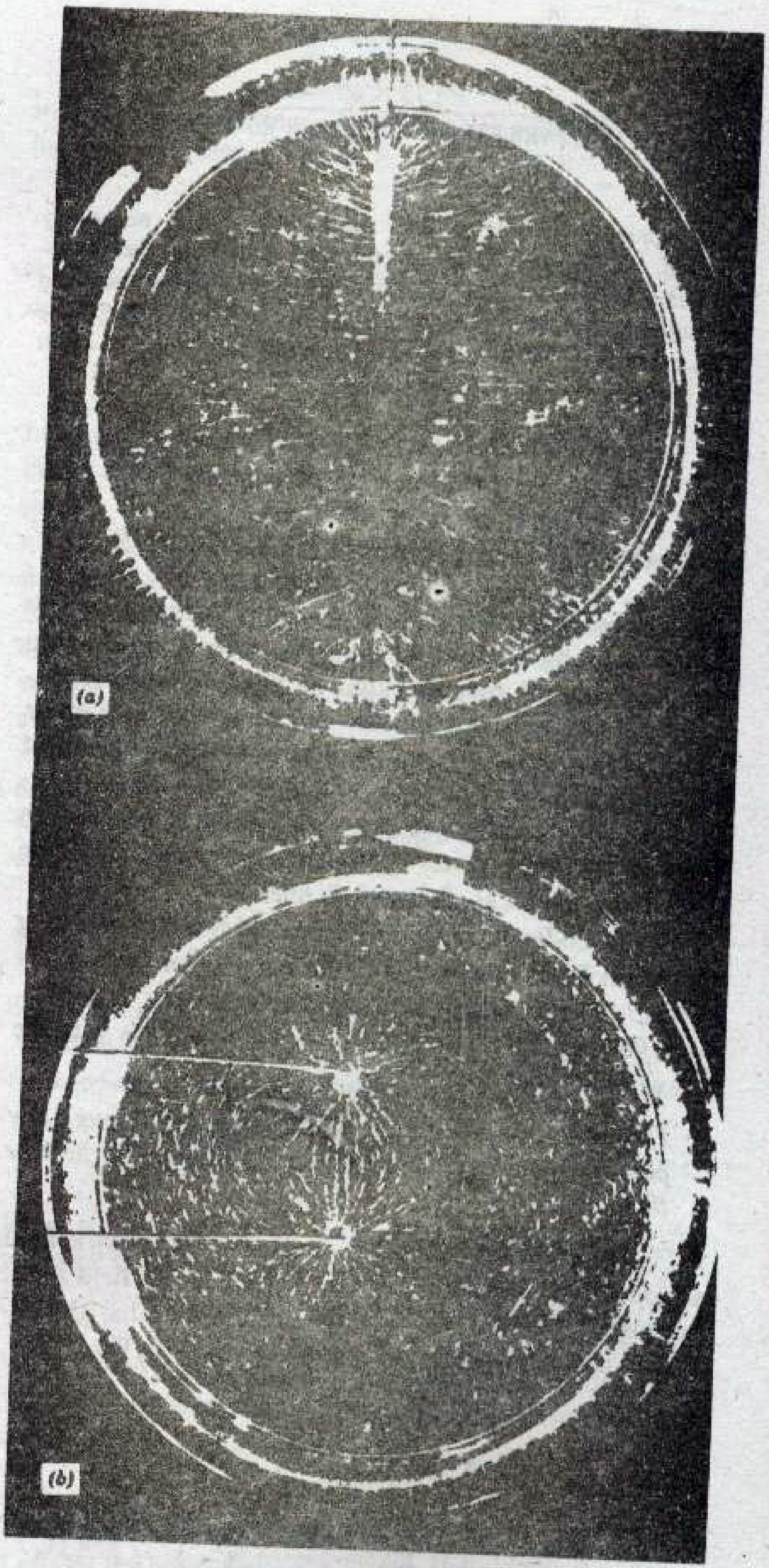

FIGURE 26-14. Photographs of the patterns of electric field lines around (*a*) a charged plate (which produces parallel field lines) and (*b*) two rods with equal and opposite charges (similar to the electric dipole of Fig. 26-12). The patterns were made visible by suspending grass seed in an insulating liquid.

to resemble that of a single charge (as in Fig. 26-10). At external points in the median plane (the plane perpendicular to the page and midway between the charges), the field is small for the dipole and directed downward, while for the equal charges the field is larger and directed radially outward, as indicated at the points P_2 in Figs. 26-12 and 26-13.

These drawings can be very useful in helping us to visualize the pattern of electric field lines. However, keep in mind that they represent only a two-dimensional "slice" through what is in reality a three-dimensional pattern. The relative spacings of the field lines in two dimensions do not strictly correspond with the spacings of the three-dimensional pattern, and the spacings of field lines in our two-dimensional drawings have no direct mathematical relationship to the magnitude of the field, other than to suggest regions where the field may be uniform or may be increasing or decreasing in magnitude.

The pattern of electric field lines can be made visible by applying an electric field to a suspension of tiny objects in an insulating fluid. Figure 26-14 shows photographs of patterns that resemble the drawings of electric field lines for a charged sheet and an electric dipole.

26-6 A POINT CHARGE IN AN ELECTRIC FIELD

In preceding sections we have considered the first part of the charge ⇄ field ⇄ charge interaction: Given a collection of charges, what is the resulting electric field? In this section and the next, we consider the second part: What happens when we put a charged particle in a known electric field?

From Eq. 26-4, we know that a particle of charge q in an electric field $\vec{E}$ experiences a force $\vec{F}$ given by

$$\vec{F} = q\vec{E}.$$

To study the motion of the particle in the electric field, all we need do is use Newton's second law, $\sum\vec{F} = m\vec{a}$, where the resultant force on the particle includes the electric force and any other forces that may act.

As we did in our original study of Newton's laws, we can achieve a simplification if we consider the case in which the force is constant. We therefore begin by considering cases in which the electric field and the corresponding electric force are uniform (that is, they do not vary with location) and constant (they do not vary with time). Such a situation can be achieved in practice in the region near a large uniform sheet of charge, as we discussed in Section 26-4. For even greater uniformity, we can use a pair of closely spaced sheets of opposite charge, obtained by connecting the terminals of a battery to a pair of parallel metal plates. In the following sample problems, we assume that the field exists only in the region between the plates and drops suddenly to zero when the particle leaves that region. In reality the field decreases rapidly over a distance that is of the order of the spacing between the plates; when this distance is small, we do not make too large an error in calculating the motion of the particle if we ignore the edge effect.

SAMPLE PROBLEM 26-5. A charged drop of oil of radius $R = 2.76\ \mu\text{m}$ and density $\rho = 918\ \text{kg/m}^3$ is maintained in equilibrium under the combined influence of its weight and a downward uniform electric field of magnitude $E = 1.65 \times 10^6$ N/C (Fig. 26-15). (*a*) Calculate the magnitude and sign of the charge on the drop. Express the result in terms of the elementary charge e. (*b*) The drop is exposed to a radioactive source that emits electrons. Two electrons strike the drop and are captured by it, changing its charge by two units. If the electric field remains at its constant value, calculate the resulting acceleration of the drop.

Solution (*a*) To keep the drop in equilibrium, its weight mg must be balanced by an equal electric force of magnitude qE acting upward. Because the electric field is given as being in the downward direction, the charge q on the drop must be negative for the electric force to point in a direction opposite the field. The equilibrium condition is

$$\sum\vec{F} = m\vec{g} + q\vec{E} = 0.$$

Taking y components, we obtain

$$-mg + q(-E) = 0$$

or, solving for the unknown q,

$$q = -\frac{mg}{E} = -\frac{\frac{4}{3}\pi R^3 \rho g}{E}$$

$$= -\frac{\frac{4}{3}\pi(2.76 \times 10^{-6}\ \text{m})^3(918\ \text{kg/m}^3)(9.80\ \text{m/s}^2)}{1.65 \times 10^6\ \text{N/C}}$$

$$= -4.80 \times 10^{-19}\ \text{C}.$$

If we write q in terms of the electronic charge $-e$ as $q = n(-e)$, where n is the number of electronic charges on the drop, then

$$n = \frac{q}{-e} = \frac{-4.80 \times 10^{-19}\ \text{C}}{-1.60 \times 10^{-19}\ \text{C}} = 3.$$

(*b*) If we add two additional electrons to the drop, its charge will become

$$q' = (n + 2)(-e) = 5(-1.60 \times 10^{-19}\ \text{C}) = -8.00 \times 10^{-19}\ \text{C}.$$

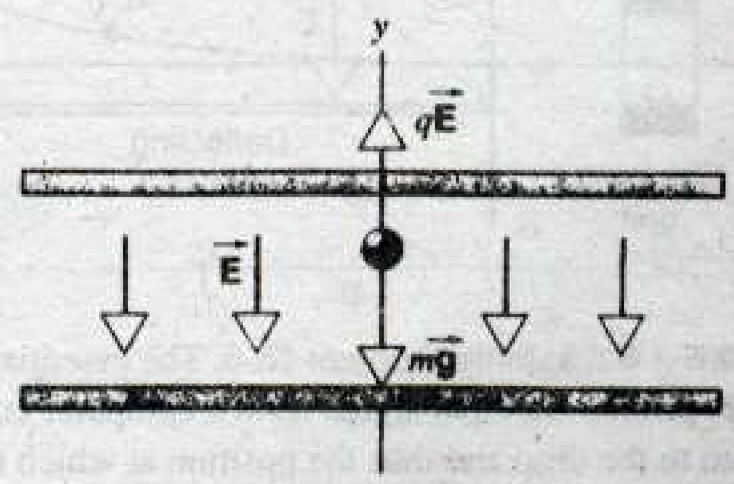

FIGURE 26-15. Sample Problem 26-5. A negatively charged drop is placed in a uniform electric field $\vec{E}$. The drop moves under the combined influence of its weight $m\vec{g}$ and the electric force $q\vec{E}$.

Newton's second law can be written

$$\sum \vec{F} = m\vec{g} + q'\vec{E} = m\vec{a}$$

and, taking y components, we obtain

$$-mg + q'(-E) = ma.$$

We can now solve for the acceleration:

$$a = -g - \frac{q'E}{m}$$

$$= -9.80\ \text{m/s}^2 - \frac{(-8.00 \times 10^{-19}\ \text{C})(1.65 \times 10^6\ \text{N/C})}{\frac{4}{3}\pi(2.76 \times 10^{-6}\ \text{m})^3(918\ \text{kg/m}^3)}$$

$$= -9.80\ \text{m/s}^2 + 16.3\ \text{m/s}^2 = +6.5\ \text{m/s}^2.$$

The drop accelerates in the positive y direction.

In this calculation, we have ignored the viscous drag force, which is usually quite important in this situation. We have, in effect, found the acceleration of the drop at the instant it acquired the extra two electrons. The drag force, which depends on the velocity of the drop, is initially zero if the drop starts from rest, but it increases as the drop begins to move, and so the acceleration of the drop will decrease in magnitude.

This experimental configuration forms the basis of the Millikan oil-drop experiment, which was used to measure the magnitude of the electronic charge. The experiment is discussed later in this section.

SAMPLE PROBLEM 26-6. Figure 26-16 shows the deflecting electrode system of an ink-jet printer. An ink drop whose mass m is 1.3×10^{-10} kg carries a charge q of -1.5×10^{-13} C and enters the deflecting plate system with a speed $v = 18$ m/s. The length L of these plates is 1.6 cm, and the magnitude of the electric field E between the plates is 1.4×10^6 N/C. What is the vertical deflection of the drop at the far edge of the plates? Ignore the varying electric field at the edges of the plates.

Solution Let t be the time of passage of the drop through the deflecting system. The vertical and the horizontal displacements are given by

$$y = \tfrac{1}{2}at^2 \quad \text{and} \quad L = vt,$$

respectively, in which a is the vertical acceleration of the drop.

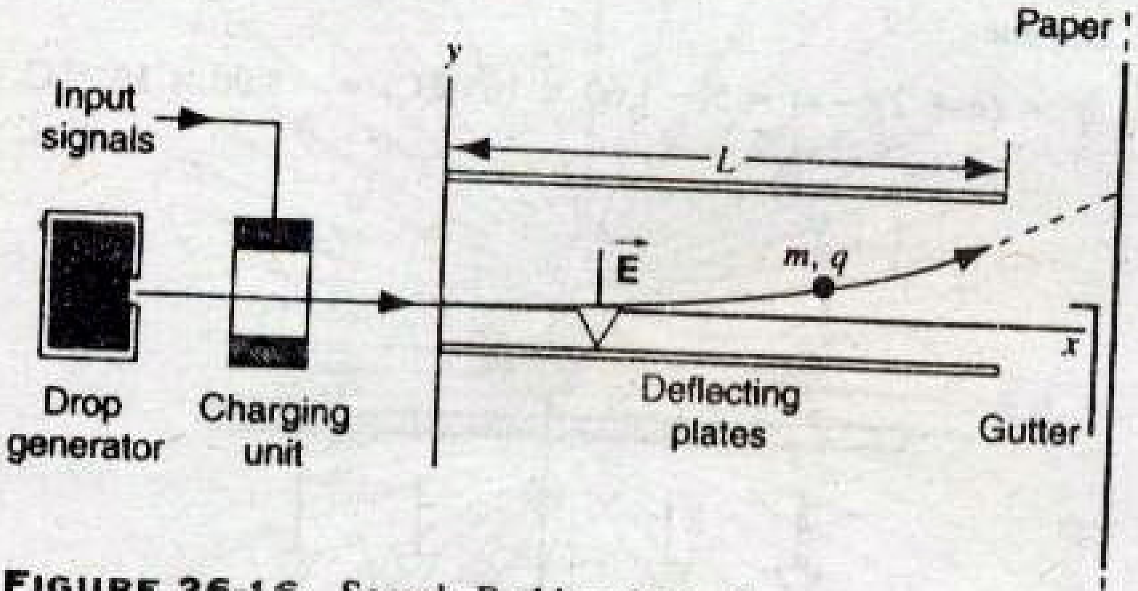

FIGURE 26-16. Sample Problem 26-6. The essential features of an ink-jet printer. An input signal from a computer controls the charge given to the drop and thus the position at which the drop strikes the paper. A transverse force from the electric field $\vec{E}$ is responsible for deflecting the drop. The drop moves in a parabolic path while it is between the plates, and it moves along a straight line (shown dashed) after it leaves the plates.

As in the previous sample problem, we can write the y component of Newton's second law as $-mg + q(-E) = ma$. As you can easily verify, the electric force acting on the drop, $-qE$, is much greater in this case than the gravitational force mg so that the acceleration of the drop can be taken to be $-qE/m$. Eliminating t between the two displacement equations and substituting this value for a leads to

$$y = \frac{-qEL^2}{2mv^2}$$

$$= \frac{-(-1.5 \times 10^{-13}\ \text{C})(1.4 \times 10^6\ \text{N/C})(1.6 \times 10^{-2}\ \text{m})^2}{(2)(1.3 \times 10^{-10}\ \text{kg})(18\ \text{m/s})^2}$$

$$= 6.4 \times 10^{-4}\ \text{m} = 0.64\ \text{mm}.$$

The deflection at the paper will be larger than this because the ink drop follows a straight-line path to the paper after leaving the deflecting region, as shown by the dashed line in Fig. 26-16. To aim the ink drops so that they form the characters well, it is necessary to control the charge q on the drops—to which the deflection is proportional—to within a few percent. In our treatment, we have again neglected the viscous drag forces that act on the drop; they are substantial at these high drop speeds.

Measuring the Elementary Charge

We know today that electric charge is quantized; that is, it appears only in integral multiples of the elementary charge e, whose currently accepted value is $1.602176462 \times 10^{-19}$ C with an experimental uncertainty that shows up only in the last two digits. This modern value, like that of nearly all fundamental constants of physics, has been obtained from a variety of interlocking and increasingly precise experiments.

How did we first learn that charge was quantized, and how was the value of e first measured? The earliest definitive answers to these questions were obtained from experiments done by the American physicist Robert A. Millikan* (1868–1953). For this and related work Millikan received the 1923 Nobel Prize in physics.

Figure 26-17 shows Millikan's apparatus. An atomizer introduces oil droplets into chamber A. Some of the drops can become charged (positive or negative) in the process. We consider a drop of charge q (assumed negative); this drop drifts into chamber C through a small hole in plate P_1.

If there is no electric field in chamber C, two forces act on the drop, its weight mg and an upwardly directed viscous drag force, whose magnitude is proportional to the speed of the falling drop. The drop quickly comes to a

*For details of Millikan's experiments, see Henry A. Boorse and Lloyd Motz (eds.), *The World of the Atom* (Basic Books, 1966), Chapter 40. For the point of view of two physicists who knew Millikan as graduate students, see "Robert A. [illegible]ikan, Physics Teacher," by Alfred Romer, *The Physics Teacher*, February 1978, p. 78, and "My Work with Millikan on the Oil-Drop Experiment," by Harvey Fletcher, *Physics Today*, June 1982, p. 43.

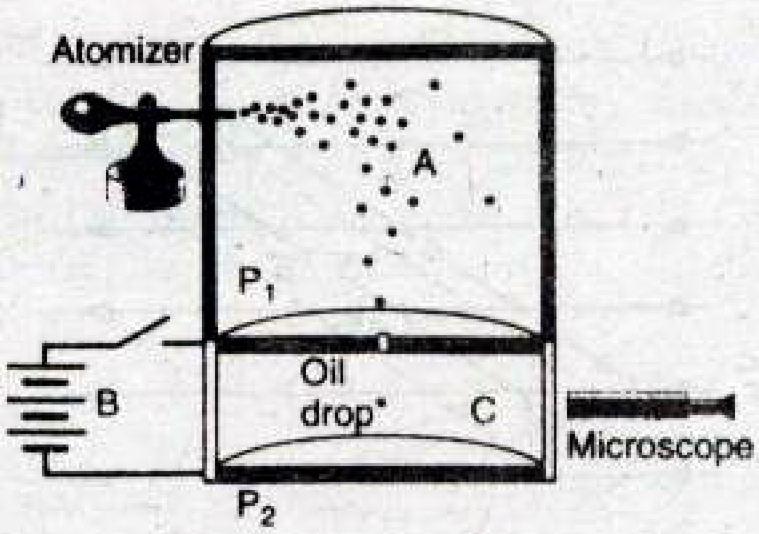

FIGURE 26-17. The Millikan oil-drop apparatus for measuring the elementary charge e. The motion of a drop is observed in chamber C, where the drop is acted on by gravity, the electric field set up by the battery B, and, if the drop is moving, a viscous drag force.

constant terminal speed v at which these two forces are just balanced.

A downward electric field $\vec{\mathbf{E}}$ is now set up in the chamber by connecting battery B between plates P_1 and P_2. A third force, $q\vec{\mathbf{E}}$, now acts on the drop. If q is negative, this force points upward, and—we assume—the drop now drifts upward, at a new terminal speed v'. In each case, the drag force points in the direction opposite to that in which the drop is moving and has a magnitude proportional to the speed of the drop. The charge q on the drop can be found from measurements of v and v'.

Millikan found that the values of q were all consistent with the relation

$$q = ne \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots .$$

That is, the charges on the drops occurred only in integer multiples of a certain fundamental quantity, the elementary charge e, which Millikan deduced to have a value of 1.64×10^{-19} C, completely consistent with the currently accepted value. Millikan's experiment provides convincing proof that charge is quantized.

Motion in Nonuniform Electric Fields (Optional)

So far we have considered only uniform fields, in which the electric field is constant in magnitude and direction over the region in which the particle moves. Often, however, we must deal with fields that are not uniform. For example, we consider a ring of positive charge, as shown in Fig. 26-18. The electric field on the axis of the ring is given by Eq. 26-18. Suppose we project a charged particle with initial speed v_0 along the z axis toward the ring from a large distance. As the particle moves along the axis, the electric field (and therefore the electric force on the particle) increases.

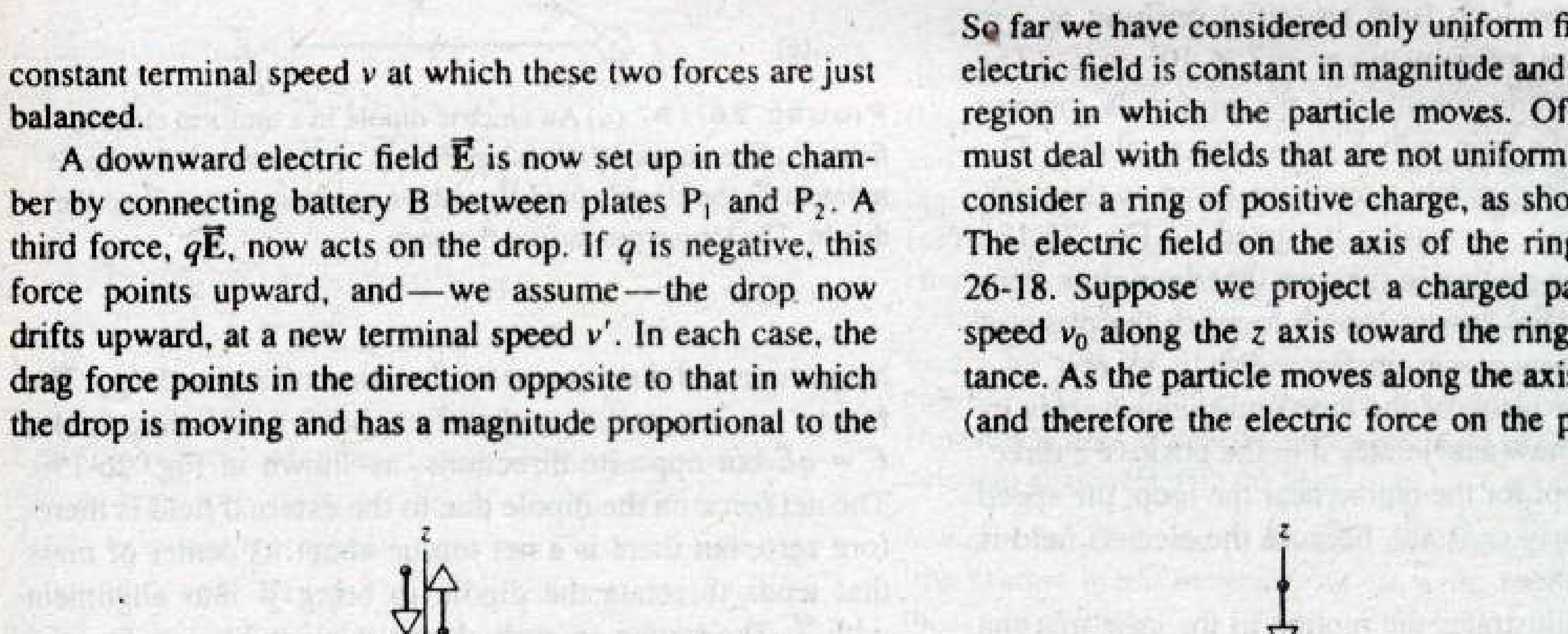

FIGURE 26-18. (*a*) The motion of a proton projected along the axis of a uniform positively charged ring. The position and velocity are shown. The proton comes instantaneously to rest at a time of about 8×10^{-7} s and reverses its motion. The points are the results of a numerical calculation; the curves are drawn through the points. (*b*) If the initial velocity of the proton is increased sufficiently, it can pass through the ring; its speed is a minimum as it passes through the center of the ring.

Neglecting gravity and considering only the electric force on the particle, how can we analyze its subsequent motion?

In such cases, we must use analytical methods for position-dependent forces similar to those discussed in Section 5-5 for time-dependent forces. An equivalent method is to follow the procedure given in Section 12-5 because, as we discuss in Chapter 28, the electrostatic force is a conservative force. Alternatively, we can use numerical techniques to find the solution by dividing the motion into infinitesimally small intervals over which we can take the acceleration to be nearly constant; an approximate solution can be obtained with a computer.

For this calculation, we use a ring of radius $R = 3$ cm and linear charge density $\lambda = +2 \times 10^{-7}$ C/m. A proton ($q = +1.6 \times 10^{-19}$ C, $m = 1.67 \times 10^{-27}$ kg) is projected along the axis of the loop from an initial position at $z = +0.5$ m with initial velocity $v_{z0} = -7 \times 10^5$ m/s. (The negative initial velocity means that the proton is moving downward toward the loop, which lies in the xy plane.) The positively charged loop exerts a repulsive force on the positively charged proton, decreasing its speed. In Fig. 26-18a we plot the resulting motion in the case that the proton does not have enough initial kinetic energy to reach the plane of the loop. The proton comes instantaneously to rest at a point just above the plane of the loop and then reverses its motion as the loop now accelerates it in the positive z direction. Note that except for the region near the loop, the speed of the proton is nearly constant, because the electric field is weak at larger distances.

Figure 26-18b illustrates the motion in the case that the proton has more than enough initial kinetic energy to reach the plane of the loop. The repulsive force slows the proton's motion but does not stop it. The proton passes through the loop, with the magnitude of its velocity reaching a minimum as it passes through the loop. Once again, far from the loop the proton moves with very nearly constant velocity. ■

26-7 A DIPOLE IN AN ELECTRIC FIELD

When we place an electric dipole (Fig. 26-5a) in an *external* electric field, the force on the positive charge will be in one direction and the force on the negative charge will be in another direction. In accounting for the net effect of these forces, it is convenient to use the dipole moment vector $\vec{\mathbf{p}}$, which has magnitude $p = qd$ and which points from the negative charge toward the positive charge. Writing the dipole moment as a vector permits us to write the fundamental relationships involving electric dipoles in a concise form.

Figure 26-19a shows a dipole in a uniform electric field $\vec{\mathbf{E}}$. (This field is *not* that of the dipole itself but is produced by an external agent not shown in the figure.) The dipole moment $\vec{\mathbf{p}}$ makes an angle θ with the direction of the field. We assume the field to be uniform, so that $\vec{\mathbf{E}}$ has the same

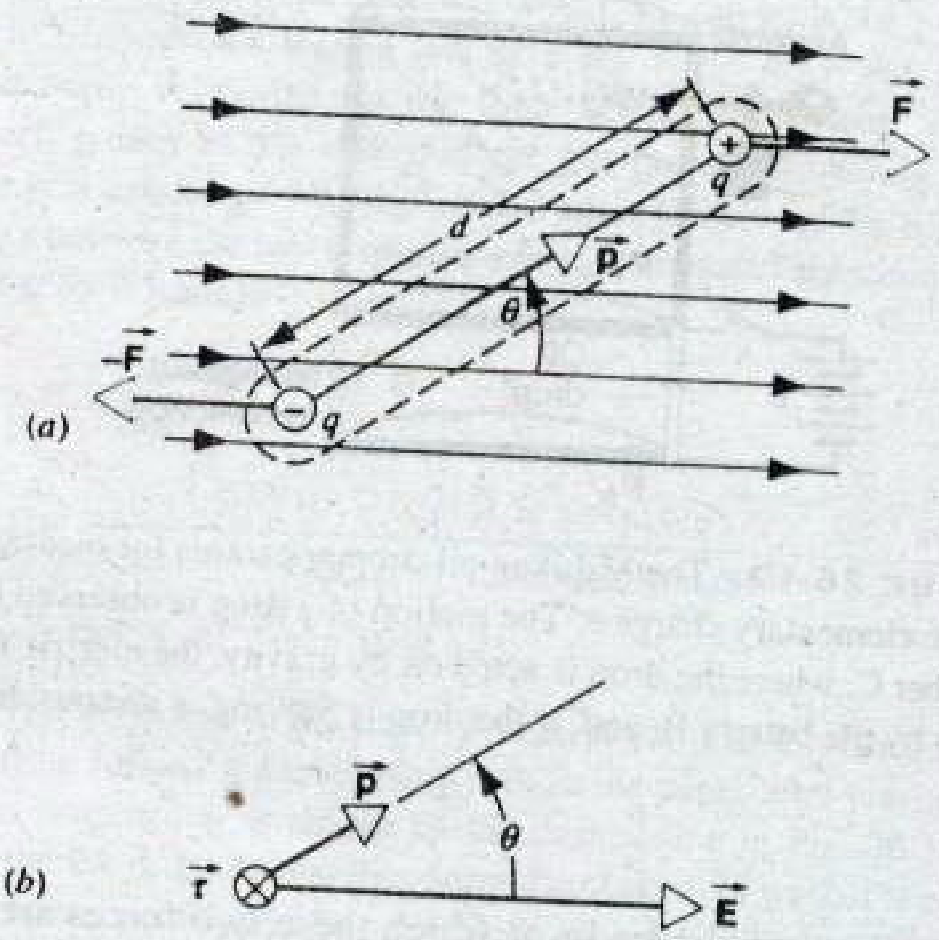

FIGURE 26-19. (a) An electric dipole in a uniform electric field. (b) The vector relationship $\vec{\tau} = \vec{\mathbf{p}} \times \vec{\mathbf{E}}$ between the dipole moment $\vec{\mathbf{p}}$, the electric field $\vec{\mathbf{E}}$, and the resultant torque $\vec{\tau}$ on the dipole. The torque points into the page.

magnitude and direction at the location of $+q$ and $-q$. The forces on $+q$ and $-q$ therefore have equal magnitudes $F = qE$ but opposite directions, as shown in Fig. 26-19a. The net force on the dipole due to the external field is therefore zero, but there is a net torque about its center of mass that tends to rotate the dipole to bring $\vec{\mathbf{p}}$ into alignment with $\vec{\mathbf{E}}$. The torque on each charge is given by $\tau = Fr_\perp$; the net torque about the center of the dipole due to the two forces has magnitude

$$\tau = F\frac{d}{2}\sin\theta + F\frac{d}{2}\sin\theta = Fd\sin\theta, \quad (26\text{-}25)$$

and its direction is perpendicular to the plane of the page and into the page, as indicated in Fig. 26-19b. We can write Eq. 26-25 as

$$\tau = (qE)d\sin\theta = (qd)E\sin\theta = pE\sin\theta. \quad (26\text{-}26)$$

Equation 26-26 can be written in vector form as

$$\vec{\tau} = \vec{\mathbf{p}} \times \vec{\mathbf{E}}, \quad (26\text{-}27)$$

which is consistent with the directional relationships for the cross product, as shown by the three vectors in Fig. 26-19b.

As is generally the case in dynamics when conservative forces act (the electrostatic force is conservative, as we discuss in Chapter 28), we can represent the system equally well using either force equations or energy equations. Let us therefore consider the work done by the electric field in turning the dipole through an angle θ. Using the appropriate expression for work in rotational motion (Eq. 11-25), the work done by the external field in turning the dipole from an initial angle θ_0 to a final angle θ is

$$W = \int dW = \int_{\theta_0}^{\theta} \vec{\tau} \cdot d\vec{\theta} = \int_{\theta_0}^{\theta} -\tau\, d\theta, \quad (26\text{-}28)$$

where $\vec{\tau}$ is the torque exerted by the external electric field. The minus sign in Eq. 26-28 is necessary because the torque τ tends to *decrease* θ; in vector terminology, $\vec{\tau}$ and $d\vec{\theta}$ are in opposite directions, so $\vec{\tau}\cdot d\vec{\theta} = -\tau\, d\theta$. Combining Eq. 26-28 with Eq. 26-26, we obtain

$$W = \int_{\theta_0}^{\theta} -pE\sin\theta\, d\theta = -pE\int_{\theta_0}^{\theta}\sin\theta\, d\theta$$
$$= pE(\cos\theta - \cos\theta_0). \qquad (26\text{-}29)$$

Since the work done by the agent that produces the external field is equal to the negative of the change in potential energy of the system of field + dipole, we have

$$\Delta U = U(\theta) - U(\theta_0) = -W = -pE(\cos\theta - \cos\theta_0). \qquad (26\text{-}30)$$

We arbitrarily define the reference angle θ_0 to be 90° and choose the potential energy $U(\theta_0)$ to be zero at that angle. At any angle θ the potential energy is then

$$U = -pE\cos\theta, \qquad (26\text{-}31)$$

which can be written in vector form as

$$U = -\vec{\mathbf{p}}\cdot\vec{\mathbf{E}}. \qquad (26\text{-}32)$$

Thus U is a minimum when $\vec{\mathbf{p}}$ and $\vec{\mathbf{E}}$ are parallel.

A water molecule has an electric dipole moment. In a microwave oven, the electric field of the microwave radiation tends to rotate the dipole moment of water molecules into alignment with the field. A free water molecule would simply oscillate back and forth about its equilibrium position, but in materials (such as food) interactions between neighboring water molecules convert the angular motion due to the torque (or, equivalently, the kinetic energy of rotation arising from the decreased potential energy of the dipole in the field) into internal energy. The direction of the electric field reverses every 2×10^{-10} s, and as the dipole moments continually try to follow the field they transfer energy that cooks the food.

We can interpret the motion of a dipole in an external field either on the basis of a torque that rotates the dipole into alignment with the field (Eq. 26-27) or a potential energy that becomes a minimum when the dipole is aligned with the field (Eq. 26-32). The choice between the two interpretations is usually based on convenience in applications to a particular problem.

SAMPLE PROBLEM 26-7. A molecule of water vapor (H_2O) has an electric dipole moment of magnitude $p = 6.2 \times 10^{-30}$ C·m. (This large dipole moment is responsible for many of the properties that make water such an important substance, such as its ability to act as an almost universal solvent.) Figure 26-20 is a representation of this molecule, showing the three nuclei and the surrounding electron distributions. The electric dipole moment $\vec{\mathbf{p}}$ is represented by a vector on the axis of symmetry. The dipole moment arises because the effective center of positive charge does not coincide with the effective center of negative charge. (A contrasting case is that of a molecule of carbon dioxide, CO_2. Here

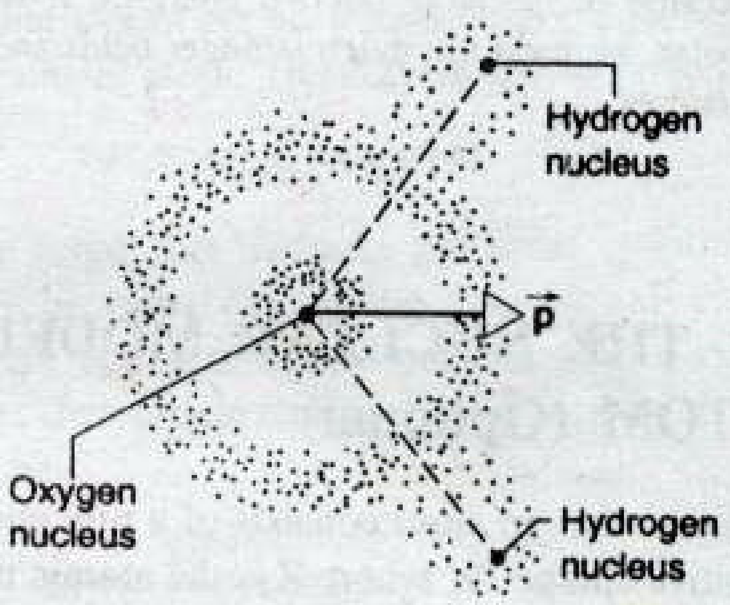

FIGURE 26-20. A molecule of H_2O, showing the three nuclei, the electron distributions, and the electric dipole moment vector $\vec{\mathbf{p}}$.

the three atoms are joined in a straight line, with a carbon in the middle and oxygens on either side. The center of positive charge and the center of negative charge coincide at the center of mass of the molecule, and the electric dipole moment of CO_2 is zero.) (*a*) How far apart are the effective centers of positive and negative charge in a molecule of H_2O? (*b*) What is the maximum torque on a molecule of H_2O in a typical laboratory electric field of magnitude 1.5×10^4 N/C? (*c*) Suppose the dipole moment of a molecule of H_2O is initially pointing in a direction opposite to the field. How much work is done by the electric field in rotating the molecule into alignment with the field?

Solution (*a*) There are 10 electrons and, correspondingly, 10 positive charges in this molecule. We can write, for the magnitude of the dipole moment,

$$p = qd = (10e)(d),$$

in which d is the separation we are seeking and e is the elementary charge. Thus

$$d = \frac{p}{10e} = \frac{6.2 \times 10^{-30}\ \text{C}\cdot\text{m}}{(10)(1.60 \times 10^{-19}\ \text{C})}$$
$$= 3.9 \times 10^{-12}\ \text{m} = 3.9\ \text{pm}.$$

This is about 4% of the OH bond distance in this molecule.

(*b*) As Eq. 26-26 shows, the torque is a maximum when $\theta = 90°$. Substituting this value in that equation yields

$$\tau = pE\sin\theta = (6.2 \times 10^{-30}\ \text{C}\cdot\text{m})(1.5 \times 10^4\ \text{N/C})(\sin 90°)$$
$$= 9.3 \times 10^{-26}\ \text{N}\cdot\text{m}.$$

(*c*) The work done in rotating the dipole from $\theta_0 = 180°$ to $\theta = 0°$ is given by Eq. 26-29,

$$W = pE(\cos\theta - \cos\theta_0)$$
$$= pE(\cos 0° - \cos 180°)$$
$$= 2pE = (2)(6.2 \times 10^{-30}\ \text{C}\cdot\text{m})(1.5 \times 10^4\ \text{N/C})$$
$$= 1.9 \times 10^{-25}\ \text{J}.$$

By comparison, the average translational contribution to the internal energy ($=\frac{3}{2}kT$) of a molecule at room temperature is 6.2×10^{-21} J, which is 33,000 times larger. For the conditions of this problem, thermal agitation would overwhelm the tendency of the dipoles to align themselves with the field. That is, if we had a collection of molecules at room temperature with randomly oriented dipole moments, the application of an electric field of this magnitude would have a negligible influence on aligning the dipole moments, because of the large internal energies. If we wish to align the dipoles, we must use much stronger fields and/or much lower temperatures.

26-8 THE NUCLEAR MODEL OF THE ATOM (Optional)

Today we know that an atom consists of a tiny nucleus carrying a positive charge Ze, where Z is the atomic number of the atom. The nucleus is surrounded by a much larger volume containing Z electrons, each carrying a charge of $-e$, so that the atom as a whole is electrically neutral. We also know that the nucleus contains a very large fraction (typically greater than 99.995%) of the mass of the atom.

In the early years of the 20th century these facts were not known, and there was much speculation about the structure of the atom and especially about the distribution of its positive charge. According to one theory that was popular at that time, the positive charge is distributed more or less uniformly throughout the entire spherical volume of the atom. This model of the structure of the atom is called the *Thomson model* after J. J. Thomson, who proposed it. (Thomson was the first to measure the charge-to-mass ratio of the electron and is therefore often credited as the discoverer of the electron.) It is also called the "plum pudding" model, because the electrons are imbedded throughout the diffuse sphere of positive charge like raisins in a plum pudding.

One way of testing this model is to determine the electric field of the atom by probing it with a beam of positively charged projectiles that pass nearby. The particles in the beam are deflected or *scattered* by the electric field of the atom. In the following discussion, we consider only the effect on the projectile of the sphere of positive charge. We assume that the projectile is both much *less* massive than the atom and much *more* massive than an electron. In this way the electrons have a negligible effect on the scattering of the projectile, and the atom can be assumed to remain at rest while the projectile is deflected.

We can estimate the deflection for a Thomson-model atom, in which the positive charge is uniformly distributed throughout the volume of the atom. The electric field due to a uniform sphere of positive charge was given by Eq. 26-6 for points outside the sphere of charge and by Eq. 26-24 for points inside. Let us calculate the electric field at the surface, which, as Fig. 26-8 shows, is the *largest* possible field that this distribution can produce. We consider a heavy atom such as gold, which has a positive charge Q of $79e$ and a radius R of about 1.0×10^{-10} m. Neglecting the electrons, the electric field at $r = R$ due to the positive charges is

$$E_{max} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2} = 1.1 \times 10^{13}\ \text{N/C}.$$

For the projectiles in our experiment, let us use a beam of alpha particles, which have a positive charge q of $2e$ and a mass m of 6.6×10^{-27} kg. Alpha particles are nuclei of helium atoms, which are emitted in certain radioactive decay processes. A typical kinetic energy for such a particle might be about $K = 6$ MeV or 9.6×10^{-13} J. At this energy you can easily verify that the particle has a speed of about 1.7×10^7 m/s.

Let the particle pass near the surface of the atom, where it experiences the largest electric field that this atom could exert. The corresponding force on the particle is

$$F = qE_{max} = 3.5 \times 10^{-6}\ \text{N}.$$

Figure 26-21 shows a schematic diagram of a scattering experiment. The actual calculation of the deflection is relatively complicated, but we can make some approximations that simplify the calculation and permit a rough estimate of the maximum deflection. Let us assume that the above force is constant and acts only during the time Δt it takes the projectile to travel a distance equal to a diameter of the atom, as indicated in Fig. 26-21. This time interval is

$$\Delta t = \frac{2R}{v} = 1.2 \times 10^{-17}\ \text{s}.$$

The force gives the particle a transverse acceleration a, which produces a transverse velocity Δv given by

$$\Delta v = a\,\Delta t = \frac{F}{m}\Delta t = 6.4 \times 10^3\ \text{m/s}.$$

The particle will be deflected by a small angle θ that can be estimated to be about

$$\theta = \tan^{-1}\frac{\Delta v}{v} = 0.02°.$$

This type of scattering experiment was first done by Ernest Rutherford and his collaborators at the University of

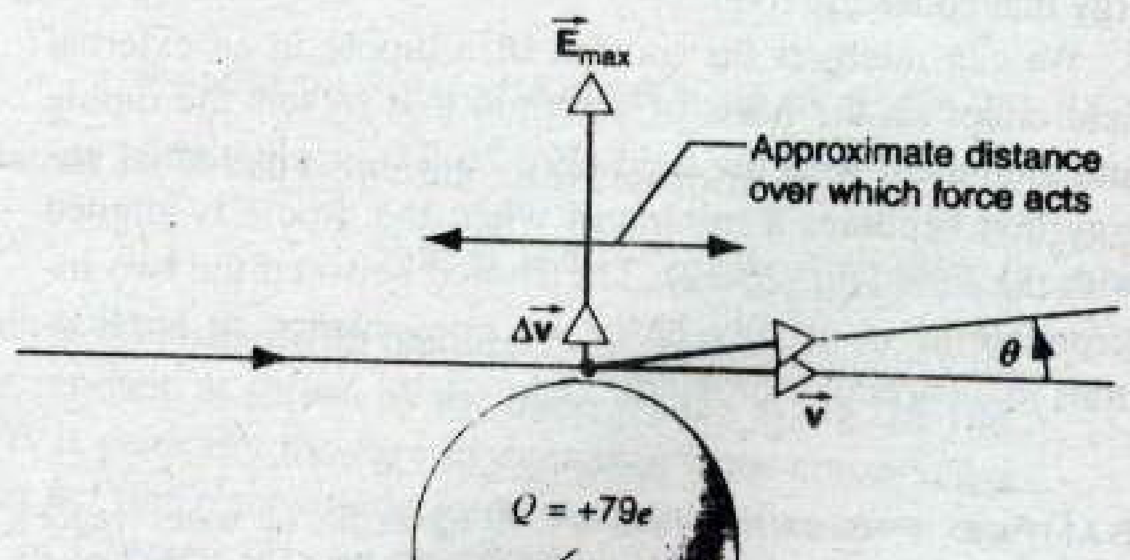

FIGURE 26-21. The scattering of a positively charged projectile passing near the surface of an atom, represented by a uniform sphere of positive charge. The electric field on the projectile causes a transverse deflection by an angle θ.

Manchester (England) in 1911. They passed a beam of alpha particles through a thin gold foil and determined the relative probability for alpha particles to be scattered through various angles θ relative to their original direction. Of course they could not control how the alpha particles passed through any particular atom; in fact, rather than grazing the edge, most alpha particles would pass through the volume of the atom pictured in Fig. 26-21 and (according to the Thomson model) be deflected by less than the maximum angle we have calculated.

The results of the experiment showed that, although nearly all of the alpha particles were deflected by angles no greater than a few hundredths of a degree, a small number (perhaps 1 in 10^4) were deflected by angles greater than 90°. This result is in complete disagreement with the Thomson model and led Rutherford to comment: "It was quite the most incredible event that ever happened to me in my life. It was almost as incredible as if you had fired a 15-inch shell at a piece of tissue paper and it came back and hit you."

Based on this kind of scattering experiment, Rutherford concluded that the positive charge of an atom was *not* diffused throughout a sphere of the same size as the atom, but instead was concentrated in a tiny region (the *nucleus*) near the center of the atom. In the case of a gold atom, the nucleus has a radius of about 7×10^{-15} m (7 fm), roughly 10^{-4} times smaller than the radius of the atom. That is, the nucleus occupies a volume only 10^{-12} that of the atom!

Let us calculate the maximum electric field and the corresponding force on an alpha particle that passes close to the surface of the nucleus. If we regard the nucleus as a uniform spherical ball of charge $Q = 79e$ and radius $R = 7$ fm, the maximum electric field is

$$E_{\max} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2} = 2.3 \times 10^{21} \text{ N/C}.$$

This is more than eight orders of magnitude larger than the electric field that would act on a particle at the surface of a plum-pudding model atom. The corresponding force is

$$F = qE_{\max} = 740 \text{ N}.$$

This is a huge force! Let us make the same simplification we did in our previous calculation and assume that this force is constant and acts on the particle only during the time Δt it takes the particle to travel a distance equal to one nuclear diameter:

$$\Delta t = \frac{2R}{v} = 8.2 \times 10^{-22} \text{ s}.$$

The corresponding change in the velocity of the particle can be estimated to be

$$\Delta v = a\,\Delta t = \frac{F}{m}\Delta t = 9 \times 10^7 \text{ m/s}.$$

This is comparable in magnitude to the velocity itself. We conclude that a nuclear atom can produce an electric field that is sufficiently large to reverse the motion of the projectile.

Based on the nuclear model of the atom, Rutherford was able to derive an exact formula for the number of particles scattered at any particular angle, and the experiments showed perfect agreement with this formula. He was also able to use the formula to determine the atomic number Z of the target atoms. Moreover, by using particles of a higher energy that actually penetrate the nucleus (see Sample Problem 25-7), this method can also be used to determine the nuclear radius.

This classic and painstaking series of experiments and their brilliant interpretation laid the foundation for modern atomic and nuclear physics, and Rutherford is generally credited as the founder of these fields. ■

MULTIPLE CHOICE

26-1 What Is a Field?

26-2 The Electric Field

1. The electric field is defined in Eq. 26-3 in terms of q_0, a small *positive* charge. If instead the definition were in terms of a small *negative* charge of the same magnitude, then compared to the *original* field, the newly defined electric field
   (A) would point in the same direction, and have the same magnitude.
   (B) would point in the opposite direction, but have the same magnitude.
   (C) would point in the same direction, but have a different magnitude.
   (D) would point in the opposite direction, and have a different magnitude.

26-3 The Electric Field of a Point Charge

2. A point charge $+q$ is located at the origin, and a point charge $+2q$ is located at $x = a$, where a is positive.
   (*a*) Which of the following statements is true?
   (A) Close to the charges, the electric field can be zero off the x axis.
   (B) Close to the charges, the magnitude of the electric field can be a maximum off the x axis.
   (C) The electric field can be zero somewhere between the charges.
   (D) The electric field can be zero on the x axis at finite points not between the charges.
   (*b*) In which of the following regions might there exist a point where the electric field has zero magnitude?

(A) $-\infty < x < 0$ (B) $0 < x < a$
(C) $a < x < \infty$
(D) E does not vanish in the region $-\infty < x < \infty$.

3. A point charge $+q$ is located at the origin, and a point charge $-2q$ is located at $x = a$, where a is positive.
(*a*) Which of the following statements is true?
(A) Close to the charges, the electric field can be zero off the x axis.
(B) Close to the charges, the magnitude of the electric field can be a maximum off the x axis.
(C) The electric field can be zero between the charges.
(D) The electric field can be zero on the x axis at finite points not between the charges.
(*b*) In which of the following regions might there exist a point where the electric field has zero magnitude?
(A) $-\infty < x < 0$ (B) $0 < x < a$
(C) $a < x < \infty$
(D) E does not vanish in the region $-\infty < x < \infty$.

26-4 Electric Field of Continuous Charge Distributions

4. Consider the magnitude of the electric field $E(z)$ on the axis of a uniform ring of charge.
(*a*) $E(z)$ will have its largest value where
(A) $z = 0$. (B) $0 < |z| < \infty$.
(C) $|z| = \infty$. (D) (A) and (C) are correct.
(*b*) $E(z)$ can be zero where
(A) $z = 0$. (B) $0 < |z| < \infty$.
(C) $|z| = \infty$. (D) (A) and (C) are correct.

5. Consider the magnitude of the electric field $E(z)$ on the axis of a uniform disk of charge.
(*a*) $E(z)$ will have its largest value where
(A) $z = 0$. (B) $0 < |z| < \infty$.
(C) $|z| = \infty$. (D) (A) and (C) are correct.
(*b*) $E(z)$ can be zero where
(A) $z = 0$. (B) $0 < |z| < \infty$.
(C) $|z| = \infty$. (D) (A) and (C) are correct.

26-5 Electric Field Lines

6. Figure 26-22 shows the electric field lines around an electric dipole. Which of the arrows best represents the electric field at point P?

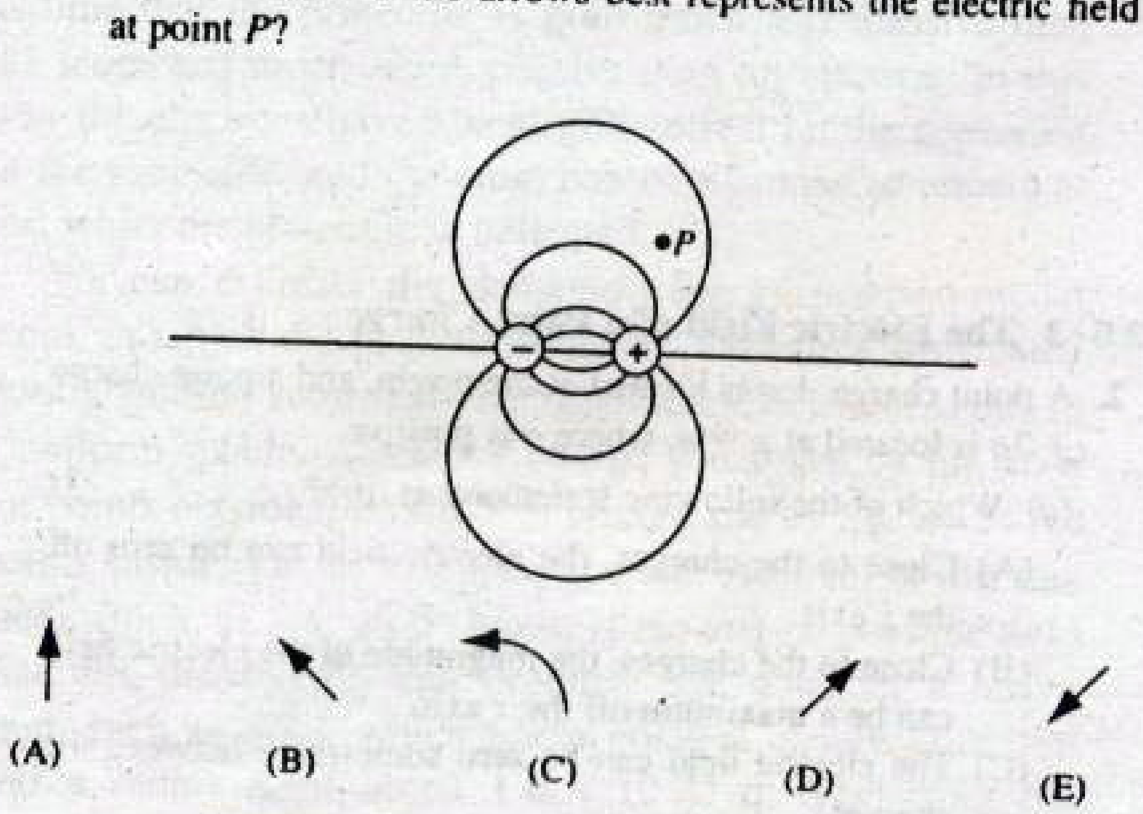

FIGURE 26-22. Multiple-choice question 6.

7. Figure 26-23 shows the electric field lines around three point charges, A, B, and C. (*a*) Which charges are positive? (*b*) Which charge has the largest magnitude? (*c*) In which region or regions of the picture could the electric field be zero?
(A) near A (B) near B
(C) near C (D) nowhere

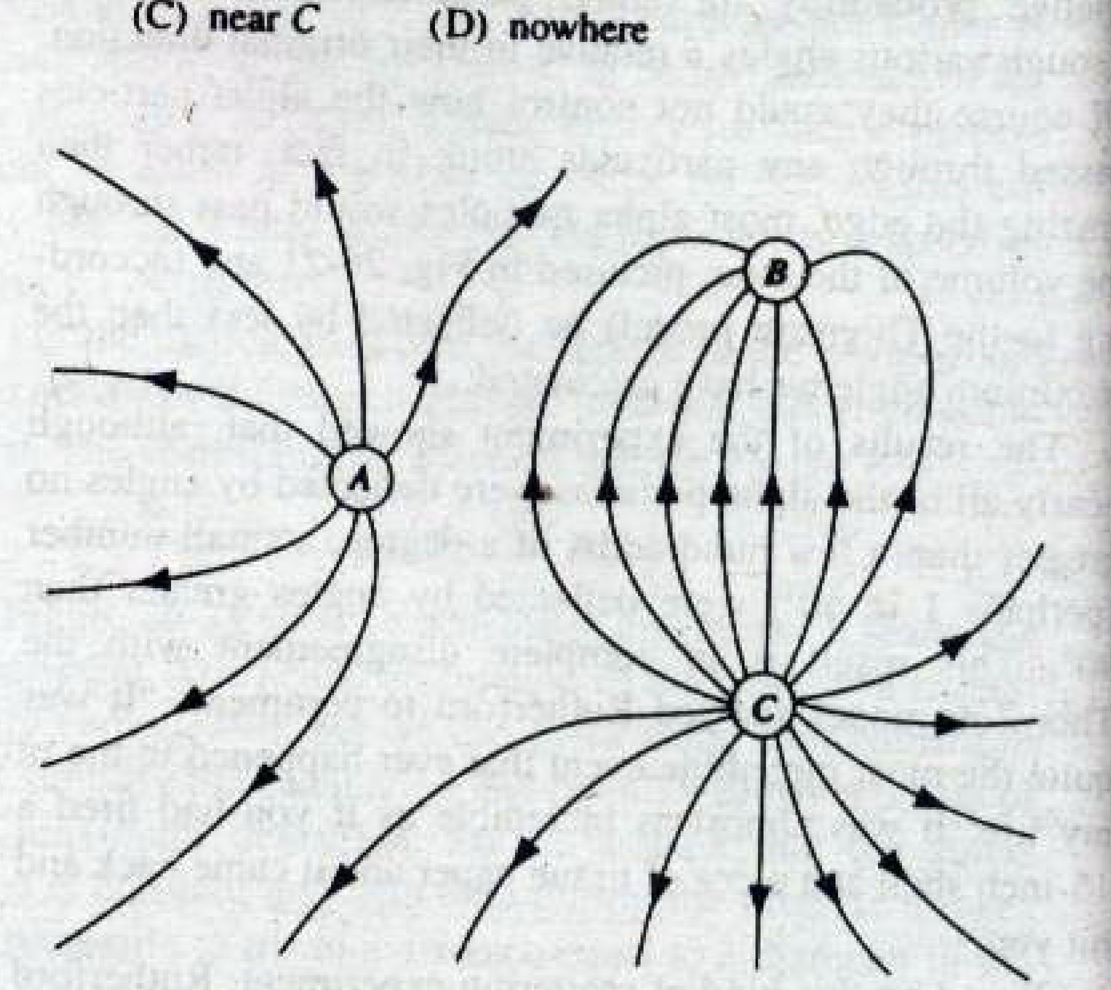

FIGURE 26-23. Multiple-choice question 7.

26-6 A Point Charge in an Electric Field

8. Three small spheres x, y, and z carry charges of equal magnitudes and with signs shown in Fig. 26-24. They are placed at the vertices of an isosceles triangle with the distance between x and y equal to the distance between x and z. Spheres y and z are held in place but sphere x is free to move on a frictionless surface.
(*a*) What is the direction of the electric force on sphere x at the point shown in the figure?
(*b*) Which path will sphere x take when released?

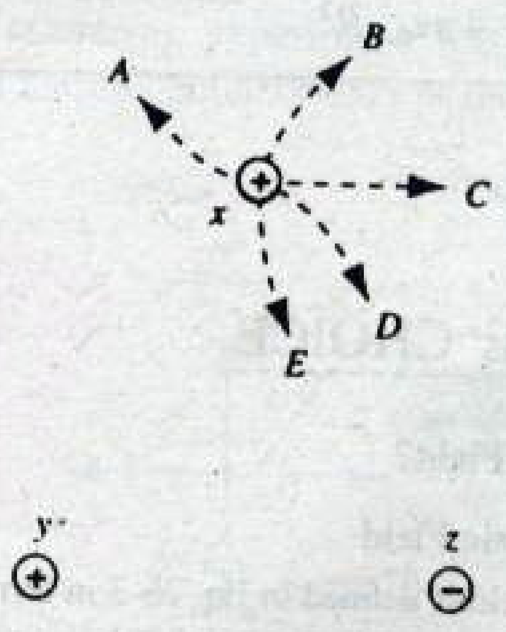

FIGURE 26-24. Multiple-choice question 8.

9. An electron is located in the *uniform* electric field established between parallel charged positive and negative plates. Where would the electron experience the greatest electrostatic force?
(A) When the electron is closer to the positive plate
(B) When the electron is closer to the negative plate
(C) When the electron is midway between the plates
(D) The electron experiences the same force regardless of the location between the plates.

10. The following measurements were made of the charge (in units of 10^{-19} C) on a series of charged droplets. What is the largest possible fundamental unit of charge that can be deduced from these data?

48	19.2	28.8
9.6	38.4	24

(A) 1.6×10^{-19} C (B) 4.8×10^{-19} C
(C) 9.6×10^{-19} C (D) 48×10^{-19} C

26-7 A Dipole in an Electric Field

11. The electric field in a certain region of space obeys $E_y \neq 0$, $E_x = E_z = 0$ and $\partial\vec{\mathbf{E}}/\partial x \neq 0$, $\partial\vec{\mathbf{E}}/\partial y = \partial\vec{\mathbf{E}}/\partial z = 0$.

(*a*) The net force on an electric dipole oriented parallel to the x axis in this field is
(A) directed along the x axis.
(B) directed along the y axis.
(C) directed along the z axis.
(D) None of the above

(*b*) The net torque on an electric dipole parallel to the x axis in this field is
(A) directed along the x axis.
(B) directed along the y axis.
(C) directed along the z axis.
(D) None of the above

26-8 The Nuclear Model of the Atom

QUESTIONS

1. Name as many scalar fields and vector fields as you can.

2. (*a*) In the gravitational attraction between the Earth and a stone, can we say that the Earth lies in the gravitational field of the stone? (*b*) How is the gravitational field due to the stone related to that due to the Earth?

3. A positively charged ball hangs from a long silk thread. We wish to measure E at a point in the same horizontal plane as that of the hanging charge. To do so, we put a positive test charge q_0 at the point and measure F/q_0. Will F/q_0 be less than, equal to, or greater than E at the point in question?

4. In exploring electric fields with a test charge, we have often assumed, for convenience, that the test charge was positive. Does this really make any difference in determining the field? Illustrate in a simple case of your own devising.

5. Electric field lines never cross. Why?

6. In Fig. 26-13, why do the field lines around the edge of the figure appear, when extended backward, to radiate uniformly from the center of the figure?

7. A point charge is moving in an electric field at right angles to the field lines. Does any force act on it?

8. In Fig. 26-14, why should grass seeds line up with electric field lines? Grass seeds normally carry no electric charge. (See "Demonstration of the Electric Fields of Current-Carrying Conductors," by O. Jefimenko, *American Journal of Physics*, January 1962, p. 19.)

9. What is the origin of "static cling," a phenomenon that sometimes affects clothes as they are removed from a dryer?

10. Two point charges of unknown magnitude and sign are a distance d apart. The electric field is zero at one point between them, on the line joining them. What can you conclude about the charges?

11. Two point charges of unknown magnitude and sign are placed a distance d apart. (*a*) If it is possible to have $E = 0$ at any point not between the charges but on the line joining them, what are the necessary conditions and where is the point located? (*b*) Is it possible, for any arrangement of two points charges, to find two points (neither at infinity) at which $E = 0$? If so, under what conditions?

12. Two point charges of unknown sign and magnitude are fixed a distance d apart. Can we have $E = 0$ for off-axis points (excluding infinity)? Explain.

13. In Sample Problem 26-3, a charge placed at point P in Fig. 26-4 is in equilibrium because no force acts on it. Is the equilibrium stable (*a*) for displacements along the line joining the charges and (*b*) for displacements at right angles to this line?

14. In Fig. 26-12, the force on the lower charge points up and is finite. The crowding of the field lines, however, suggests that E is infinitely great at the site of this (point) charge. A charge immersed in an infinitely great field should have an infinitely great force acting on it. What is the solution to this dilemma?

15. A point charge q of mass m is released from rest in a nonuniform field. (*a*) Will it necessarily follow the electric field line that passes through the release point? (*b*) Under what circumstances, if any, will a charged particle follow the electric field lines?

16. A positive and a negative charge of the same magnitude lie on a long straight line. What is the direction of $\vec{\mathbf{E}}$ for points on this line that lie (*a*) between the charges, (*b*) outside the charges in the direction of the positive charge, (*c*) outside the charges in the direction of the negative charge, and (*d*) off the line but in the median plane of the charges?

17. In the median plane of an electric dipole, is the electric field parallel or antiparallel to the electric dipole moment $\vec{\mathbf{p}}$?

18. In what way does Eq. 26-12 fail to represent the field lines of Fig. 26-12 if we relax the requirement that $x \gg d$?

19. (*a*) Two identical electric dipoles are placed in a straight line, as shown in Fig. 26-25*a*. What is the direction of the electric force on each dipole due to the presence of the other? (*b*) Suppose that the dipoles are rearranged as in Fig. 26-25*b*. What now is the direction of the force?

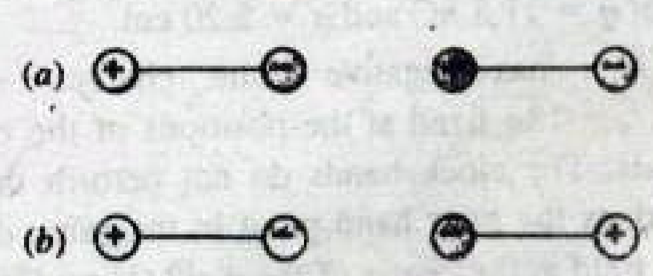

FIGURE 26-25. Question 19.

20. Compare the way E varies with r for (*a*) a point charge, (*b*) a dipole, and (*c*) a quadrupole.

21. What mathematical difficulties would you encounter if you were to calculate the electric field of a charged ring (or disk) at points not on the axis?

22. Equation 26-20 shows that E_z has the same value for all points in front of an infinite uniformly charged sheet. Is this reasonable? One might think that the field should be stronger near the sheet because the charges are so much closer.

23. Describe, in your own words, the purpose of the Millikan oil-drop experiment.

24. How does the sign of the charge on the oil drop affect the operation of the Millikan experiment?

25. Why did Millikan not try to balance electrons in his apparatus instead of oil drops?

26. You turn an electric dipole end for end in a uniform electric field. How does the work you do depend on the initial orientation of the dipole with respect to the field?

27. For what orientations of an electric dipole in a uniform electric field is the potential energy of the dipole (*a*) the greatest and (*b*) the least?

28. An electric dipole is placed in a nonuniform electric field. Is there a net force on the dipole?

29. An electric dipole is placed at rest in a uniform external electric field, as in Fig. 26-19*a*, and released. Discuss its motion.

30. An electric dipole has its dipole moment $\vec{p}$ aligned with a uniform external electric field $\vec{E}$. (*a*) Is the equilibrium stable or unstable? (*b*) Discuss the nature of the equilibrium if $\vec{p}$ and $\vec{E}$ point in opposite directions.

31. An atom is normally electrically neutral. Why then should an alpha particle be deflected by the atom under any circumstances?

EXERCISES

26-1 What Is a Field?

26-2 The Electric Field

1. An electron is accelerated eastward at 1.84×10^9 m/s² by an electric field. Determine the magnitude and direction of the electric field.

2. Humid air breaks down (its molecules become ionized) in an electric field of 3.0×10^6 N/C. What is the magnitude of the electric force on (*a*) an electron and (*b*) an ion (with a single electron missing) in this field?

3. An alpha particle, the nucleus of a helium atom, has a mass of 6.64×10^{-27} kg and a charge of $+2e$. What are the magnitude and direction of the electric field that will balance its weight?

4. In a uniform electric field near the surface of the Earth, a particle having a charge of -2.0×10^{-9} C is acted on by a downward electric force of 3.0×10^{-6} N. (*a*) Find the magnitude of the electric field. (*b*) What are the magnitude and direction of the electric force exerted on a proton placed in this field? (*c*) What is the gravitational force on the proton? (*d*) What is the ratio of the electric force to the gravitational force in this case?

26-3 The Electric Field of a Point Charge

5. What is the magnitude of a point charge chosen so that the electric field 75.0 cm away has the magnitude 2.30 N/C?

6. Calculate the dipole moment of an electron and a proton that are 4.30 nm apart.

7. Calculate the magnitude of the electric field, due to an electric dipole of dipole moment 3.56×10^{-29} C·m, at a point 25.4 nm away along the bisector axis.

8. Find the electric field at the center of the square of Fig. 26-26. Assume that $q = 11.8$ nC and $a = 5.20$ cm.

9. A clock face has negative point charges $-q, -2q, -3q, \ldots, -12q$ fixed at the positions of the corresponding numerals. The clock hands do not perturb the field. At what time does the hour hand point in the same direction as the electric field at the center of the dial? (Hint: Consider diametrically opposite charges.)

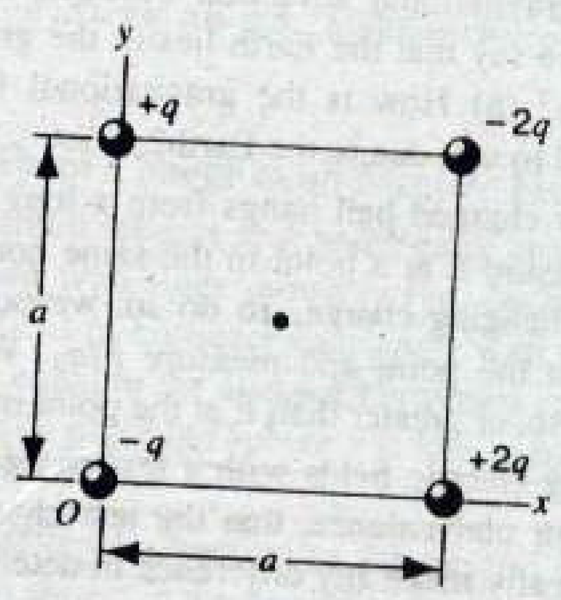

FIGURE 26-26. Exercise 8.

10. In Fig. 26-5, assume that both charges are positive. Show that the magnitude of E at point P in that figure, assuming $x \gg d$, is given by

$$E = \frac{1}{4\pi\epsilon_0}\frac{2q}{x^2}.$$

11. One type of electric quadrupole is formed by four charges located at the vertices of a square of side $2a$. Point P lies a distance x from the center of the quadrupole on a line parallel to two sides of the square as shown in Fig. 26-27. For $x \gg a$, show that the electric field at P is approximately given by

$$E = \frac{3(2qa^2)}{2\pi\epsilon_0 x^4}.$$

(Hint: Treat the quadrupole as two dipoles.)

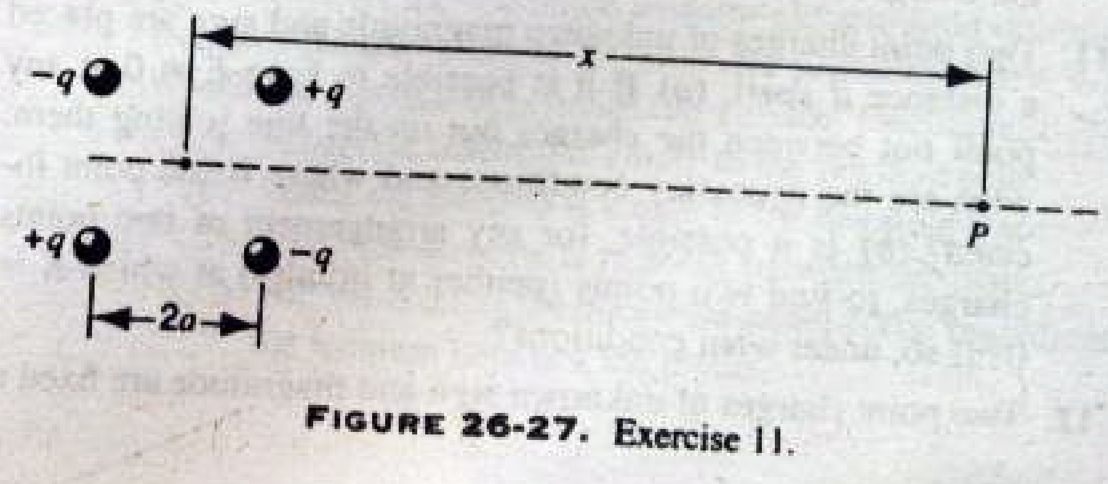

FIGURE 26-27. Exercise 11.

26-4 Electric Field of Continuous Charge Distributions

12. Show that Eq. 26-19, for the electric field of a charged disk at points on its axis, reduces to the field of a point charge for $z \gg R$.

13. At what distance along the axis of a charged disk of radius R is the electric field strength equal to one-half the value of the field at the surface of the disk at the center?

14. At what distance along the axis of a charged ring of radius R is the axial electric field strength a maximum?

15. (*a*) What total charge q must a disk of radius 2.50 cm carry so that the electric field on the surface of the disk at its center equals the value at which air breaks down electrically, producing sparks? See Table 26-1. (*b*) Suppose that each atom at the surface has an effective cross-sectional area of 0.015 nm^2. How many atoms are at the disk's surface? (*c*) The charge in (*a*) results from some of the surface atoms carrying one excess electron. What fraction of the surface atoms must be so charged?

16. A thin glass rod is bent into a semicircle of radius r. A charge $+q$ is uniformly distributed along the upper half and a charge $-q$ is uniformly distributed along the lower half, as shown in Fig. 26-28. Find the electric field $\mathbf{E}$ at P, the center of the semicircle.

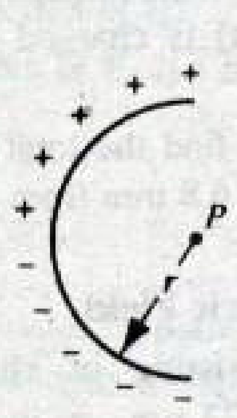

FIGURE 26-28. Exercise 16.

17. Measured values of the electric field E a distance z along the axis of a charged plastic disk are given here:

z (cm)	E (10^7 N/C)
0	2.043
1	1.732
2	1.442
3	1.187
4	0.972
5	0.797

Calculate (*a*) the radius of the disk and (*b*) the charge on it.

18. An insulating rod of length L has charge $-q$ uniformly distributed along its length, as shown in Fig. 26-29. (*a*) What is the linear charge density of the rod? (*b*) Find the electric field at point P a distance a from the end of the rod. (*c*) If P were very far from the rod compared to L, the rod would look like a point charge. Show that your answer to (*b*) reduces to the electric field of a point charge for $a \gg L$.

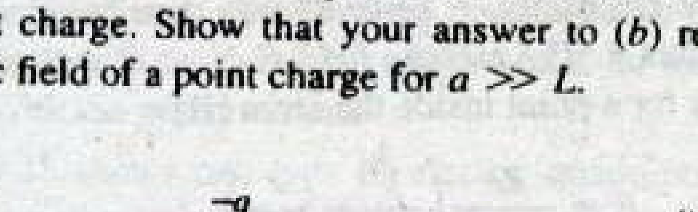

FIGURE 26-29. Exercise 18.

19. Sketch qualitatively the field lines associated with three long parallel lines of charge in a perpendicular plane. Assume that the intersections of the lines of charge with such a plane form an equilateral triangle (Fig. 26-30) and that each line of charge has the same linear charge density λ.

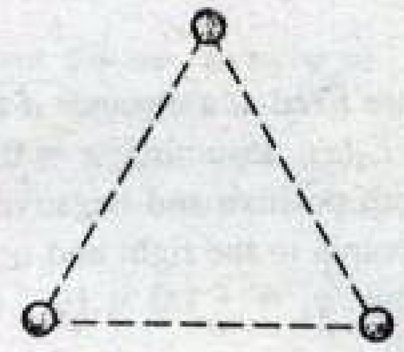

FIGURE 26-30. Exercise 19.

26-5 Electric Field Lines

20. Figure 26-31 shows field lines of an electric field; the line spacing perpendicular to the page is the same everywhere. (*a*) If the magnitude of the field at A is 40 N/C, what force does an electron at that point experience? (*fb*) What is the magnitude of the field at B?

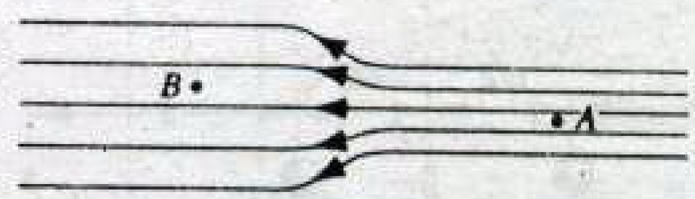

FIGURE 26-31. Exercise 20.

21. Sketch qualitatively the field lines associated with a thin, circular, uniformly charged disk of radius R. (Hint: Consider as limiting cases points very close to the disk, where the electric field is perpendicular to the surface, and points very far from it, where the electric field is like that of a point charge.)

22. Sketch qualitatively the field lines associated with two separated point charges $+q$ and $-2q$.

23. Three charges are arranged in an equilateral triangle as in Fig. 26-32. Consider the field lines due to $+Q$ and $-Q$ and from them identify the direction of the force that acts on $+q$ because of the presence of the other two charges. (Hint: See Fig. 26-12.)

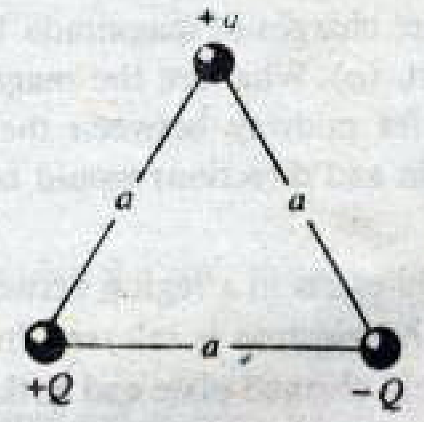

FIGURE 26-32. Exercise 23.

24. (a) In Fig. 26-33, locate the point (or points) at which the electric field is zero. (b) Sketch qualitatively the field lines.

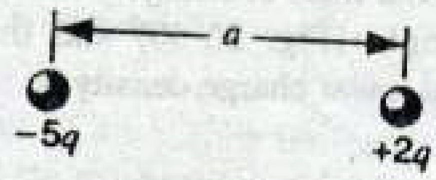

FIGURE 26-33. Exercise 24.

25. Two point charges are fixed at a distance d apart on the x axis (Fig. 26-34). Plot $E_x(x)$, assuming $x = 0$ at the left-hand charge. Consider both positive and negative values of x. Plot E_x as positive if $\vec{E}$ points to the right and negative if $\vec{E}$ points to the left. Assume $q_1 = +1.0 \times 10^{-6}$ C, $q_2 = +3.0 \times 10^{-6}$, and $d = 10$ cm.

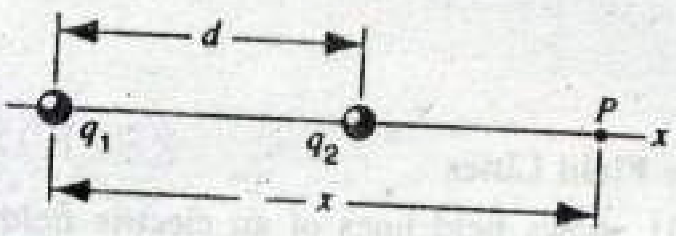

FIGURE 26-34. Exercise 25.

26. Charges $+q$ and $-2q$ are fixed a distance d apart as in Fig. 26-35. (a) Find $\vec{E}$ at points A, B, and C. (b) Sketch roughly the electric field lines.

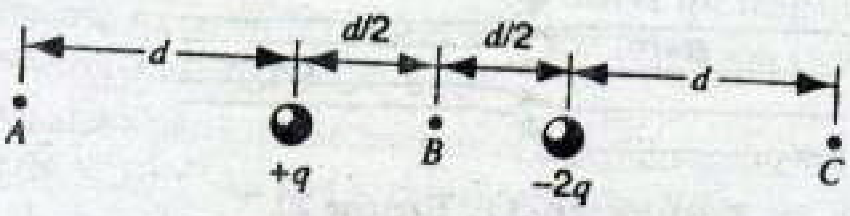

FIGURE 26-35. Exercise 26.

26-6 A Point Charge in an Electric Field

27. An electron moving with a speed of 4.86×10^6 m/s is shot parallel to a uniform electric field of strength 1030 N/C arranged so as to retard its motion. (a) How far will the electron travel in the field before coming (momentarily) to rest and (b) how much time will elapse? (c) If the electric field ends abruptly after 7.88 mm, what fraction of its initial kinetic energy will the electron lose in traversing it?

28. One weapon being considered for anti-missile defense uses particle beams. For example, a proton beam striking an enemy missile could render it harmless. Such beams can be produced in "guns" using electric fields to accelerate the charged particles. (a) What acceleration would a proton experience if the electric field is 2.16×10^4 N/C? (b) What speed would the proton attain if the field acts over a distance of 1.22 cm?

29. Two equal and opposite charges of magnitude 1.88×10^{-7} C are held 15.2 cm apart. (a) What are the magnitude and direction of $\vec{E}$ at a point midway between the charges? (b) What force (magnitude and direction) would act on an electron placed there?

30. A uniform electric field exists in a region between two oppositely charged plates. An electron is released from rest at the surface of the negatively charged plate and strikes the surface of the opposite plate, 1.95 cm away, 14.7 ns later. (a) What is the speed of the electron as it strikes the second plate? (b) What is the magnitude of the electric field?

31. In Millikan's experiment, a drop of radius 1.64 μm and density 0.851 g/cm³ is balanced when an electric field of 1.92×10^5 N/C is applied. Find the charge on the drop, in terms of e.

32. Two point charges of magnitudes $q_1 = 2.16$ μC and $q_2 = 85.3$ nC are 11.7 cm apart. (a) Find the magnitude of the electric field that each produces at the site of the other. (b) Find the magnitude of the force on each charge.

33. In a particular early run (1911), Millikan observed that the following measured charges, among others, appeared at different times on a single drop:

6.563×10^{-19} C	13.13×10^{-19} C	19.71×10^{-19} C
8.204×10^{-19} C	16.48×10^{-19} C	22.89×10^{-19} C
11.50×10^{-19} C	18.08×10^{-19} C	26.13×10^{-19} C

What value for the quantum of charge e can be deduced from these data?

34. A uniform vertical field $\vec{E}$ is established in the space between two large parallel plates. A small conducting sphere of mass m is suspended in the field from a string of length L. Find the period of this pendulum when the sphere is given a charge $+q$ if the lower plate (a) is charged positively and (b) is charged negatively.

35. In Sample Problem 26-6, find the total deflection of the ink drop on striking the paper 6.8 mm from the end of the deflection plates; see Fig. 26-16.

26-7 A Dipole in an Electric Field

36. An electric dipole, consisting of charges of magnitude 1.48 nC separated by 6.23 μm, is in an electric field of strength 1100 N/C. (a) What is the magnitude of the electric dipole moment? (b) What is the difference in potential energy corresponding to dipole orientations parallel and antiparallel to the field?

37. An electric dipole consists of charges $+2e$ and $-2e$ separated by 0.78 nm. It is in an electric field of strength 3.4×10^6 N/C. Calculate the magnitude of the torque on the dipole when the dipole moment is (a) parallel, (b) at a right angle, and (c) opposite to the electric field.

38. A charge $q = 3.16$ μC is 28.5 cm from a small dipole along its perpendicular bisector. The force on the charge equals 5.22×10^{-16} N. Show on a diagram (a) the direction of the force on the charge and (b) the direction of the force on the dipole. Determine (c) the magnitude of the force on the dipole and (d) the dipole moment of the dipole.

26-8 The Nuclear Model of the Atom

39. In a 1911 paper, Ernest Rutherford said: In order to form some idea of the forces required to deflect an alpha particle through a large angle, consider an atom containing a point positive charge Ze at its center and surrounded by a distribution of negative electricity, $-Ze$ uniformly distributed within a sphere of radius R. The electric field E . . . at a distance r from the center for a point inside the atom [is]

$$E = \frac{Ze}{4\pi\epsilon_0}\left(\frac{1}{r^2} - \frac{r}{R^3}\right).$$

Verify this equation.

40. Figure 26-36 shows a Thomson atom model of helium ($Z = 2$). Two electrons, at rest, are embedded inside a uniform sphere of positive charge $2e$. Find the distance d between the electrons so that the configuration is in static equilibrium.

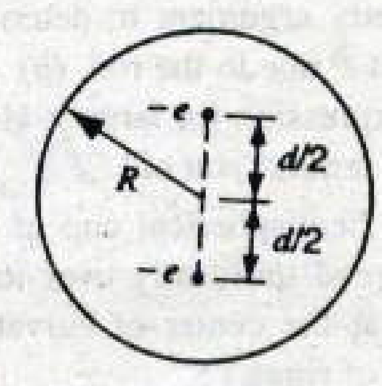

FIGURE 26-36. Exercise 40.

PROBLEMS

1. In Fig. 26-5, consider a point that is a distance z from the center of a dipole along its axis. (*a*) Show that, at large values of z, the magnitude of the electric field is given by

$$E = \frac{1}{2\pi\epsilon_0}\frac{p}{z^3}.$$

(Compare with the field at a point on the perpendicular bisector.) (*b*) What is the direction of $\vec{E}$?

2. Show that the components of $\vec{E}$ due to a dipole are given, at distant points, by

$$E_x = \frac{1}{4\pi\epsilon_0}\frac{3pxz}{(x^2+z^2)^{5/2}}, \quad E_z = \frac{1}{4\pi\epsilon_0}\frac{p(2z^2-x^2)}{(x^2+z^2)^{5/2}},$$

where x and z are coordinates of point P in Fig. 26-37. Show that this general result includes the special results of Eq. 26-12 and Problem 1.

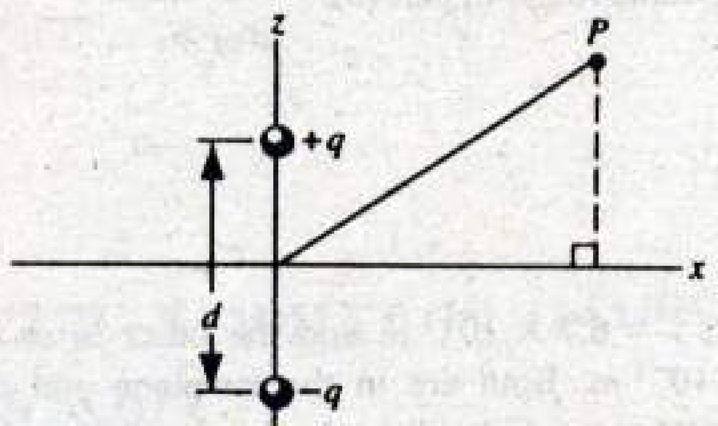

FIGURE 26-37. Problem 2.

3. Consider the ring of charge of Section 26-4. Suppose that the charge q is not distributed uniformly over the ring but that charge q_1 is distributed uniformly over half the circumference and charge q_2 is distributed uniformly over the other half. Let $q_1 + q_2 = q$. (*a*) Find the component of the electric field at any point on the axis directed along the axis and compare with the uniform case. (*b*) Find the component of the electric field at any point on the axis perpendicular to the axis and compare with the uniform case.

4. Figure 26-38 shows one type of electric quadrupole. It consists of two dipoles whose effects at external points do not quite cancel. Show that the value of E on the axis of the quadrupole for points a distance z from its center (assume $z \gg d$) is given by

$$E = \frac{3Q}{4\pi\epsilon_0 z^4},$$

where Q ($= 2qd^2$) is called the *quadrupole moment* of the charge distribution.

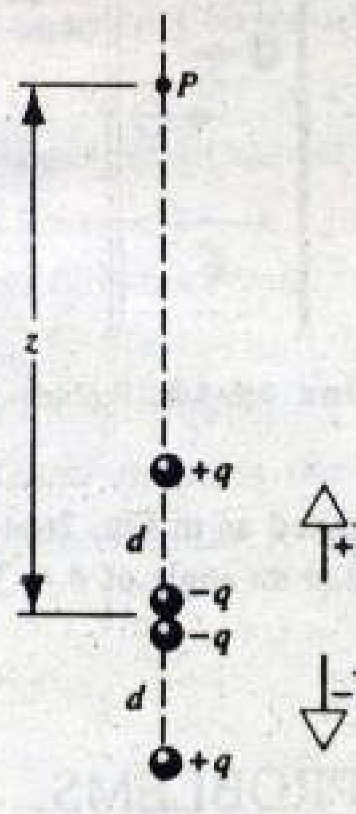

FIGURE 26-38. Problem 4.

5. Construct a distribution of point charges along the x axis so that, far from the charges, the electric field along the x axis falls off as $1/r^6$.

6. A "semi-infinite" insulating rod (Fig. 26-39) carries a constant charge per unit length of λ. Show that the electric field at the point P makes an angle of 45° with the rod and that this result is independent of the distance R.

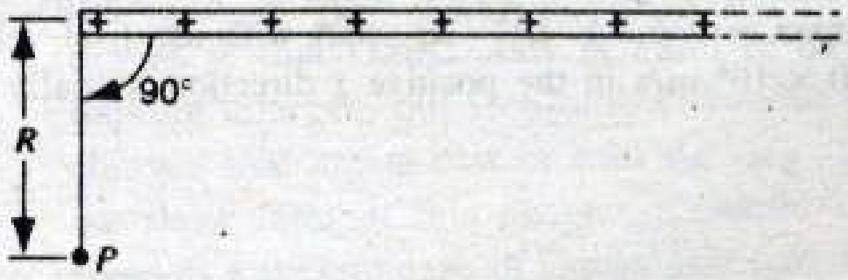

FIGURE 26-39. Problem 6.

7. A thin nonconducting rod of finite length L carries a uniform linear charge density $+\lambda$ on the top half and a uniform charge density $-\lambda$ on the bottom half; compare to Fig. 26-6. (*a*) Use a symmetry argument to determine the direction of the electric field at *P* due to the rod. (*b*) Find $\vec{E}$ at *P*. (*c*) Take the limit of this expression for large *y*. How does it depend on *y*? What does this remind you of?

8. A nonconducting hemispherical cup of inner radius *R* has a total charge *q* spread uniformly over its inner surface. Find the electric field at the center of curvature. (Hint: Consider the cup as a stack of rings.)

9. Assume that the exponent in Coulomb's law is not 2 but *n*. Show that for $n \neq 2$ it is impossible to construct lines that will have the properties listed for electric field lines in Section 26-5. For simplicity, treat an isolated point charge.

10. Two large parallel copper plates are 5.00 cm apart and have a uniform electric field between them as depicted in Fig. 26-40. An electron is released from the negative plate at the same time that a proton is released from the positive plate. Neglect the force of the particles on each other and find their distance from the positive plate when they pass each other. Does it surprise you that you need not know the electric field to solve this problem?

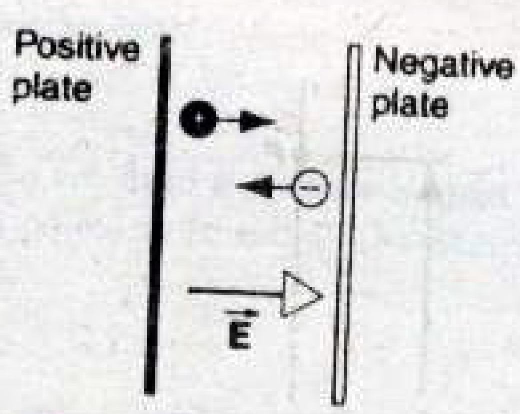

FIGURE 26-40. Problem 10.

11. An electron is projected as in Fig. 26-41 at a speed of $v_0 = 5.83 \times 10^6$ m/s and at an angle of $\theta = 39.0°$; $E = 1870$ N/C (directed upward), $d = 1.97$ cm, and $L = 6.20$ cm. Will the electron strike either of the plates? If it strikes a plate, which plate does it strike and at what distance from the left edge?

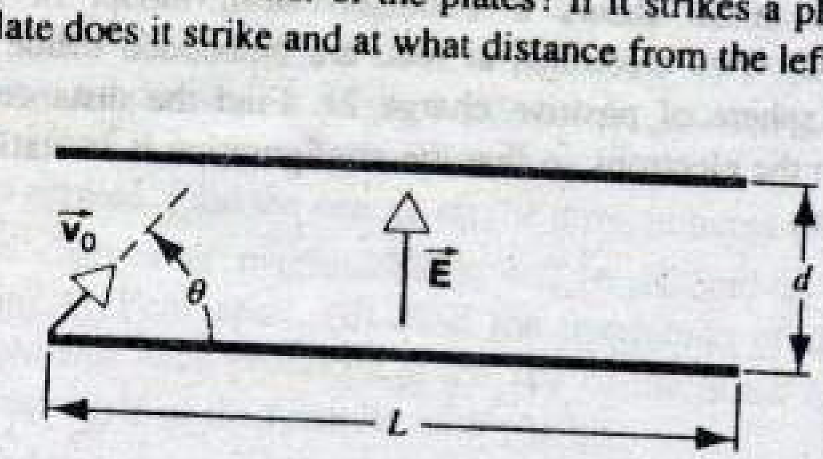

FIGURE 26-41. Problem 11.

12. An electron is constrained to move along the axis of the ring of charge discussed in Section 26-4. Show that the electron can perform small oscillations, through the center of the ring with a frequency given by

$$\omega = \sqrt{\frac{eq}{4\pi\epsilon_0 mR^3}}.$$

13. Find the work required to turn an electric dipole end for end in a uniform electric field $\vec{E}$, in terms of the dipole moment $\vec{p}$ and the initial angle θ_0 between $\vec{p}$ and $\vec{E}$.

14. Find the frequency of oscillation of an electric dipole, of moment *p* and rotational inertia *I*, for small amplitudes of oscillation about its equilibrium position in a uniform electric field *E*.

15. Two equal positive point charges $+q$ are located at $z = +a/2$ and $z = -a/2$. (*a*) Derive an expression for dE_z/dz for points along the *z* axis, and evaluate dE_z/dz in the limit $z \ll a/2$. (*b*) Show that the force on a small dipole placed at this point, its axis along the line joining the charges, is given by $F = p(dE_z/dz)$, where *p* is the dipole moment and dE_z/dz is the limiting value found in part (*a*).

COMPUTER PROBLEMS

1. A ring of radius $r = 1.0$ m has a nonuniform charge density given by $\lambda = (2.0\ \mu\text{C/m})(2 + \sin\theta)$. Numerically find the coordinates of a point where the electric field vanishes.

2. The charge density on a rod of length *L* centered on the *x* axis is given by $\lambda = (1.0\ \mu\text{C/m})\sin^2(\pi x/L)$. Numerically generate a plot of the electric field lines in the *xy* plane.

3. Consider two particles that exert electric forces on each other. Each accelerates in response to the electric field of the other, and as their positions change the forces they exert also change. Two identical particles, each with charge $q = +1.9 \times 10^{-9}$ C and mass $m = 6.1 \times 10^{-15}$ kg, start with identical velocities of 3.0×10^4 m/s in the positive *x* direction. Initially one is at $x = 0$, $y = 6.7 \times 10^{-3}$ m and the other is at $x = 0$, $y = -6.7 \times 10^{-3}$ m. Both are in the *xy* plane and continue to move in that plane. Consider only the electric forces they exert on each other. (*a*) Use a computer program to plot the trajectories from time $t = 0$ to $t = 1.0 \times 10^{-6}$ s. Because the situation is symmetric you need calculate only the position and velocity of one of the charges. Use symmetry to find the position and velocity of the other at the beginning of each integration interval. Use $\Delta t = 1 \times 10^{-8}$ s for the integration interval. (*b*) Now suppose that one of the particles has charge $q = -1.9 \times 10^{-9}$ C, but all other conditions are the same. Plot the trajectories from $t = 0$ to $t = 5.0 \times 10^{-7}$ s.

CHAPTER 27

GAUSS' LAW

Coulomb's law can always be used to calculate the electric field $\vec{\mathbf{E}}$ *for any discrete or continuous distribution of charges at rest. The sums or integrals might be complicated (and a computer might be needed to evaluate them numerically), but the resulting electric field can always be found.*

Some cases discussed in the previous chapter used simplifying arguments based on the symmetry of the physical situation. For example, in calculating the electric field at points on the axis of a charged circular loop, we used a symmetry argument to deduce that components of $\vec{\mathbf{E}}$ *perpendicular to the axis must vanish. In this chapter we discuss an alternative to Coulomb's law, called* Gauss' law, *that provides a more useful and instructive approach to calculating the electric field in situations having certain symmetries.*

The number of situations that can directly be analyzed using Gauss' law is small, but those cases can be done with extraordinary ease. Although Gauss' law and Coulomb's law give identical results in the cases in which both can be used, Gauss' law is considered a more fundamental equation than Coulomb's law. It is fair to say that whereas Coulomb's law provides the workhorse of electrostatics, Gauss' law provides the insight.

27-1 WHAT IS GAUSS' LAW ALL ABOUT?

So far, everything we have done in electrostatics has been based on Coulomb's law, Eq. 25-4, which gives the electric force between point charges. Starting with Coulomb's law, which is essentially a mathematical representation of an experimental observation, we *defined* the electric field of a point charge q so that $\vec{\mathbf{E}} = \vec{\mathbf{F}}/q_0$, where $\vec{\mathbf{F}}$ is the force exerted on q_0 by q. By generalizing to charge distributions that can be considered as assemblies of many infinitesimal point charges, we were able to find the electric field of several different charge distributions, such as a line or a disk.

Gauss' law provides us with another way to calculate electric fields. It is equivalent to Coulomb's law for point charges, which means that everything we have done so far using Coulomb's law could also be done if we had started with Gauss' law instead.

Why do we need Gauss' law, if Coulomb's law is sufficient to calculate electric fields for any static arrangements of charges? One answer is that Gauss' law offers a much simpler way to calculate electric fields in situations with a high degree of symmetry, such as a spherical charge distribution. Another answer is that by writing Gauss' law rather than Coulomb's law as the fundamental law for electrostatics we can develop a system of equations for all electromagnetic phenomena that illustrate more clearly the relationship between electric and magnetic fields. A third reason is that Gauss' law is valid in the case of rapidly moving charges, but Coulomb's law can be used only for charges that are at rest or moving very slowly. Finally, as we show later in this chapter, Coulomb's law can be derived as a special case of Gauss' law, and thus Gauss' law is more general than Coulomb's law. For these reasons, Gauss' law is considered to be more fundamental than Coulomb's law and is included as one of the four basic equations of electromagnetism (Maxwell's equations, which we discuss in Chapter 38).

Before we introduce Gauss' law, we first need to define and discuss a new quantity, the *flux of the electric field.* The flux is a mathematical property of any field represented by vectors that is determined by the *surface integral* of the field vector over a particular area. There is also a geometrical interpretation of the flux that is based on the number of field lines that pass through the area.

27-2 THE FLUX OF A VECTOR FIELD

The word "flux" comes from a Latin word meaning "to flow," and you can consider the flux of a vector field to be a measure of the flow or penetration of the field vectors through an imaginary fixed element of surface in the field. Later we consider the flux of the electric field, but for now we consider a more familiar example, the velocity field of a flowing fluid.

Imagine a stream of fluid in steady flow, in which we represent the flow by specifying the velocity vector at each point. Figure 27-1 shows a uniform flow; the velocity vectors are parallel throughout the fluid. Suppose we place into the stream a wire bent into a square loop of area A. In Fig. 27-1*a*, the loop is placed so that its plane is perpendicular to the direction of flow. We define the flux Φ of the velocity field so that its magnitude is given by

$$|\Phi| = vA, \qquad (27\text{-}1)$$

where v is the magnitude of the velocity at the location of the loop. The flux has units of m^3/s and might be considered to represent the rate at which fluid passes through the loop; in terms of the field concept (and for the purpose of introducing Gauss' law), however, it is convenient to consider the flux as a measure of the *number of field lines passing through the loop.*

In Fig. 27-1*b*, the loop has been rotated so that its plane is no longer perpendicular to the direction of the velocity. Note that the number of lines of the velocity field passing through the loop is smaller in Fig. 27-1*b* than in Fig. 27-1*a*. The projected area of the square is $A \cos \theta$, and by examining Fig. 27-1*b* you should convince yourself that the number of field lines passing through the inclined loop of area A is the same as the number of field lines passing through the smaller loop of area $A \cos \theta$ perpendicular to the stream. Thus the magnitude of the flux in the situation of Fig. 27-1*b* is

$$|\Phi| = vA \cos \theta. \qquad (27\text{-}2)$$

If the loop were rotated so that the fluid velocity were parallel to its surface, as in Fig. 27-1*c*, the flux would be zero, corresponding to $\theta = 90°$ in Eq. 27-2. Note that in this case no field lines pass through the loop.

Gauss' law, as we shall see, concerns the net flux through a *closed* surface. We must therefore distinguish between positive and negative flux penetrating a surface. The right side of Eq. 27-2 can be expressed in terms of the dot product between $\vec{v}$ and a vector $\vec{A}$ whose magnitude is the area of the surface and whose direction is perpendicular to the surface (Fig. 27-1*d*). However, since the normal to a surface can point either in the direction shown in Fig. 27-1*d* or in the reverse direction, we must have a way to specify that direction; otherwise the sign of Φ will not be clearly defined. By convention, we choose the direction of $\vec{A}$ to be that of the *outward normal* from a closed surface. Thus flux *leaving* the volume enclosed by the surface is considered positive, and flux *entering* the volume is considered negative. With this choice, we can then write the flux for a closed surface consisting of several individual surfaces (Fig. 27-1*e*, for example) as

$$\Phi = \sum \vec{v} \cdot \vec{A}, \qquad (27\text{-}3)$$

where $\vec{v}$ is the velocity vector at the surface. The sum is carried out over all the individual surfaces that make up a

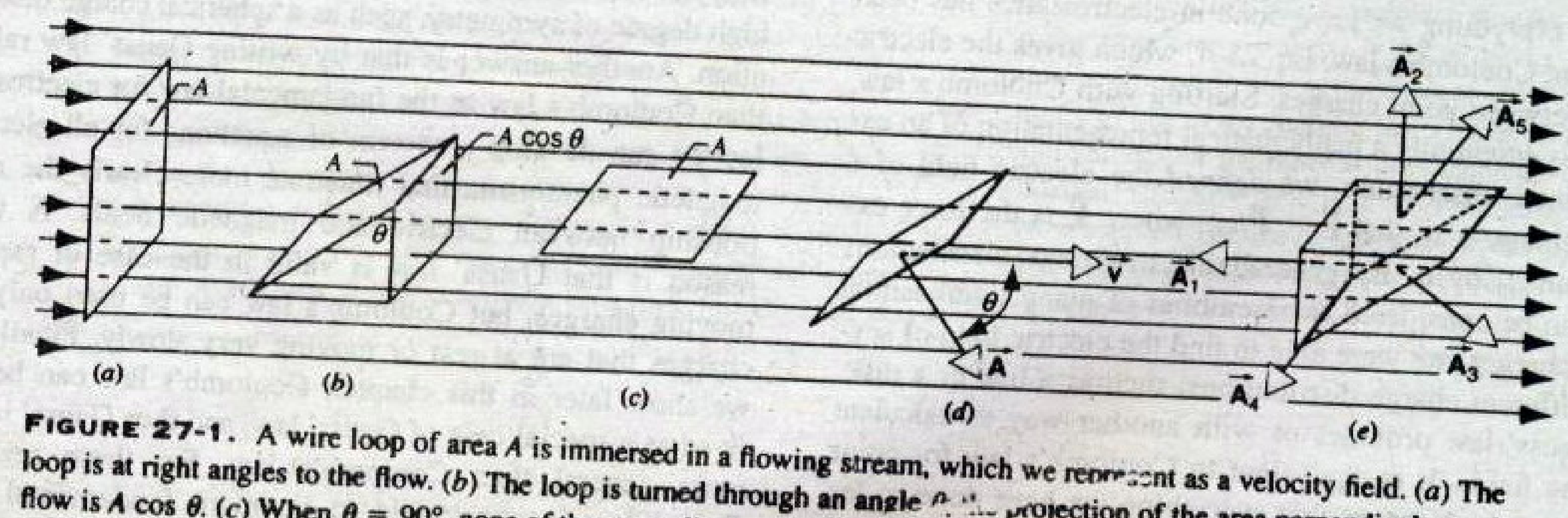

FIGURE 27-1. A wire loop of area A is immersed in a flowing stream, which we represent as a velocity field. (*a*) The loop is at right angles to the flow. (*b*) The loop is turned through an angle θ; the projection of the area perpendicular to the flow is $A \cos \theta$. (*c*) When $\theta = 90°$, none of the streamlines pass through the plane of the loop. (*d*) The area of the loop is represented by a vector $\vec{A}$ perpendicular to the plane of the loop. The angle between $\vec{A}$ and the flow velocity $\vec{v}$ is θ. (*e*) A closed surface made of five plane surfaces. The area $\vec{A}$ of each surface is represented by the outward normal.

closed surface. The flux is a scalar quantity, because it is defined in terms of the dot product of two vectors.

SAMPLE PROBLEM 27-1. Consider the closed surface of Fig. 27-1*e*, which shows a volume enclosed by five surfaces (1, 2, and 3, which are parallel to the surfaces of Figs. 27-1*a*, 27-1*c*, 27-1*b*, respectively, along with 4 and 5, which are parallel to the streamlines). Assuming the velocity field is uniform, so that it has the same magnitude and direction everywhere, find the total flux through the closed surface.

Solution Using Eq. 27-3 we can write the total flux as the sum of the values of the flux through each of the five separate surfaces:

$$\Phi = \vec{v}\cdot\vec{A}_1 + \vec{v}\cdot\vec{A}_2 + \vec{v}\cdot\vec{A}_3 + \vec{v}\cdot\vec{A}_4 + \vec{v}\cdot\vec{A}_5.$$

Note that for surface 1 the angle between the *outward normal* $\vec{A}_1$ and the velocity $\vec{v}$ is 180°, so that the dot product $\vec{v}\cdot\vec{A}_1$ can be written $-vA_1$. The contributions from surfaces 2, 4, and 5 all vanish, because in each case (as shown in Fig. 27-1*e*) the vector $\vec{A}$ is perpendicular to $\vec{v}$. For surface A_3, the flux can be written $vA_3\cos\theta$, and thus the total flux is

$$\Phi = -vA_1 + 0 + vA_3\cos\theta + 0 + 0 = -vA_1 + vA_3\cos\theta.$$

However, from the geometry of Fig. 27-1*e* we conclude that $A_3\cos\theta = A_1$, and as a result we obtain

$$\Phi = 0.$$

That is, the total flux through the closed surface is zero.

The result of the previous sample problem should not be surprising if we remember that the velocity field is an equivalent way of representing the actual flow of material particles in the stream. Every field line that enters the closed surface of Fig. 27-1*e* through surface 1 leaves through surface 3. Equivalently, we can state that, for the closed surface shown in Fig. 27-1*e*, the net amount of fluid entering the volume enclosed by the surface is equal to the net amount of fluid leaving the volume. This is to be expected *for any closed surface* if there are within the volume no *sources* or *sinks* of fluid—that is, locations at which new fluid is created or flowing fluid is trapped. If there were a source within the volume (such as a melting ice cube that introduced additional fluid into the stream), then more fluid would leave the surface than entered it, and the total flux would be positive. If there were a sink within the volume, then more fluid would enter than would leave, and the net flux would be negative. The net positive or negative flux through the surface depends on the strength of the source or sink (that is, on the volume rate at which fluid leaves the source or enters the sink). For example, if a melting solid inside the surface released 1 cm^3 of fluid per second into the stream, then we would find the net flux through the closed surface to be $+1\ cm^3/s$.

Figure 27-1 showed the special case of a uniform field and planar surfaces. We can easily generalize these concepts to a nonuniform field and to surfaces of arbitrary shape and orientation. Any arbitrary surface can be divided into infinitesimal elements of area dA that are approximately plane surfaces. The direction of the vector $d\vec{A}$ is that of the outward normal to this infinitesimal element. The field has a value $\vec{v}$ at the site of this element, and the net flux is found by adding the contributions of all such elements—that is, by integrating over the entire surface.

$$\Phi = \int \vec{v}\cdot d\vec{A}. \tag{27-4}$$

The conclusions we derived above remain valid in this general case: if Eq. 27-4 is evaluated over a closed surface, then the flux is (1) *zero* if the surface encloses no sources or sinks, (2) *positive* and equal in magnitude to their strength if the surface contains only sources, or (3) *negative* and equal in magnitude to their strength if the surface contains only sinks. If the surface encloses both sources *and* sinks, the net flux can be zero, positive, or negative, depending on the relative strength of the sources and sinks.

In the next section we apply similar considerations to the flux of another vector field, namely, the electric field $\vec{E}$. As you might anticipate, when we discuss electrostatics the sources or sinks of the field are positive or negative charges, and the strengths of the sources or sinks are proportional to the magnitudes of the charges. Gauss' law relates the flux of the electric field through a closed surface, calculated by analogy with Eq. 27-4, to the net electric charge enclosed by the surface.

27-3 THE FLUX OF THE ELECTRIC FIELD

Imagine the field lines in Fig. 27-1 to represent an electric field of charges at rest rather than a velocity field. Even though nothing is flowing in the electrostatic case, we still use the concept of flux. The definition of electric flux is similar to that of velocity flux, with $\vec{E}$ replacing $\vec{v}$ wherever it appears. In analogy with Eq. 27-3, we define the flux of the electric field Φ_E as

$$\Phi_E = \sum \vec{E}\cdot\vec{A}. \tag{27-5}$$

As was the case with the velocity flux, the flux Φ_E can be considered as a measure of the number of lines of the electric field that passes through the surface. The subscript E on Φ_E reminds us that we are speaking of the *electric* flux and serves to distinguish electric from magnetic flux, which we consider in Chapter 34. Equation 27-5 applies, as did Eq. 27-3, only to cases in which $\vec{E}$ is constant in magnitude and direction over each area $\vec{A}$ included in the sum.

Like the velocity flux, the flux of the electric field is a scalar. Its units are, from Eq. 27-5, $N\cdot m^2/C$.

Gauss' law deals with the flux of the electric field through a closed surface. To define Φ_E more generally, particularly in cases in which $\vec{E}$ is not uniform, consider

Fig. 27-2, which shows an arbitrary closed surface immersed in a nonuniform electric field. Let us divide the surface into small squares of area ΔA, each of which is small enough so that it may be considered to be plane. Each element of area can be represented as a vector $\Delta\vec{A}$ whose magnitude is the area ΔA. The direction of $\Delta\vec{A}$ is taken as the outward-drawn normal to the surface, as in Fig. 27-1. Since the squares have been made very small, $\vec{E}$ may be taken as constant for all points on a given square.

The vectors $\vec{E}$ and $\Delta\vec{A}$ that characterize each square make an angle θ with each other. Figure 27-2 shows an enlarged view of three squares on the surface, marked *a*, *b*, and *c*. Note that at *a*, $\theta > 90°$ ($\vec{E}$ points in); at *b*, $\theta = 90°$ ($\vec{E}$ is parallel to the surface); and at *c*, $\theta < 90°$ ($\vec{E}$ points out).

A provisional definition of the total flux of the electric field over the surface is, by analogy with Eq. 27-5,

$$\Phi_E = \sum \vec{E}\cdot\Delta\vec{A}, \tag{27-6}$$

which instructs us to add up the scalar quantity $\vec{E}\cdot\Delta\vec{A}$ for all elements of area into which the surface has been divided. For points such as *a* in Fig. 27-2 the contribution to the flux is negative; at *b* it is zero, and at *c* it is positive. Thus if $\vec{E}$ is everywhere outward ($\theta < 90°$), each $\vec{E}\cdot\Delta\vec{A}$ is positive, and Φ_E for the entire surface is positive. If $\vec{E}$ is everywhere inward ($\theta > 90°$), each $\vec{E}\cdot\Delta\vec{A}$ is negative, and Φ_E for the surface is negative. Whenever $\vec{E}$ is everywhere parallel to a surface ($\theta = 90°$), each $\vec{E}\cdot\Delta\vec{A}$ is zero, and Φ_E for the surface is zero.

The exact definition of electric flux is found in the differential limit of Eq. 27-6. Replacing the sum over the surface by an integral over the surface yields

$$\Phi_E = \int \vec{E}\cdot d\vec{A}. \tag{27-7}$$

This *surface integral* indicates that the surface in question is to be divided into infinitesimal elements of area $d\vec{A}$ and that the scalar quantity $\vec{E}\cdot d\vec{A}$ is to be evaluated for each element and summed over the entire surface. In the case of Gauss' law, we are concerned with evaluating this integral over a *closed* surface. In this case the integral sign is written with a circle $\oint$ as a reminder.

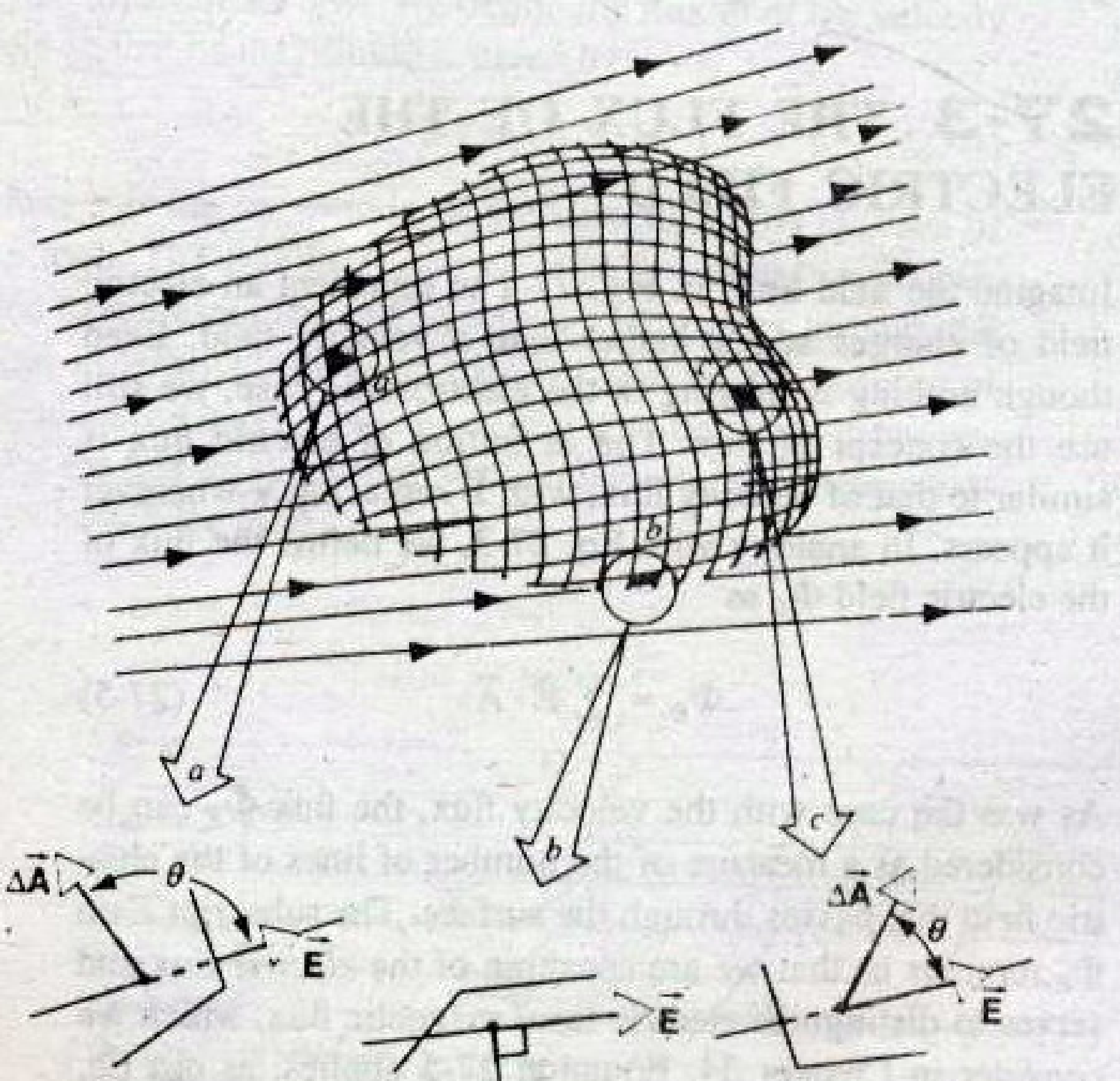

FIGURE 27-2. A surface of arbitrary shape immersed in a nonuniform electric field $\vec{E}$. The surface is divided into small elements of area $\Delta\vec{A}$. The relationship between the vectors $\vec{E}$ and $\Delta\vec{A}$ is shown for three different elements (*a*, *b*, and *c*).

SAMPLE PROBLEM 27-2. Figure 27-3 shows a hypothetical closed cylinder of radius R immersed in a uniform electric field $\vec{E}$, the cylinder axis being parallel to the field. What is Φ_E for this closed surface?

Solution The flux Φ_E can be written as the sum of three terms, an integral over (*a*) the left cylinder cap, (*b*) the cylindrical surface, and (*c*) the right cap. Thus, from Eq. 27-7, written for a closed surface,

$$\Phi_E = \oint \vec{E}\cdot d\vec{A}$$

$$= \int_a \vec{E}\cdot d\vec{A} + \int_b \vec{E}\cdot d\vec{A} + \int_c \vec{E}\cdot d\vec{A}.$$

For the left cap, the angle θ for all points is 180°, $\vec{E}$ has a constant value, and the vectors $d\vec{A}$ are all parallel. Thus

$$\int_a \vec{E}\cdot d\vec{A} = \int E\,dA\cos 180° = -E\int dA = -EA,$$

where A ($= \pi R^2$) is the area of the left cap. Similarly, for the right cap,

$$\int_c \vec{E}\cdot d\vec{A} = +EA,$$

the angle θ for all points being 0 here. Finally, for the cylinder wall,

$$\int_b \vec{E}\cdot d\vec{A} = 0,$$

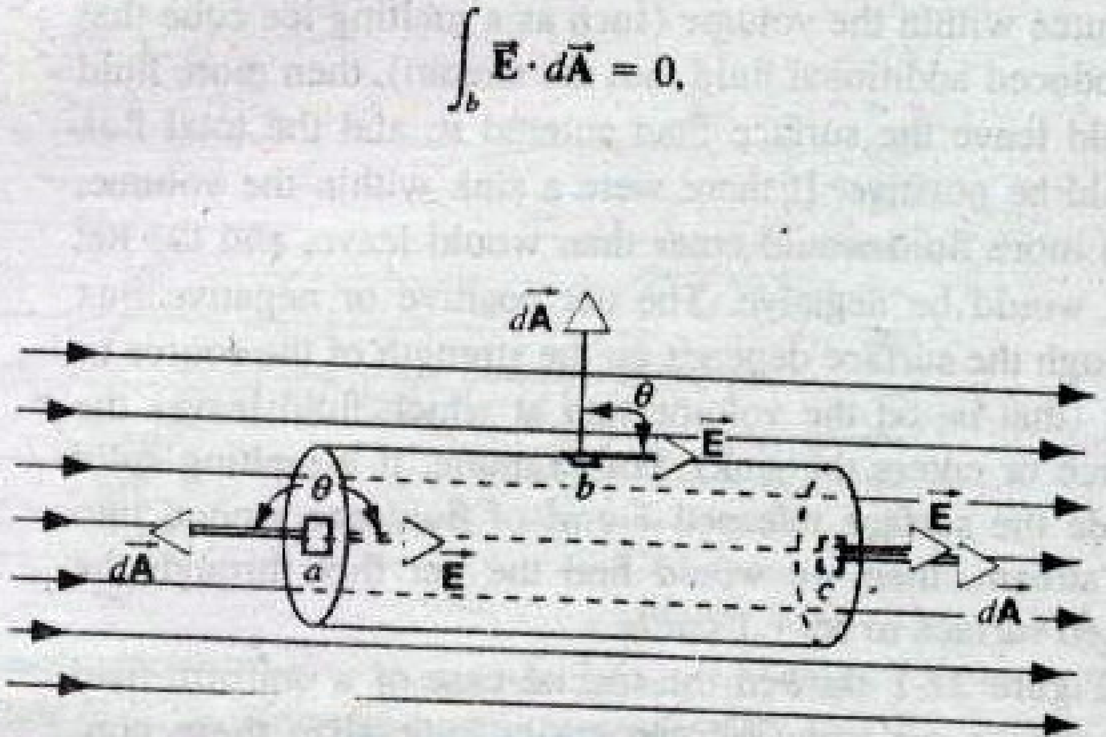

FIGURE 27-3. Sample Problem 27-2. A closed cylinder is immersed in a uniform electric field $\vec{E}$ parallel to its axis.

because $\theta = 90°$; hence $\vec{E} \cdot d\vec{A} = 0$ for all points on the cylindrical surface. Thus the total flux is

$$\Phi_E = -EA + 0 + EA = 0.$$

This result is expected, because there are no charges within the closed surface of Fig. 27-5. Lines of (constant) $\vec{E}$ enter at the left and emerge at the right, just as in Fig. 27-1*e*.

Flux and Lines of Field

To illustrate the relationship between flux and number of electric field lines passing through a closed surface, let us agree that each unit of charge q will be represented by a specific number of field lines, for example six lines, as shown in Fig. 27-4.* Six field lines point away from a charge $+q$, and six field lines terminate on a charge $-q$. If each charge is surrounded by a closed surface, then the electric flux through the surface surrounding the positive charge is $+6$ units, and the electric flux through the surface surrounding the negative charge is -6 units. (We count $+1$ arbitrary unit of flux for field lines that pass outward through the surface and -1 unit for field lines that pass inward through the surface.) No matter how big or how small is the surface surrounding each charge, six field lines always penetrate the surface and the flux is six units.

In Fig. 27-4*b*, the field line at the bottom of the drawing passes through the surface three times. Moving toward the charge from outside the surface, the first time the field line enters the surface we count -1, because it is going inward through the surface; the second time we count $+1$ because it is now going outward, and the third passage inward through the surface gives us another -1. The net contribution to the flux through the surface for this field line is -1, and the net flux for the entire surface is -6 units. No matter what the shape of the surface is or how it is stretched or distorted, the net flux through the surface is the same and is determined only by how much charge the surface encloses.

Figure 27-5 shows a closed surface enclosing charges $+q$ and $-q$. The net flux through the surface is zero, because for every field line from the positive charge passing outward through the surface, there is a field line from the negative charge passing inward through the surface. Because both charges have the same magnitude, the total number of field lines is zero, and therefore the net flux is zero.

Now suppose you are told that there are $+30$ field lines (or $+30$ units of flux) passing through an arbitrary closed surface. Can you determine how much charge is contained within the surface and where it is located inside the surface? We know that the *net* charge inside the surface is $+5q$, but we do not know whether one particle contains the full charge $+5q$, or whether there are two particles of charges $+6q$ and $-q$, three particles of $+8q$, $+4q$, and $-7q$, or any one of an infinite number of other possibilities. Also, the charge or charges can be located anywhere inside the surface and still produce the same $+30$ units of flux. If we know only the flux, we know the net amount of charge inside the surface, but we cannot deduce anything about the size or locations of the charges, and we therefore cannot deduce anything about the electric field, either on the surface or elsewhere in space.

If, however, we draw a spherical surface, and if we know that the flux is distributed uniformly over the surface, then we can conclude that all of the charge is located at the center of the sphere in a single particle of charge $+5q$, and knowing the magnitude and location of this charged particle we can then deduce the electric field at any location. We therefore conclude that

The relationship between the total flux through a closed surface and the net charge enclosed by the surface is always valid, but we can use this relationship to deduce

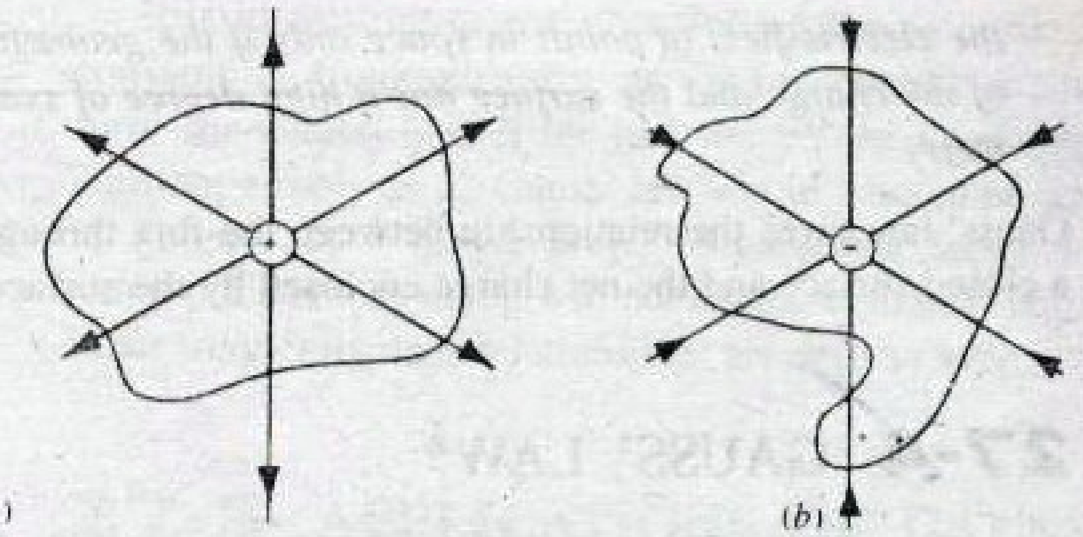

FIGURE 27-4. (*a*) Six field lines pass through an arbitrary closed surface surrounding a positive charge $+q$. (*b*) Six field lines enter the surface surrounding a negative charge $-q$.

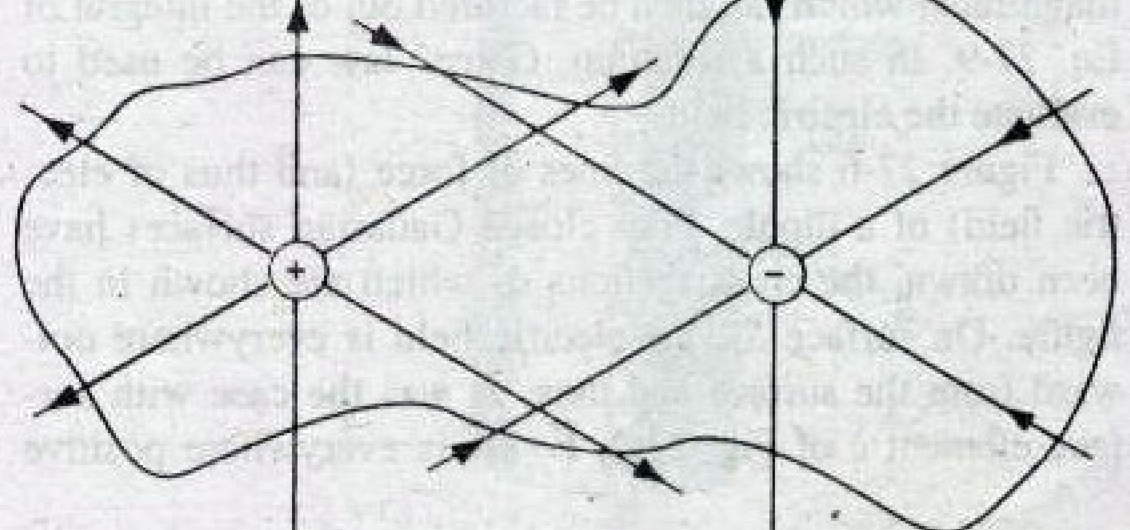

FIGURE 27-5. If the net charge enclosed by a surface is zero, then the total number of field lines (and the total electric flux) passing through the surface is zero.

*For convenience, we represent the field line diagrams in two dimensions rather than three. In three dimensions, the diagrams are often more complex and care must be taken to represent the field pattern associated with the charges. For a discussion see "Electric Field Line Diagrams Don't Work," by A. Wolf, S. J. Van Hook, and E. R. Weeks, *American Journal of Physics*, June 1996, p. 714.

the electric field at points in space only if the geometry of the charge and the surface has a high degree of symmetry.

Gauss' law gives the relationship between the flux through a closed surface and the net charge enclosed by the surface.

27-4 GAUSS' LAW*

Now that we have defined the flux of the electric field vector through a *closed* surface, we are ready to write Gauss' law. Let us suppose we have a collection of positive and negative charges, which establish an electric field $\vec{E}$ throughout a certain region of space. We construct in that space an imaginary closed surface called a *Gaussian surface,* which may or may not enclose some of the charges. Gauss' law, which relates the total flux Φ_E through this surface to the *net* charge q enclosed by the surface, can be stated as

$$\epsilon_0 \Phi_E = q \qquad (27\text{-}8)$$

or

$$\epsilon_0 \oint \vec{E} \cdot d\vec{A} = q. \qquad (27\text{-}9)$$

The circle on the integral sign indicates that the integral is to be carried out over a *closed* surface. We see that Gauss' law predicts that Φ_E is zero for the surface considered in Sample Problem 27-2, because the surface encloses no charge.

As discussed in Section 26-5, the magnitude of the electric field is proportional to the number of field lines crossing an element of area perpendicular to the field. The integral in Eq. 27-9 essentially counts the number of field lines passing through the surface. It is entirely reasonable that the number of field lines passing through a surface should be proportional to the net charge enclosed by the surface, as Eq. 27-9 requires.

The choice of the Gaussian surface is arbitrary. It is usually chosen so that the symmetry of the distribution gives, on at least part of the surface, an electric field of constant magnitude, which can then be factored out of the integral of Eq. 27-9. In such a situation, Gauss' law can be used to evaluate the electric field.

Figure 27-6 shows the lines of force (and thus of electric field) of a dipole. Four closed Gaussian surfaces have been drawn, the cross sections of which are shown in the figure. On surface S_1, the electric field is everywhere outward from the surface and thus, as was the case with surface element c of Fig. 27-2, $\vec{E} \cdot d\vec{A}$ is everywhere positive

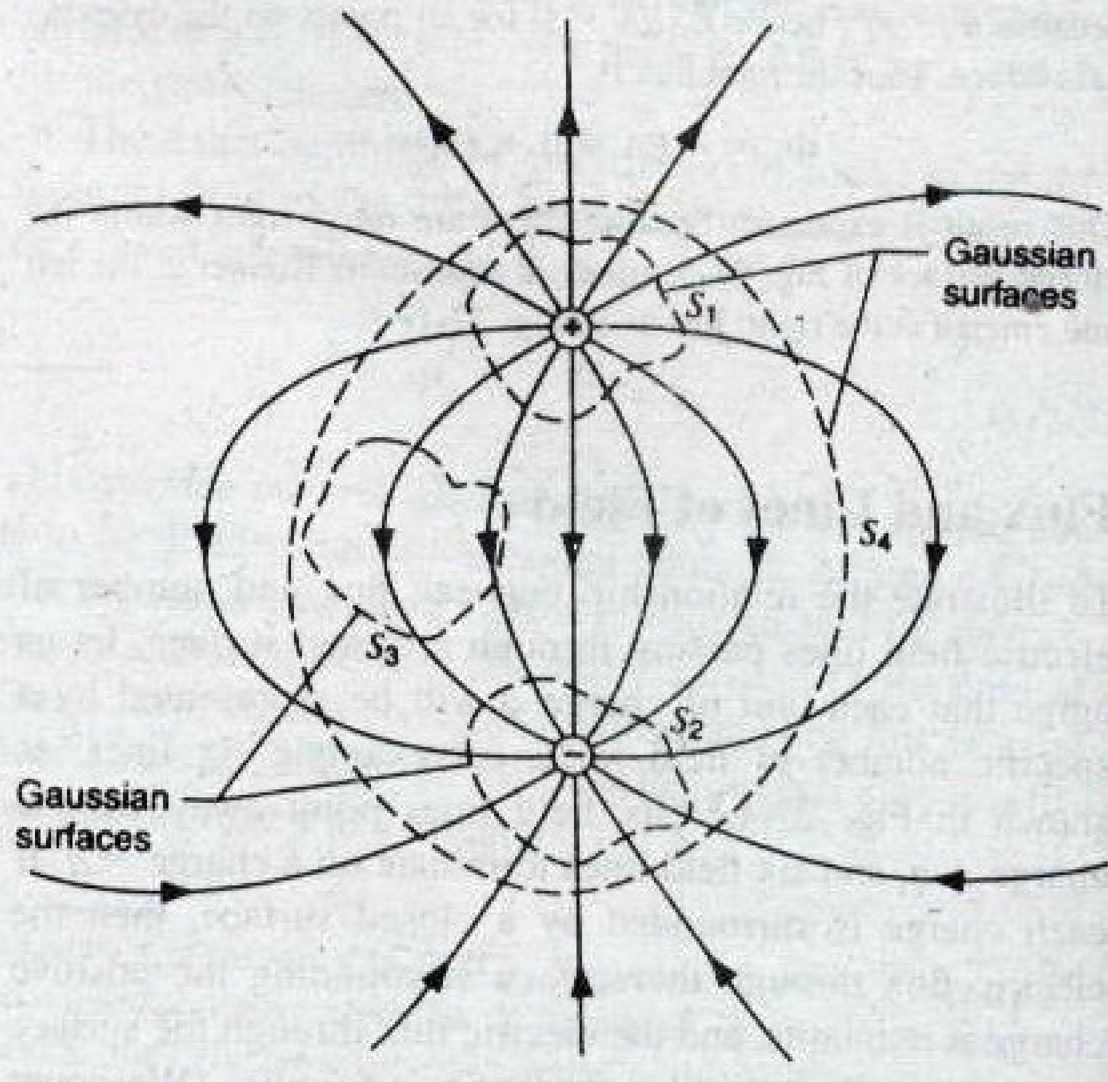

FIGURE 27-6. Two equal and opposite charges and the lines that represent the electric field in their vicinity. The cross sections of four closed Gaussian surfaces are shown.

on S_1. When we evaluate the integral of Eq. 27-9 over the entire closed surface, we get a positive result. Equation 27-9 then demands that the surface must enclose a net positive charge, as is the case. In Faraday's terminology, more lines of force leave the surface than enter it, so it must enclose a net positive charge.

On surface S_2 of Fig. 27-6, on the other hand, the electric field is everywhere entering the surface. Like surface element a in Fig. 27-2, $\vec{E} \cdot d\vec{A}$ is negative for every element of area, and the integral of Eq. 27-9 gives a negative value, which indicates that the surface encloses a net negative charge (as is the case). More lines of force enter the surface than leave it.

Surface S_3 encloses no charge at all, so according to Gauss' law the total flux through the surface must be zero. This is consistent with Fig. 27-6, which shows that as many lines of force enter the top of the surface as leave the bottom. This is no accident; you can draw a surface in Fig. 27-6 of any irregular shape, and as long as it encloses neither of the charges, the number of field lines that enter the surface equals the number that leave the surface.

Surface S_4 also encloses no *net* charge, since we assumed the magnitudes of the two charges to be equal. Once again, the total flux through the surface should be zero. Some of the field lines are wholly contained within the surface and therefore do not contribute to the flux *through* the surface. However, since every field line that leaves the positive charge eventually terminates on the negative charge, every line from the positive charge that breaks the surface in the outward direction has a corresponding line that breaks the surface in the inward direction as it seeks the negative charge. The total flux is therefore zero.

*Karl Friedrich Gauss (1777–1855) was a German mathematician who made substantial discoveries in number theory, geometry, and probability. He also contributed to astronomy and to measuring the size and shape of the Earth. See "Gauss," by Ian Stewart, *Scientific American*, July 1977, p. 122, for a fascinating account of the life of this remarkable mathematician.

Gauss' Law and Coulomb's Law

Coulomb's law can be deduced from Gauss' law and symmetry considerations. To do so, let us apply Gauss' law to an isolated positive point charge q as in Fig. 27-7. Although Gauss' law holds for any surface whatever, we choose a spherical surface of radius r centered on the charge. The advantage of this surface is that, from symmetry, $\vec{\mathbf{E}}$ must be perpendicular to the surface, so the angle θ between $\vec{\mathbf{E}}$ and $d\vec{\mathbf{A}}$ is zero everywhere on the surface. Moreover, $\vec{\mathbf{E}}$ has the same magnitude everywhere on the surface. *Constructing a Gaussian surface that takes advantage of such a symmetry is of fundamental importance in applying Gauss' law.*

In Fig. 27-7 both $\vec{\mathbf{E}}$ and $d\vec{\mathbf{A}}$ at any point on the Gaussian surface are directed radially outward, so the quantity $\vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$ becomes simply $E\,dA$. Gauss' law (Eq. 27-9) thus reduces to

$$\epsilon_0 \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \epsilon_0 \oint E\,dA = q.$$

Because E has the same magnitude for all points on the sphere, it can be factored from inside the integral sign, which gives

$$\epsilon_0 E \oint dA = q.$$

The integral is simply the total surface area of the sphere, $4\pi r^2$. We therefore obtain

$$\epsilon_0 E(4\pi r^2) = q$$

or

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}. \qquad (27\text{-}10)$$

Equation 27-10 gives the magnitude of the electric field $\vec{\mathbf{E}}$ at any point a distance r from an isolated point charge q and is identical to Eq. 26-6, which was obtained from Coulomb's law. Thus by choosing a Gaussian surface with the proper symmetry, we obtain Coulomb's law from Gauss' law. These two laws can be regarded as equivalent for our applications, but (as we discussed in Section 27-1) Gauss' law is more generally applicable and so is regarded as a more fundamental equation of electromagnetism.

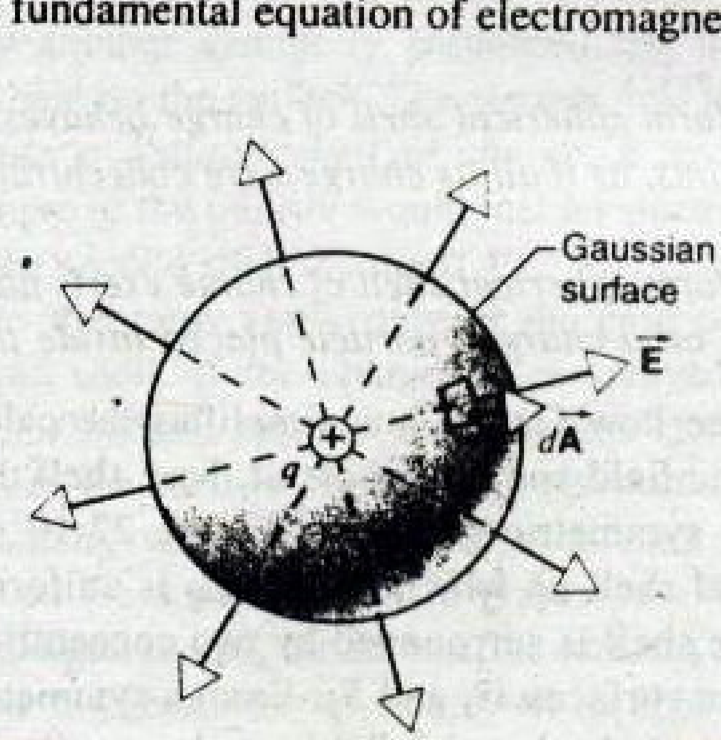

FIGURE 27-7. A spherical Gaussian surface of radius r surrounding a positive point charge q.

It is interesting to note that writing the proportionality constant in Coulomb's law as $1/4\pi\epsilon_0$ permits a simpler form for Gauss' law. If we had written the Coulomb law constant simply as K, Gauss' law would have to be written as $(1/4\pi K)\Phi_E = q$. We prefer to leave the factor 4π in Coulomb's law so that it will not appear in Gauss' law or in other frequently used relations that are derived later.

27-5 APPLICATIONS OF GAUSS' LAW

Gauss' law can be used to calculate $\vec{\mathbf{E}}$ if the symmetry of the charge distribution is high. One example of this calculation, the field of a point charge, has already been discussed in connection with Eq. 27-10. Here we present other examples.

Infinite Line of Charge

Figure 27-8 shows a section of an infinite line of charge of constant positive linear charge density (charge per unit length) λ. We would like to find the electric field at a distance r from the line.

In Section 26-4 we discussed the symmetry arguments that lead us to conclude that the electric field in this case can have only a radial component. The problem therefore has cylindrical symmetry, and so as a Gaussian surface we choose a circular cylinder of radius r and length h, closed at each end by plane caps normal to the axis. E is constant over the cylindrical surface and perpendicular to the surface. The flux of $\vec{\mathbf{E}}$ through this surface is $E(2\pi rh)$, where $2\pi rh$ is the area of the surface. There is no flux through the circular caps because $\vec{\mathbf{E}}$ here is parallel to the surface at every point, so that $\vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 0$ everywhere on the caps.

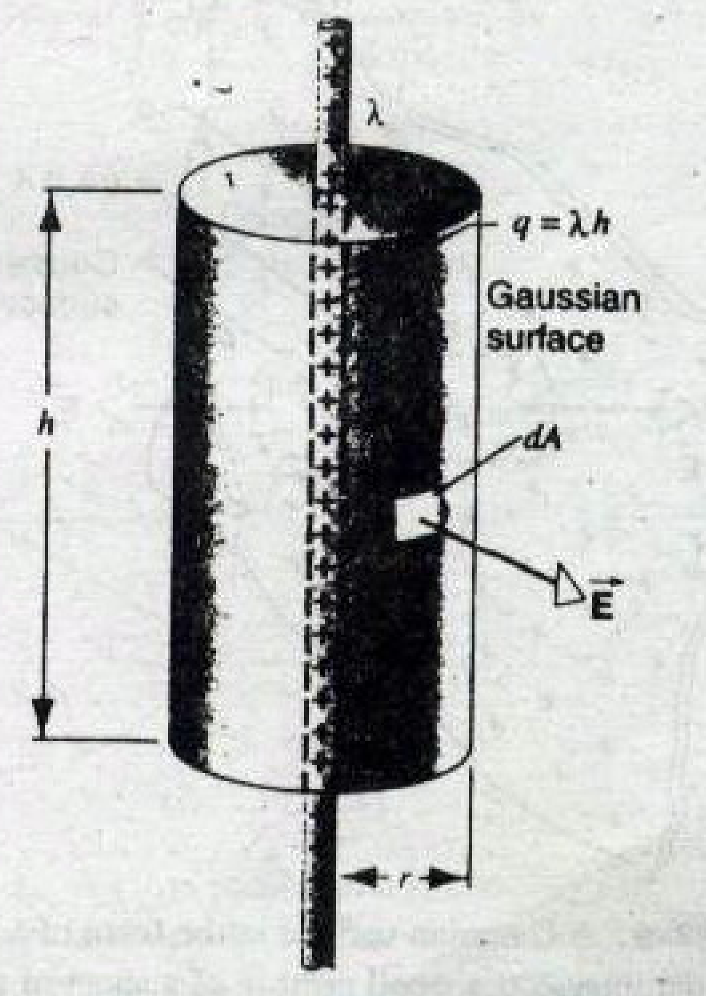

FIGURE 27-8. A Gaussian surface in the shape of a closed cylinder surrounds a portion of an infinite line of positive charge.

The charge q enclosed by the Gaussian surface of Fig. 27-8 is λh. Gauss' law (Eq. 27-9) then gives

$$\epsilon_0 \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = q$$

$$\epsilon_0 E(2\pi rh) = \lambda h,$$

or

$$E = \frac{\lambda}{2\pi\epsilon_0 r}, \qquad (27\text{-}11)$$

in agreement with Eq. 26-17.

Note how much simpler is the solution using Gauss' law than that using integration methods, as in Chapter 26. Note too that the solution using Gauss' law is possible only if we choose our Gaussian surface to take full advantage of the cylindrical symmetry of the electric field set up by a long line of charge. We are free to choose any closed surface, such as a cube or a sphere (see Exercise 24), for a Gaussian surface. Even though Gauss' law holds for all such surfaces, they are not all useful for the problem at hand; only the cylindrical surface of Fig. 27-8 is appropriate in this case.

Gauss' law has the property that it provides a useful technique for calculation only in problems that have a certain degree of symmetry, but in these problems the solutions are strikingly simple.

Infinite Sheet of Charge

Figure 27-9 shows a portion of a thin, nonconducting, infinite sheet of charge of constant positive surface charge density σ (charge per unit area). We calculate the electric field at points near the sheet.

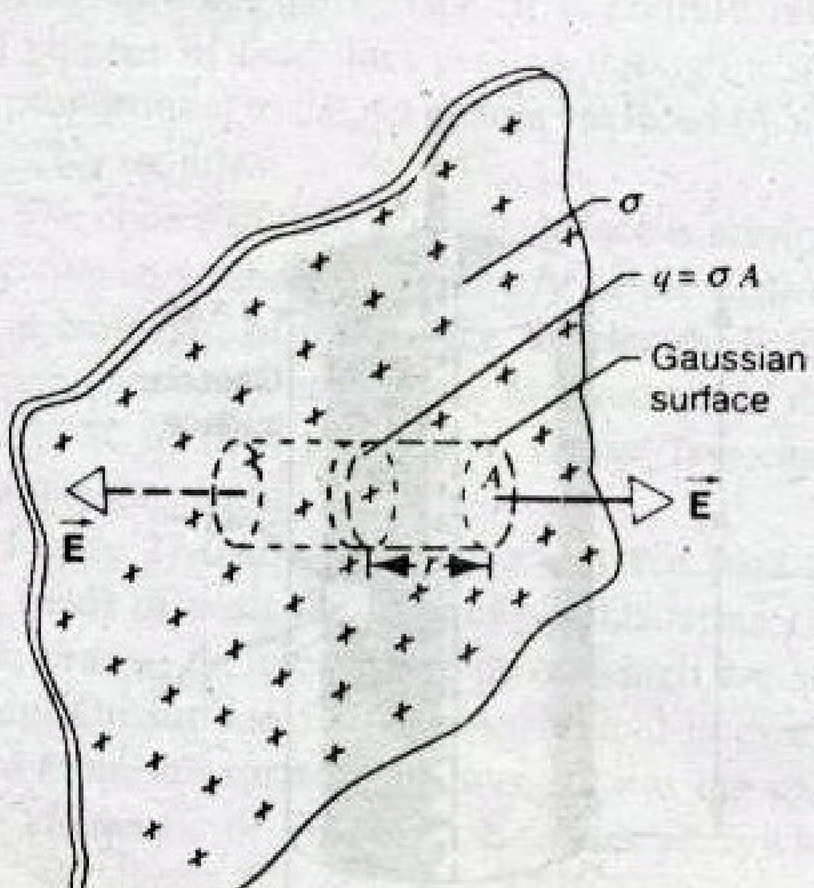

FIGURE 27-9. A Gaussian surface in the form of a small closed cylinder intersects a small portion of a sheet of positive charge. The field is perpendicular to the sheet, and so only the end caps of the Gaussian surface contribute to the flux.

A convenient Gaussian surface is a closed cylinder of cross-sectional area A, arranged to pierce the plane as shown. From symmetry, we can conclude that $\vec{\mathbf{E}}$ points at right angles to the end caps and away from the plane. Since $\vec{\mathbf{E}}$ does not pierce the cylindrical surface, there is no contribution to the flux from the curved wall of the cylinder. We assume that the end caps are equidistant from the sheet. From symmetry the field has the same magnitude at the end caps. The flux through each end cap is EA and is positive for both. Gauss' law gives

$$\epsilon_0 \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = q$$

$$\epsilon_0(EA + EA) = \sigma A,$$

where σA is the enclosed charge. Solving for E, we obtain

$$E = \frac{\sigma}{2\epsilon_0}. \qquad (27\text{-}12)$$

Note that E is the same for all points on each side of the sheet.

Although an infinite sheet of charge cannot exist physically, this derivation is still useful in that Eq. 27-12 gives approximately correct results for real (not infinite) charge sheets if we consider only points that are far from the edges and whose distance from the sheet is small compared to the dimensions of the sheet. In fact, Eq. 27-12 agrees with Eq. 26-20, which we obtained by considering points close to a circular disk of charge.

A Spherical Shell of Charge

In Section 25-5 we used the similarity between electrostatic and gravitational forces to establish two properties of the forces exerted by uniformly charged spherical shells. Then in Section 26-4 we used those properties of the electrostatic force to deduce the electric field due to a uniformly charged spherical shell at points inside or outside the shell.

We can summarize the *shell theorems* for electric fields as follows:

1. *A uniform spherical shell of charge behaves, for external points, as if all its charge were concentrated at its center.*
2. *A uniform spherical shell of charge exerts no electrical force on a charged particle placed inside the shell.*

Let us see how Gauss' law simplifies the calculation of the electric field and the proofs of these shell theorems in this very symmetric geometry. Figure 27-10 shows a thin spherical shell on which a charge q is uniformly distributed. The shell is surrounded by two concentric spherical Gaussian surfaces, S_1 and S_2. From a symmetry argument, we conclude that the field can have only a radial component E_r. (Assume that there were a nonradial component, and suppose that someone rotated the shell

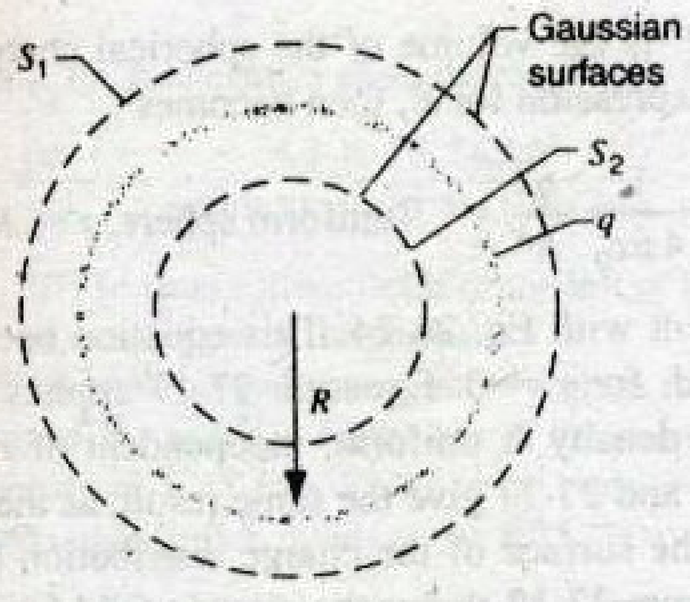

FIGURE 27-10. A cross section of a thin uniformly charged shell of total charge q. The shell is surrounded by two closed spherical Gaussian surfaces, one inside the shell and another outside the shell.

through some angle about a diameter when your back was turned. When you turned back, you could use a probe of the electric field—say, a test charge—to learn that the electric field had changed direction, even though the charge distribution was the same as before the rotation. Clearly this is a contradiction. Would this symmetry argument hold if the charge were *not* uniformly distributed over the surface?) Applying Gauss' law to surface S_1, for which $r > R$, gives

$$\epsilon_0 E_r(4\pi r^2) = q,$$

or

$$E_r = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \quad \text{(spherical shell, } r > R\text{)}, \quad (27\text{-}13)$$

just as it did in connection with Fig. 27-7. Thus *the uniformly charged shell behaves like a point charge for all points outside the shell.* This proves the first shell theorem.

Applying Gauss' law to surface S_2, for which $r < R$, leads directly to

$$E_r = 0 \quad \text{(spherical shell, } r < R\text{)}, \quad (27\text{-}14)$$

because this Gaussian surface encloses no charge and because E_r (by another symmetry argument) has the same value everywhere on the surface. *The electric field therefore vanishes inside a uniform shell of charge;* a test charge placed anywhere in the interior would feel no electric force. This proves the second shell theorem.

These two theorems apply only in the case of a *uniformly* charged shell. If the charges were sprayed on the surface in a nonuniform manner, such that the charge density varied over the surface, these theorems would not apply. The symmetry would be lost, and as a result $\vec{\mathbf{E}}$ could not be removed from the integral in Gauss' law. The flux would remain equal to q/ϵ_0 for all exterior surfaces and zero for all interior surfaces, but we would not be able to make such a direct connection with $\vec{\mathbf{E}}$ as we can in the uniform case. In contrast to the uniformly charged shell, the field would *not* be zero throughout the interior.

Spherically Symmetric Charge Distribution

Figure 27-11 shows a cross section of a spherical distribution of charge of radius R. Here the charge is distributed throughout the spherical volume. We do not assume that the volume charge density ρ (the charge per unit volume) is a constant; however, we make the restriction that ρ at any point depends *only* on the distance of the point from the center, a condition called *spherical symmetry* That is, ρ may be a function of r, but not of any angular coordinate. Let us find an expression for E for points outside (Fig. 27-11*a*) and inside (Fig. 27-11*b*) the charge distribution.

Any spherically symmetric charge distribution, such as that of Fig. 27-11, can be regarded as a nest of concentric thin shells. The volume charge density ρ may vary from one shell to the next, but we make the shells so thin that we can assume ρ is uniform on any particular shell. We can use the results of the previous subsection to calculate the contribution of each shell to the total electric field. The electric field from each thin shell has only a radial component, and thus the total electric field of the sphere can likewise have only a radial component. (This conclusion also follows from a symmetry argument but would not hold if the charge distribution lacked spherical symmetry—that is, if ρ depended on direction.)

Let us calculate the radial component of the electric field at points that lie at a distance r greater than the radius R of the sphere, as shown in Fig. 27-11*a*. Each concentric shell, with a charge dq, contributes a radial component dE_r to the electric field according to Eq. 27-13. The total field is the total of all such components, and because all components to the field are radial, we must compute only an algebraic sum rather than a vector sum. The sum over all the shells then gives

$$E_r = \int dE_r = \int \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}$$

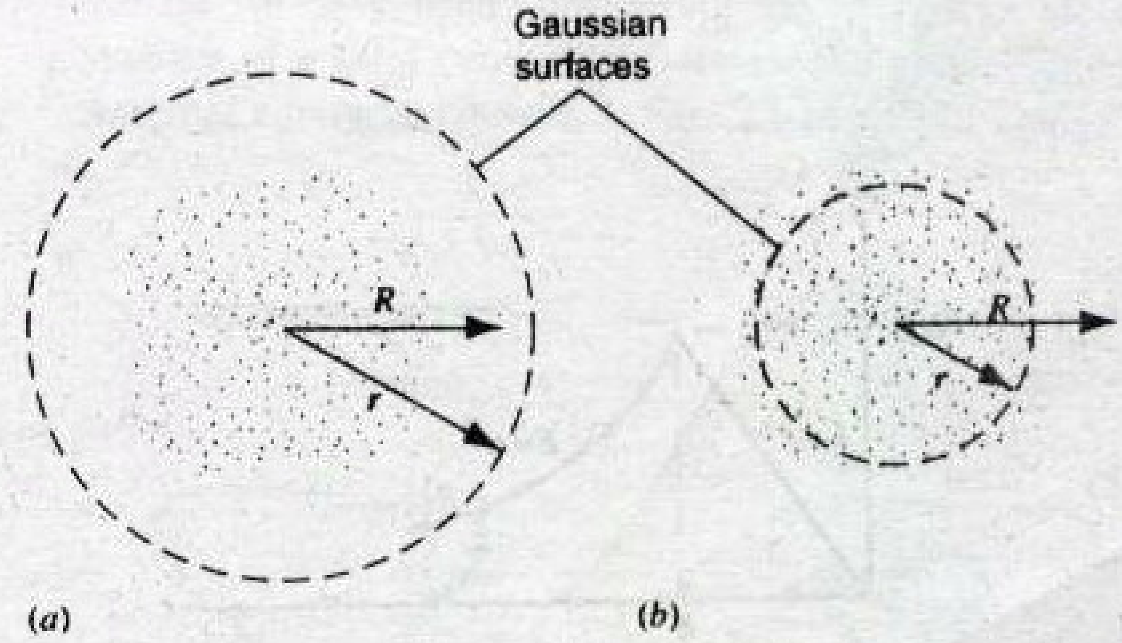

FIGURE 27-11. A cross section of a spherically symmetric charge distribution, in which the volume charge density may vary with r in this assumed nonconducting material. Closed spherical Gaussian surfaces have been drawn (*a*) outside the distribution and (*b*) within the distribution.

or, since r is constant in the integral over q,

$$E_r = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}, \qquad (27\text{-}15)$$

where q is the total charge of the sphere. Thus for points outside a spherically symmetric distribution of charge, the electric field has the value that it would have if the charge were concentrated at its center. This result is similar to the gravitational case proved in Section 14-5. Both results follow from the inverse square nature of the corresponding force laws.

We now consider the electric field for points inside the charge distribution. Figure 27-11*b* shows a spherical Gaussian surface of radius $r < R$. Gauss' law gives

$$\epsilon_0 \oint \vec{E}\cdot d\vec{A} = \epsilon_0 E_r(4\pi r^2) = q'$$

or

$$E_r = \frac{1}{4\pi\epsilon_0}\frac{q'}{r^2}, \qquad (27\text{-}16)$$

in which q' is that part of q contained *within* the sphere of radius r. According to the second shell theorem, the part of q that lies outside this sphere makes no contribution to $\vec{E}$ at radius r.

To continue this calculation, we must know the charge q' that is within the radius r; that is, we must know $\rho(r)$. Let us consider the special case in which the sphere is uniformly charged, so that the charge density ρ has the same value for all points within a sphere of radius R and is zero for all points outside this sphere. For points inside such a uniform sphere of charge, the fraction of charge within r is equal to the fraction of the volume within r, and so

$$\frac{q'}{q} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}$$

or

$$q' = q\left(\frac{r}{R}\right)^3,$$

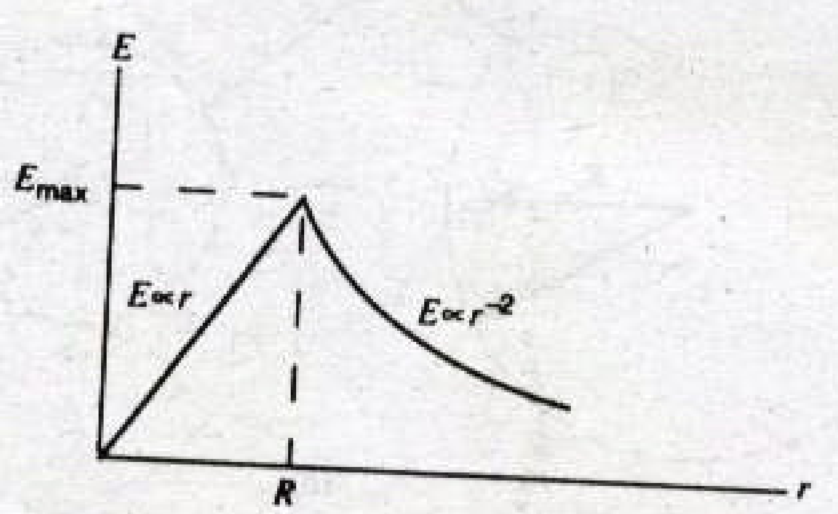

FIGURE 27-12. The variation with radius of the electric field due to a uniform spherical distribution of charge of radius R. The variation for $r > R$ applies to *any* spherically symmetric charge distribution, whereas that for $r < R$ applies *only* to a uniform distribution.

where $\frac{4}{3}\pi R^3$ is the volume of the spherical charge distribution. The expression for E_r then becomes

$$E_r = \frac{1}{4\pi\epsilon_0}\frac{qr}{R^3} \qquad \text{(uniform sphere, } r < R\text{)}. \quad (27\text{-}17)$$

in agreement with Eq. 26-24. This equation becomes zero, as it should, for $r = 0$. Equation 27-17 applies *only* when the charge density is uniform, independent of r. Note that Eqs. 27-15 and 27-17 give the same result, as they must, for points on the surface of the charge distribution (that is, for $r = R$). Figure 27-12 shows the electric field for points with $r < R$ (given by Eq. 27-17) and for points with $r > R$ (given by Eq. 27-15).

SAMPLE PROBLEM 27-3. Figure 27-13*a* shows portions of two large sheets of charge with uniform surface charge densities of $\sigma_+ = +6.8\ \mu C/m^2$ and $\sigma_- = -4.3\ \mu C/m^2$. Find the electric field $\vec{E}$ to the left of the sheets, between the sheets, and to the right of the sheets.

Solution Our strategy is to deal with each sheet separately and then to add the resulting electric fields by using the superposition principle. For the positive sheet we have, from Eq. 27-12,

$$E_+ = \frac{\sigma_+}{2\epsilon_0} = \frac{6.8\times10^{-6}\ \text{C/m}^2}{(2)(8.85\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)} = 3.84\times10^5\ \text{N/C}.$$

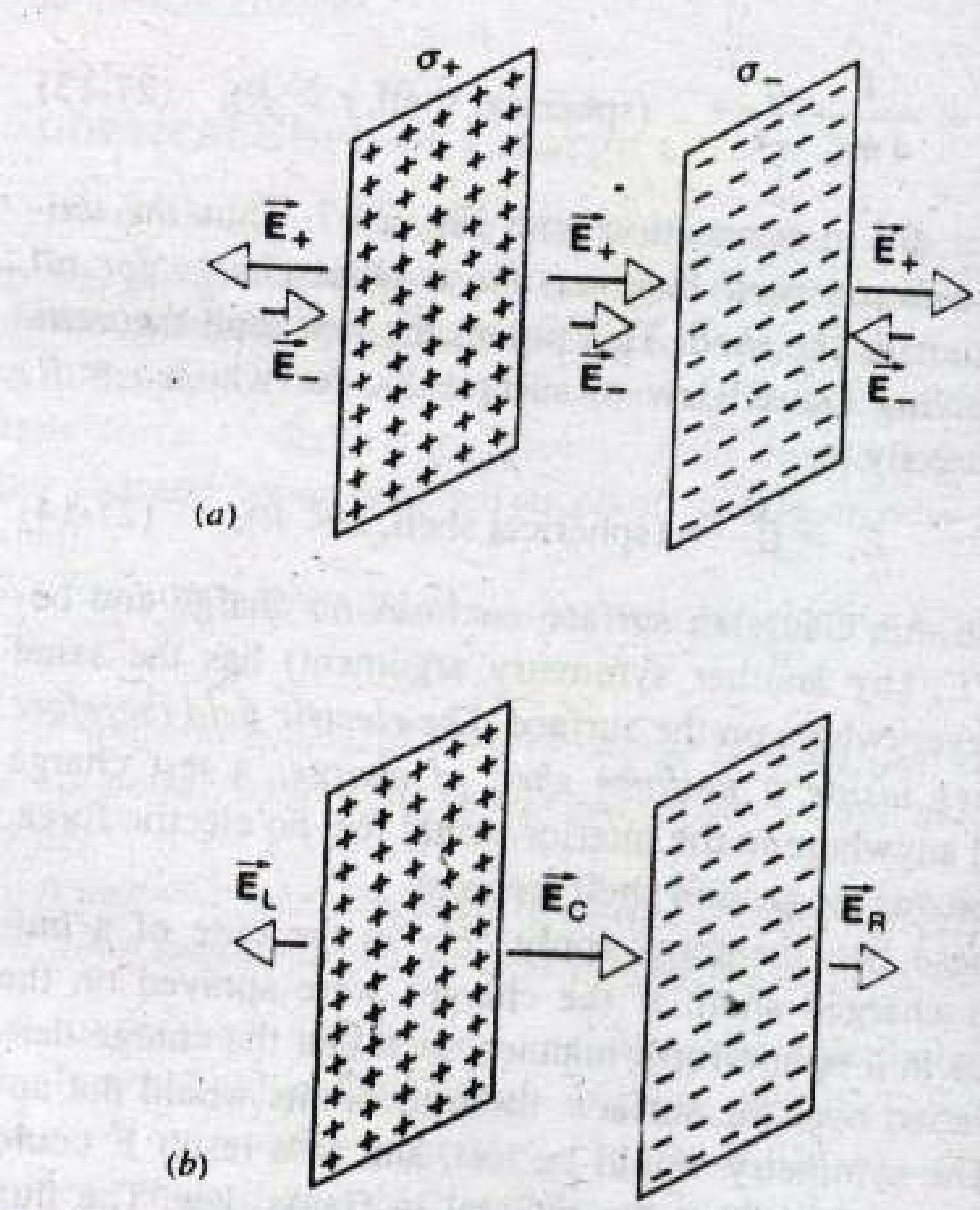

FIGURE 27-13. Sample Problem 27-3. (*a*) Two large parallel sheets of charge carry different charge distributions σ_+ and σ_-. The fields $\vec{E}_+$ and $\vec{E}_-$ would be set up by each sheet if the other were not present. (*b*) The net fields in the nearby regions to the left (L), center (C), and right (R) of the sheets, calculated from the vector sum of $\vec{E}_+$ and $\vec{E}_-$ in each region.

Similarly, for the negative sheet the magnitude of the field is

$$E_- = \frac{|\sigma_-|}{2\epsilon_0} = \frac{4.3 \times 10^{-6}\ \mathrm{C/m^2}}{(2)(8.85 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2})} = 2.43 \times 10^5\ \mathrm{N/C}.$$

Figure 27-13*a* shows these fields to the left of the sheets, between them, and to the right of the sheets.

The resultant fields in these three regions follow from the vector sums of $\mathbf{E}_+$ and $\mathbf{E}_-$. To the left of the sheets, we have (taking components of $\mathbf{E}$ in Fig. 27-13 to be positive if $\mathbf{E}$ points to the right and negative if $\mathbf{E}$ points to the left)

$$E_L = -E_+ + E_- = -3.84 \times 10^5\ \mathrm{N/C} + 2.43 \times 10^5\ \mathrm{N/C} = -1.4 \times 10^5\ \mathrm{N/C}.$$

The resultant (negative) electric field in this region points to the left, as Fig. 27-13*b* shows. To the right of the sheets, the electric field has this same magnitude but points to the right in Fig. 27-13*b*.

Between the sheets, the two fields add to give

$$E_C = E_+ + E_- = 3.84 \times 10^5\ \mathrm{N/C} + 2.43 \times 10^5\ \mathrm{N/C} = 6.3 \times 10^5\ \mathrm{N/C}.$$

Outside the sheets, the electric field behaves like that due to a single sheet whose surface charge density is $\sigma_+ + \sigma_-$ or $+2.5 \times 10^{-6}\ \mathrm{C/m^2}$. The field pattern of Fig. 27-13*b* bears this out. In Exercises 14 and 15 you can investigate the case in which the two surface charge densities are equal in magnitude but opposite in sign and also the case in which they are equal in both magnitude and sign.

27-6 GAUSS' LAW AND CONDUCTORS

We have seen that by using Gauss' law we can find the electric field for several highly symmetric charge distributions. We can also use Gauss' law to deduce the properties of conductors carrying a net electric charge. One such property is

An excess charge placed on an isolated conductor moves entirely to the outer surface of the conductor. None of the excess charge is found within the body of the conductor.

Let us review what occurs when we place a quantity of electric charge on an isolated conductor. These charges can in principle be deposited anywhere in the conductor, even deep within its interior. Initially there is an electric field in the interior of the conductor due to the charges. This electric field results in forces on the charges that cause them to redistribute themselves. Very quickly (within 10^{-9} s) the electric field becomes zero, and the charges stop moving. This is the condition we describe as *electrostatic equilibrium*. If the field in the interior were nonzero, the conduction electrons in the metal would experience a force, and moving charges (an electric current) would be observed. Since we do not observe such currents, we conclude that the electric field is zero in the interior.

Keep in mind that here we are considering only an "isolated" conductor—that is, a conductor that is free from all external influences. A wire carrying a current cannot be considered an isolated conductor, because it must be connected to an external agent such as a battery. The electric field in such a wire is not zero, the wire is *not* in electrostatic equilibrium, and the conclusions of this section do not apply to the wire.

If we accept that the electric field in the interior of the conductor is zero under electrostatic conditions, then Gauss' law directly implies that the charge on the conductor must reside on its outside surface. Figure 27-14*a* shows a conductor of arbitrary shape, perhaps a lump of copper, carrying a net charge q and hanging from an insulating thread. A Gaussian surface has been drawn just inside the outer surface of the conductor.

If the electric field is zero everywhere inside the conductor, it is zero everywhere on our Gaussian surface, which lies entirely inside the conductor. This means that the flux through the Gaussian surfaces is zero. Gauss' law then allows us to conclude that the net charge enclosed by the Gaussian surface must be zero. If there is no charge inside the Gaussian surface, it must be outside that surface, which means that the charge must be on the actual outer surface of the conductor.

Why is the electric field zero inside the conductor? Suppose that we could somehow "freeze" the charges on the surface, perhaps by embedding them in a thin plastic coating, while we removed the conductor completely, leaving only a thin shell of charge. The electric field would not change at all—it would remain zero everywhere inside the shell. This shows that the electric field is set up by the charges and not by the conductor. The conductor merely provides a pathway so that the charges can easily move to take positions where they set up a net electric field of zero inside the conductor.

The Charge on Interior Surfaces

So far we have been considering the charge on the outer surface of a solid conductor. Suppose the conductor has an internal cavity, as shown in Fig. 27-14*b*. Will charge also

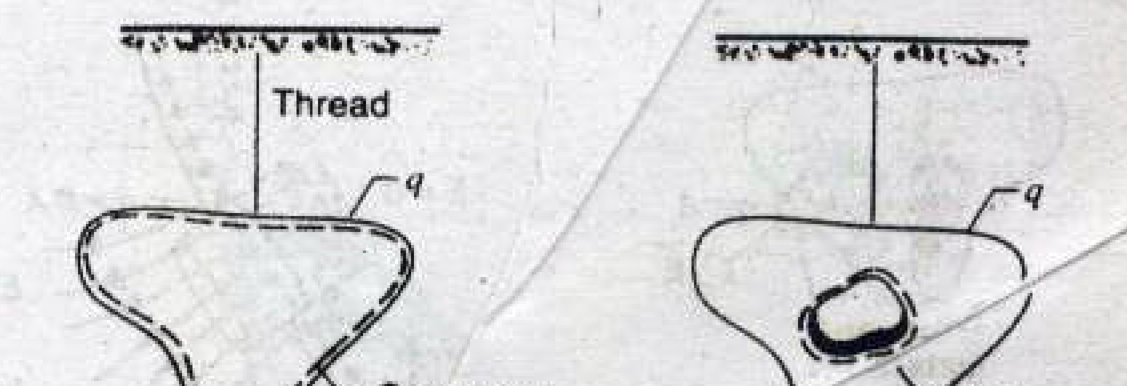

FIGURE 27-14. (*a*) An isolated metallic conductor carrying a charge q hangs from a thread. A Gaussian surface has been drawn just inside the surface of the conductor. (*b*) An internal cavity in the conductor is surrounded by a different Gaussian surface.

appear on the surface of this cavity? It is reasonable to suppose that scooping out electrically neutral material to form the cavity should not change the distribution of charge on the outer surface or the electric field in the interior. We can use Gauss' law for a quantitative proof.

We draw a Gaussian surface surrounding the cavity, close to its surface but inside the conductor, as shown in Fig. 27-14*b*. Because $\vec{\mathbf{E}} = 0$ everywhere inside the conductor, there can be no flux through this Gaussian surface. Therefore, according to Gauss' law, the surface can enclose no net charge, and so there can be no charge on the surface of the interior cavity for an isolated conductor.

If an object with a charge q' is placed inside the cavity (so that we can no longer regard our conductor as isolated), Gauss' law still requires that the net charge inside the Gaussian surface is zero. In this case, a charge $-q'$ must be attracted to the surface of the cavity to keep the net charge zero within the Gaussian surface. If the outer conductor originally carried a net charge q, then a charge of $q + q'$ will appear on its outer surface, so that the net charge does not change.

The Electric Field Outside the Conductor

Although the excess charge on an isolated conductor moves entirely to its surface, that charge—except for an isolated spherical conductor—does not in general distribute itself uniformly over that surface. Put another way, the surface charge density σ $(= dq/dA)$ varies from point to point over the surface.

We can use Gauss' law to find a relation—at any surface point—between the surface charge density σ at that point and the electric field $\vec{\mathbf{E}}$ just outside the surface at that same point. Figure 27-15*a* shows a squat cylindrical Gaussian surface, the (small) area of its two end caps being A. The end caps are parallel to the surface, one lying entirely inside the conductor and the other entirely outside. The short cylindrical walls are perpendicular to the surface of the conductor. An enlarged view of the Gaussian surface is shown in Fig. 27-15*b*.

The electric field just outside a charged isolated conductor in electrostatic equilibrium must be at right angles to the surface of the conductor. If this were not so, there would be a component of $\vec{\mathbf{E}}$ lying in the surface and this component would set up surface currents that would redistribute the surface charges, violating our assumption of electrostatic equilibrium. Thus $\vec{\mathbf{E}}$ is perpendicular to the surface of the conductor, and the flux through the exterior end cap of the Gaussian surface of Fig. 27-15*b* is EA. The flux through the interior end cap is zero, because $\vec{\mathbf{E}} = 0$ for all interior points of the conductor. The flux through the cylindrical walls is also zero because the lines of $\vec{\mathbf{E}}$ are parallel to the surface, so they cannot pierce it. The charge q enclosed by the Gaussian surface is σA.

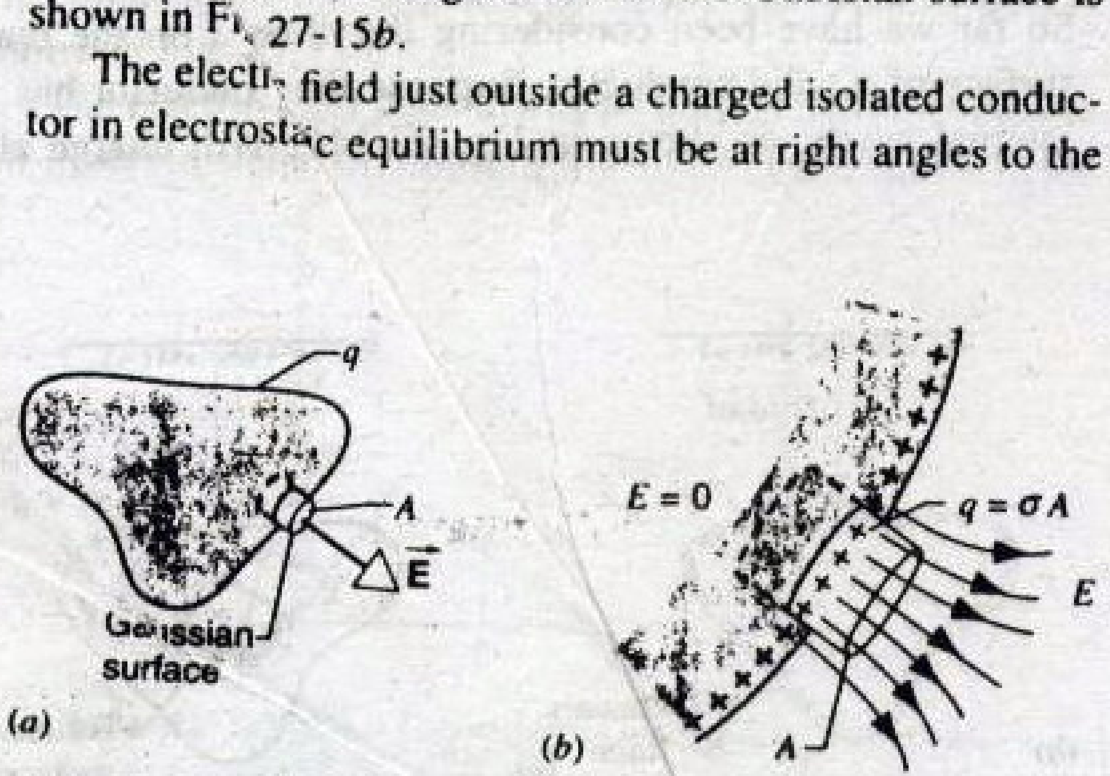

FIGURE 27-15. (*a*) A small Gaussian surface has been placed at the surface of a charged conductor. (*b*) An enlarged view of the Gaussian surface, which encloses a charge q equal to σA.

The total flux can then be calculated as

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

$$= \int_{\text{outer cap}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} + \int_{\text{inner cap}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} + \int_{\text{side walls}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

$$= EA + 0 + 0 = EA.$$

The electric field can now be found by using Gauss' law:

$$\epsilon_0 \Phi_E = q,$$

and substituting the values for the flux and the enclosed charge q $(= \sigma A)$, we find

$$\epsilon_0 EA = \sigma A$$

or

$$E = \frac{\sigma}{\epsilon_0}. \qquad (27\text{-}18)$$

Compare this result with Eq. 27-12 for the electric field near a sheet of charge: $E = \sigma/2\epsilon_0$. The electric field near a conductor is *twice* the field we would expect if we considered the conductor to be a sheet of charge, even for points very close to the surface, where the immediate vicinity *does* look like a sheet of charge. How can we understand the difference between the two cases?

A sheet of charge can be constructed by spraying charges on one side of a thin layer of plastic. The charges stick where they land and are not free to move. We cannot charge a conductor in the same way. We can imagine the surface of the conductor to be divided into two sections: the region near where we wish to find the electric field and the remainder of the conductor. If we are sufficiently close to the conductor in Fig. 27-15, the region near the Gaussian surface can be approximated as a sheet of charge, and it contributes an amount $E = \sigma/2\epsilon_0$ to the electric field. However, the charge on the rest of the conductor can be shown to contribute an identical amount to the electric field. The total electric field is the sum of the two contributions, or σ/ϵ_0.

We can see this most directly in the case of a thin conducting plate. Suppose the plate has area A. If we spray charge q anywhere on the plate, the charge will distribute itself over *both* surfaces of the plate, as in Fig. 27-16. We

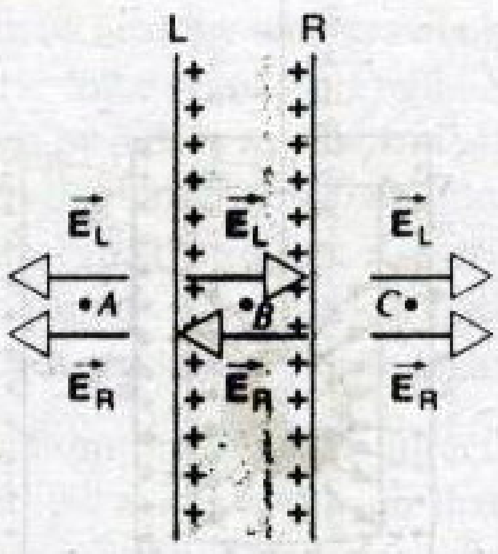

FIGURE 27-16. The electric charge near a thin conducting sheet. Note that both surfaces have charges on them. The fields $\vec{E}_L$ and $\vec{E}_R$ due, respectively, to the charges on the left and right surfaces reinforce at points A and C, and they cancel at points B in the interior of the sheet.

therefore expect to find charge $q/2$ and a charge density $\sigma = q/2A$ on each surface of the plate. We can consider each surface of the plate as a sheet of charge, which (according to Eq. 27-12) establishes an electric field $E = \sigma/2\epsilon_0 = q/4A\epsilon_0$. Close to the plate (at points A or C in Fig. 27-16) the fields due to the left and right surfaces are equal, and they add to give a total electric field of $E = q/4A\epsilon_0 + q/4A\epsilon_0 = q/2A\epsilon_0 = \sigma/\epsilon_0$. In the interior of the plate (point B), the fields are in the opposite direction and sum to zero, as expected for the interior of a conductor.

Suppose that now we bring a second plate, carrying a charge $-q$, to the vicinity of the first plate. Now the original conductor can no longer be considered as "isolated," and the charge on its outer surface is no longer uniformly distributed. There is an attraction between the positive charges on one plate and the negative charges on the other that draws the charges to the surfaces of the plates facing one another (Fig. 27-17). Each surface has charge q (instead of $q/2$) and charge density $\sigma = q/A$. Regarded as a sheet of charge, each surface sets up an electric field $E = \sigma/2\epsilon_0 = q/2A\epsilon_0$, according to Eq. 27-12. In the region between them, the positive and negative plates give contributions to the electric field of equal magnitudes and identical directions, so the net electric field between the plates is $E = \sigma/\epsilon_0 = q/A\epsilon_0$. This is the electric field of a parallel-plate capacitor, which is what is shown in Fig. 27-17.

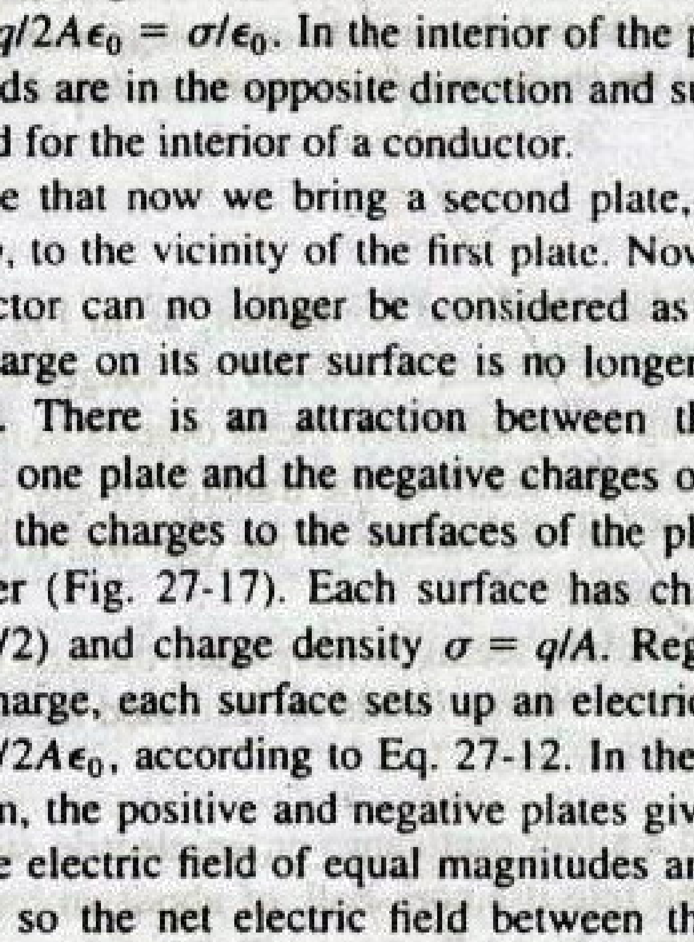

FIGURE 27-17. Two thin conducting plates carry equal and opposite charges. $\vec{E}_+$ is the field due to the positively charged plate, and $\vec{E}_-$ is the field set up by the negatively charged plate.

SAMPLE PROBLEM 27-4. The electric field just above the surface of the charged drum of a photocopying machine has a magnitude E of 2.3×10^5 N/C. What is the surface charge density on the drum if it is a conductor?

Solution From Eq. 27-18 we have

$$\sigma = \epsilon_0 E = (8.85 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2})(2.3 \times 10^5\ \mathrm{N/C}) = 2.0 \times 10^{-6}\ \mathrm{C/m^2} = 2.0\ \mu\mathrm{C/m^2}.$$

SAMPLE PROBLEM 27-5. The magnitude of the average electric field normally present in the Earth's atmosphere just above the surface of the Earth is about 150 N/C, directed downward (radially inward, toward the center of the Earth). What is the total net surface charge carried by the Earth? Assume the Earth to be a conductor.

Solution Lines of force terminate on negative charges so that, if the Earth's electric field points downward, its average surface charge density σ must be negative. From Eq. 27-18 we find

$$\sigma = \epsilon_0 E = (8.85 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2})(-150\ \mathrm{N/C}) = -1.33 \times 10^{-9}\ \mathrm{C/m^2}.$$

The Earth's total charge q is the surface charge density multiplied by $4\pi R^2$, the surface area of the (presumed spherical) Earth. Thus

$$q = \sigma 4\pi R^2 = (-1.33 \times 10^{-9}\ \mathrm{C/m^2})(4\pi)(6.37 \times 10^6\ \mathrm{m})^2 = -6.8 \times 10^5\ \mathrm{C} = -680\ \mathrm{kC}.$$

SAMPLE PROBLEM 27-6. A long hollow cylindrical conductor (inner radius a, outer radius b) is surrounded by a long coaxial cylindrical conducting shell (inner radius c, outer radius d), as shown in Fig. 27-18. The inner conductor carries a positive charge $2q$, and the outer conductor carries a charge $-3q$. Find the charge that resides on each surface of the two conductors.

Solution Gauss' law gives similar results in the cylindrical and spherical geometries. In particular, the electric field due to the outer conductor in the region $r < c$ is zero, just as we proved for the spherical shell of charge. The charges on the outer conductor produce no electric field at the location of the inner conductor, which can thus be regarded as "isolated" for this discussion. If we

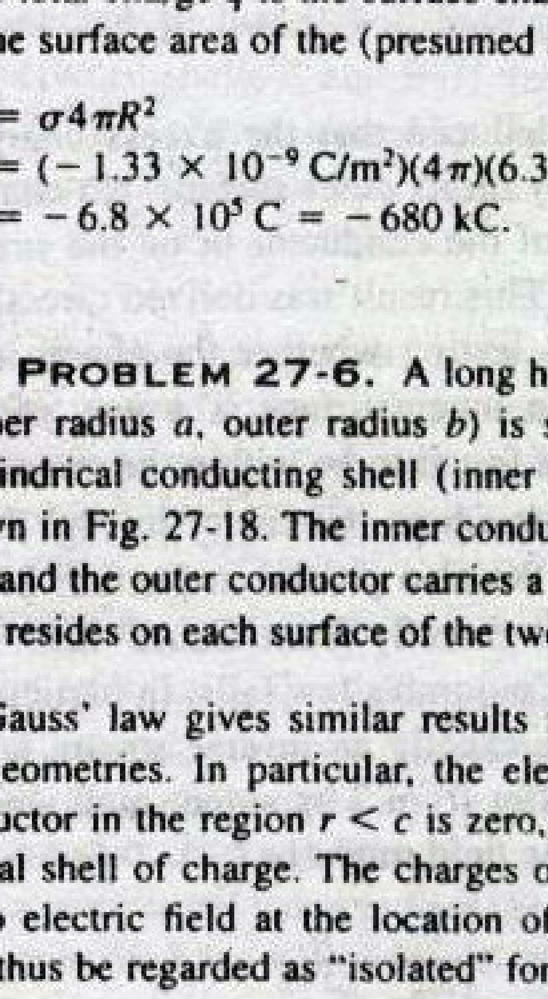

FIGURE 27-18. Sample Problem 27-6. Two coaxial cylindrical conducting shells.

treat the inner conductor as isolated, we conclude that charge must reside entirely on its outer surface. Thus there is no charge on surface a and a positive charge $2q$ on surface b.

If we were to draw a coaxial cylindrical Gaussian surface through the interior of the outer cylinder ($c < r < d$), we can use Gauss' law to conclude that the flux through the Gaussian surface is zero. The flux through the curved part of the Gaussian surface is zero, because $E = 0$ everywhere inside the conductor, and the flux through the flat ends of the surface is also zero, because the field for $b < r < c$ must be radial and therefore parallel to the flat surfaces. This means, according to Gauss' law, that the total charge inside the Gaussian surface must be zero. We know that there is a charge $2q$ on the inner conductor, so to make the net charge zero there must be a charge $-2q$ on surface c. Since the total charge on the outer cylinder is $-3q$, the remaining charge of $-q$ must appear on surface d.

Note that the outer conductor is influenced by the charge on the inner conductor and cannot be considered as isolated, so the charge on that conductor does not all reside on its outer surface.

27-7 EXPERIMENTAL TESTS OF GAUSS' LAW AND COULOMB'S LAW

In Section 27-6, we deduced that the excess charge in a conductor must lie only on its outer surface. No charge can be within the volume of the conductor or on the surface of an empty inner cavity. This result was derived directly from Gauss' law. Therefore, testing whether the charge does in fact lie entirely on the outer surface is a way of testing Gauss' law. If charge is found to be within the conductor or on an interior surface (such as the cavity in Fig. 27-14b), then Gauss' law fails. We also proved in Section 27-4 that Coulomb's law follows directly from Gauss' law. Thus if Gauss' law fails, then Coulomb's law fails. In particular, the force law might not be exactly an inverse square law. The exponent of r might differ from 2 by some small amount δ, so that the radial electric field might be

$$E_r = \frac{1}{4\pi\epsilon_0}\frac{q}{r^{2+\delta}}. \qquad (27\text{-}19)$$

in which δ is exactly zero if Coulomb's law and Gauss' law hold.

The direct measurement of the force between two charges, described in Chapter 25, does not have the precision necessary to test whether δ is zero beyond a few percent. The observation of the charge within a conductor provides the means for a test that, as we shall see, is far more precise.

In principle, the experiment follows a procedure illustrated in Fig. 27-19. A charged metal ball hangs from an insulating thread and is lowered into a metal can that rests on an insulating stand. When the ball is touched to the inside of the can, the two objects form a *single conductor*, and, if Gauss' law is valid, all the charge from the ball must go to

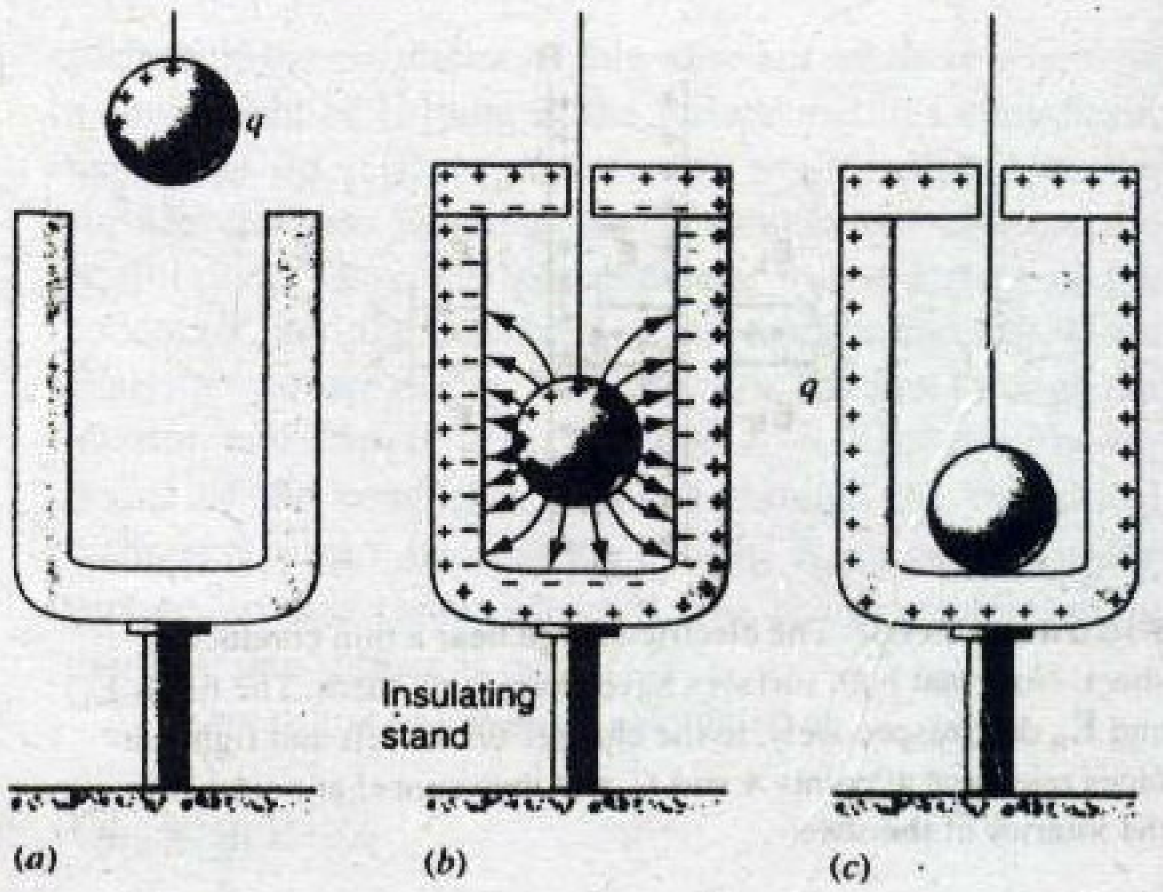

FIGURE 27-19. An arrangement conceived by Benjamin Franklin to show that the charge placed on a conductor moves to its surface. (a) A charged metal ball is lowered into an uncharged metal can. (b) The ball is inside the can and a cover is added. The field lines between the ball and the uncharged can are shown. The ball attracts charges of the opposite sign to the inside of the can. (c) When the ball touches the can, they form a single conductor, and the net charge flows to the outer surface. The ball can then be removed from the can and shown to be completely uncharged, thus proving that the charge must have been transferred entirely to the can.

the outside of the combined conductor, as in Fig. 27-19c. When the ball is removed, it should no longer carry any charge. Touching other insulated metal objects to the inside of the can should not result in the transfer of any charge to the objects. Only on the outside of the can will it be possible to transfer charge.

Benjamin Franklin seems to have been the first to notice that there can be no charge inside an insulated metal can. In 1755 he wrote to a friend:

> I electrified a silver pint cann, on an electric stand, and then lowered into it a cork-ball, of about an inch in diameter, hanging by a silk string, till the cork touched the bottom of the cann. The cork was not attracted to the inside of the cann as it would have been to the outside, and though it touched the bottom, yet when drawn out, it was not found to be electrified by that touch, as it would have been by touching the outside. The fact is singular. You require the reason; I do not know it. . . .

About 10 years later Franklin recommended this "singular fact" to the attention of his friend Joseph Priestley (1733–1804). In 1767 (about 20 years before Coulomb's experiments) Priestley checked Franklin's observation and, with remarkable insight, realized that the inverse square law of force followed from it. Thus the indirect approach is not only more accurate than the direct approach of Section 25-4 but was also carried out earlier.

Priestley, reasoning by analogy with gravitation, said that the fact that no electric force acted on Franklin's cork ball when it was surrounded by a deep metal can is similar to the fact (see Section 14-5) that no gravitational force acts on a particle inside a spherical shell of matter; if gravitation obeys an inverse square law, perhaps the electrical force does also. Considering Franklin's experiment, Priestley reasoned:

> May we not infer from this that the attraction of electricity is subject to the same laws with that of gravitation and is therefore according to the squares of the distances; since it is easily demonstrated that were the earth in the form of a shell, a body in the inside of it would not be attracted to one side more than another?

Note how knowledge of one subject (gravitation) helps in understanding another (electrostatics).

Michael Faraday also carried out experiments designed to show that excess charge resides on the outside surface of a conductor. In particular, he built a large metal-covered box, which he mounted on insulating supports and charged with a powerful electrostatic generator. In Faraday's words:

> I went into the cube and lived in it, and using lighted candles, electrometers, and all other tests of electrical states, I could not find the least influence upon them . . . though all the time the outside of the cube was very powerfully charged, and large sparks and brushes were darting off from every part of its outer surface.

Coulomb's law is vitally important in physics, and if δ in Eq. 27-19 is not zero, there are serious consequences for our understanding of electromagnetism and quantum physics. The best way to measure δ is to find out *by experiment* whether an excess charge, placed on an isolated conductor, does or does not move *entirely* to its outside surface.

Modern experiments, carried out with remarkable precision, have shown that if δ in Eq. 27-19 is not zero it is certainly very, very small. Table 27-1 summarizes the results of the most important of these experiments.

Figure 27-20 is a drawing of the apparatus used by Plimpton and Lawton to measure δ. It consists in principle of two concentric metal shells, A and B, the former being 1.5 m in diameter. The inner shell contains a sensitive electrometer E connected so that it will indicate whether any charge moves between shells A and B. If the shells are connected electrically, any charge placed on the shell assembly should reside entirely on shell A if Gauss' law—and thus Coulomb's law—is correct as stated.

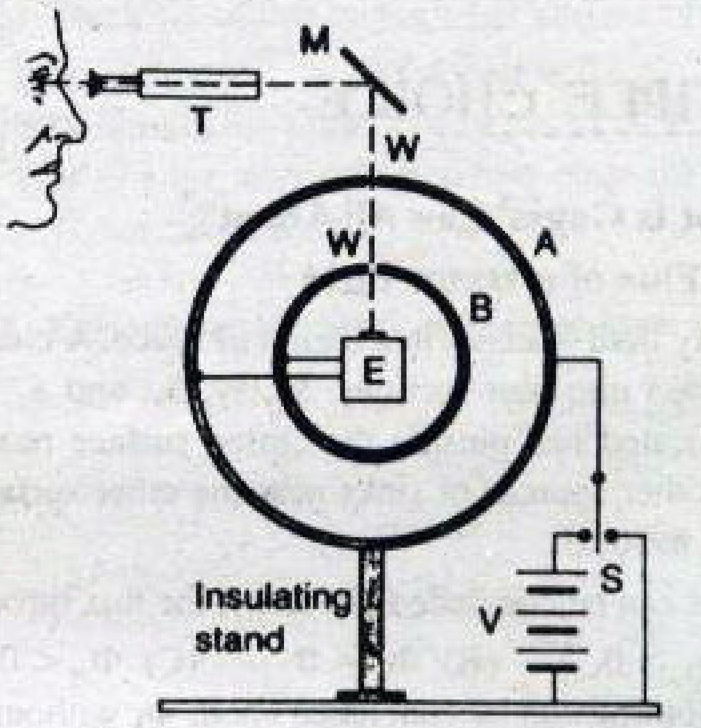

FIGURE 27-20. A modern and more precise version of the apparatus of Fig. 27-19, also designed to verify that charge resides only on the outside surface of a conductor. Charge is placed on sphere A by throwing switch S to the left, and the sensitive electrometer E is used to search for any charge that might move to the inner sphere B. It is expected that all the charge will remain on the outer surface (sphere A).

By throwing switch S to the left, a substantial charge could be placed on the sphere assembly by the battery V. If any of this charge moved to shell B, it would have to pass through the electrometer and would cause a deflection, which could be observed optically using telescope T, mirror M, and windows W.

However, when the switch S was thrown alternately from left to right, connecting the shell assembly either to the battery or to the ground, no effect was observed. Knowing the sensitivity of their electrometer, Plimpton and Lawton calculated that δ in Eq. 27-19 differs from zero by no more than 2×10^{-9}, a very small value indeed. Yet since their experiment, the limits on δ have been improved by more than seven orders of magnitude by experimenters using more detailed and precise versions of this basic apparatus.

TABLE 27-1 Tests of Coulomb's Inverse Square Law

Experimenters	*Date*	*δ (Eq. 27-19)*
Franklin	1755	
Priestley	1767	. . . according to the squares . . .
Robison	1769	<0.06
Cavendish	1773	<0.02
Coulomb	1785	a few percent at most
Maxwell	1873	$<5 \times 10^{-5}$
Plimpton and Lawton	1936	$<2 \times 10^{-9}$
Bartlett, Goldhagen, and Phillips	1970	$<1.3 \times 10^{-13}$
Williams, Faller, and Hill	1971	$<1.0 \times 10^{-16}$

MULTIPLE CHOICE

27-1 What Is Gauss' Law All About?

27-2 The Flux of a Vector Field

1. A velocity field $\vec{v}$ exists in a region of space. A closed surface S is divided into four sections, S_1, S_2, S_3, and S_4. There is a source located just outside the closed surface near S_1; there may be other sources or sinks near the other surfaces S_n, but none are inside S.

   (*a*) What can be concluded about Φ_1, the flux through S_1?

   (A) $\Phi_1 > 0$ (B) $\Phi_1 = 0$ (C) $\Phi_1 < 0$
   (D) Nothing can be concluded about Φ_1 without additional information.

   (*b*) Which of the following statements is correct about the flux through the four surfaces?

   (A) At least one of the Φ_n must be negative.
   (B) At least one of the Φ_n must be positive.
   (C) At least one of the Φ_n must be zero.
   (D) If A is correct, then so is B.
   (E) Either A or B could be correct, but not both.

   (*c*) Measurements indicate that $\Phi_1 + \Phi_2 > 0$. From this information we can conclude that

   (A) $\Phi_3 = \Phi_4$. (B) $\Phi_3 = -\Phi_4$. (C) $\Phi_3 > \Phi_4$.
   (D) $\Phi_3 < -\Phi_4$.

27-3 The Flux of the Electric Field

2. The flux through a flat surface of area A in a uniform $\vec{E}$ field is a maximum when

   (A) the surface is parallel to $\vec{E}$.
   (B) the surface is perpendicular to $\vec{E}$.
   (C) the surface is shaped like a rectangle.
   (D) the surface is shaped like a square.

3. A closed spherical surface of radius a is in a uniform electric field $\vec{E}$. What is the electric flux Φ_E through the surface?

   (A) $\Phi_E = 4\pi a^2 E$ (B) $\Phi_E = \pi a^2 E$
   (C) $\Phi_E = 0$
   (D) Φ_E cannot be determined without additional knowledge.

27-4 Gauss' Law

4. Consider two concentric spherical surfaces, S_1 with radius a and S_2 with radius $2a$, both centered on the origin. There is a charge $+q$ at the origin, and no other charges. Compare the flux Φ_1 through S_1 with the flux Φ_2 through S_2.

   (A) $\Phi_1 = 4\Phi_2$. (B) $\Phi_1 = 2\Phi_2$.
   (C) $\Phi_1 = \Phi_2$. (D) $\Phi_1 = \Phi_2/2$.

5. An imaginary closed spherical surface S of radius R is centered on the origin. A positive charge is originally at the origin, and the flux through the surface is Φ_E. The positive charge is slowly moved from the origin to a point $R/2$ away from the origin. In doing so the flux through S

   (A) increases to $4\Phi_E$. (B) increases to $2\Phi_E$.
   (C) remains the same. (D) decreases to $\Phi_E/2$.
   (E) decreases to $\Phi_E/4$.

6. Under what conditions can the electric flux Φ_E be found through a closed surface?

   (A) If the magnitude of $\vec{E}$ is known everywhere on the surface.
   (B) If the total charge inside the surface is specified.
   (C) If the total charge outside the surface is specified.
   (D) Only if the location of each point charge inside the surface is specified.

7. An imaginary, closed, spherical surface S of radius R is centered on the origin. A positive charge $+q$ is originally at the origin, and the flux through the surface is Φ_E. Three additional charges are now added along the x axis: $-3q$ at $x = -R/2$, $+5q$ at $x = R/2$, and $+4q$ at $x = 3R/2$. The flux through S is now

   (A) $2\Phi_E$. (B) $3\Phi_E$. (C) $6\Phi_E$. (D) $7\Phi_E$.
   (E) Φ_E cannot be determined, because the problem is no longer symmetric.

27-5 Applications of Gauss' Law

8. A dipole lies on the x axis, with the positive charge $+q$ at $x = +d/2$, and the negative charge at $-d/2$. The electric flux Φ_E through the yz plane midway between the charges

   (A) is zero. (B) depends on d.
   (C) depends on q. (D) depends on both q and d.

9. The surface in Multiple-choice question 8 is shifted closer to the positive charge. As the surface moves, the flux Φ_E through the surface

   (A) increases. (B) decreases.
   (C) remains the same.

10. The surface in Multiple-choice question 8 is instead *rotated* so that the normal to the surface is no longer parallel to the x axis. As the surface moves, the flux Φ_E through the surface

    (A) increases. (B) decreases.
    (C) remains the same.

11. In which of the following problems would Gauss' law be useful?

    (A) Finding the electric field at various points on the surface of a uniformly charged cylinder of finite length
    (B) Finding the electric flux through the end surface of a uniformly charged cylinder
    (C) Finding the electric field at various points on the surface of a uniformly charged cube
    (D) Finding the electric flux through one side of a uniformly charged cube

27-6 Gauss' Law and Conductors

12. A hollow conducting ball has a single positive charge $+q$ fixed at the center. The ball has no net charge.

    (*a*) The charge on the inner surface of the ball is

    (A) $+2q$. (B) $+q$. (C) $-q$. (D) 0.

    (*b*) The charge on the outer surface of the ball is

    (A) $+2q$. (B) $+q$. (C) $-q$. (D) 0.

13. Suppose that a net charge $+q$ is placed on the ball of Multiple-choice question 12; the point charge remains at its center.

    (*a*) The charge on the inner surface of the ball is

    (A) $+2q$. (B) $+q$. (C) $-q$. (D) 0.

    (*b*) The charge on the outer surface of the ball is

    (A) $+2q$. (B) $+q$. (C) $-q$. (D) 0.

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14. The positive charge at the center of the ball in Multiple-choice question 13 is moved off center closer to the inner surface, but it does not touch the inner surface.

(*a*) The total charge on the inner surface of the ball will

(A) increase. (B) decrease.
(C) remain the same.
(D) change, depending on how close the ball gets to the inner surface.

(*b*) The total charge on the outer surface of the ball will

(A) increase. (B) decrease.
(C) remain the same.
(D) change, depending on how close the ball gets to the inner surface.

27-7 Experimental Tests of Gauss' Law and Coulomb's Law

QUESTIONS

1. What is the basis for the statement that lines of electric force begin and end only on electric charges?
2. Positive charges are sometimes called "sources" and negative charges "sinks" of electric field. How would you justify this terminology? Are there sources and/or sinks of gravitational field?
3. By analogy with Φ_E, how would you define the flux Φ_g of a gravitational field? What is the flux of the Earth's gravitational field through the boundaries of a room, assumed to contain no matter? Through a spherical surface closely surrounding the Earth? Through a spherical surface the size of the Moon's orbit?
4. Consider the Gaussian surface that surrounds part of the charge distribution shown in Fig. 27-21. (*a*) Which of the charges contribute to the electric field at point *P*? (*b*) Would the value obtained for the flux through the surface, calculated using only the field due to q_1 and q_2, be greater than, equal to, or less than that obtained using the total field?

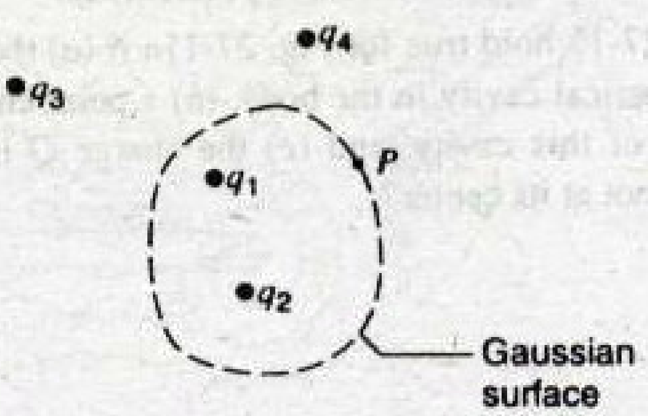


FIGURE 27-21. Question 4.

5. Suppose that an electric field in some region is found to have a constant direction but to be decreasing in strength in that direction. What do you conclude about the charge in the region? Sketch the lines of force.
6. Is it precisely true that Gauss' law states that the total number of lines of force crossing any closed surface in the outward direction is proportional to the net positive charge enclosed within the surface?
7. A point charge is placed at the center of a spherical Gaussian surface. Is Φ_E changed (*a*) if the surface is replaced by a cube of the same volume; (*b*) if the sphere is replaced by a cube of one-tenth the volume; (*c*) if the charge is moved off-center in the original sphere, still remaining inside; (*d*) if the charge is moved just outside the original sphere; (*e*) if a second charge is placed near, and outside, the original sphere; and (*f*) if a second charge is placed inside the Gaussian surface?
8. In Gauss' law $\epsilon_0 \oint \vec{E} \cdot d\vec{A} = q$, is $\vec{E}$ necessarily the electric field attributable to the charge *q*?
9. A surface encloses an electric dipole. What can you say about Φ_E for this surface?
10. Suppose that a Gaussian surface encloses no net charge. Does Gauss' law require that $\vec{E}$ equal zero for all points on the surface? Is the converse of this statement true; that is, if $\vec{E}$ equals zero everywhere on the surface, does Gauss' law require that there be no net charge inside?
11. Is Gauss' law useful in calculating the field due to three equal charges located at the corners of an equilateral triangle? Explain why or why not.
12. A total charge *Q* is distributed uniformly througnout a cube of edge length *a*. Is the resulting electric field at an external point *P*, a distance *r* from the center *C* of the cube, given by $E = Q/4\pi\epsilon_0 r^2$? See Fig. 27-22. If not, can *E* be found by constructing a "concentric" cubical Gaussian surface? If not, explain why not. Can you say anything about *E* if $r \gg a$?

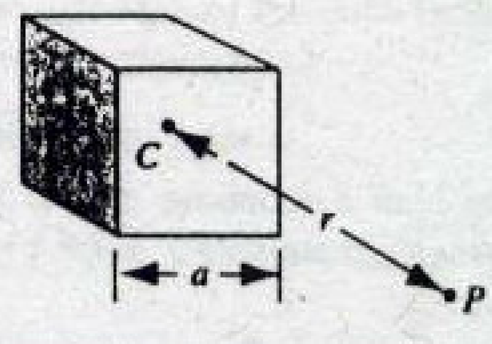


FIGURE 27-22. Question 12.

13. Is $\vec{E}$ necessarily zero inside a charged rubber balloon if the balloon is (*a*) spherical or (*b*) sausage-shaped? For each shape, assume the charge to be distributed uniformly over the surface. How would the situation change, if at all, if the balloon has a thin layer of conducting paint on its outside surface?
14. A spherical rubber balloon carries a charge that is uniformly distributed over its surface. As the balloon is blown up, how does *E* vary for points (*a*) inside the balloon, (*b*) at the surface of the balloon, and (*c*) outside the balloon?

15. In Section 27-4 we have seen that Coulomb's law can be derived from Gauss' law. Does this necessarily mean that Gauss' law can be derived from Coulomb's law?
16. Would Gauss' law hold if the exponent in Coulomb's law were not exactly 2?
17. A large, insulated, hollow conductor carries a positive charge. A small metal ball carrying a negative charge of the same magnitude is lowered by a thread through a small opening in the top of the conductor, allowed to touch the inner surface, and then withdrawn. What then is the charge on (*a*) the conductor and (*b*) the ball?
18. Can we deduce from the argument of Section 27-6 that the electrons in the wires of a house wiring system move along the surfaces of those wires? If not, why not?
19. In Section 27-6, we assumed that $\mathbf{E}$ equals zero everywhere inside an isolated conductor. However, there are certainly very large electric fields inside the conductor, at points close to the electrons or to the nuclei. Does this invalidate the proof of Section 27-4? Explain.
20. Does Gauss' law, as applied in Section 27-6, require that all the conduction electrons in an insulated conductor reside on the surface?
21. A positive point charge q is located at the center of a hollow metal sphere. What charges appear on (*a*) the inner surface and (*b*) the outer surface of the sphere? (*c*) If you bring an (uncharged) metal object near the sphere, will it change your answers to (*a*) or (*b*)? Will it change the way charge is distributed over the sphere?
22. If a charge $-q$ is distributed uniformly over the surface of a thin, insulated, spherical metal shell of radius a, there will be no electric field inside. If now a point charge $+q$ is placed at the center of the sphere, there will be no external field. This point charge can be displaced a distance $d < a$ from the center, but that gives the system a dipole moment and creates an external field. How do you account for the energy appearing in this external field?
23. How can you remove completely the excess charge from a small conducting body?
24. Explain why the spherical symmetry of Fig. 27-7 restricts us to a consideration of $\mathbf{E}$ that has only a radial component at any point. (Hint: Imagine other components, perhaps along the equivalent of longitude or latitude lines on the Earth's surface. Spherical symmetry requires that these look the same from any perspective. Can you invent such field lines that satisfy this criterion?)
25. Explain why the symmetry of Fig. 27-8 restricts us to a consideration of $\mathbf{E}$ that has only a radial component at any point. Remember, in this case, that the field must not only look the same at any point along the line but must also look the same if the figure is turned end for end.
26. The total charge on a charged infinite rod is infinite. Why is not E also infinite? After all, according to Coulomb's law, if q is infinite, so is E.
27. Explain why the symmetry of Fig. 27-9 restricts us to a consideration of $\mathbf{E}$ that has only a component directed away from the sheet. Why, for example, could $\mathbf{E}$ not have a component parallel to the sheet? Remember, in this case, that the field must not only look the same at any point along the sheet in any direction but must also look the same if the sheet is rotated about a line perpendicular to the sheet.
28. The field due to an infinite sheet of charge is uniform, having the same strength at all points no matter how far from the surface charge. Explain how this can be, given the inverse square nature of Coulomb's law.
29. As you penetrate a uniform sphere of charge, E should decrease because less charge lies inside a sphere drawn through the observation point. On the other hand, E should increase because you are closer to the center of this charge. Which effect dominates, and why?
30. Given a spherically symmetric charge distribution (not of uniform radial density of charge), is E necessarily a maximum at the surface? Comment on various possibilities.
31. Does Eq. 27-15 hold true for Fig. 27-11*a* if (*a*) there is a concentric spherical cavity in the body, (*b*) a point charge Q is at the center of this cavity, and (*c*) the charge Q is inside the cavity but not at its center?

EXERCISES

27-1 What Is Gauss' Law All About?

27-2 The Flux of a Vector Field

27-3 The Flux of the Electric Field

1. The square surface shown in Fig. 27-23 measures 3.2 mm on each side. It is immersed in a uniform electric field with $E = 1800$ N/C. The field lines make an angle of 65° with the "outward pointing" normal, as shown. Calculate the flux through the surface.

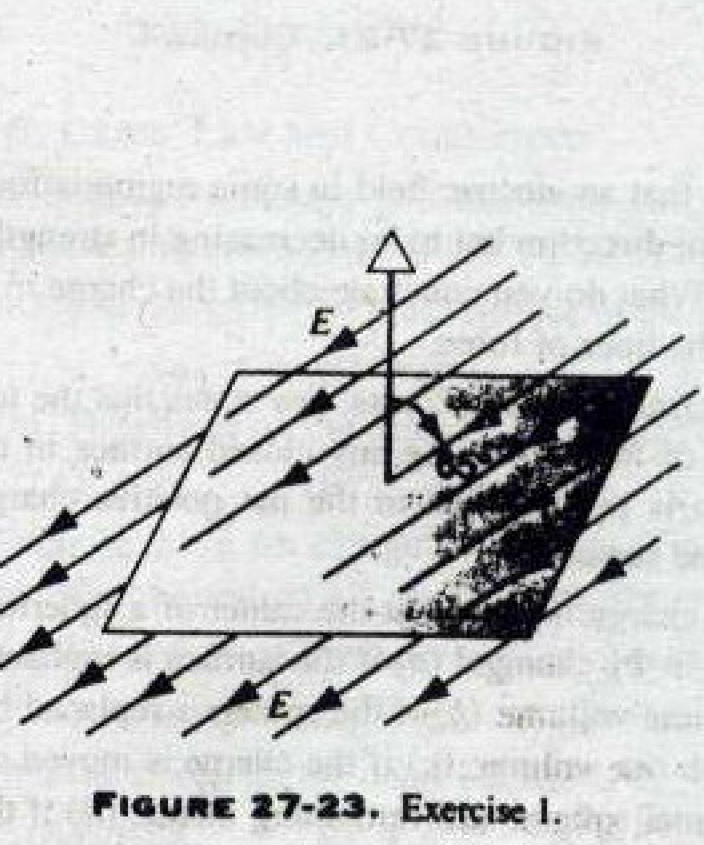


FIGURE 27-23. Exercise 1.

2. A cube with 1.4-m edges is oriented as shown in Fig. 27-24 in a region of uniform electric field. Find the electric flux through the right face if the electric field is given by (*a*) (6 N/C)$\hat{\mathbf{i}}$, (*b*) (− 2 N/C)$\hat{\mathbf{j}}$, and (*c*) (− 3 N/C)$\hat{\mathbf{i}}$ + (4 N/C)$\hat{\mathbf{k}}$. (*d*) Calculate the total flux through the cube for each of these fields.

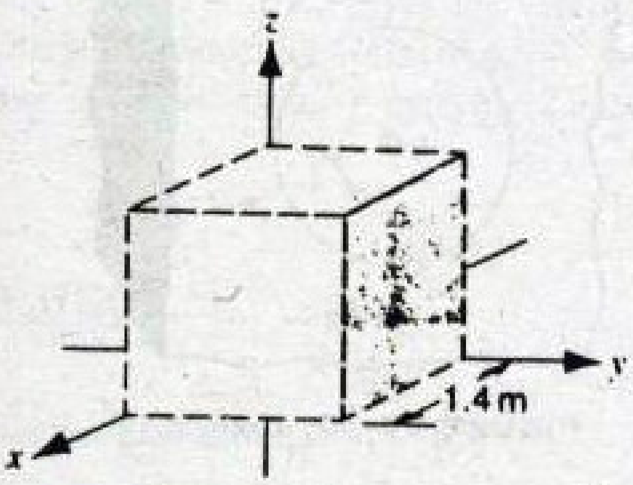


FIGURE 27-24. Exercise 2.

3. Calculate Φ_E through (*a*) the flat base and (*b*) the curved surface of a hemisphere of radius *R*. The field $\mathbf{E}$ is uniform and parallel to the axis of the hemisphere, and the lines of $\mathbf{E}$ enter through the flat base. (Use the outward pointing normal.)

27-4 Gauss' Law

4. Charge on an originally uncharged insulated conductor is separated by holding a positively charged rod very closely nearby, as in Fig. 27-25. Calculate the flux for the five Gaussian surfaces shown. Assume that the induced negative charge on the conductor is equal to the positive charge *q* on the rod.

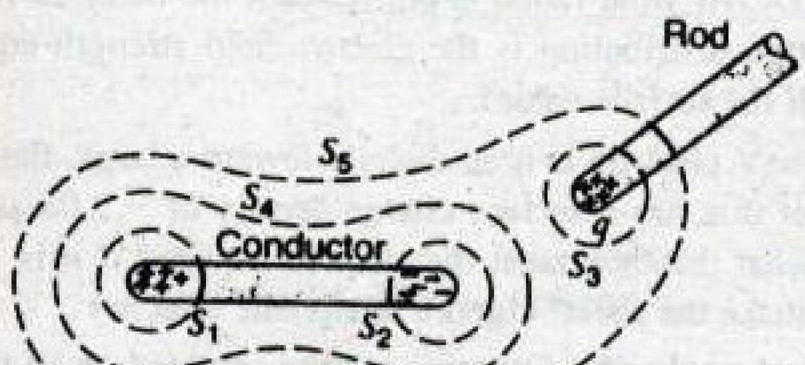


FIGURE 27-25. Exercise 4.

5. A point charge of 1.84 μC is at the center of a cubical Gaussian surface 55 cm on edge. Find Φ_E through the surface.

6. The net electric flux through each face of a die (one member of a pair of dice) has magnitude in units of 10^3 N·m^2/C equal to the number *N* of spots on the face (1 through 6). The flux is inward for *N* odd and outward for *N* even. What is the net charge inside the die?

7. A point charge $+q$ is a distance $d/2$ from a square surface of side *d* and is directly above the center of the square as shown in Fig. 27-26. Find the electric flux through the square. (Hint: Think of the square as one face of a cube with edge *d*.)

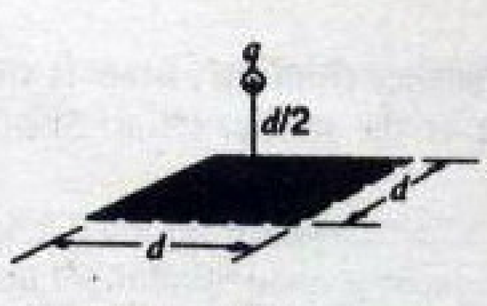


FIGURE 27-26. Exercise 7.

8. A butterfly net is in a uniform electric field *E* as shown in Fi ure 27-27. The rim, a circle of radius *a*, is aligned perpendic lar to the field. Find the electric flux through the netting, rel tive to the outward normal.

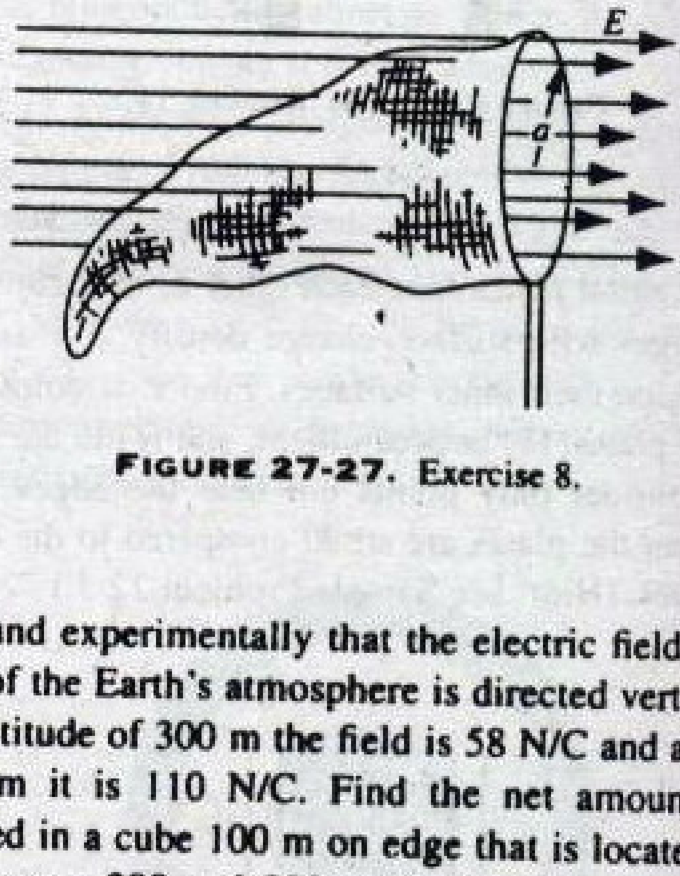


FIGURE 27-27. Exercise 8.

9. It is found experimentally that the electric field in a certai region of the Earth's atmosphere is directed vertically down At an altitude of 300 m the field is 58 N/C and at an altitud of 200 m it is 110 N/C. Find the net amount of charg contained in a cube 100 m on edge that is located at an alti tude between 200 and 300 m. Neglect the curvature of th Earth.

10. Find the net flux through the cube of Exercise 2 and Figur 27-14 if the electric field is given by (*a*) $\mathbf{E}$ = (3 N/C·m)*y*$\hat{\mathbf{j}}$ and (*b*) (− 4 N/C)$\hat{\mathbf{i}}$ + [6 N/C + (3 N/C·m)*y*]$\hat{\mathbf{j}}$. (*c*) In each case, how much charge is inside the cube?

11. A point charge *q* is placed at one corner of a cube of edge *a*. What is the flux through each of the cube faces? (Hint: Use Gauss' law and symmetry arguments.)

27-5 Applications of Gauss' Law

12. An infinite line of charge produces a field of 4.52 × 10^4 N/C at a distance of 1.96 m. Calculate the linear charge density.

13. (*a*) The drum of the photocopying machine in Sample Problem 27-4 has a length of 42 cm and a diameter of 12 cm. What is the total charge on the drum? (*b*) The manufacturer wishes to produce a desktop version of the machine. This requires reducing the size of the drum to a length of 28 cm and a diameter of 8.0 cm. The electric field at the drum surface must remain unchanged. What must be the charge on this new drum?

14. Two thin, large, nonconducting sheets of positive charge face each other as in Figure 27-28. What is $\mathbf{E}$ at points (*a*) to the left of the sheets, (*b*) between them, and (*c*) to the right of the sheets? Assume the same surface charge density σ for each sheet. Consider only points not near the

edges whose distance from the sheets is small compared to the dimensions of the sheets. (Hint: See Sample Problem 27-3.)

FIGURE 27-28. Exercise 14.

15. Two large metal plates face each other as in Figure 27-29 and carry charges with surface charge density $+\sigma$ and $-\sigma$, respectively, on their inner surfaces. Find E at points (*a*) to the left of the plates, (*b*) between them, and (*c*) to the right of the plates. Consider only points not near the edges whose distances from the plates are small compared to the dimensions of the plates. (Hint: See Sample Problem 27-3.)

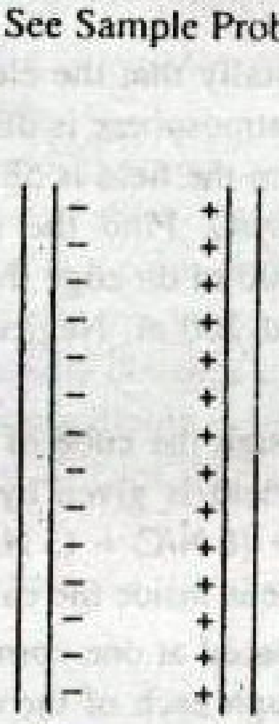


FIGURE 27-29. Exercise 15.

16. An electron remains stationary in an electric field directed downward in the Earth's gravitational field. If the electric field is due to charge on two large, parallel, conducting plates, oppositely charged and separated by 2.3 cm, what is the surface charge density, assumed to be uniform, on the plates?

17. A very long, straight, thin wire carries -3.60 nC/m of fixed negative charge. The wire is to be surrounded by a uniform cylinder of positive charge, radius 1.50 cm, coaxial with the wire. The volume charge density ρ of the cylinder is to be selected so that the net electric field outside the cylinder is zero. Calculate the required positive charge density ρ.

18. Figure 27-30 shows a point charge $q = 126$ nC at the center of a spherical cavity of radius 3.66 cm in a piece of metal. Use Gauss' law to find the electric field (*a*) at point P_1 halfway from the center to the surface, and (*b*) at point P_2.

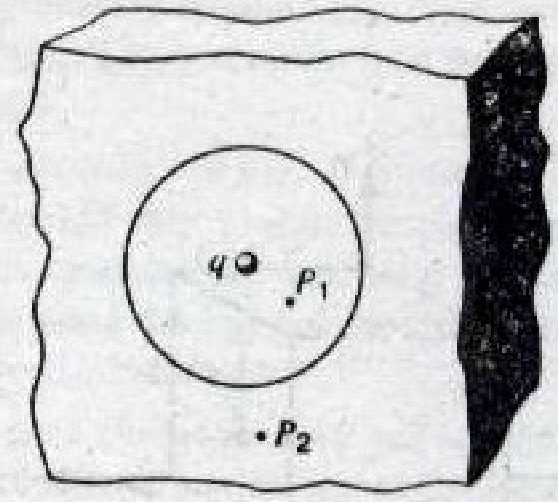


FIGURE 27-30. Exercise 18.

19. A proton orbits with a speed $v = 294$ km/s just outside a charged sphere of radius $r = 1.13$ cm. Find the charge on the sphere.

20. Two charged, concentric, thin, spherical shells have radii of 10.0 cm and 15.0 cm. The charge on the inner shell is 40.6 nC and that on the outer shell is 19.3 nC. Find the electric field (*a*) at $r = 12.0$ cm, (*b*) at $r = 22.0$ cm, and (*c*) at $r = 8.18$ cm from the center of the shells.

21. Two long, charged, concentric cylinders have radii of 3.22 and 6.18 cm. The surface charge density on the inner cylinder is 24.1 $\mu C/m^2$ and that on the outer cylinder is $-18.0\ \mu C/m^2$. Find the electric field at (*a*) $r = 4.10$ cm and (*b*) $r = 8.20$ cm.

22. Positive charge is distributed uniformly throughout a long, nonconducting, cylindrical shell of inner radius R and outer radius $2R$. At what radial depth beneath the outer surface of the charge distribution is the electric field strength equal to one-half the surface value?

23. A 115-keV electron is fired directly toward a large, flat, plastic sheet that has a surface charge density of $-2.08\ \mu C/m^2$. From what distance must the electron be fired if it is just to fail to strike the sheet? (Ignore relativistic effects.)

24. Construct a spherical Gaussian surface centered on an infinite line of charge, calculate the flux through the sphere, and thereby show that Gauss' law is satisfied.

25. Charge is distributed uniformly throughout an infinitely long cylinder of radius R. (*a*) Show that E at a distance r from the cylinder axis ($r < R$) is given by

$$E = \frac{\rho r}{2\epsilon_0},$$

where ρ is the volume charge density. (*b*) What result do you obtain for $r > R$?

27-6 Gauss' Law and Conductors

26. A uniformly charged conducting sphere of 1.22 m radius has a surface charge density of 8.13 $\mu C/m^2$. (*a*) Find the charge on the sphere. (*b*) What is the total electric flux leaving the surface of the sphere? (*c*) Calculate the electric field at the surface of the sphere.

27. Space vehicles traveling through the Earth's radiation belts collide with trapped electrons. Since in space there is no ground,

the resulting charge buildup can become significant and can damage electronic components, leading to control-circuit upsets and operational anomalies. A spherical metallic satellite 1.3 m in diameter accumulates 2.4 μC of charge in one orbital revolution. (*a*) Find the surface charge density. (*b*) Calculate the resulting electric field just outside the surface of the satellite.

28. Equation 27-18 ($E = \sigma/\epsilon_0$) gives the electric field at points near a charged conducting surface. Apply this equation to a conducting sphere of radius r, carrying a charge q on its surface, and show that the electric field outside the sphere is the same as the field of a point charge at the position of the center of the sphere.

29. A metal plate 8.0 cm on a side carries a total charge of 6.0 μC. (*a*) Using the infinite plate approximation, calculate the electric field 0.50 mm above the surface of the plate near the plate's center. (*b*) Estimate the field at a distance of 30 m.

27-7 Experimental Tests of Gauss' Law and Coulomb's Law

PROBLEMS

1. Gauss' law for gravitation is

$$\frac{1}{4\pi G}\Phi_g = \frac{1}{4\pi G}\oint \vec{g}\cdot d\vec{A} = -m,$$

where m is the enclosed mass and G is the universal gravitation constant. Derive Newton's law of gravitation from this. What is the significance of the negative sign?

2. The electric field components in Fig. 27-31 are $E_y = by^{1/2}$, $E_x = E_z = 0$, in which $b = 8830$ N/C $\cdot$ m$^{1/2}$. Calculate (*a*) the flux Φ_E through the cube and (*b*) the charge within the cube. Assume that $a = 13.0$ cm.

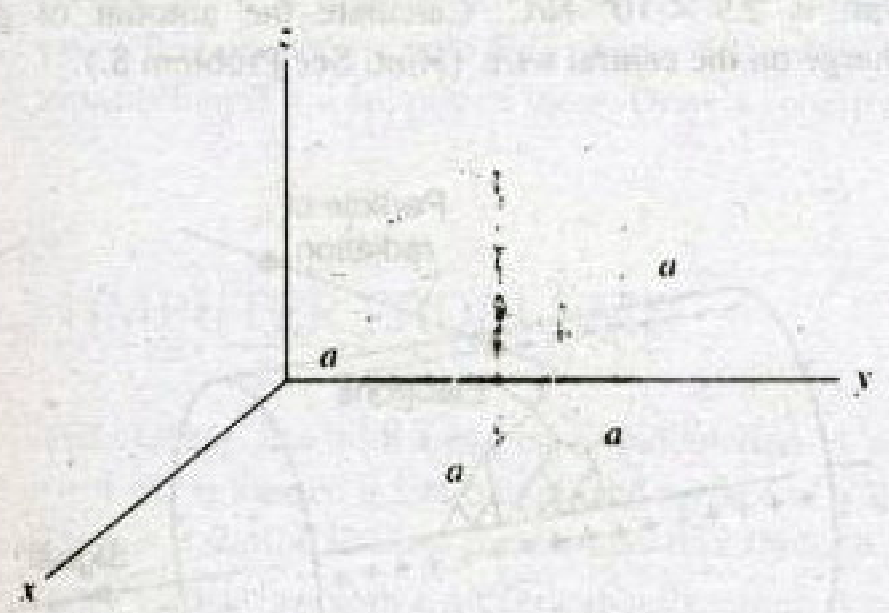


FIGURE 27-31. Problem 2.

3. A small sphere whose mass m is 1.12 mg carries a charge q = 19.7 nC. It hangs in the Earth's gravitational field from a silk thread that makes an angle $\theta = 27.4°$ with a large, uniformly charged, nonconducting sheet as in Fig. 27-32. Calculate the uniform charge density σ for the sheet.

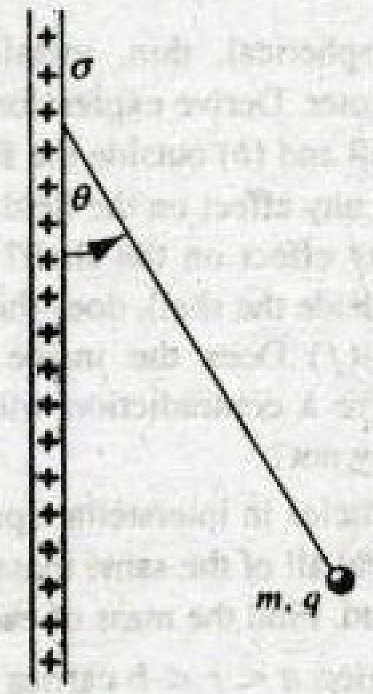


FIGURE 27-32. Problem 3.

4. Figure 27-33 shows a charge $+q$ arranged as a uniform conducting sphere of radius a and placed at the center of a spherical conducting shell of inner radius b and outer radius c. The outer shell carries a charge of $-q$. Find $E(r)$ at locations (*a*) within the sphere ($r < a$), (*b*) between the sphere and the shell ($a < r < b$), (*c*) inside the shell ($b < r < c$), and (*d*) outside the shell ($r > c$). (*e*) What charges appear on the inner and outer surfaces of the shell?

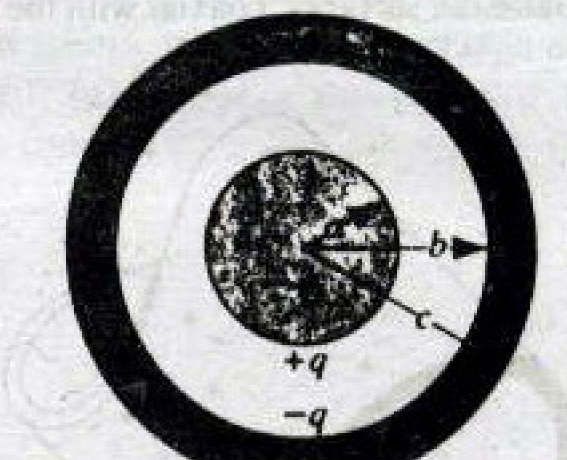


FIGURE 27-33. Problem 4.

5. A very long conducting cylinder (length L) carrying a total charge $+q$ is surrounded by a conducting cylindrical shell (also of length L) with total charge $-2q$, as shown in cross section in Fig. 27-34. Use Gauss' law to find (*a*) the electric field at points outside the conducting shell, (*b*) the distribution of the charge on the conducting shell, and (*c*) the electric field in the region between the cylinders.

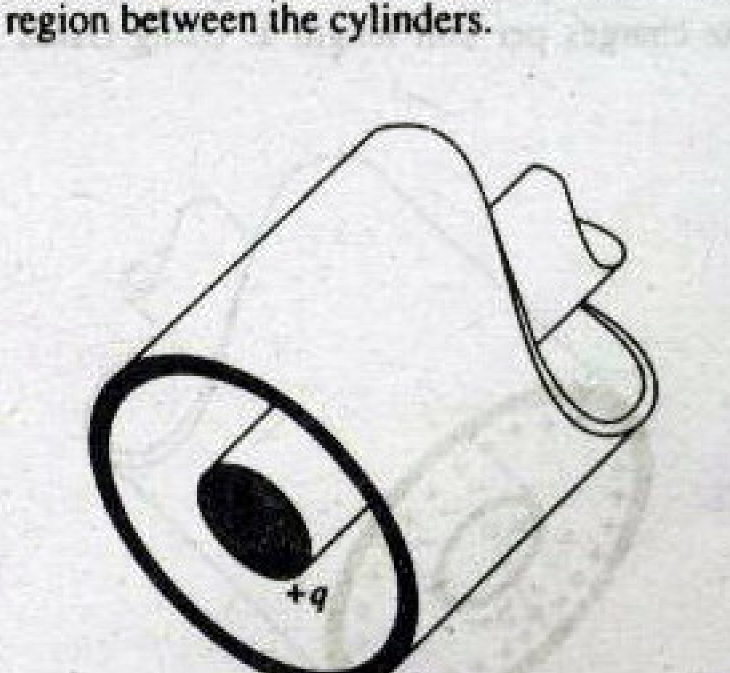


FIGURE 27-34. Problem 5.

6. A large, flat, nonconducting surface carries a uniform charge density σ. A small circular hole of radius R has been cut in the middle of the sheet, as shown in Fig. 27-35. Ignore fringing of the field lines around all edges and calculate the electric field at point P, a distance z from the center of the hole along its axis. (Hint: See Eq. 26-19 and use the principle of superposition.)

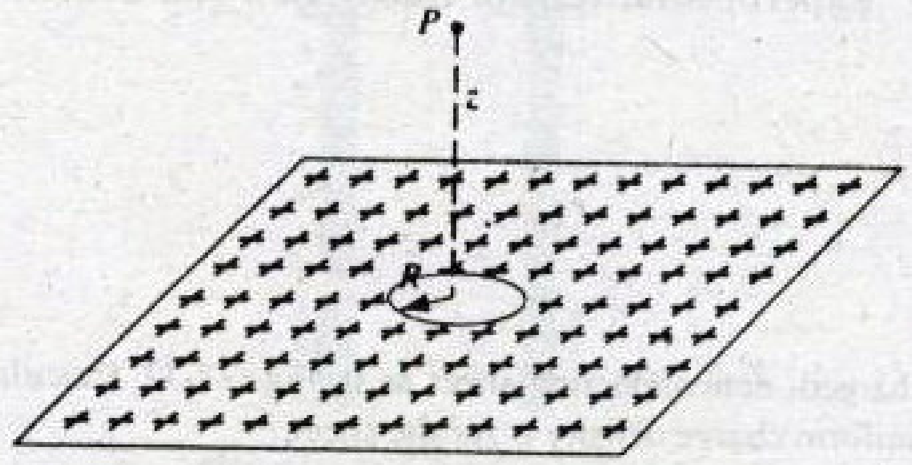


FIGURE 27-35. Problem 6.

7. Figure 27-36 shows a section through a long, thin-walled metal tube of radius R, carrying a charge per unit length λ on its surface. Derive expressions for E for various distances r from the tube axis, considering both (*a*) $r > R$ and (*b*) $r < R$. (*c*) Plot your results for the range $r = 0$ to $r = 5.0$ cm, assuming that $\lambda = 2.0 \times 10^{-8}$ C/m and $R = 3.0$ cm. (Hint: Use cylindrical Gaussian surfaces, coaxial with the metal tube.)

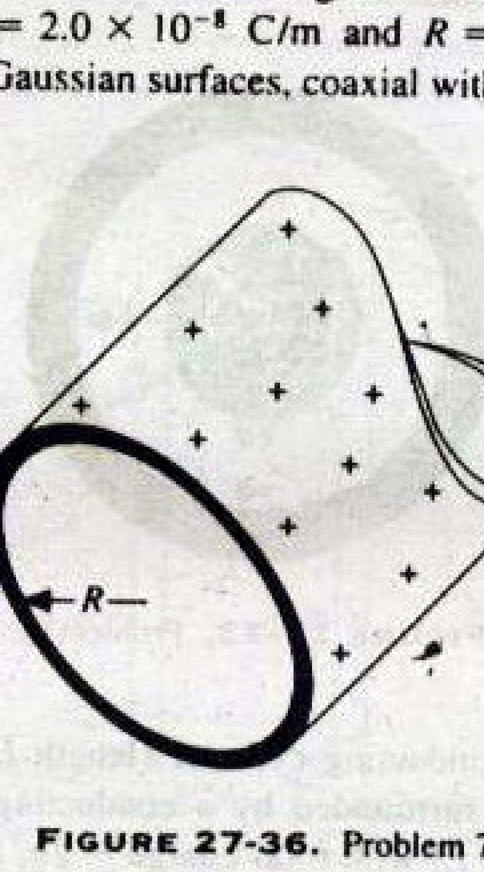


FIGURE 27-36. Problem 7.

8. Figure 27-37 shows a section through two long, thin, concentric cylinders of radii a and b. The cylinders carry equal and opposite charges per unit length λ. Using Gauss' law, prove

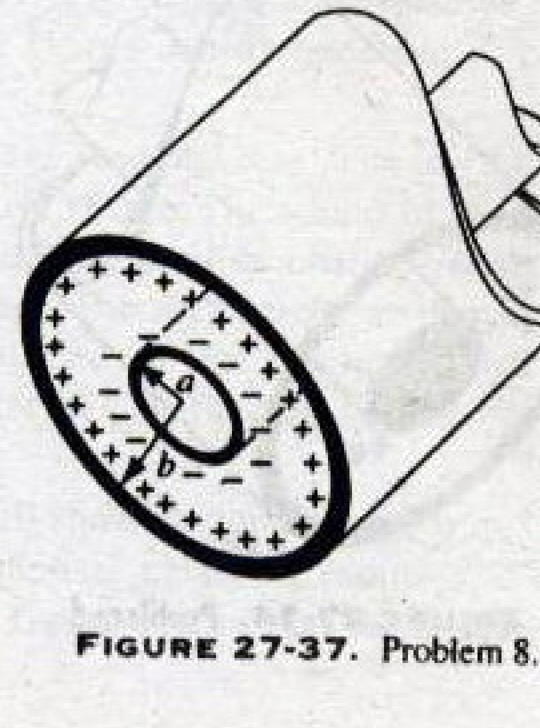


FIGURE 27-37. Problem 8.

(*a*) that $E = 0$ for $r < a$ and (*b*) that between the cylinders E is given by

$$E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}.$$

9. In the geometry of Problem 8 a positron revolves in a circular path between and concentric with the cylinders. Find its kinetic energy, in electron-volts. Assume that $\lambda = 30$ nC/m. (Why do you not need to know the radii of the cylinders?)

10. Figure 27-38 shows a Geiger counter, used to detect ionizing radiation. The counter consists of a thin central wire, carrying positive charge, surrounded by a concentric circular conducting cylinder, carrying an equal negative charge. Thus a strong radial electric field is set up inside the cylinder. The cylinder contains a low-pressure inert gas. When a particle of radiation enters the tube through the cylinder walls, it ionizes a few of the gas atoms. The resulting free electrons are drawn to the positive wire. However, the electric field is so intense that, between collisions with the gas atoms, they have gained energy sufficient to ionize these atoms also. More free electrons are thereby created, and the process is repeated until the electrons reach the wire. The "avalanche" of electrons is collected by the wire, generating a signal recording the passage of the incident particle of radiation. Suppose that the radius of the central wire is 25 μm, the radius of the cylinder 1.4 cm, and the length of the tube 16 cm. The electric field at the cylinder wall is 2.9×10^4 N/C. Calculate the amount of positive charge on the central wire. (Hint: See Problem 8.)

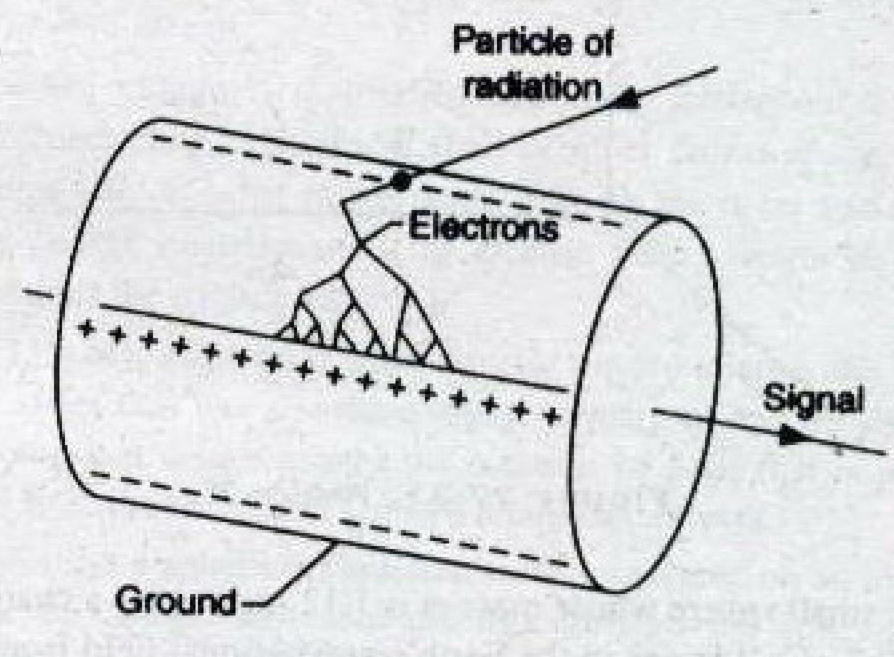


FIGURE 27-38. Problem 10.

11. An uncharged, spherical, thin, metallic shell has a point charge q at its center. Derive expressions for the electric field (*a*) inside the shell and (*b*) outside the shell, using Gauss' law. (*c*) Has the shell any effect on the field due to q? (*d*) Has the presence of q any effect on the shell? (*e*) If a second point charge is held outside the shell, does this outside charge experience a force? (*f*) Does the inside charge experience a force? (*g*) Is there a contradiction with Newton's third law here? Why or why not?

12. Charged dust particles in interstellar space, each carrying one excess electron and all of the same mass, form a stable, spherical, uniform cloud. Find the mass of each particle.

13. The spherical region $a < r < b$ carries a charge per unit volume of $\rho = A/r$, where A is a constant. At the center ($r = 0$) of the enclosed cavity is a point charge q. What should be the value of A so that the electric field in the region $a < r < b$ has constant magnitude?

14. A spherical region carries a uniform charge per unit volume ρ. Let $\vec{r}$ be the vector from the center of the sphere to a general point P within the sphere. (*a*) Show that the electric field at P is given by $\vec{E} = \rho\vec{r}/3\epsilon_0$. (*b*) A spherical cavity is created in the sphere, as shown in Fig. 27-39. Using superposition concepts, show that the electric field at all points within the cavity is $\vec{E} = \rho\vec{a}/3\epsilon_0$ (uniform field), where $\vec{a}$ is the vector connecting the center of the sphere with the center of the cavity. Note that both these results are independent of the radii of the sphere and the cavity.

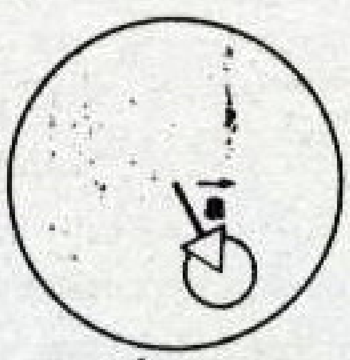


FIGURE 27-39. Problem 14.

15. Show that stable equilibrium under the action of electrostatic forces alone is impossible. (Hint: Assume that at a certain point P in an electric field $\vec{E}$, a charge $+q$ would be in stable equilibrium if it were placed there. Draw a spherical Gaussian surface about P, imagine how $\vec{E}$ must point on this surface, and apply Gauss' law to show that the assumption leads to a contradiction.) This result is known as Earnshaw's theorem.

16. A plane slab of thickness d has a uniform volume charge density ρ. Find the magnitude of the electric field at all points in space both (*a*) inside and (*b*) outside the slab, in terms of x, the distance measured from the median plane of the slab.

17. A solid nonconducting sphere of radius R carries a nonuniform charge distribution, with charge density $\rho = \rho_S r/R$, where ρ_S is a constant and r is the distance from the center of the sphere. Show that (*a*) the total charge on the sphere is $Q = \pi\rho_S R^3$ and (*b*) the electric field inside the sphere is given by

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^4}r^2.$$

18. An insulated conductor of arbitrary shape carries a net charge of $+10\ \mu C$. Inside the conductor is a hollow cavity within which is a point charge $q = +3.0\ \mu C$. What is the charge (*a*) on the cavity wall and (*b*) on the outer surface of the conductor?

19. A conducting sphere carrying charge Q is surrounded by a spherical conducting shell. (*a*) What is the net charge on the inner surface of the shell? (*b*) Another charge q is placed outside the shell. Now what is the net charge on the inner surface of the shell? (*c*) If q is moved to a position between the shell and the sphere, what is the net charge on the inner surface of the shell? (*d*) Are your answers valid if the sphere and shell are not concentric?

COMPUTER PROBLEMS

1. Verify Gauss' law with a numerical calculation. A point charge $q = 1$ nC is located 0.5 m outside the surface of a sphere of radius $r = 1.0$ m. Calculate the electric flux through the sphere.

2. Verify Gauss' law with a numerical calculation. A point charge $q = 1$ nC is located halfway between the center and the surface of a sphere of radius $r = 1.0$ m. Calculate the electric flux through the sphere.

3. A point charge of $q = 1.0\ \mu C$ is located on the axis of a cylindrical surface of radius $r = 0.5$ m and length $L = 3.0$ m. The point charge is 1.0 m from one end, and 2.0 m from the other. (*a*) Numerically calculate the electric flux through the curved portion of the cylinder. (*b*) Analytically verify your answer. (Note: No integration is required here!)

CHAPTER 28

ELECTRIC POTENTIAL ENERGY AND POTENTIAL

In Chapters 11 through 13 we learned that methods based on energy concepts offered new insights in understanding mechanics and often provided simplifications in solving mechanics problems. In Chapter 14 we used methods based on potential energy in situations involving the gravitational force to determine such properties as the motions of satellites and planets.

In this chapter we introduce the energy method to the study of electrostatics. We begin with electric potential energy, which we shall find can characterize an electrostatic force just as gravitational potential energy can characterize a gravitational force. We then generalize to the concept of electric potential and show how to find the electric potential for various discrete and continuous charge distributions.

28-1 POTENTIAL ENERGY

Many electrical phenomena are associated with the transfer of large quantities of energy. For example, when a lightning flash strikes the Earth from a cloud, an energy of typically 10^8 J is released in the form of light, sound, heat, and shock wave. Where does this energy come from, and how is it stored in clouds? To understand this question, we must consider the energy associated with electrical forces.

The electrostatic force law is very similar to the gravitational force law:

$$F = \frac{1}{4\pi\epsilon_0}\frac{|q_1||q_2|}{r^2} \qquad \text{electrostatic,} \qquad (28\text{-}1a)$$

$$F = G\frac{m_1 m_2}{r^2} \qquad \text{gravitational.} \qquad (28\text{-}1b)$$

Both forces depend on the inverse square of the separation distance between the two objects. When an object moves from place to place under the gravitational force of another object (which we assume to remain at rest), the work done by the gravitational force on the first object depends only on the starting and finishing points and does not depend on the path taken between the points. In Section 12-1 we described a force that has this special property as a *conservative force*, and we concluded in Section 12-2 that for a conservative force we could define a *potential energy*. The difference in potential energy ΔU of the system as the object moves from its initial to its final position is equal to the negative of the work done by the force:

$$\Delta U = U_f - U_i = -W_{if} = -\int_i^f \vec{F}\cdot d\vec{s}, \qquad (28\text{-}2)$$

where W_{if} is the work done by the force $\vec{F}$ when the object moves from i to f. In the case of the gravitational force, we showed in Section 14-6 that, when an object with mass m_2 moves from a distance r_i from mass m_1 to a distance r_f from m_1, the potential energy difference is

$$\Delta U = -Gm_1m_2\left(\frac{1}{r_f} - \frac{1}{r_i}\right). \qquad (28\text{-}3)$$

This potential energy difference is associated with the entire system consisting of m_1 and m_2, not with either object alone.

Because of the similarity of the electrostatic and gravitational force laws, we can make the same conclusion about the electrostatic force that we did about the gravitational force: *The electrostatic force is conservative, and therefore there is a potential energy associated with the configuration* *(the relative locations of the objects) of a system in which electrostatic forces act.*

Why is this approach useful for electrostatic forces? In mechanics, we learned that there are two ways to analyze problems. One approach is based on force (a vector) and allows us to determine the position and velocity of an object at every point of its motion. The other approach is based on energy (a scalar) and allows us to determine how a system changes in moving from a certain initial state to a certain final state. We will find in a similar way that both approaches are useful when we study interactions between charged objects.

There is one important property in which the electrostatic force differs from the gravitational force: the gravitational force is always attractive, whereas (depending on the relative signs of the charges) electrostatic forces can be either attractive or repulsive. This difference can affect the sign of the potential energy, but it in no way changes our argument based on the similarity of the two forces.

28-2 ELECTRIC POTENTIAL ENERGY

In this section we use the electrostatic force discussed in Chapter 25 to obtain the electric potential energy due to the interaction between two electric charges, and we extend the calculation to include the case of a collection of more than two charges.

Accepting the conclusion of the previous section that the electrostatic force is conservative, we can calculate the change in potential energy when a charge q_2 moves from point a to point b subject to the force due to another charge q_1 at rest. Let us assume for the present that both charges are positive. Figure 28-1 shows the geometry of the process. We have simplified the problem slightly by assuming that the motion from a to b is along the line that connects q_1 and q_2. (Later we shall generalize to other kinds of displacements.) We take the origin to be at the location of q_1, and we let r represent the location of q_2 relative to this origin. In Eq. 28-2, the vector $d\vec{s}$ represents an infinitesimal displacement along the direction of motion from a to b. The force $\vec{F}$ and the displacement $d\vec{s}$ are always parallel for this motion, and so $\vec{F}\cdot d\vec{s} = F\,ds$. For the motion shown in Fig. 28-1, $ds = dr$ because the displacement is always in the direction of r. With these substitutions, Eq. 28-2 becomes

$$\Delta U = -\int_a^b \vec{F}\cdot d\vec{s} = -\int_a^b F\,dr = -\int_{r_a}^{r_b} \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\,dr. \quad (28\text{-}4)$$

Carrying out the integral, we obtain

$$\Delta U = U_b - U_a = \frac{1}{4\pi\epsilon_0} q_1 q_2 \left(\frac{1}{r_b} - \frac{1}{r_a}\right). \quad (28\text{-}5)$$

Equation 28-5 is valid whether the motion of q_2 is toward or away from q_1. If q_2 moves toward q_1, then $r_b < r_a$ and $\Delta U > 0$; that is, the potential energy increases if the charges move closer together. If q_2 moves away from q_1, then $r_b > r_a$ and $\Delta U < 0$; that is, the potential energy decreases if the charges move further apart.

Equation 28-5 also remains valid whether the signs of the charges are positive or negative. If both charges are negative, clearly we obtain the same result. If the charges have opposite signs (one positive and the other negative), then the force between them is attractive. With the force vector in Fig. 28-1 in the opposite direction, we have

$$\vec{F}\cdot d\vec{s} = -F\,ds = -F\,dr = -\frac{1}{4\pi\epsilon_0}\frac{|q_1||q_2|}{r^2}\,dr$$
$$= \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\,dr, \quad (28\text{-}6)$$

where the last step can be made because $q_1 q_2 = -|q_1||q_2|$ when one of the charges is negative and one is positive. This gives exactly the same integrand as Eq. 28-4, and so the integral gives the same result.

When the charges have opposite signs, so that $q_1 q_2$ is negative in Eq. 28-5, then $\Delta U < 0$ when the charges move closer together and $\Delta U > 0$ when the charges move further apart.

Suppose we move q_2 in a direction that is *not* along the line connecting q_1 and q_2. Figure 28-2 shows q_2 moving from a to b along an arc of a circle of radius r centered at q_1. Along this path, $\vec{F}$ is always perpendicular to $d\vec{s}$, and so $\vec{F}\cdot d\vec{s} = 0$ throughout the path. The electrostatic force does no work along this path, and so $\Delta U = 0$.

To move q_2 between arbitrary points a and b, as in Fig. 28-3, we can choose a variety of possible paths. Along

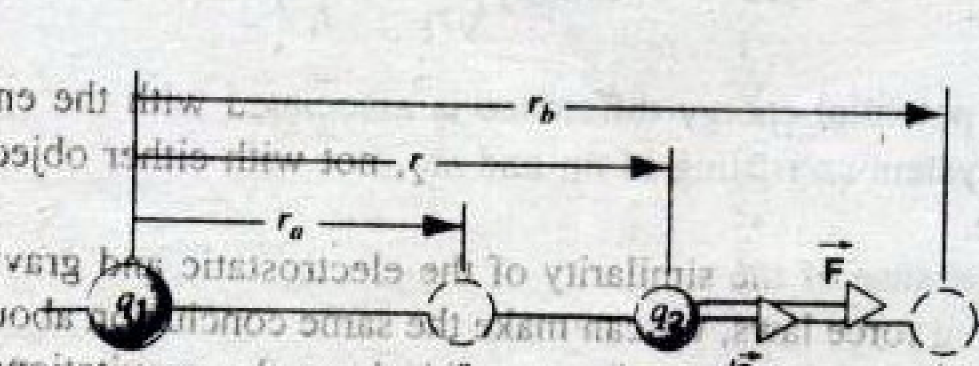


FIGURE 28-1. A charged particle q_2 moves from a to b under the influence of the electrostatic force $\vec{F}$ exerted by q_1. The points a and b lie along the line connecting q_1 and q_2.

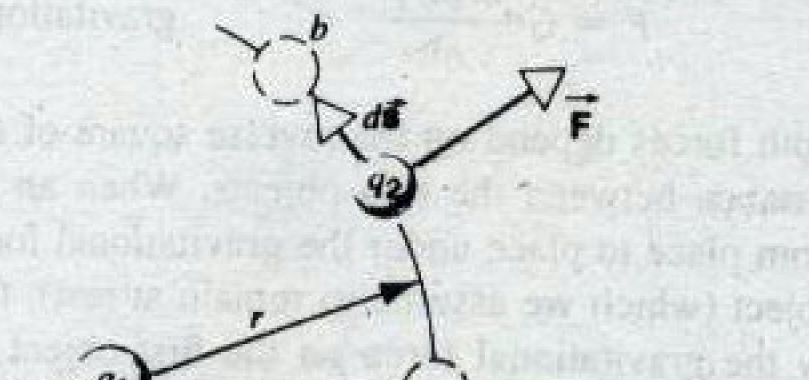


FIGURE 28-2. The motion of q_2 from a to b is now along a path of constant radius r.

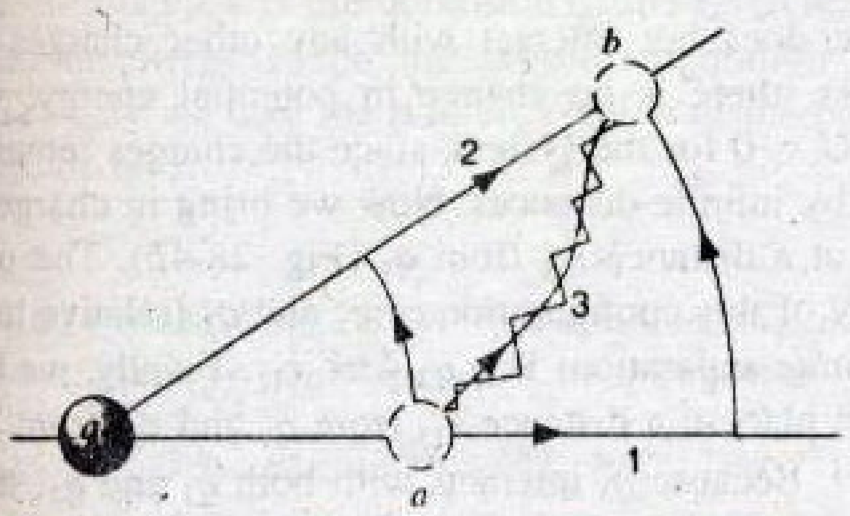


FIGURE 28-3. q_2 moves between arbitrary points a and b along several possible paths.

paths 1 and 2, ΔU is given by Eq. 28-5 for the radial (straight) parts of the paths and $\Delta U = 0$ for the tangential (curved) parts of the paths. The arbitrary path 3 can be broken into a series of short radial and tangential steps. Along each tangential step $\Delta U = 0$, while the total ΔU along all the radial steps is given by Eq. 28-5. Our conclusion is that Eq. 28-5 gives ΔU for any path between point a, which is a distance r_a from q_1, and point b, which is a distance r_b from q_1, no matter where the points may be located. This is consistent with our assertion that the electrostatic force is conservative, which means that the work and therefore the change in potential energy in moving from a to b does not depend on the path.

So far we have been discussing the *difference* in potential energy between two points: $\Delta U = U_b - U_a$. We can extend the discussion to define the potential energy at a single point b by choosing a reference point a of potential energy and assigning a reference value to the potential energy U_a at that point. Often it is convenient to choose the reference point to correspond to an infinite separation between the charges, and we generally choose the reference value $U_a = 0$. Then, letting point b represent any point where the separation is r, Eq. 28-5 becomes

$$U(r) = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r}. \qquad (28\text{-}7)$$

In this expression, U is positive whenever q_1 and q_2 have the same sign, corresponding to a repulsive force, and U is negative whenever q_1 and q_2 have opposite signs, corresponding to an attractive force. If we compare Eq. 28-7 with the corresponding expression given in Eq. 14-17 for the gravitational potential energy, $U(r) = -Gm_1m_2/r$, we note that the gravitational potential energy is always negative, because the gravitational force is always attractive. This agrees with the negative value of the electrostatic potential energy when the charges have opposite signs and the force is attractive.

Conservation of Energy in Electrostatics

In an isolated system of two charges, the total mechanical energy $E = K + U$ is conserved. Let us assume that q_1 is held in a fixed position and q_2 is released from rest at a certain distance from q_1. If the two charges have the same sign, then $\Delta U < 0$ as q_2 is pushed away from q_1 by the repulsive force. Conservation of total mechanical energy then requires that $\Delta K > 0$, so the speed of q_2 must increase. If instead we propel q_2 toward q_1 with a certain initial kinetic energy, then $\Delta U > 0$ as the separation decreases; conservation of energy then requires that $\Delta K < 0$, and so the speed of q_2 decreases. These conclusions are reversed if the charges have opposite signs so that the force is attractive.

Here is another way to view energy conservation in a system of two charges. Let us assume that the two charges have the same sign. We begin with the charges at rest separated by a very large distance, and we move q_2 and place it at rest a certain distance from q_1. To accomplish this task, the external agent that moves q_2 must exert a force to oppose the electrostatic repulsion between q_1 and q_2. In doing so the agent does positive external work on the system, and so the energy of the system increases. The energy of the system has increased by an amount ΔU as a result of the work done by the external agent. Put another way, the external agent has stored energy in the system, in exact analogy to the storage of energy when an external agent compresses a spring. By releasing the charges, we could recover the stored energy as kinetic energy of the moving charges.

If instead the charges have opposite signs so that the electrostatic force is attractive, then the external agent does negative external work on the system to move q_2 from a large separation to place it at rest in a location with a smaller separation from q_1. This work decreases the energy stored in the system and so is not recoverable. (Without the external agent, q_2 would on its own accelerate toward q_1; the agent must expend energy in restraining q_2 in order to place it at rest at the specified location.)

If q_1 and q_2 have opposite signs and begin with a small separation, then the external agent must do positive work equal to ΔU in order to separate the charges to a large distance. When this concept is applied to atoms or molecules, this energy may be called the *binding energy*, the *ionization energy*, or the *dissociation energy*. This quantity represents the external energy we must supply, for example, to remove an electron from an atom or to divide a molecule such as KCl into K^+ and Cl^- ions.

SAMPLE PROBLEM 28-1. Two protons in the nucleus of a ^{238}U atom are 6.0 fm apart. What is the potential energy associated with the electric force that acts between these two particles?

Solution From Eq. 28-5, with $q_1 = q_2 = +1.60 \times 10^{-19}$ C, we obtain

$$U = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r} = \frac{(8.99 \times 10^9\ \mathrm{N\cdot m^2/C^2})(1.60 \times 10^{-19}\ \mathrm{C})^2}{6.0 \times 10^{-15}\ \mathrm{m}}$$

$$= 3.8 \times 10^{-14}\ \mathrm{J} = 2.4 \times 10^5\ \mathrm{eV} = 240\ \mathrm{keV},$$

where we take $U = 0$ for the configuration in which the protons are far apart. The two protons do not fly apart because they are held together by the attractive *strong force* that binds the nucleus together. Unlike the electric force, there is no simple potential energy function that represents the strong force.

SAMPLE PROBLEM 28-2. Two objects, one with mass $m_1 = 0.0022$ kg and charge $q_1 = +32\ \mu C$ and the other with mass $m_2 = 0.0039$ kg and charge $q_2 = -18\ \mu C$, are initially a distance 4.6 cm apart. With object 1 held in a fixed position, object 2 is released from rest. What is the speed of object 2 when the separation between the objects is 2.3 cm? Assume that the objects behave like point charges.

Solution As the charges move closer together, with only the electrostatic force acting, the reduction in potential energy must be balanced by a corresponding increase in the kinetic energy. Let the initial condition be the instant that object 2 is released (with $K_i = 0$) and the final condition be the instant that their separation is 2.3 cm. Then conservation of energy gives $U_i + K_i = U_f + K_f$, or (with $K_i = 0$)

$$K_f = U_i - U_f = -\Delta U = -\frac{q_1 q_2}{4\pi\epsilon_0}\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$$

$$= -(8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2})(32 \times 10^{-6}\ \mu C)$$

$$\times (-18 \times 10^{-6}\ \mu C)\left(\frac{1}{0.023\ \mathrm{m}} - \frac{1}{0.046\ \mathrm{m}}\right)$$

$$= 113\ \mathrm{J}.$$

$$v_f = \sqrt{\frac{2K_f}{m_2}} = \sqrt{\frac{2(113\ \mathrm{J})}{0.0039\ \mathrm{kg}}} = 240\ \mathrm{m/s}.$$

If we instead hold object 2 fixed and release object 1, when the separation reaches 2.3 cm the kinetic energy will have the same value of 113 J, because the energy is a property of the entire system. If we released both particles from rest and allowed them to fall together, at a separation of 2.3 cm the total kinetic energy of the two particles would be 113 J. We could find the velocity of each particle using conservation of momentum.

Potential Energy of a System of Charges

Suppose we have three charges (q_1, q_2, q_3) separated by infinite distances from one another. In this configuration, $U = 0$. We wish to find the potential energy of the configuration that results after the three charges are brought closer to one another.

Let us bring the first charge q_1 in from infinity and place it at rest at the location shown in Fig. 28-4a. Since this charge does not interact with any other charges in the process, there is no change in potential energy; we still have $U = 0$ for the system, since the charges remain separated by infinite distances. Now we bring in charge q_2 and fix it at a distance r_{12} from q_1 (Fig. 28-4b). The potential energy of this configuration of q_1 and q_2 (relative to $U = 0$ at infinite separation) is $q_1q_2/4\pi\epsilon_0 r_{12}$. Finally, we bring in q_3 and place it a distance r_{13} from q_1 and r_{23} from q_2 (Fig. 28-4c). Because q_3 interacts with both q_1 and q_2, there are two additional contributions to the potential energy of this final configuration: $q_1q_3/4\pi\epsilon_0 r_{13}$ (interaction of q_1 and q_3) and $q_2q_3/4\pi\epsilon_0 r_{23}$ (interaction of q_2 and q_3). The total electric potential energy of the entire system is

$$U = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r_{12}} + \frac{1}{4\pi\epsilon_0}\frac{q_1q_3}{r_{13}} + \frac{1}{4\pi\epsilon_0}\frac{q_2q_3}{r_{23}}. \quad (28\text{-}8)$$

As Eq. 28-8 makes clear, the potential energy is a property of the system, not of any individual charge.

We could continue the process to assemble any arbitrary distribution of charges. The resulting total potential energy of any such system is independent of the order in which we assemble the charges.

From this example you can see the advantage of using an energy method to analyze this system: the sum involved in Eq. 28-8 is an *algebraic* sum of scalars. If we tried to calculate the electric field associated with a collection of three charges, we would have a more complicated *vector* sum to evaluate.

Implicit in this process is the assumption that the principle of superposition is valid. Previously we have applied this principle, which states that the interaction of any two charges is independent of the presence of other charges, to analyze vector sums. Here we see that a similar result applies to the scalar terms; for example, the potential energy term that describes the interaction of q_1 and q_3 is independent of the presence of q_2.

As we discussed above, if the external agent does positive work in assembling the charges from infinite separation (opposing a repulsive force in the process), the total potential energy calculated using Eq. 28-8 is positive. The external agent has in effect stored energy in the system of charges. If the charges are released from their positions, they will tend to fly apart, and the potential energy will decrease as the kinetic energy increases. If the total potential energy is negative, the external agent has done negative work in assembling the system of charges. In this case, the external agent must supply additional energy in the form of work to disassemble the system of charges and move them to infinite separation.

This view of potential energy can be summarized as follows:

> *The electric potential energy of a system of fixed point charges at rest is equal to the work that must be done by an external agent to assemble the system, bringing each charge in from an infinite distance where it is also at rest.*

FIGURE 28-4. A system of three charges is assembled from initially infinite separations.

Implicit in this view is the definition of the reference point of potential energy to be the infinite separation of the charges, where we take the reference value of the potential energy to be zero.

SAMPLE PROBLEM 28-3. In the system shown in Fig. 28-4, assume that $r_{12} = r_{13} = r_{23} = d = 12$ cm, and that

$$q_1 = +q, \quad q_2 = -4q, \quad \text{and} \quad q_3 = +2q,$$

where $q = 150$ nC. What is the potential energy of the system? Assume that $U = 0$ when the charges are infinitely far apart.

Solution Using Eq. 28-8, we obtain

$$U = \frac{1}{4\pi\epsilon_0}\left(\frac{(+q)(-4q)}{d} + \frac{(+q)(+2q)}{d} + \frac{(-4q)(+2q)}{d}\right)$$

$$= -\frac{10q^2}{4\pi\epsilon_0 d}$$

$$= -\frac{(8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2})(10)(150 \times 10^{-9}\ \mathrm{C})^2}{0.12\ \mathrm{m}}$$

$$= -1.7 \times 10^{-2}\ \mathrm{J} = -17\ \mathrm{mJ}.$$

The negative potential energy in this case means that negative work would be done by an external agent to assemble this structure, starting with the three charges infinitely separated and at rest. Put another way, an external agent would have to do +17 mJ of work to dismantle the structure completely.

28-3 ELECTRIC POTENTIAL

Imagine a charge q fixed at the origin of a coordinate system. We take another charge q_0, which we call our "test charge," and we move it from r_a to r_b under the influence of the force due to q. The change in potential energy ΔU of this two-charge system is given by Eq. 28-5.

If we were to use a test charge twice as large, we would obtain twice the change in potential energy; a test charge three times as large would give three times the potential energy change.

The potential energy change is directly proportional to the size of the test charge. Put another way, the quantity $\Delta U/q_0$ is independent of the size of the test charge and is characteristic only of the central charge q. This quantity turns out to be extremely useful in analyzing a wide variety of electrostatic problems, even those that involve more complicated assemblies of charges. We define the *electric potential difference* ΔV to be the *electric potential energy difference per unit test charge:*

$$\Delta V = \frac{\Delta U}{q_0} \qquad (28\text{-}9)$$

or

$$V_b - V_a = \frac{U_b - U_a}{q_0}. \qquad (28\text{-}10)$$

Like the potential energy, the electric potential is a scalar. Usually we will refer to electric potential simply as "potential."

Using the relation between work and potential energy given in Eq. 28-2, we can write the definition of potential difference as

$$\Delta V = -\frac{W_{ab}}{q_0}, \qquad (28\text{-}11)$$

where W_{ab} is the work done by the electrostatic force exerted by q on q_0 when the test charge moves from a to b.

By defining a suitable choice of the reference point of potential energy (such as $U_a = 0$ for an infinite initial separation of the charges), we obtained in the previous section an expression (Eq. 28-7) for the potential energy of a particular configuration rather than the change in potential energy for a change in configuration. We can do the same for electric potential. Only differences in potential have physical significance, so we are free to choose the zero point and its reference value at our convenience. When the potential is taken to be zero at points that are infinitely far from q, the electric potential is

$$V = \frac{U}{q_0}. \qquad (28\text{-}12)$$

In a complicated arrangement of many charges, the potential V may be positive, negative, or zero. The potential at a point near an isolated positive charge is positive. If we were to move a positive test charge from infinity to that point, the charge would move from a location where $V = 0$ to a location where $V > 0$. Thus $\Delta V > 0$ and (according to Eq. 28-9) $\Delta U > 0$, indicating that the electric force on the test charge has done negative work. Similarly, the potential at a point near an isolated negative charge is negative; the electric force does positive work when we move a positive test charge from infinity to that point.

If the potential is zero at a point, no net work is done by the electric force as the test charge moves in from infinity to that point, although the test charge may pass through regions where it experiences attractive or repulsive electric forces. *A potential of zero at a point does not necessarily mean that the electric force is zero at that point.*

The SI unit of potential that follows from Eq. 28-9 is the joule per coulomb. This combination is given the name of *volt* (V):

$$1\ \text{volt} = 1\ \text{joule/coulomb}. \qquad (28\text{-}13)$$

The common name of "voltage" is often used for the potential at a point, and we often speak of "voltage difference" instead of potential difference. When you touch the two probes of a voltmeter to two points in an electric circuit, you are measuring the voltage difference or potential difference (in volts) between those points.

We have already discussed that the electric force is conservative, and so the potential energy difference when a test charge is moved between any two points depends only on the locations of the points and not on the path taken to move from one point to the other. Equation 28-9 therefore suggests that the potential difference is similarly path independent: the potential difference between any two points in an electric field is independent of the path through which the test ch rge moves in traveling from one point to the other.

Fo. any arbitrary potential difference ΔV, no matter what the arrangement of charges that produces it, we can write Eq. 28-9 as

$$\Delta U = q\,\Delta V. \tag{28-14}$$

This equation indicates that when *any* charge q moves between two points whose potential difference is ΔV, the system experiences a change in potential energy ΔU given by Eq. 28-14. The potential difference ΔV is set up by other charges that are fixed at rest, so that the motion of q does not change ΔV. In using Eq. 28-14, we see from Eq. 28-13 that if ΔV is expressed in volts and q is in coulombs, then ΔU comes out in joules.

From Eq. 28-14, you can see that the *electron-volt*, which we have introduced previously as a unit of energy, follows directly from the definition of potential or potential difference. If ΔV is expressed in volts and q in units of the elementary charge e, then ΔU is expressed in electron-volts (eV). For example, consider a system in which a carbon atom from which all six electrons have been removed ($q = +6e$) moves through a change in potential of $\Delta V = +20$ kV. The change in potential energy is

$$\Delta U = q\,\Delta V = (+6e)(+20\text{ kV}) = +120\text{ keV}.$$

Doing such calculations in units of eV is a great convenience when dealing with atoms or nuclei, in which the charge is easily expressed in terms of e.

Keep in mind that *potential differences* are of fundamental concern and that Eq. 28-12 depends on the arbitrary assignment of the value zero to the potential at the reference position (infinity); this reference potential could equally well have been chosen as any other value—say, -100 V. Similarly, any other agreed-upon point could be chosen as a reference position. In many problems the Earth is taken as a reference of potential and is assigned the value zero. The location of the reference point and the value of the potential there are chosen for convenience; other choices would change the potential everywhere by the same amount but would not change the potential difference between any two points.

SAMPLE PROBLEM 28-4. An alpha particle ($q = +2e$) in a nuclear accelerator moves from one terminal at a potential of $V_a = +6.5 \times 10^6$ V to another at a potential of $V_b = 0$. (*a*) What is the corresponding change in the potential energy of the system? (*b*) Assuming that the terminals and their charges do not move and that no external forces act on the system, what is the change in kinetic energy of the particle?

Solution (*a*) From Eq. 28-14, we have

$$\begin{aligned}\Delta U = U_b - U_a &= q(V_b - V_a)\\ &= (+2)(1.6 \times 10^{-19}\text{ C})(0 - 6.5 \times 10^6\text{ V})\\ &= -2.1 \times 10^{-12}\text{ J}.\end{aligned}$$

(*b*) If no external force acts on the system, then its mechanical energy $E = U + K$ must remain constant. That is, $\Delta E = \Delta U + \Delta K = 0$, and so

$$\Delta K = -\Delta U = +2.1 \times 10^{-12}\text{ J}.$$

The alpha particle gains a kinetic energy of 2.1×10^{-12} J, in the same way that a particle falling in the Earth's gravitational field gains kinetic energy.

To see the simplifications that result, try working this problem again with the energies expressed in units of eV.

28-4 CALCULATING THE POTENTIAL FROM THE FIELD

So far we have characterized electric charges and their interactions using four different properties: electric force, electric field, electric potential energy, and electric potential. Table 28-1 shows these four properties. Two of them are vectors (force and field), and two are scalars (potential energy and potential). Two of them characterize the interactions of two particles with one another (force and potential energy), and two of them represent the effect at a point in space due to a single charge or collection of charges (field and potential). The double arrows in the table show that the quantities in adjacent boxes of the table can be calculated from one another; for example, $\vec{\mathbf{E}}$ from $\vec{\mathbf{F}}$ (Eq. 26-3), U from $\vec{\mathbf{F}}$ (Eq. 2°-4), and V from U (Eq. 28-12). Now we examine the fourth connection—namely, that between V and $\vec{\mathbf{E}}$.

The connection between V and $\vec{\mathbf{E}}$ follows directly from the definition of potential in Eq. 28-11: $\Delta V = -W_{ab}/q_0$. Suppose we move a test charge q_0 from a to b in an electric field $\vec{\mathbf{E}}$. Calculating the work done by the electric force $\vec{\mathbf{F}} = q_0\vec{\mathbf{E}}$, we obtain

$$\Delta V = \frac{-W_{ab}}{q_0} = \frac{-\int_a^b \vec{\mathbf{F}}\cdot d\vec{\mathbf{s}}}{q_0} = \frac{-\int_a^b q_0\vec{\mathbf{E}}\cdot d\vec{\mathbf{s}}}{q_0}$$

or

$$\Delta V = V_b - V_a = -\int_a^b \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}}. \tag{28-15}$$

If the electric field is along the direction of $d\vec{\mathbf{s}}$, then the integral in Eq. 28-15 will be positive, and the potential difference will be negative; that is, $V_b < V_a$. The electric field

TABLE 28-1 Properties of Electric Charges

	Vector Description		*Scalar Description*
Interaction between two charges	Force $\vec{\mathbf{F}}$	⟷	Potential energy U
	↕		↕
Effect of one charge or group of charges at a point in space	Field $\vec{\mathbf{E}}$	⟷	Potential V

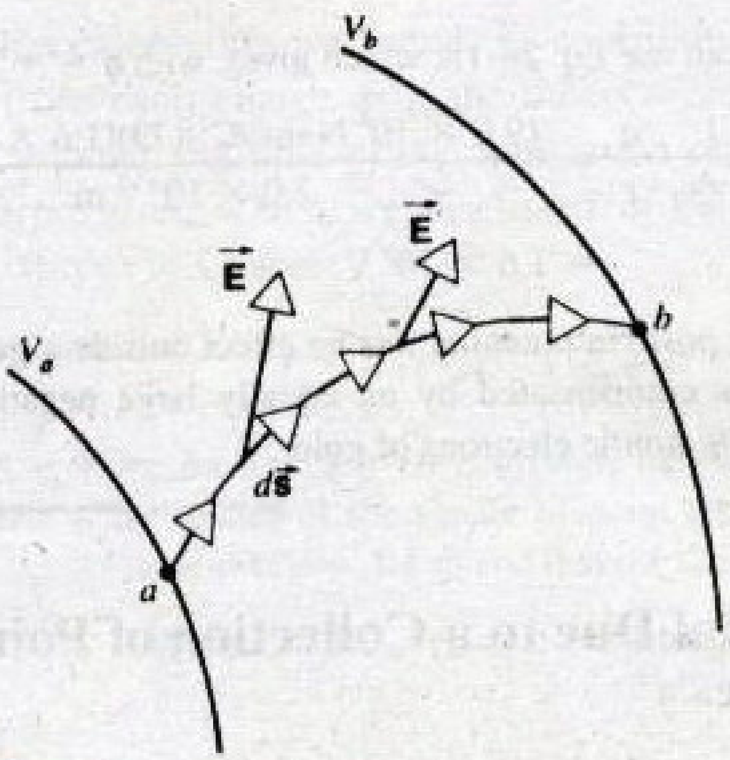


FIGURE 28-5. The potential difference between a and b can be found by calculating the line integral of $\vec{E}$ along the path ab.

would move a positively charged particle from a region of higher potential to a region of lower potential or a negatively charged particle in the opposite direction.

An integral of the form of Eq. 28-15 is called a *line integral.* Figure 28-5 illustrates the calculation of a line integral. We integrate from a to b along any convenient path; we know that the potential difference is a path-independent quantity, so we get the same result from Eq. 28-15 no matter what path we choose. In general the magnitude and direction of $\vec{E}$ may change from point to point along the path. At each step of the path, we find the dot product between $\vec{E}$ and the path increment $d\vec{s}$ (which essentially gives the component of $\vec{E}$ along the path), and we add up these dot products for the entire path.

As we did in Section 28-3, we may wish to find the potential at a point, relative to some chosen reference potential, rather than the potential difference given by Eq. 28-15. If we choose the reference point to be at infinity and define $V = 0$ as the reference, then Eq. 28-15 gives for the potential at point P

$$V_P = -\int_{\infty}^{P} \vec{E}\cdot d\vec{s}. \qquad (28\text{-}16)$$

SAMPLE PROBLEM 28-5. In Fig. 28-6, a test charge q_0 moves through a uniform electric field $\vec{E}$ from a to b along the path acb. Find the potential difference between a and b.

Solution For the path ac we have, from Eq. 28-15,

$$V_c - V_a = -\int_a^c \vec{E}\cdot d\vec{s} = -\int_a^c E\,ds\cos(\pi-\theta)$$

$$= E\cos\theta\int_a^c ds.$$

The integral is the length of the line ac, which is $L/\cos\theta$. Thus

$$V_c - V_a = E\cos\theta\frac{L}{\cos\theta} = EL.$$

Points b and c have the same potential, because no work is done in moving a charge between them, $\vec{E}$ and $d\vec{s}$ being at right angles for all points on the line cb. Thus

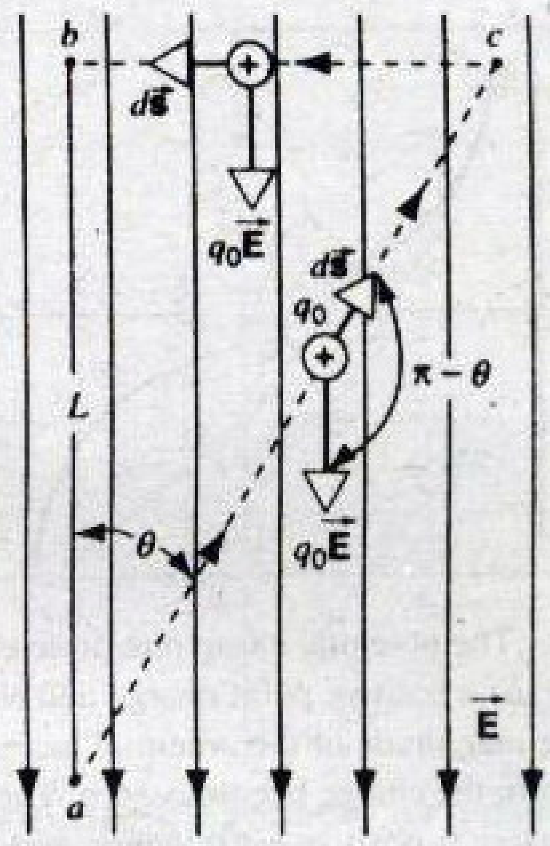


FIGURE 28-6. Sample Problem 28-5. A test charge q_0 moves along the path acb through the uniform electric field $\vec{E}$.

$$V_b - V_a = (V_b - V_c) + (V_c - V_a) = 0 + EL = EL.$$

This is the same value derived for a direct path connecting a and b, a result to be expected because the potential difference between two points is independent of path.

28-5 POTENTIAL DUE TO POINT CHARGES

In this section we will use the results of the previous sections to obtain the potential for various arrangements of point charges. In the next section we discuss the potential due to continuous charge distributions.

We first consider the potential due to a positive point charge q. Let a test charge q_0 move from point a to point b in the vicinity of q. We wish to use the test charge to find the potential difference between points a and b due to q. We can use the geometry of Fig. 28-1, with q_1 replaced by q and q_2 replaced by q_0.

We have already found the potential energy difference ΔU for this situation, which was given by Eq. 28-5 for two point charges. Writing Eq. 28-5 for charges q and q_0, and using Eq. 28-9 for the potential difference, we find

$$V_b - V_a = \frac{U_b - U_a}{q_0} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_b} - \frac{1}{r_a}\right). \qquad (28\text{-}17)$$

As we discussed in Section 28-2, Eq. 28-5 holds even if points a and b do not lie on the same line. Equation 28-17 is valid for the potential difference between any two points a and b.

Instead of the potential difference between two points, we can find the potential at a single point in the vicinity of q. Equation 28-7 gives the potential energy U due to the interaction of two point charges. The reference point for this expression is taken at infinity, where we define $U = 0$. We

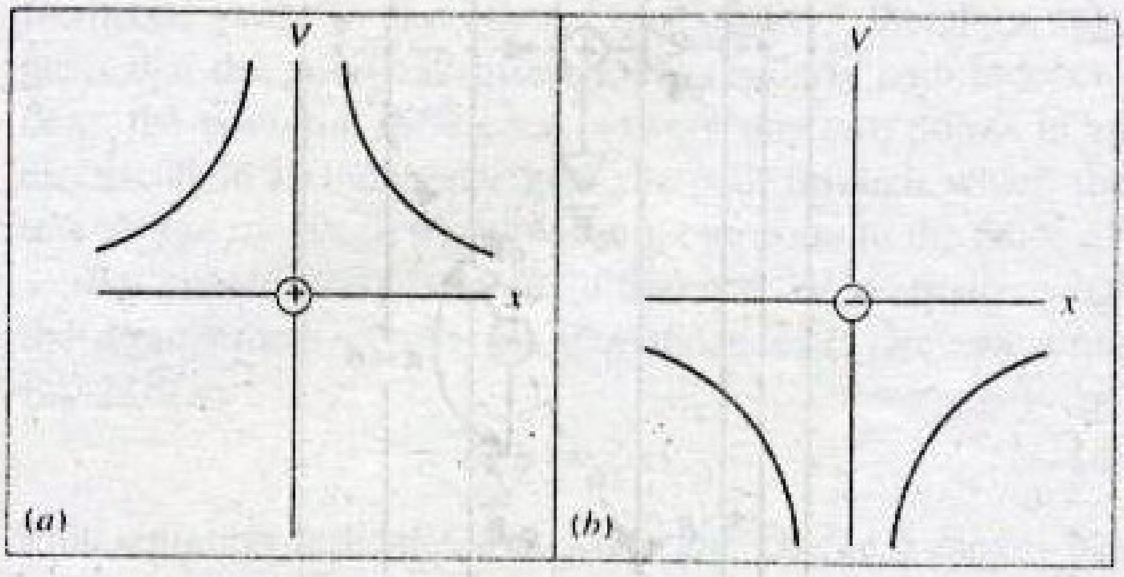


FIGURE 28-7. The potential along one dimension (chosen to be the x axis) for (a) a positive point charge and (b) a negative point charge. The magnitude of the potential increases to infinity as the distance from the charge becomes zero. The potential for a single positive charge is positive everywhere, and for a single negative charge the potential is negative everywhere.

can use Eq. 28-7, written for charges q and q_0, to find the potential at a point using Eq. 28-12 for the potential:

$$V = \frac{U}{q_0} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}, \qquad (28\text{-}18)$$

for any point at a distance r from q. Note that Eq. 28-18 could have been obtained directly from Eq. 28-17 by imposing the reference condition with $V_a = 0$ at $r_a = \infty$.

Equation 28-18 shows that the potential for a single positive point charge is zero at large distances and grows to large positive values as we approach the charge ($r \to 0$). If q is negative, the potential grows to large negative values as we approach the charge. Note that these results do not depend at all on the sign of the test charge q_0 we use in the calculation. Figure 28-7 shows the potential as a function of the distance from the charge for a positive and a negative point charge.

SAMPLE PROBLEM 28-6. What must be the magnitude of an isolated positive point charge for the electric potential at 15 cm from the charge to be +120 V? Assume that $V = 0$ at infinity.

Solution Solving Eq. 28-18 for q yields

$$q = 4\pi\epsilon_0 rV = (4\pi)(8.9 \times 10^{-12}\ \mathrm{C^2/N\cdot m^2})(0.15\ \mathrm{m})(120\ \mathrm{V})$$
$$= 2.0 \times 10^{-9}\ \mathrm{C} = 2.0\ \mathrm{nC}.$$

This charge is comparable to charges that can be produced by friction, such as by rubbing a balloon.

SAMPLE PROBLEM 28-7. What is the electric potential at the surface of a gold nucleus? The radius is 7.0×10^{-15} m, and the atomic number Z is 79.

Solution The nucleus, assumed to be spherically symmetric, behaves electrically for external points as if it were a point charge. Thus we can use Eq. 28-18, which gives, with $q = +79e$,

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r} = \frac{(9.0 \times 10^9\ \mathrm{N\cdot m^2/C^2})(79)(1.6 \times 10^{-19}\ \mathrm{C})}{7.0 \times 10^{-15}\ \mathrm{m}}$$
$$= 1.6 \times 10^7\ \mathrm{V}.$$

This large positive potential has no effect outside a gold *atom* because it is compensated by an equally large negative potential from the 79 atomic electrons of gold.

Potential Due to a Collection of Point Charges

Suppose we have a collection of N point charges q_1, q_2, . . . , q_N located at various fixed points (Fig. 28-8). We wish to find the potential at an arbitrary point P due to this collection of charges. The procedure is to calculate the potential at P due to each charge as if the others were not present and then add the resulting potentials to get the total potential. That is,

$$V = V_1 + V_2 + \cdots + V_N$$
$$= \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1} + \frac{1}{4\pi\epsilon_0}\frac{q_2}{r_2} + \cdots + \frac{1}{4\pi\epsilon_0}\frac{q_N}{r_N}, \qquad (28\text{-}19)$$

which we can write in compact form as

$$V = \frac{1}{4\pi\epsilon_0}\sum_{n=1}^{N}\frac{q_n}{r_n}. \qquad (28\text{-}20)$$

In these expressions, q_n is the value (magnitude and sign) of the nth charge and r_n is the distance of the nth charge from the point P where we wish to find the potential.

We could use Eq. 28-20, for instance, to find the work done when we bring a test charge q_0 in from infinity to point P in Fig. 28-8. For this calculation we see the advantage of using the potential, which is a scalar, rather than the force, which is a vector. To find the net force on a test charge at P, it would be necessary to find a vector sum. The scalar calculation of the potential is much simpler.

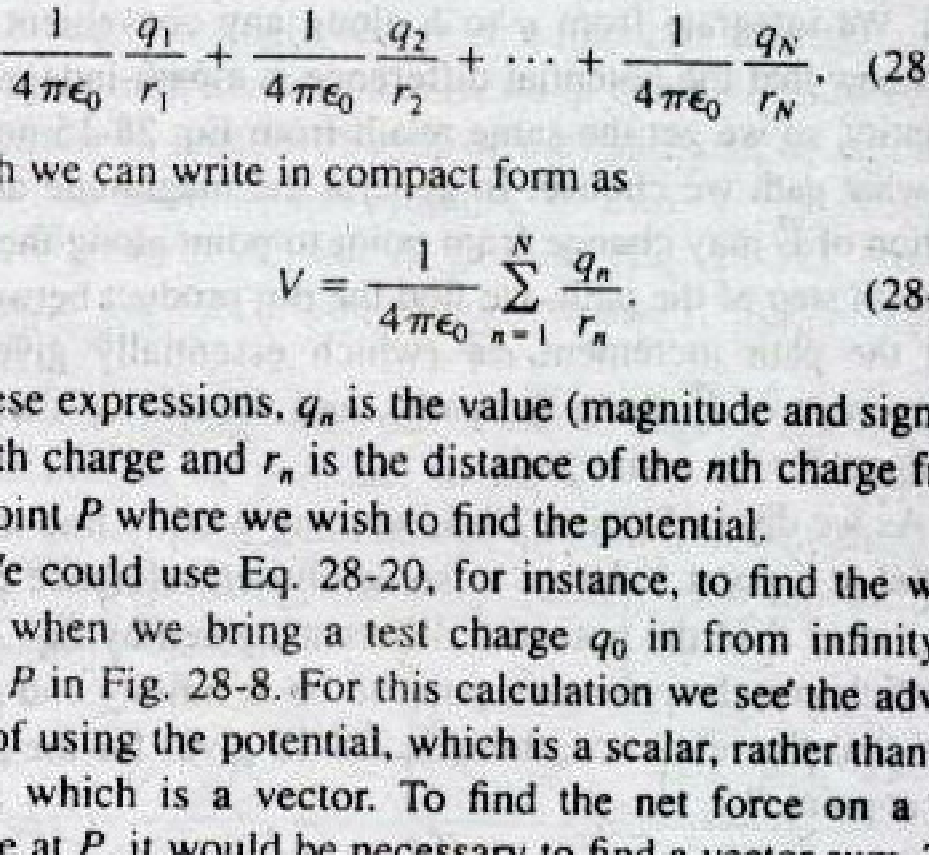


FIGURE 28-8. A collection of point charges.

In this calculation we found the contribution to the potential from each charge as if the others were not present. This is another example of the application of the principle of superposition, which we discussed in connection with electric forces in Chapter 25.

SAMPLE PROBLEM 28-8. Calculate the potential at point P, located at the center of the square of point charges shown in Fig. 28-9*a*. Assume that $d = 1.3$ m and that the charges are

$$q_1 = +12\text{ nC}, \quad q_3 = +31\text{ nC},$$
$$q_2 = -24\text{ nC}, \quad q_4 = +17\text{ nC}.$$

Solution From Eq. 28-20 we have

$$V = \frac{1}{4\pi\epsilon_0}\sum_n \frac{q_n}{r_n} = \frac{1}{4\pi\epsilon_0}\frac{q_1 + q_2 + q_3 + q_4}{R}.$$

The distance R of each charge from the center of the square is $d/\sqrt{2}$ or 0.919 m, so that

$$V = \frac{(8.99 \times 10^9\text{ N}\cdot\text{m}^2/\text{C}^2)(12 - 24 + 31 + 17) \times 10^{-9}\text{ C}}{0.919\text{ m}}$$
$$= 3.5 \times 10^2\text{ V}.$$

Close to any of the three positive charges in Fig. 28-9*a*, the potential can have very large positive values. Close to the single negative charge in that figure, the potential can have large negative values. There must then be other points within the boundaries of the square that have the same potential as that at point P. The dashed line in Fig. 28-9*b* connects other points in the plane that have this same value of the potential. As we discuss later in Section 28-8, such *equipotential surfaces* provide a useful way of visualizing the potentials of various charge distributions.

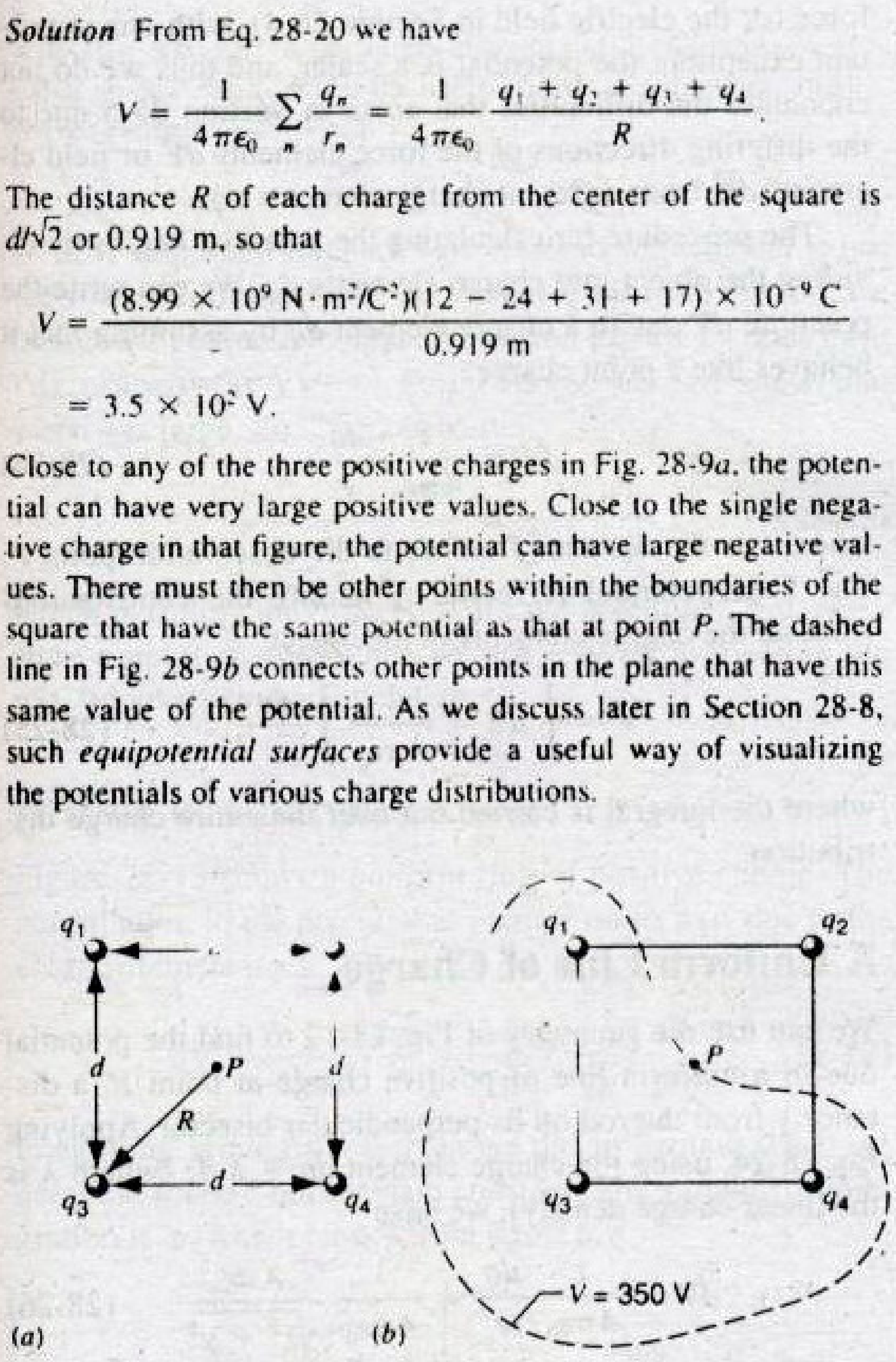


FIGURE 28-9. Sample Problem 28-8. (*a*) Four charges are held at the corners of a square. (*b*) The curve connects points that have the same potential (350 V) as the point P at the center of the square.

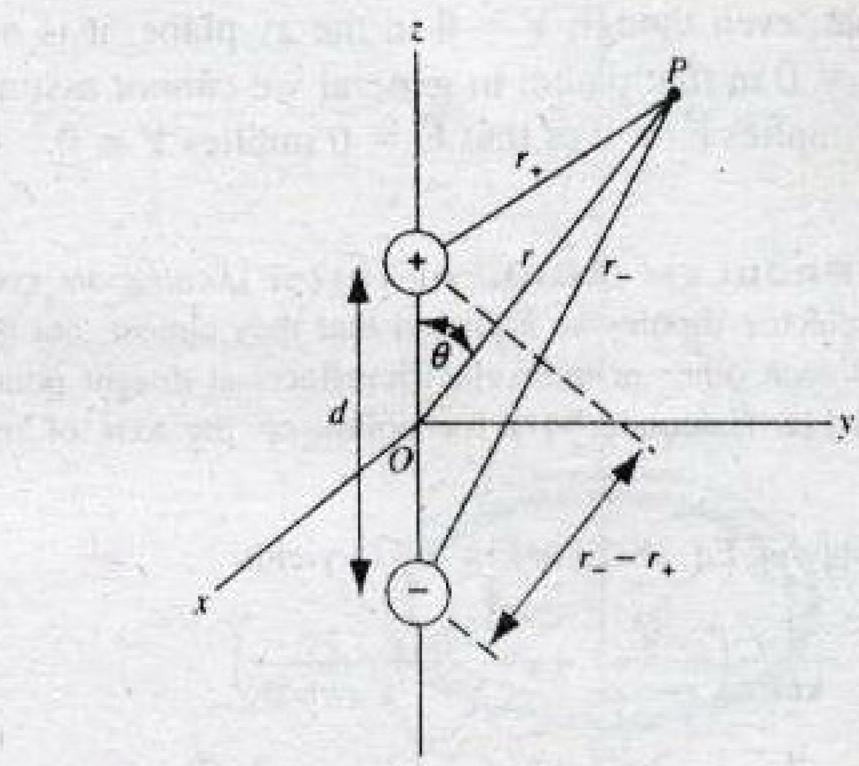


FIGURE 28-10. The geometry for calculating the potential at point P due to an electric dipole.

Potential Due to an Electric Dipole

The potential due to a electric dipole can be calculated in a straightforward way using Eq. 28-20. Figure 28-10 shows the geometry for the calculation. We place the origin of our coordinate system at the center of the dipole, and we seek the electric potential at the point P, which is located a distance r from the center of the dipole and at an angle θ from the axis of the dipole (the z axis). The distances from the positive and negative charges to P are respectively r_+ and r_-. Using Eq. 28-20, we find the potential to be

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+} + \frac{-q}{r_-}\right). \quad (28\text{-}21)$$

Equation 28-21 is the exact expression for the potential due to a dipole. However, in many applications (such as for atomic or molecular dipoles) we can obtain a more useful relationship by recognizing that our observation point P is usually very far from the dipole, compared with the distance d between the charges; that is, $r \gg d$. In this case,

$$r_- - r_+ \approx d\cos\theta \quad \text{and} \quad r_- r_+ \approx r^2,$$

and substituting these results into Eq. 28-21 we obtain

$$V = \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}, \quad (28\text{-}22)$$

where we have used Eq. 26-8 ($p = qd$) for the dipole moment. Equation 28-22 gives the potential due to a dipole at any point in space. The dipole has cylindrical symmetry for rotations about the z axis, so Eq. 28-22 is valid at points that do not lie in the plane of the diagram of Fig. 28-10.

Note that the potential due to the dipole varies as $1/r^2$. This is in contrast to the potential for a single charge, which varies (see Eq. 28-18) as $1/r$.

Equation 28-22 shows that $V = 0$ when $\theta = 90°$, which corresponds to points in the xy plane in Fig. 28-10. This means that if we move a test charge from infinity to a point in the xy plane, the dipole does no net work on the test charge. For a given r, the potential varies from positive values on the positive z axis ($\theta = 0$) to zero in the xy plane ($\theta = 90°$) to negative values on the negative z axis ($\theta = 180°$).

Note that, even though $V = 0$ in the xy plane, it is *not* true that $\vec{\mathbf{E}} = 0$ in that plane. In general we *cannot* assume that $V = 0$ implies $\vec{\mathbf{E}} = 0$ or that $\vec{\mathbf{E}} = 0$ implies $V = 0$.

SAMPLE PROBLEM 28-9. An *electric quadrupole* consists of two electric dipoles so arranged that they almost, but not quite, cancel each other in their electric effects at distant points (see Fig. 28-11). Calculate $V(r)$ for points on the axis of this quadrupole.

Solution Applying Eq. 28-20 to Fig. 28-11 yields

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r-d} + \frac{-2q}{r} + \frac{q}{r+d}\right)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{2qd^2}{r(r^2-d^2)} = \frac{1}{4\pi\epsilon_0}\frac{2qd^2}{r^3(1-d^2/r^2)}.$$

Because $d \ll r$, we can neglect d^2/r^2 compared with 1, in which case the potential becomes

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^3}, \qquad (28\text{-}23)$$

where Q $(= 2qd^2)$ is the *electric quadrupole moment* of the charge assembly of Fig. 28-11. Note that V varies (1) as $1/r$ for a point charge (see Eq. 28-18), (2) as $1/r^2$ for a dipole (see Eq. 28-22), and (3) as $1/r^3$ for a quadrupole (see Eq. 28-23).

Note too that (1) a dipole is two equal and opposite charges that do not quite coincide in space so that their electric effects at distant points do not quite cancel, and (2) a quadrupole is two equal and opposite dipoles that do not quite coincide in space so that their electric effects at distant points again do not quite cancel. We can continue to construct more complex assemblies of electric charges. This process turns out to be useful, because the electric potential of *any* charge distribution can be represented as a series of terms in increasing powers of $1/r$. The $1/r$ part, called the *monopole* term, depends on the net charge of the distribution, and the succeeding terms ($1/r^2$, the *dipole* term; $1/r^3$, the *quadrupole* term; and so on) indicate how the charge is distributed. This type of analysis is called an *expansion in multipoles*.

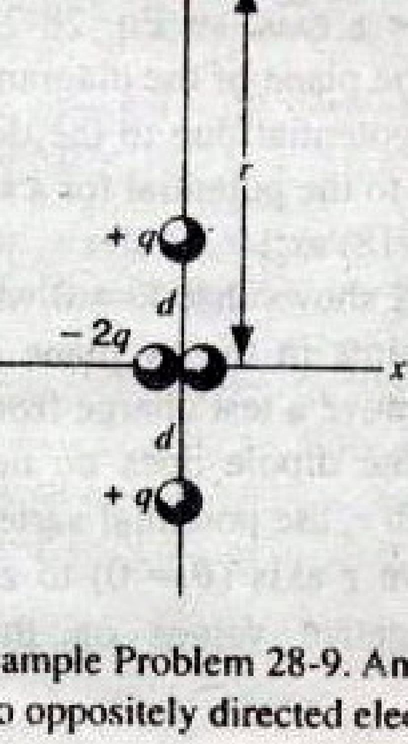


FIGURE 28-11. Sample Problem 28-9. An electric quadrupole, consisting of two oppositely directed electric dipoles.

28-6 ELECTRIC POTENTIAL OF CONTINUOUS CHARGE DISTRIBUTIONS

In Section 25-5 we introduced a procedure for calculating the force exerted by a continuous charge distribution on a point charge. We can similarly obtain the potential energy for the interaction between a continuous distribution and a point charge by calculating the potential due to the charge distribution. In this section we calculate the potential for the same three charge distributions considered in Section 25-5.

The procedure for calculating the potential for a continuous charge distribution is similar to that used to find the force (or the electric field in Section 26-4), with one important exception: the potential is a scalar, and thus we do not encounter the difficulties that arose in Section 25-5 due to the differing directions of the force elements $d\vec{\mathbf{F}}$ or field elements $d\vec{\mathbf{E}}$ from different charge elements dq.

The procedure for calculating the potential begins by dividing the object into charge elements dq. We can write the potential dV due to a charge element dq by assuming that it behaves like a point charge:

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{r}, \qquad (28\text{-}24)$$

where r is the distance from dq to the observation point P. The total potential is found by adding the contributions from all the charge elements of the object:

$$V = \int dV = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{r}, \qquad (28\text{-}25)$$

where the integral is carried out over the entire charge distribution.

A Uniform Line of Charge

We can use the geometry of Fig. 28-12 to find the potential due to a uniform line of positive charge at point P, a distance y from the rod on its perpendicular bisector. Applying Eq. 28-24, using the charge element $dq = \lambda\, dz$ (where λ is the linear charge density), we have

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{r} = \frac{1}{4\pi\epsilon_0}\frac{\lambda\, dz}{\sqrt{z^2+y^2}}. \qquad (28\text{-}26)$$

Carrying out the integration over the length L as in Eq. 28-25 and noting that y is a constant, we obtain

$$V = \frac{1}{4\pi\epsilon_0}\int_{-L/2}^{+L/2}\frac{\lambda\, dz}{\sqrt{z^2+y^2}}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(z + \sqrt{z^2+y^2}\right)\right]_{-L/2}^{+L/2}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{L/2 + \sqrt{L^2/4 + y^2}}{-L/2 + \sqrt{L^2/4+y^2}}\right], \qquad (28\text{-}27)$$

where we have used the relation $\ln A - \ln B = \ln(A/B)$ to obtain the last result.

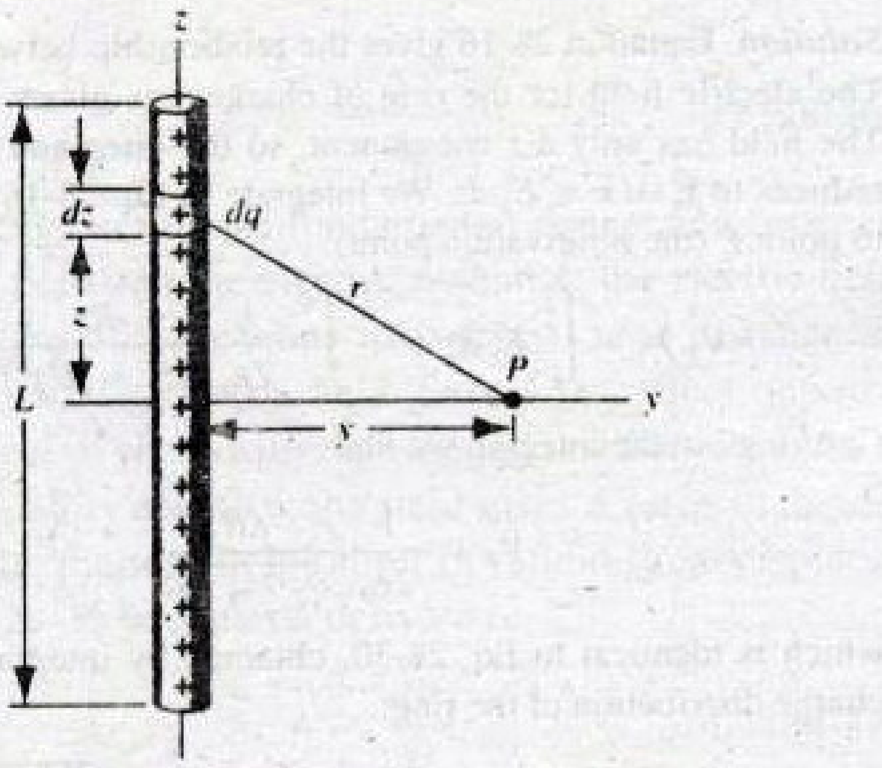


FIGURE 28-12. A uniformly charged rod. To find the potential at point P, we consider the rod to consist of many individual charge elements such as dq.

It is important to check this result to see whether it has the correct limiting value. As we move far from the rod, we expect the potential to approach 0, and Eq. 28-27 does have this property as $y \to \infty$. Furthermore, we can show that when y is large, Eq. 28-27 becomes

$$V \approx \frac{1}{4\pi\epsilon_0}\frac{\lambda L}{y} = \frac{1}{4\pi\epsilon_0}\frac{q}{y}, \qquad (28\text{-}28)$$

which is simply the expression for the potential a distance y from a point charge. When we are very far away from the rod, it looks like a point charge.

A Ring of Charge

Figure 28-13 shows a uniform ring of positive charge. The contribution to the potential at point P on its axis due to the charge element $dq = \lambda\, ds = \lambda R\, d\phi$ is

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{r} = \frac{1}{4\pi\epsilon_0}\frac{\lambda R\, d\phi}{\sqrt{R^2 + z^2}}. \qquad (28\text{-}29)$$

Integrating around the ring as we did in Section 25-5, we note that R and z both remain constant. The variable of integration is ϕ, which ranges from 0 to 2π.

$$V = \frac{1}{4\pi\epsilon_0}\frac{\lambda R}{\sqrt{R^2 + z^2}}\int_0^{2\pi} d\phi = \frac{1}{4\pi\epsilon_0}\frac{2\pi\lambda R}{\sqrt{R^2 + z^2}}. \qquad (28\text{-}30)$$

Note that as $z \to \infty$, the potential decreases to zero and for large z has the approximate value $q/4\pi\epsilon_0 z$ (where $q = 2\pi\lambda R$), as expected for a location a distance z from a point charge.

A Charged Disk

With the geometry of Fig. 28-14, we can use Eq. 28-30 to find the potential dV at point P due to the ring of radius w and charge $dq = \sigma\, dA$ with area element $dA = 2\pi w\, dw$:

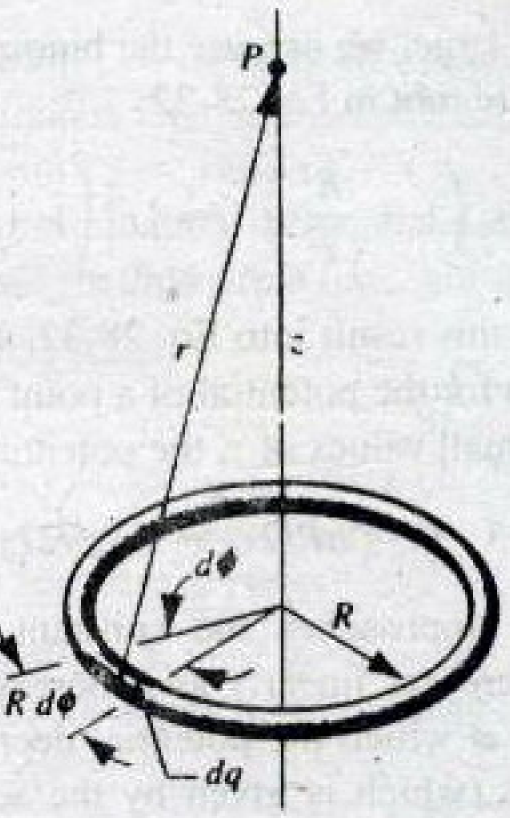


FIGURE 28-13. A uniformly charged ring. To find the potential at P, we calculate the total effect of all charge elements such as dq.

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{\sqrt{w^2 + z^2}} = \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma w\, dw}{\sqrt{w^2 + z^2}}. \qquad (28\text{-}31)$$

To sum the contributions from all the rings on the disk, we integrate as w ranges from 0 to R:

$$V = \frac{\sigma}{2\epsilon_0}\int_0^R \frac{w\, dw}{\sqrt{w^2 + z^2}} = \frac{\sigma}{2\epsilon_0}\left(\sqrt{R^2 + z^2} - |z|\right). \qquad (28\text{-}32)$$

The last term in Eq. 28-32 comes from evaluating $\sqrt{z^2}$ and is written as $|z|$ so that Eq. 28-32 remains valid for points on the z axis above the disk ($z > 0$) as well as below the disk ($z < 0$). The potential has its maximum value at the surface of the disk (where $z = 0$) and decreases as we move along the z axis in either direction.

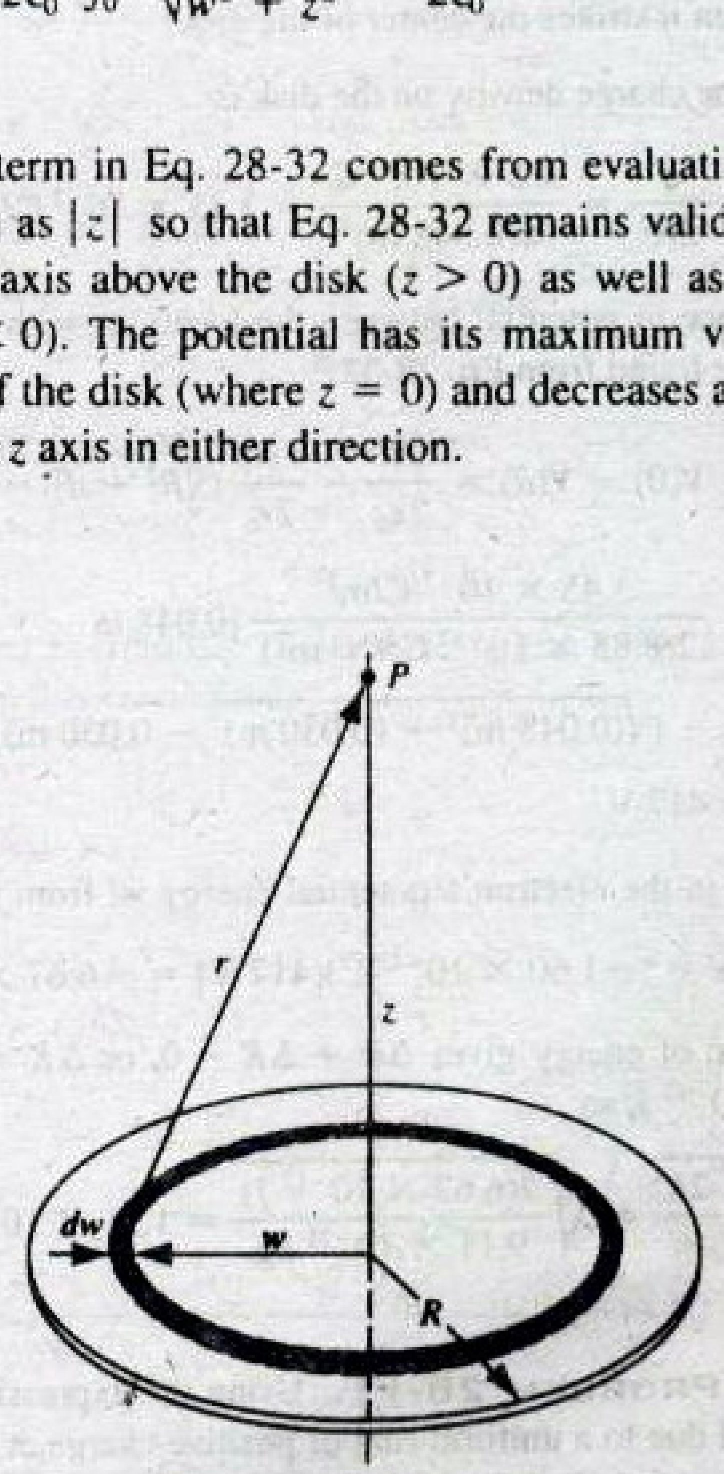


FIGURE 28-14. A disk of radius R carries a uniform charge density σ. The element of charge dq is a uniformly charged ring.

When z is large, we can use the binomial theorem to expand the square root in Eq. 28-32:

$$\sqrt{R^2 + z^2} = |z|\left(1 + \frac{R^2}{z^2}\right)^{1/2} \approx |z|\left(1 + \frac{1}{2}\frac{R^2}{z^2}\right) \quad (28\text{-}33)$$

and, inserting this result into Eq. 28-32, we find once again the expression for the potential of a point charge.

For very small values of z, the potential is

$$V = \sigma R/2\epsilon_0 - \sigma|z|/2\epsilon_0. \quad (28\text{-}34)$$

The potential approaches the constant value $\sigma R/2\epsilon_0$ as $z \to 0$ and decreases linearly as z increases in either direction. The rate at which the potential decreases as we move along the axis (which is given by the second term of Eq. 28-34) is independent of the size of the disk, for a given charge density. In fact, this term turns out to be the same for any large, flat, uniformly charged plate, no matter what its size or shape (round, square, etc.), as long as we are near its center and thus far from any edge. We will use this fact in drawing a "map" of the potential in the next section.

SAMPLE PROBLEM 28-10. A disk of radius $R = 4.8$ cm carries a total charge $q = +2.5$ nC that is uniformly distributed over its surface and held in fixed locations (consider the surface to behave like an insulator). An electron is initially at rest a distance of $d = 3.0$ cm from the disk along its axis. When the electron is released, it is attracted toward the disk. What is the speed of the electron when it strikes the center of the disk?

Solution The charge density on the disk is

$$\sigma = \frac{q}{\pi R^2} = \frac{2.5 \times 10^{-9}\text{ C}}{\pi(0.048\text{ m})^2} = 3.45 \times 10^{-7}\text{ C/m}^2.$$

The difference in potential between the locations with $z = d$ and $z = 0$ can be found from Eq. 28-32:

$$\Delta V = V(0) - V(d) = \frac{\sigma R}{2\epsilon_0} - \frac{\sigma}{2\epsilon_0}(\sqrt{R^2 + d^2} - d)$$

$$= \frac{3.45 \times 10^{-7}\text{ C/m}^2}{2(8.85 \times 10^{-12}\text{ C}^2/\text{N}\cdot\text{m}^2)}[0.048\text{ m}$$

$$-(\sqrt{(0.048\text{ m})^2 + (0.030\text{ m})^2} - 0.030\text{ m})]$$

$$= 417\text{ V}.$$

The change in the electron's potential energy is, from Eq. 28-14,

$$\Delta U = q\,\Delta V = (-1.60 \times 10^{-19}\text{ C})(417\text{ V}) = -6.67 \times 10^{-17}\text{ J}.$$

Conservation of energy gives $\Delta U + \Delta K = 0$, or $\Delta K = -\Delta U = +6.67 \times 10^{-17}$ J, so

$$v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{2(6.67 \times 10^{-17}\text{ J})}{9.11 \times 10^{-31}\text{ kg}}} = 1.21 \times 10^7\text{ m/s}.$$

SAMPLE PROBLEM 28-11. Using the expression for the electric field due to a uniform ring of positive charge at a point on its axis (the z axis), find the expression for the potential due to the ring at a point on the axis a distance z' from the ring.

Solution Equation 28-16 gives the relationship between V and $\vec{\mathbf{E}}$. The electric field for the ring of charge was given in Eq. 26-18. The field has only a z component, so the integrand of Eq. 28-16 reduces to $\vec{\mathbf{E}} \cdot d\vec{s} = E_z\,dz$. We integrate in Eq. 28-16 from infinity to point P (the observation point):

$$V_P = -\int_\infty^{z'} E_z\,dz = -\int_\infty^{z'} \frac{\lambda}{2\epsilon_0}\frac{Rz}{(z^2 + R^2)^{3/2}}\,dz.$$

Carrying out the integral, we find

$$V_P = \frac{1}{2\epsilon_0}\frac{\lambda R}{\sqrt{z'^2 + R^2}},$$

which is identical to Eq. 28-30, obtained by integrating over the charge distribution of the ring.

28-7 CALCULATING THE FIELD FROM THE POTENTIAL

In Section 28-4 we described a method for obtaining the potential difference from the electric field. Now we discuss how to do that calculation in reverse: given the potential, we can find the electric field. That is, the double arrow connecting the two lower boxes in Table 28-1 can indeed go in either direction.

Figure 28-15a shows a positive test charge q_0 as it moves from point a (where the potential is V) to point b (potential $V + \Delta V$). In the process, the electric potential energy of q_0 changes by an amount $\Delta U = q_0\,\Delta V$. In the language of forces, we would say that there is an electric field $\vec{\mathbf{E}}$ that exerts a force $\vec{\mathbf{F}} = q_0\vec{\mathbf{E}}$ on the particle. The work done by this force as the particle moves from a to b is $W = F_s\,\Delta s = q_0 E_s\,\Delta s$, where E_s and F_s are the components of $\vec{\mathbf{E}}$ and $\vec{\mathbf{F}}$ along Δs, which represents the displacement of the particle as it travels from a to b. (We assume that Δs is small, so that we can regard the force and the field as roughly constant in both magnitude and direction along ab.) The mathematical connection between the two equivalent descriptions is $W = -\Delta U$, which gives

$$q_0 E_s\,\Delta s = -q_0\,\Delta V \quad (28\text{-}35)$$

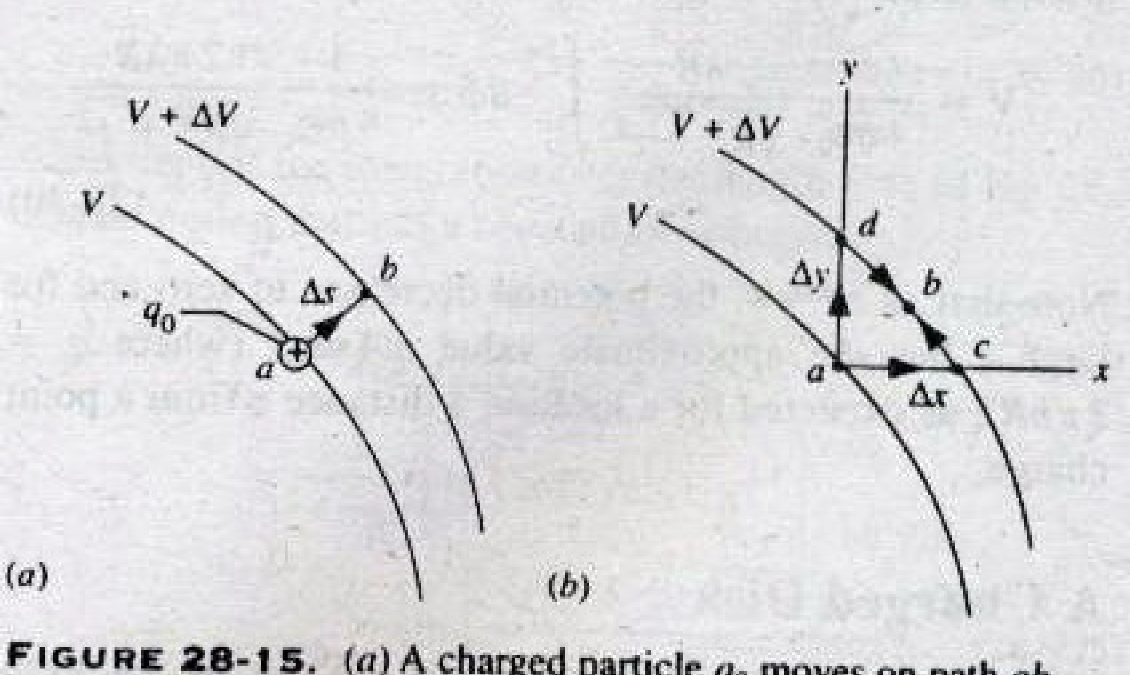


FIGURE 28-15. (a) A charged particle q_0 moves on path ab between two equipotentials. (b) The particle moves from a to b along either path acb or adb.

or

$$E_s = -\frac{\Delta V}{\Delta s}. \qquad (28\text{-}36)$$

This equation gives us the fundamental connection between the electric field and the electric potential: the electric field is the negative of the change in potential with distance. If ΔV is positive, the electric field gives a force that opposes the movement of the positively charged test particle from a to b, and if ΔV is negative, the field gives a force in the direction of the motion. In the limit of infinitesimal displacements, Eq. 28-36 becomes a derivative:

$$E_s = -\frac{dV}{ds}. \qquad (28\text{-}37)$$

The component of the electric field in any direction is the negative of the derivative of the potential with respect to a displacement in that direction.

Let us choose a different geometry for this process. Figure 28-15b shows the same process, but instead of moving the test charge from a to b directly, we move it along two different paths. Path acb takes the charge along the x axis from a to c and then along the path from c to b, which has been chosen so that the potential has the same value $V + \Delta V$ everywhere between c and b. The work done by the electric field along cb is zero, because the potential does not change (see Eq. 28-11). The work done by the electric field along ac is $F_x \Delta x = q_0 E_x \Delta x$. Because the change in potential energy is independent of path, we have again from $W = -\Delta U$

$$q_0 E_x \Delta x = -q_0 \Delta V \quad \text{or} \quad E_x = -\frac{\Delta V}{\Delta x}.$$

If we move the particle on path adb, the work is $F_y \Delta y = q_0 E_y \Delta y$ along ad and zero along db (which has again been chosen so that the potential has the same value $V + \Delta V$ everywhere between d and b). Because the net change in potential energy along adb is also ΔV, we obtain

$$q_0 E_y \Delta y = -q_0 \Delta V \quad \text{or} \quad E_y = -\frac{\Delta V}{\Delta y}.$$

A similar result would be obtained for E_z from a three-dimensional calculation.

If we take the limit as the path lengths become very small, the differences become derivatives, and we can write the most general relationship between $\vec{E}$ and V as

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}. \qquad (28\text{-}38)$$

If $V(x, y, z)$ is known at all points in space for a particular charge distribution, then we can find the components of $\vec{E}$ by taking partial derivatives of V with respect to each of the coordinates.*

* The symbol $\partial V/\partial x$ denotes a *partial derivative*. In taking this derivative of the function $V(x, y, z)$, the quantity x is to be viewed as a variable and y and z are to be regarded as constants. Similar considerations hold for $\partial V/\partial y$ and $\partial V/\partial z$.

We therefore have two methods of calculating the electric field for continuous charge distributions; one based on integrating Coulomb's law (Eqs. 26-13 to 26-15) and another based on differentiating the potential (Eq. 28-38). In practice, the second method often turns out to be less difficult.

SAMPLE PROBLEM 28-12. Using Eq. 28-32 for the potential on the axis of a uniformly charged disk, derive an expression for the electric field at axial points.

Solution From symmetry, $\vec{E}$ must lie along the axis of the disk (the z axis). Using Eq. 28-38, we have (assuming $z > 0$)

$$E_z = -\frac{\partial V}{\partial z} = -\frac{\sigma}{2\epsilon_0}\frac{d}{dz}[(z^2 + R^2)^{1/2} - z]$$

$$= \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right),$$

in agreement with the result obtained by direct integration, Eq. 26-19.

SAMPLE PROBLEM 28-13. Figure 28-16 shows a (distant) point P in the field of a dipole located at the origin of an xz coordinate system. Calculate $\vec{E}$ as a function of position.

Solution From symmetry, $\vec{E}$ at points in the plane of Fig. 28-16 lies in this plane and can be expressed in terms of its components E_x and E_z, E_y being zero. Let us first express the potential in rectangular coordinates rather than polar coordinates, making use of

$$r = (x^2 + z^2)^{1/2} \quad \text{and} \quad \cos\theta = \frac{z}{(x^2 + z^2)^{1/2}}.$$

V is given by Eq. 28-22:

$$V = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}.$$

Substituting for r^2 and $\cos\theta$, we obtain

$$V = \frac{p}{4\pi\epsilon_0}\frac{z}{(x^2 + z^2)^{3/2}}.$$

We find E_z from Eq. 28-38, recalling that x is to be treated as a constant in this calculation,

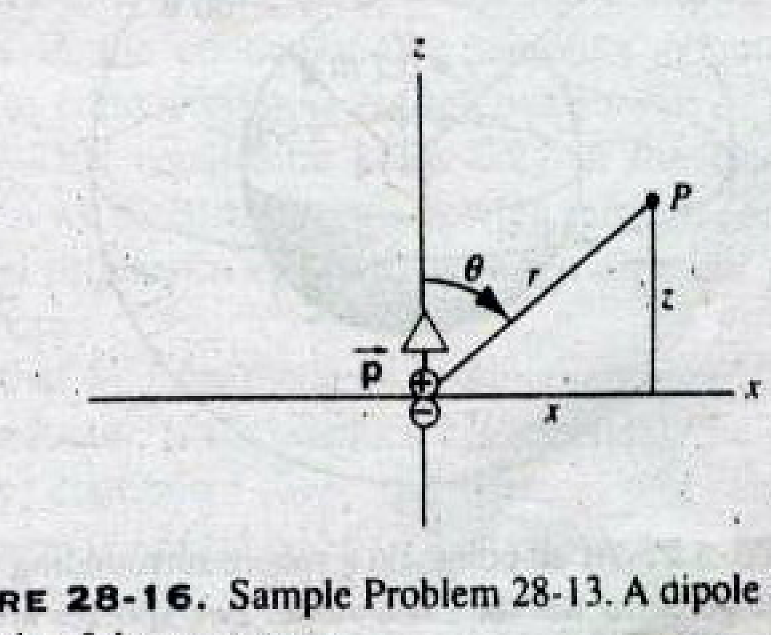


FIGURE 28-16. Sample Problem 28-13. A dipole is located at the origin of the xz system.

$$E_z = -\frac{\partial V}{\partial z} = -\frac{p}{4\pi\epsilon_0}\frac{(x^2+z^2)^{3/2} - z[\frac{3}{2}(x^2+z^2)^{1/2}](2z)}{(x^2+z^2)^3}$$

$$= -\frac{p}{4\pi\epsilon_0}\frac{x^2-2z^2}{(x^2+z^2)^{5/2}}. \qquad (28\text{-}39)$$

Putting $x = 0$ describes distant points along the dipole axis (that is, the z axis), and the expression for E_z reduces to

$$E_z = \frac{1}{4\pi\epsilon_0}\frac{2p}{z^3}.$$

This result agrees exactly with that found in Chapter 26, Problem 1 for the field along the dipole axis. Note that along the z axis, $E_x = 0$ from symmetry.

Putting $z = 0$ in Eq. 28-39 gives E_z for distant points in the median plane of the dipole:

$$E_z = -\frac{1}{4\pi\epsilon_0}\frac{p}{x^3},$$

which agrees exactly with Eq. 26-12 for, again from symmetry, E_x equals zero in the median plane. The negative sign in this equation indicates that $\vec{E}$ points in the negative z direction. You can carry out a similar procedure to find E_x, and you should obtain a result that agrees with that of Problem 2 of Chapter 26.

28-8 EQUIPOTENTIAL SURFACES

Consider a point charge $q = 1.11$ nC. Using Eq. 28-18, we can find the potential due to this charge to be 100 V at a distance 0.1 m from the charge. Because there is no directionality associated with potential, its value is 100 V at that distance in any direction. This is indicated in Fig. 28-17. At any point on the sphere of radius 0.1 m surrounding q, the potential is 100 V. On a second sphere of radius 0.2 m, the potential everywhere has the value 50 V.

A surface on which the potential has the same value everywhere, such as one of the spheres in Fig. 28-17, is called an *equipotential surface.* No net work is done by electric forces when we move a test charge from any point on an equipotential surface to any other point on the same surface, because $\Delta V = 0$. Even if the path moves off the surface, no net work is done as long as the path starts and finishes on the same equipotential surface. The amount of work done by electrical forces when a test charge moves from one equipotential surface to another depends only on the potential difference between the two surfaces; the work is independent of the starting and finishing locations on the two surfaces—the same work is done when the charge moves from *any* point on the first surface to *any* point on the second surface.

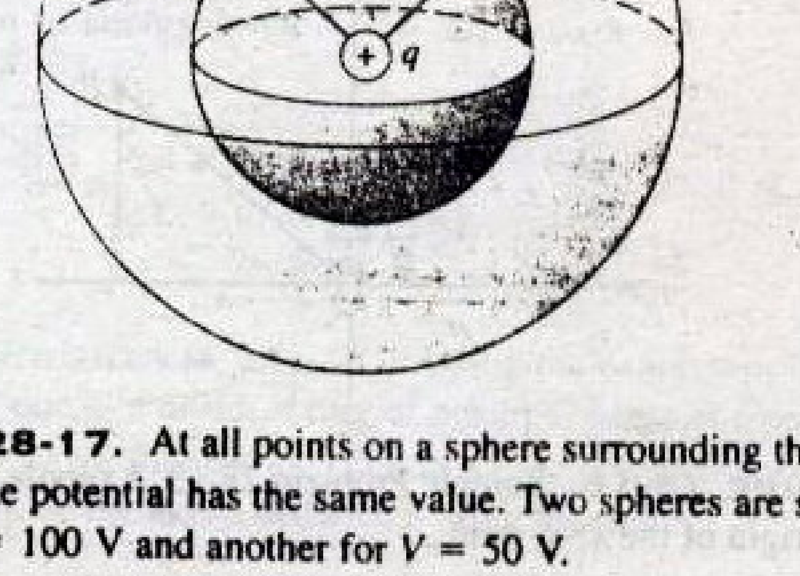


FIGURE 28-17. At all points on a sphere surrounding the charge q, the potential has the same value. Two spheres are shown, one for $V = 100$ V and another for $V = 50$ V.

Figure 28-18 shows portions of a *family* of equipotential surfaces that might be associated with a certain charge distribution. The work done by electric forces when a charged particle moves along path 1 is zero because that path starts and ends on the same equipotential surface. The work done along path 2 is zero for the same reason. The work is not zero along paths 3 and 4, but it has the same value for both these paths because they connect the same pair of equipotential surfaces. That is, paths 3 and 4 connect points with the same potential difference ($V_B - V_A$). If we move a charge q from *any* point on surface A to *any* point on surface B, the work done by the electrostatic force is, according to Eq. 28-11, $W_{AB} = -q(V_B - V_A)$.

Field Lines and Equipotential Surfaces

In Section 26-5 we discussed a different graphical method of describing a charge distribution, based on electric field lines. The mathematical relationship between $\vec{E}$ and V that we derived in Section 28-7 suggests a relationship between the graphical representations as well.

Suppose we release a positive charge from rest at point b on the equipotential $V + \Delta V$ in Fig. 28-15. In the language of potential, we say that the particle would "fall" through the potential difference ΔV toward the equipotential V. We can also regard the particle as being accelerated by an electric field that is present in the region between the equipotential surfaces. The electric field must be perpendicular to the equipotential surface at point b. If this were not so, then there would be a component of the electric field along the equipotential surface, which would do work on a

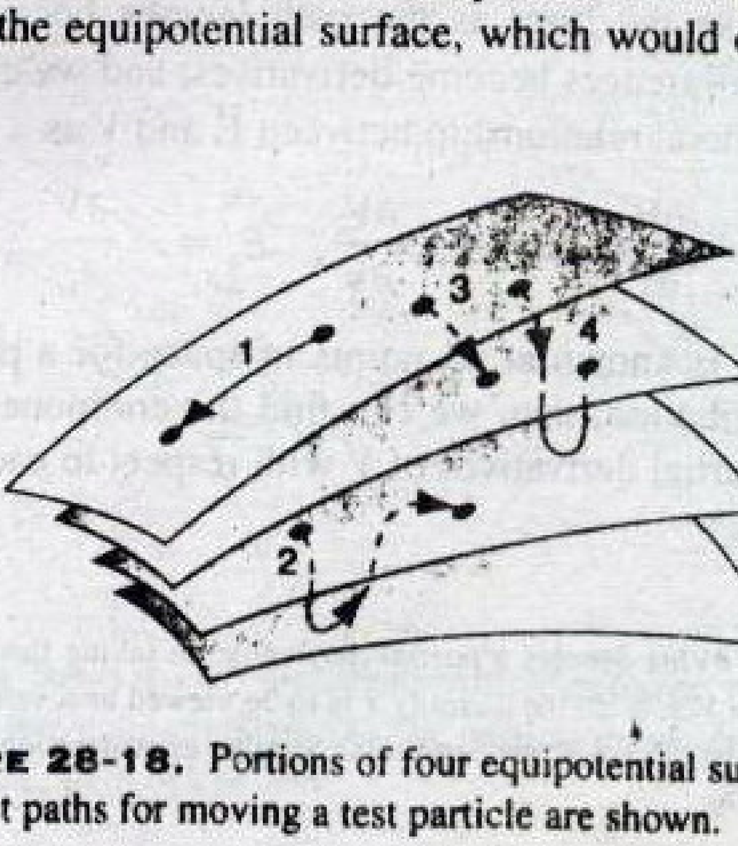


FIGURE 28-18. Portions of four equipotential surfaces. Four different paths for moving a test particle are shown.

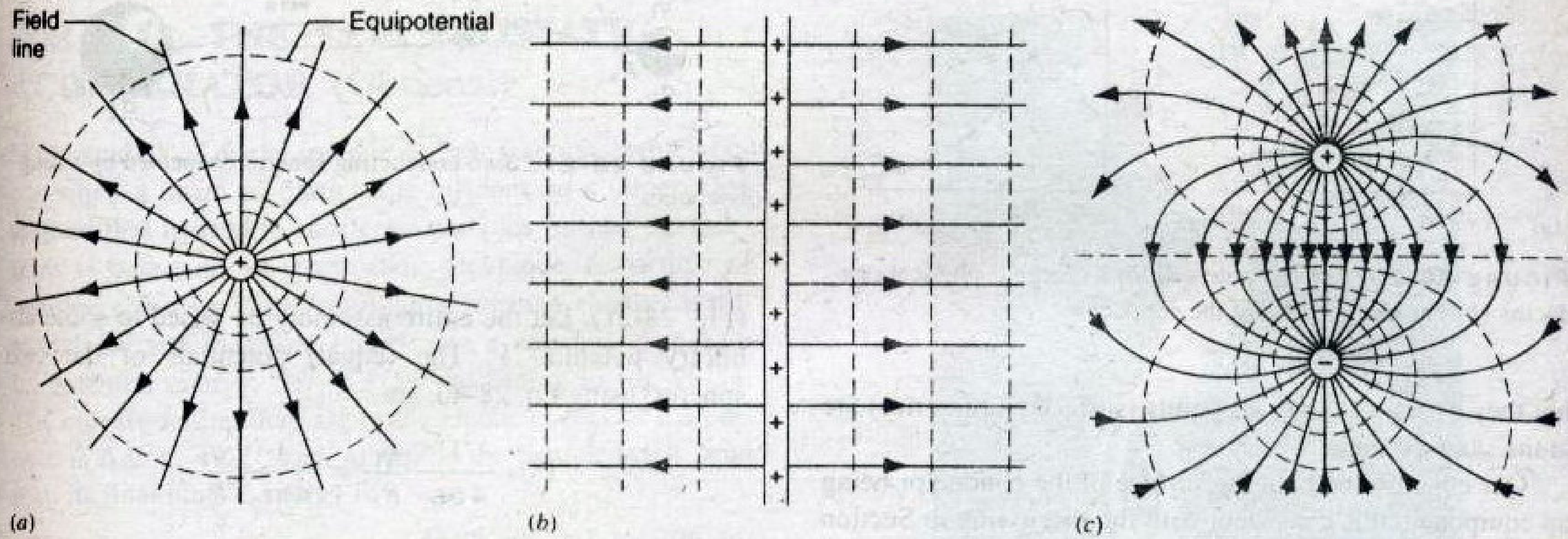


FIGURE 28-19. Electric field lines (solid lines) and cross sections of equipotential surfaces (dashed lines) for (*a*) a positive point charge, (*b*) an infinite sheet of positive charge, viewed along its edge, and (*c*) an electric dipole.

particle that moved along the surface. This, however, would violate the definition of an equipotential as a surface of constant potential, along which we can move a charged particle freely with no work. We conclude that *the electric field lines must everywhere be perpendicular to the equipotential surfaces.*

It is also possible to reach the same conclusion from Eq. 28-37, $E_s = -dV/ds$. There will be one direction for ds in Fig. 28-15 in which the value of the quantity $-dV/ds$ is a maximum, which means that E_s is also a maximum in that direction. That maximum value is E, the magnitude of the electric field at that point, and the direction for which E_s has its maximum is the direction of the electric field. Equivalently, we can draw at point b a circle of radius ds. One point on the circle will be the closest to the next equipotential and therefore will represent the largest value of $-dV$. The direction from b to that point is perpendicular to the equipotential surface at b and represents the direction of the electric field at b.

If we know the pattern of equipotential surfaces for a particular charge distribution, we can find the field lines by drawing perpendiculars to the equipotentials. Figure 28-19 shows the combined equipotentials and field lines for three cases we have already considered: the point charge, the infinite sheet of charge, and the dipole. These drawings represent the electric field lines of Figs. 26-10, 26-11, and 26-12 with superimposed equipotential surfaces. Note that the field lines are perpendicular to the equipotentials wherever they cross.

28-9 THE POTENTIAL OF A CHARGED CONDUCTOR

In Section 27-6 we deduced two properties of an isolated charged conductor: (1) the electric field is zero in its interior, and (2) the charge resides on the outer surface of the conductor. A third important property of a charged conductor results from considering its electric potential.

Suppose we have a conductor of arbitrary shape, to which we transfer a net charge. The charges are free to move and will quickly distribute themselves on the outer surface of the conductor until they are in equilibrium. In effect, the charges of the same sign repel one another until they reach a distribution in which the average distance between them is as large as possible, so that the potential energy of the arrangement of charges reaches a minimum value.

If the charges are in equilibrium on the surface of the conductor, then its surface must be an equipotential. If this were not so, some parts of the surface would be at higher or lower potentials than other parts. Positive charges would then migrate toward regions of low potential and negative charges toward regions of high potential. However, this contradicts our assertion that the charges are in equilibrium, and therefore the surface must be an equipotential.

If the electric field is zero in the interior of the conductor, then we may move a test charge along any path in the interior or from the surface to the interior and the net work done on the test charge by the surface charges will be zero. This means that the potential difference between any two points is zero, and thus the potential has the same value at all points in the conductor. We therefore obtain a third property of conductors: *the entire conductor is at the same potential.* This conclusion holds only in the electrostatic case; when we discuss currents flowing through conductors, a potential difference can exist between different points in the conductor.

Note that we have made no assumptions about the shape of the conductor. If the conductor is spherical, the charge is uniformly distributed over the surface. For conductors whose shape is nonspherical, the charge density is not uniform over the surface, but the surface is still an equipotential. Even in a conductor with internal cavities, whether or

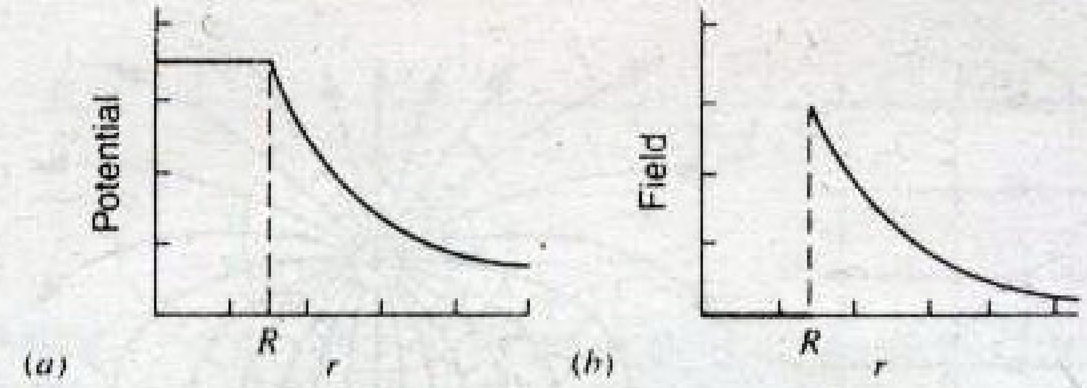


FIGURE 28-20. (*a*) The potential for a charged spherical conductor. (*b*) The electric field of the conductor.

not they contain charge, all points (surface and interior) are at the same potential.

Our conclusion about the surface of the conductor being an equipotential is consistent with the discussion in Section 28-8, in which we concluded that electric field lines are always perpendicular to equipotential surfaces. In Section 27-6 we used Gauss' law to determine that the electric field just outside the surface of the conductor is perpendicular to the surface, which must be true if the surface of the conductor is an equipotential.

We can derive explicit results for the case of a solid spherical conductor that carries a uniformly distributed total charge q on its surface. In Section 25-5, we discussed a property of a uniformly charged spherical shell: the force on an external charge is the same as if the shell were replaced with a point charge at its center. This property allows us to use the point-charge expressions for the electric potential (Eq. 28-18) and electric field (Eq. 26-6) at locations where $r > R$.

Inside the shell, the force on a point charge is zero, which means that the potential must have the same value everywhere in the conductor, including the surface. The value at the surface is found from Eq. 28-18 evaluated for $r = R$, and so the potential in the interior is

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{R}, \qquad r < R. \tag{28-40}$$

Figure 28-20 shows the field and potential for an isolated charged spherical conductor. The field is zero for $r < R$ and decreases like $1/r^2$ for $r > R$. The potential is constant for $r < R$ and falls off like $1/r$ for $r > R$.

Corona Discharge (Optional)

Although the surface charge is distributed uniformly on a spherical conductor, this will *not* be the case on conductors of arbitrary shape.* Near sharp points or edges, the surface charge density—and thus the electric field just outside the surface—can reach very high values.

To see qualitatively how this occurs, consider two conducting spheres of different radii connected by a fine wire

FIGURE 28-21. Two conducting spheres connected by a long fine wire.

(Fig. 28-21). Let the entire assembly be raised to some arbitrary potential V. The (equal) potentials of the two spheres, using Eq. 28-40, are

$$V = \frac{1}{4\pi\epsilon_0}\frac{q_1}{R_1} = \frac{1}{4\pi\epsilon_0}\frac{q_2}{R_2},$$

which yields

$$\frac{q_1}{q_2} = \frac{R_1}{R_2}. \tag{28-41}$$

We assume the spheres to be so far apart that the charge on one does not affect the distribution of charge on the other.

The ratio of the surface charge densities of the two spheres is

$$\frac{\sigma_1}{\sigma_2} = \frac{q_1/4\pi R_1^2}{q_2/4\pi R_2^2} = \frac{q_1 R_2^2}{q_2 R_1^2}.$$

Combining this result with Eq. 28-41 gives

$$\frac{\sigma_1}{\sigma_2} = \frac{R_2}{R_1}. \tag{28-42}$$

Equation 28-42 suggests that the smaller sphere has the larger surface charge density. Recalling that for an external charge, the field is the same as if we replaced the sphere with a point charge at its center, we can express the field just outside the surface of the sphere as

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} = \frac{\sigma}{\epsilon_0}. \tag{28-43}$$

According to Eq. 28-42, the surface charge density is larger for the sphere of smaller radius, and thus the field is also larger just outside the sphere of smaller radius. *The smaller the radius of the sphere, the larger is the electric field just outside its surface.*

Near a sharp conductor (that is, one of very small radius) the electric field may be large enough to ionize molecules in the surrounding air; as a result the normally nonconducting air can conduct and carry charge away from the conductor. Such an effect is called a *corona discharge.* Electrostatic paint sprayers use a corona discharge to transfer charge to droplets of paint, which are then accelerated by an electric field. Photocopy machines based on the xerography process use a wire to produce a corona discharge that transfers charge to a selenium-covered surface; the charge is neutralized on regions where light strikes the surface, and the remaining charged areas attract a fine black powder that forms the image. ■

* See "The Lightning-rod Fallacy," by Richard H. Price and Ronald J. Crowley, *American Journal of Physics*, September 1985, p. 843, for a careful discussion of this phenomenon.

28-10 THE ELECTROSTATIC ACCELERATOR (Optional)

Many studies of nuclei involve nuclear reactions, which occur when a beam of particles is incident on a target. One method that is used to accelerate particles for nuclear reactions is based on an electrostatic technique. A particle of positive charge q "falls" through a negative change in potential ΔV and therefore experiences a negative change in its potential energy, $\Delta U = q\,\Delta V$, according to Eq. 28-14. The corresponding increase in the kinetic energy of the particle is $\Delta K = -\Delta U$, and, assuming the particle starts from rest, its final kinetic energy is

$$K = -q\,\Delta V. \qquad (28\text{-}44)$$

For ionized atoms, q is normally positive. To obtain the highest energy possible for the beam, we would like to have the largest difference in potential. For applications of interest in nuclear physics, particles with kinetic energies of millions of electron-volts (MeV) are required to overcome the Coulomb force of repulsion between the incident and target particles. Kinetic energies of MeV require potential differences of millions of volts.

An electrostatic device that can produce such large potential differences is illustrated in Fig. 28-22. A small conducting sphere of radius a and carrying charge q is located inside a larger shell of radius b that carries charge Q. A conducting path is momentarily established between the two conductors, and the charge q then moves entirely to the outer conductor, no matter how much charge Q is already residing there (because the charge on a conductor always moves to its outer surface). If there is a convenient mechanism for replenishing the charge q on the inner sphere from an external supply, the charge Q on the outer sphere and its potential can, in principle, be increased without limit. In practice, the terminal potential is limited by sparking that occurs through air (Fig. 28-23).

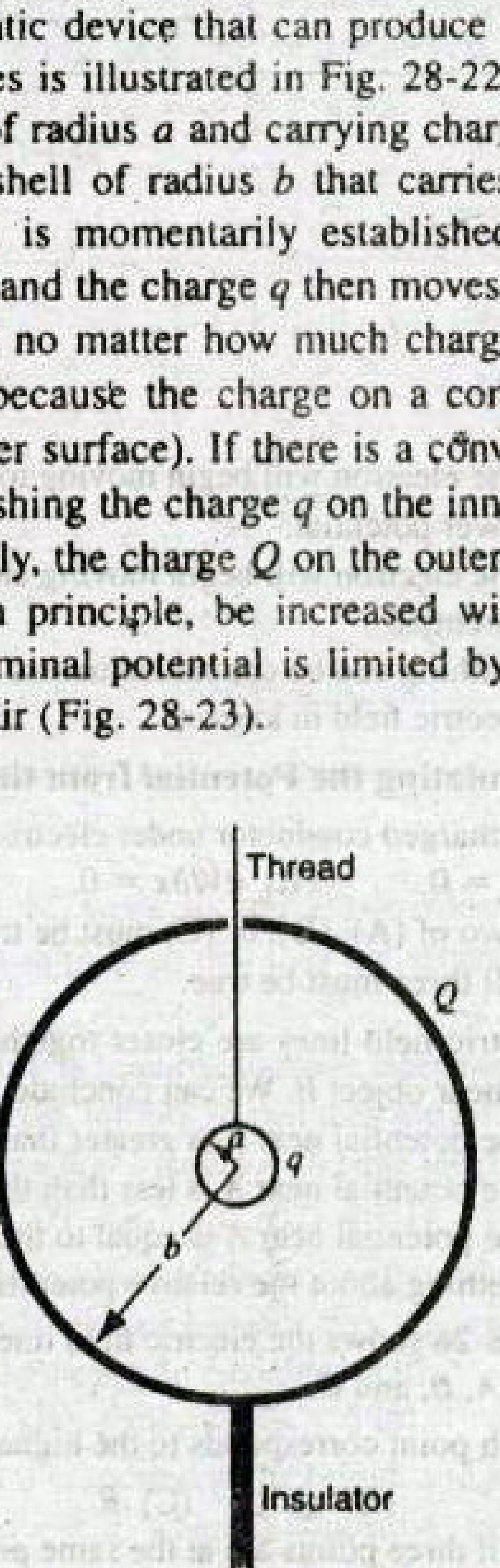
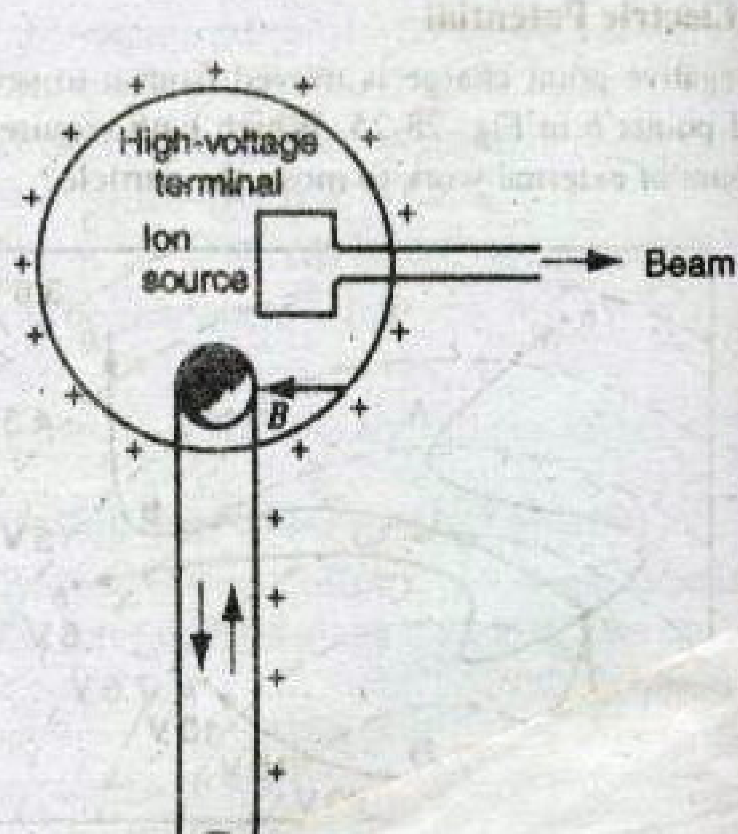


FIGURE 28-22. A small charged sphere is suspended i- larger charged spherical shell.

FIGURE 28-23. An electrostatic generator, with a potential 2.7 million volts, causing sparking due to conduction through ai

This well-known principle of electrostatics was first a plied to accelerating nuclear particles by Robert J. Van Graaff in the early 1930s, and the accelerator has becor known as a *Van de Graaff accelerator.* Potentials of sever million volts were easily achieved, the limiting potent coming from the leakage of charge through the insulati supports or breakdown of air (or the high-pressure insul ing gas) surrounding the high-voltage terminal.

Figure 28-24 shows the basic design of the Van Graaff accelerator. Charge is sprayed from a sharp



FIG

(called a corona point) at A onto a moving belt made of insulating material (often rubber). The belt carries the charge into the high-voltage terminal, where it is removed by another corona point B and travels to the outer conductor. Inside the terminal is a source of positive ions, for example, nuclei of hydrogen (protons) or helium (alpha particles). The ions "fall" from the high potential, gaining a kinetic energy of several MeV in the process. The terminal is enclosed in a tank that contains insulating gas to prevent sparking.

A clever variation of this basic design makes use of the same high voltage to accelerate ions twice, thereby gaining an additional increase in kinetic energy. A source of *negative* ions, made by adding an electron to a neutral atom, is located outside the terminal. These negative ions "fall toward" the positive potential of the terminal. Inside the high-voltage terminal, the beam passes through a chamber consisting of a gas or thin foil, which is designed to remove or strip several electrons from the negative ions, turning them into positive ions which then "fall from" the positive potential. Such "tandem" Van de Graaff accelerators currently use a terminal voltage of 25 million volts to accelerate ions such as carbon or oxygen to kinetic energies in excess of 100 MeV.

SAMPLE PROBLEM 28-14. Calculate the potential difference between the two spheres illustrated in Fig. 28-22.

Solution The potential difference $V(b) - V(a)$ has two contributions: one from the small sphere and one from the large spherical shell. These can be calculated independently and added algebraically. Let us first consider the large shell. Figure 28-20*a* shows that the potential at all interior points has the same value as the potential on the surface. Thus the contribution of the large shell to the difference $V(b) - V(a)$ is 0.

All that remains then is to evaluate the difference considering only the small sphere. For all points external to the small sphere, we can treat it as a point charge, and the potential difference can be found from Eq. 28-17:

$$V(b) - V(a) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right).$$

This expression gives the difference in potential between the inner sphere and the outer shell. Note that this is *independent of the charge Q on the outer shell.* If q is positive, the difference will always be negative, indicating that the outer shell will always be at a lower potential. If positive charge is permitted to flow between the spheres, it will always flow from higher to lower potential—that is, from the inner to the outer sphere—no matter how much charge already resides on the outer spherical shell.

MULTIPLE CHOICE

28-1 Potential Energy

28-2 Electric Potential Energy

28-3 Electric Potential

1. A negative point charge is moved from a to several possible final points b in Fig. 28-25. Which path requires the greatest amount of external work to move the particle?

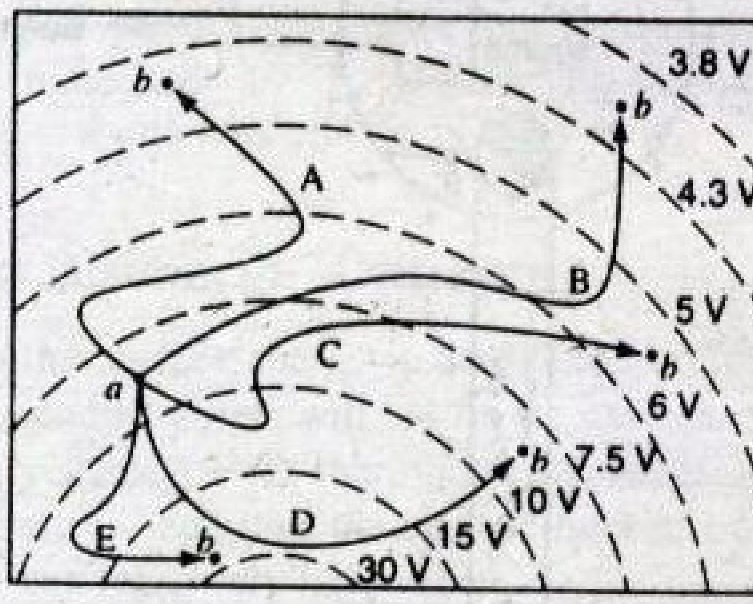


FIGURE 28-25. Multiple-choice question 1.

2. An electron is released from rest in a region of space with a nonzero electric field. Which of the following statements is true?
   (A) The electron will begin moving toward a region of higher potential.
   (B) The electron will begin moving toward a region of lower potential.
   (C) The electron will begin moving along a line of constant potential.
   (D) Nothing can be concluded unless the direction of the electric field in known.

28-4 Calculating the Potential from the Field

3. Inside a charged conductor under electrostatic conditions,
   (A) $V = 0$. (B) $\partial V/\partial x = 0$. (C) $\partial^2 V/\partial x^2 = 0$.
   (D) Two of (A), (B), or (C) must be true.
   (E) All three must be true.

4. The electric field lines are closer together near object A than they are near object B. We can conclude
   (A) the potential near A is greater than the potential near B.
   (B) the potential near A is less than the potential near B.
   (C) the potential near A is equal to the potential near B.
   (D) nothing about the relative potentials near A or B.

5. Figure 28-26 shows the electric field lines around three point charges, A, B, and C.
   (*a*) Which point corresponds to the highest potential?
   (A) P (B) Q (C) R
   (D) All three points are at the same potential.
   (*b*) Which point corresponds to the lowest potential?
   (A) P (B) Q (C) R
   (D) All three points are at the same potential.

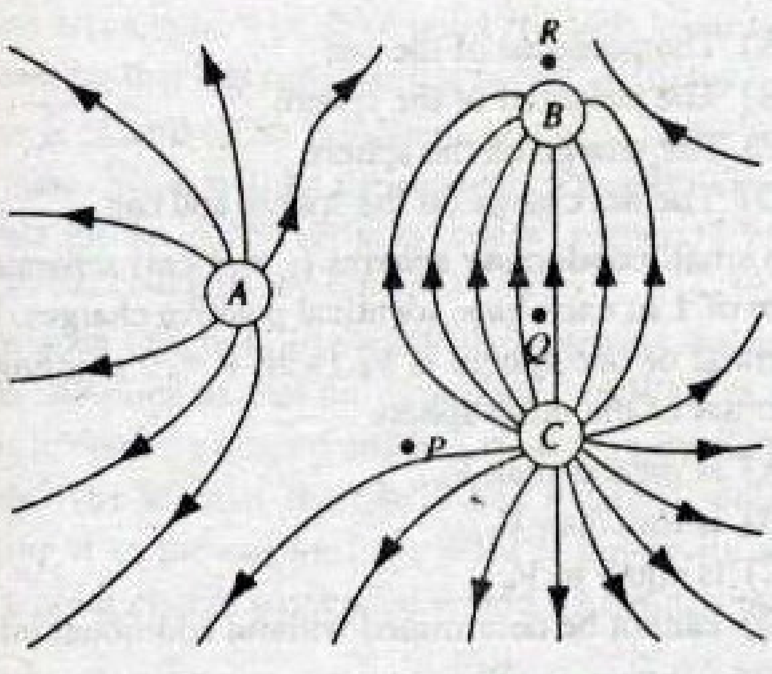


FIGURE 28-26. Multiple-choice question 5.

28-5 Potential Due to Point Charges

6. A single positive point charge q is located as shown in Fig. 28-27a, and the potential at point P is V_0 (with $V = 0$ at infinity).

(a) A second charge $q' = +q$ is placed equidistant from P as shown in Fig. 28-27b. The potential at P is now

(A) $4V_0$. (B) $2V_0$. (C) $\sqrt{2}V_0$.
(D) $V_0/2$. (E) 0.

(b) Instead of a positive charge, a negative charge $q' = -q$ is located as shown in Fig. 28-27b. The potential at P is now

(A) $4V_0$. (B) $2V_0$. (C) $\sqrt{2}V_0$.
(D) $V_0/2$. (E) 0.

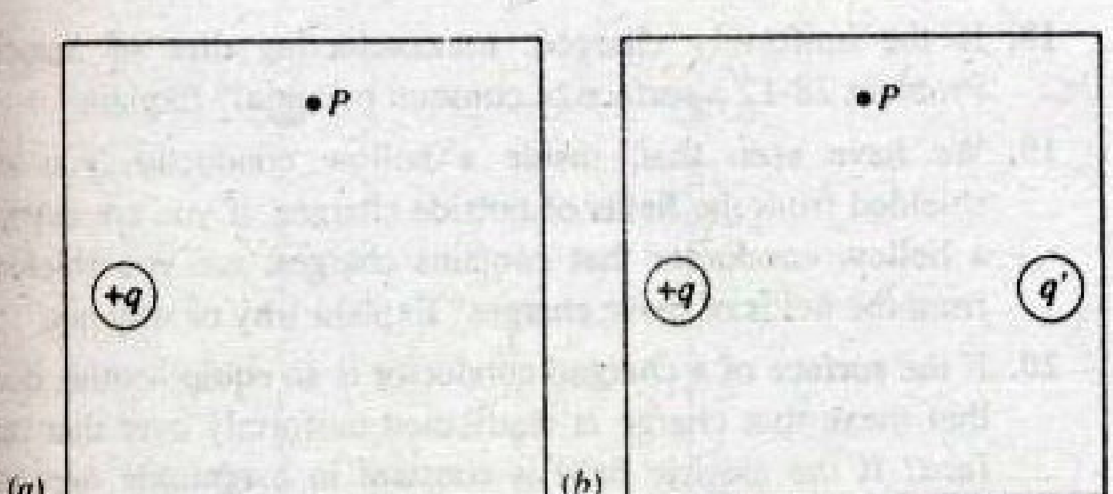


FIGURE 28-27. Multiple-choice question 6.

7. It requires 1 mJ of work to move two identical positive charges $+q$ from infinity so that they are separated by a distance a.

(a) How much work is required to move three identical positive charges $+q$ from infinity so that they are arranged at the vertices of an equilateral triangle with edge length a?

(A) 2 mJ (B) 3 mJ (C) 4 mJ (D) 9 mJ

(b) How much work is required to move four identical positive charges $+q$ from infinity so that they are arranged at the vertices of a tetrahedron with edge length a?

(A) 3 mJ (B) 4 mJ (C) 6 mJ (D) 16 mJ

8. A point charge $+q$ is located at the origin, and a point charge $+2q$ is located at $x = a$, where a is positive; here $V(\infty) = 0$.

(a) Which of the following statements is true?

(A) Close to the charges, the electric potential can be zero off the x axis.
(B) The magnitude of the electric potential will be a maximum on the x axis.
(C) The electric potential can be zero in the region between the charges.
(D) The electric potential can be zero only on the x axis.

(b) In which of the following regions on the x axis might there exist a point where the electric potential is zero?

(A) $-\infty < x < 0$ (B) $0 < x < a$
(C) $a < x < \infty$
(D) V does not vanish in the region $-\infty < x < \infty$.

9. A point charge $+q$ is located at the origin, and a point charge $-2q$ is located at $x = a$, where a is positive; here $V(\infty) = 0$.

(a) Which of the following statements is true?

(A) Close to the charges, the electric potential can be zero off the x axis.
(B) Close to the charges, the magnitude of the electric potential can be a maximum off the x axis.
(C) The electric potential can be zero only between the charges.
(D) The electric potential can be zero only on the x axis.

(b) In which of the following region or regions might there exist a point where the electric potential is zero?

(A) $-\infty < x < 0$ (B) $0 < x < a$
(C) $a < x < \infty$
(D) V does not vanish in the region $-\infty < x < \infty$.

28-6 Electric Potential of Continuous Charge Distributions

10. Consider the electric potential $V(z)$ on the axis of a uniform ring of positive charge; here $V(\infty) = 0$.

(a) $V(z)$ will have its largest value where

(A) $z = 0$. (B) $0 < |z| < \infty$.
(C) $|z| = \infty$. (D) (A) and (C) are correct.

(b) $|V(z)|$ can be zero where

(A) $z = 0$. (B) $0 < |z| < \infty$.
(C) $|z| = \infty$. (D) (A) and (C) are correct.

11. Consider the electric potential $V(z)$ on the axis of a uniform disk of positive charge; here $V(\infty) = 0$.

(a) $V(z)$ will have its largest value where

(A) $z = 0$. (B) $0 < |z| < \infty$.
(C) $|z| = \infty$. (D) (A) and (C) are correct.

(b) $V(z)$ can be zero where

(A) $z = 0$. (B) $0 < |z| < \infty$.
(C) $|z| = \infty$. (D) (A) and (C) are correct.

28-7 Calculating the Field from the Potential

12. A small positive charge located at the origin experiences an electrostatic force directed along the x axis. One can conclude that at the origin

(A) $V \neq 0$. (B) $\partial V/\partial x \neq 0$. (C) $\partial^2 V/\partial x^2 \neq 0$.
(D) Two of (A), (B), or (C) must be true.
(E) All three must be true.

13. An electric dipole parallel to the x axis and located at the origin experiences an electrostatic force directed along the x axis. One can conclude that at the origin

(A) $V \neq 0$. (B) $\partial V/\partial x \neq 0$. (C) $\partial^2 V/\partial x^2 \neq 0$.
(D) Two of (A), (B), or (C) must be true.
(E) All three must be true.

28-8 Equipotential Surfaces

14. Which of the following is always true for the electric flux Φ_E through a closed equipotential surface?
    (A) $\Phi_E = 0$ (B) $\Phi_E > 0$ (C) $\Phi_E < 0$
    (D) Φ_E is proportional to the net charge inside the surface.

28-9 The Potential of a Charged Conductor

15. A small conducting sphere originally has a charge $+q$. The sphere is lowered into a conducting can. (*a*) Which of the following quantities are fixed as the sphere is lowered, but before it touches the can? (There may be more than one correct answer.)
    (A) The potential of the can
    (B) The potential of the sphere
    (C) The charge on the sphere
    (D) The net charge on the sphere and can

    (*b*) The ball touches the can. Which of the following quantities are the same both before and after the ball touches the can? (There may be more than one correct answer.)
    (A) The potential of the can
    (B) The potential of the sphere
    (C) The charge on the sphere
    (D) The net charge on the sphere and can

16. Two small conducting spheres ($r = 1$ cm) separated by a di[s]tance of 1 m each have identical positive charges. The elec[tric] potential of one sphere is V_0 (with $V = 0$ at infinity). (*a*) T[he] potential of the other sphere
    (A) is greater than V_0.
    (B) is less than V_0.
    (C) is equal to V_0.
    (D) cannot be determined without additional informatio[n].

    (*b*) The spheres are brought closer together until they touch. The electric potential of the two spheres is now V, where
    (A) $V = V_0$. (B) $V_0 < V < 2V_0$.
    (C) $V = 2V_0$. (D) $2V_0 < V$.

28-10 The Electrostatic Accelerator

QUESTIONS

1. Are we free to call the potential of the Earth + 100 V instead of zero? What effect would such an assumption have on measured values of (*a*) potentials and (*b*) potential differences?
2. What would happen to you if you were on an insulated stand and your potential were increased by 10 kV with respect to the Earth?
3. Why is the electron-volt often a more convenient unit of energy than the joule?
4. How would a proton-volt compare with an electron-volt? The mass of a proton is 1840 times that of an electron.
5. Does the amount of work per unit charge required to transfer electric charge from one point to another in an electrostatic field depend on the amount of charge transferred?
6. Distinguish between potential difference and difference of potential energy. Give examples of statements in which each term is used properly.
7. Estimate the combined energy of all the electrons striking the screen of a cathode ray oscilloscope in 1 second.
8. Why is it possible to shield a room against electrical forces but not against gravitational forces?
9. Suppose the Earth has a net charge that is not zero. Why is it still possible to adopt the Earth as a standard reference point of potential and to assign the potential $V = 0$ to it?
10. Can there be a potential difference between two conductors that carry like charges of the same magnitude?
11. Give examples of situations in which the potential of a charged body has a sign opposite to that of its charge.
12. Can two different equipotential surfaces intersect?
13. An electrical worker was accidentally electrocuted and a newspaper account reported: "He accidentally touched a high-voltage cable and 20,000 V of electricity surged through his body." Criticize this statement.
14. Advice to mountaineers caught in lightning and thunderstorms is (*a*) get rapidly off peaks and ridges, and (*b*) put both feet together and crouch in the open, only the feet touching the ground. What is the basis of this good advice?
15. If $\vec{E}$ equals zero at a given point, must V equal zero for th[at] point? Give some examples to prove your answer.
16. If you know $\vec{E}$ only at a given point, can you calculate V a[t] that point? If not, what additional information do you need?
17. In Fig. 28-18, is the electric field E greater at the left or at the right of the figure?
18. Is the uniformly charged, nonconducting disk of Sample Problem 28-12 a surface of constant potential? Explain.
19. We have seen that, inside a hollow conductor, you are shielded from the fields of outside charges. If you are outside a hollow conductor that contains charges, are you shielded from the fields of these charges? Explain why or why not.
20. If the surface of a charged conductor is an equipotential, does that mean that charge is distributed uniformly over that surface? If the electric field is constant in magnitude over the surface of a charged conductor, does that mean that the charge is distributed uniformly?
21. In Section 28-9 we were reminded that charge delivered to the inside of an isolated conductor is transferred entirely to the outer surface of the conductor, no matter how much charge is already there. Can you keep this up forever? If not, what stops you?
22. Why can an isolated atom not have a permanent electric dipole moment?
23. Ions and electrons act like condensation centers; water droplets form around them in air. Explain why.
24. If V equals a constant throughout a given region of space, what can you say about $\vec{E}$ in that region?
25. In Chapter 14 we saw that the gravitational field strength is zero inside a spherical shell of matter. The electrical field strength is zero not only inside an isolated charged spherical conductor but inside an isolated conductor of any shape. Is the gravitational field strength inside, say, a cubical shell of matter zero? If not, in what respect is the analogy not complete?
26. How can you ensure that the electric potential in a given region of space will have a constant value?

27. Devise an arrangement of three point charges, separated by finite distances, that has zero electric potential energy.

28. A charge is placed on an insulated conductor in the form of a perfect cube. What will be the relative charge density at various points on the cube (surfaces, edges, corners)? What will happen to the charge if the cube is in air?

29. We have seen (Section 28-9) that the potential inside a conductor is the same as that on its surface. (*a*) What if the conductor is irregularly shaped and has an irregularly shaped cavity inside? (*b*) What if the cavity has a small "worm hole" connecting it to the outside? (*c*) What if the cavity is closed but has a point charge suspended within it? Discuss the potential within the conducting material and at different points within the cavities.

30. An isolated conducting spherical shell carries a negative charge. What will happen if a positively charged metal object is placed in contact with the shell interior? Discuss the three cases in which the positive charge is (*a*) less than, (*b*) equal to, and (*c*) greater than the negative charge in magnitude.

31. An uncharged metal sphere suspended by a silk thread is placed in a uniform external electric field. What is the magnitude of the electric field for points inside the sphere? Is your answer changed if the sphere carries a charge?

EXERCISES

Quarks i) up ii) down iii) strange iv) down v) Top vi) charm

28-1 Potential Energy

28-2 Electric Potential Energy

1. In the quark model of fundamental particles, a proton is composed of three quarks: two "up" quarks, each having charge $+\frac{2}{3}e$, and one "down" quark, having charge $-\frac{1}{3}e$. Suppose that the three quarks are equidistant from each other. Take the distance to be 1.32×10^{-15} m and calculate (*a*) the potential energy of the interaction between the two "up" quarks and (*b*) the total electric potential energy of the system.

2. Derive an expression for the work required by an external agent to put the four charges together as indicated in Fig. 28-28. Each side of the square has length *a*.

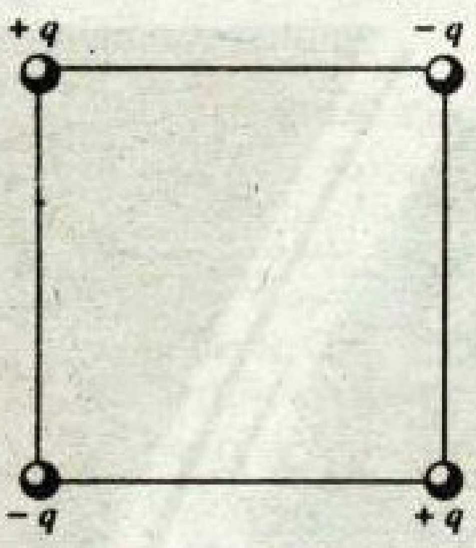


FIGURE 28-28. Exercise 2.

3. A decade before Einstein published his theory of relativity, J. J. Thomson proposed that the electron might be made up of small parts and that its mass is due to the electrical interaction of the parts. Furthermore, he suggested that the energy equals mc^2. Make a rough estimate of the electron mass in the following way: assume that the electron is composed of three identical parts that are brought in from infinity and placed at the vertices of an equilateral triangle having sides equal to the classical radius of the electron, 2.82×10^{-15} m. (*a*) Find the total electric potential energy of this arrangement. (*b*) Divide by c^2 and compare your result to the accepted electron mass (9.11×10^{-31} kg). The result improves if more parts are assumed; see Problem 2. Today, the electron is thought to be a single, indivisible particle.

4. The charges shown in Fig. 28-29 are fixed in space. Find the value of the distance *x* so that the electric potential energy of the system is zero.

0.146m

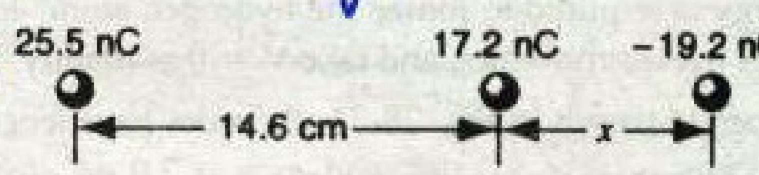


FIGURE 28-29. Exercise 4.

5. Figure 28-30 shows an idealized representation of a ^{238}U nucleus ($Z = 92$) on the verge of fission. Calculate (*a*) the repulsive force acting on each fragment and (*b*) the mutual electric potential energy of the two fragments. Assume that the fragments are equal in size and charge, spherical, and just touching. The radius of the initially spherical ^{238}U nucleus is 8.0 fm. Assume that the material out of which nuclei are made has a constant density.

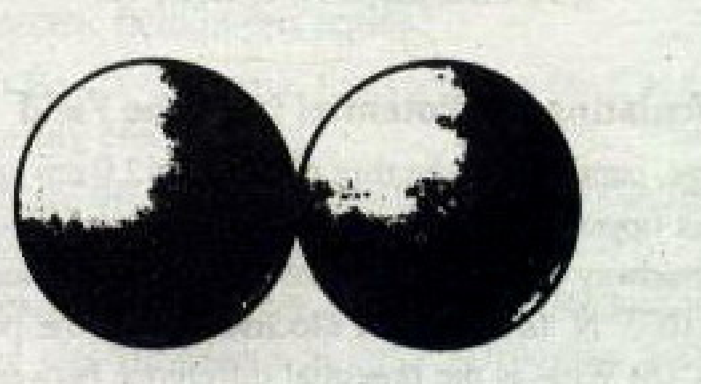


FIGURE 28-30. Exercise 5.

28-3 Electric Potential

6. Two parallel, flat, conducting surfaces of spacing $d = 1.0$ cm have a potential difference ΔV of 10.3 kV. An electron is projected from one plate directly toward the second. What is the initial velocity of the electron if it comes to rest just at the surface of the second plate? Ignore relativistic effects.

7. In a typical lightning flash, the potential difference between discharge points is about 1.0×10^9 V and the quantity of charge transferred is about 30 C. (*a*) How much energy is released? (*b*) If all the energy released could be used to accelerate a 1200-kg automobile from rest, what would be the final speed of the automobile? (*c*) If it could be used to melt ice, how much ice would it melt at 0°C?

8. The electric potential difference between discharge points during a particular thunderstorm is 1.23×10^9 V. What is the magnitude of the change in the electric potential energy of an electron that moves between these points? Give your answer in (*a*) joules and (*b*) electron-volts.

9. A particle of charge q is kept in a fixed position at a point P and a second particle of mass m, having the same charge q, is initially held at rest a distance r_1 from P. The second particle is then released and is repelled from the first one. Determine its speed at the instant it is a distance r_2 from P. Let $q = 3.1\ \mu C$, $m = 18$ mg, $r_1 = 0.90$ mm, and $r_2 = 2.5$ mm.

10. An electron is projected with an initial speed of 3.44×10^5 m/s directly toward a proton that is essentially at rest. If the electron is initially a great distance from the proton, at what distance from the proton is its speed instantaneously equal to twice its initial value?

11. Calculate (*a*) the electric potential established by the nucleus of a hydrogen atom at the average distance of the circulating electron ($r = 5.29 \times 10^{-11}$ m); (*b*) the electric potential energy of the atom when the electron is at this radius; and (*c*) the kinetic energy of the electron, assuming it to be moving in a circular orbit of this radius centered on the nucleus. (*d*) How much energy is required to ionize the hydrogen atom? Express all energies in electron-volts, and take $V = 0$ at infinity.

12. In the rectangle shown in Fig. 28-31, the sides have lengths 5.0 cm and 15 cm, $q_1 = -5.0\ \mu C$, and $q_2 = +2.0\ \mu C$. (*a*) What are the electric potentials at corner B and at corner A? (Assume $V = 0$ at infinity.) (*b*) How much external work is required to move a third charge $q_3 = +3.0\ \mu C$ from B to A along a diagonal of the rectangle? (*c*) In this process, is the external work converted into electrostatic potential energy or vice versa? Explain.

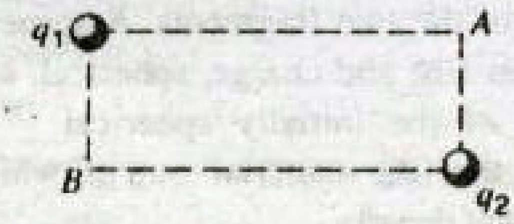


FIGURE 28-31. Exercise 12.

28-4 Calculating the Potential from the Field

13. Two large, parallel, conducting plates are 12.0 cm apart and carry equal but opposite charges on their facing surfaces. An electron placed midway between the two plates experiences a force of 3.90×10^{-15} N. (*a*) Find the electric field at the position of the electron. (*b*) What is the potential difference between the plates?

14. An infinite sheet of charge has a charge density $\sigma = 0.12\ \mu C/m^2$. How far apart are the equipotential surfaces whose potentials differ by 48 V?

15. A Geiger counter has a metal cylinder 2.10 cm in diameter along whose axis is stretched a wire 1.34×10^{-4} cm in diameter. If 855 V is applied between them, find the electric field at the surface of (*a*) the wire and (*b*) the cylinder. (Hint: Use the result of Problem 10, Chapter 27.)

16. In the Millikan oil-drop experiment (see Section 26-6), an electric field of 1.92×10^5 N/C is maintained at balance across two plates separated by 1.50 cm. Find the potential difference between the plates.

28-5 Potential Due to Point Charges

17. A gold nucleus contains a positive charge equal to that of 79 protons and has a radius of 7.0 fm; see Sample Problem 28-7. An alpha particle (which consists of two protons and two neutrons) has a kinetic energy K at points far from the nucleus and is traveling directly toward it. The alpha particle just touches the surface of the nucleus where its velocity is reversed in direction. (*a*) Calculate K. (*b*) The actual alpha particle energy used in the experiment of Rutherford and his collaborators that led to the discovery of the concept of the atomic nucleus was 5.0 MeV. What do you conclude?

18. Compute the escape speed for an electron from the surface of a uniformly charged sphere of radius 1.22 cm and total charge 1.76×10^{-15} C. Neglect gravitational forces.

19. A point charge has $q = +1.16\ \mu C$. Consider point A, which is 2.06 m distant, and point B, which is 1.17 m distant in a direction diametrically opposite, as in Fig. 28-32*a*. (*a*) Find the potential difference $V_A - V_B$. (*b*) Repeat if points A and B are located as in Fig. 28-32*b*.

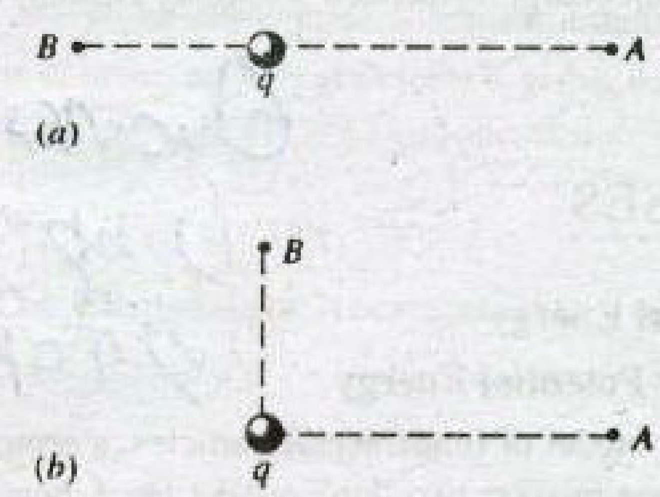


FIGURE 28-32. Exercise 19.

20. Much of the material comprising Saturn's rings (see Fig. 28-33) is in the form of tiny dust grains having radii on the order of 1.0 μm. These grains are in a region containing a dilute ionized gas, and they pick up excess electrons. If the electric potential at the surface of a grain is -400 V (relative to $V = 0$ at infinity), how many excess electrons has it picked up?

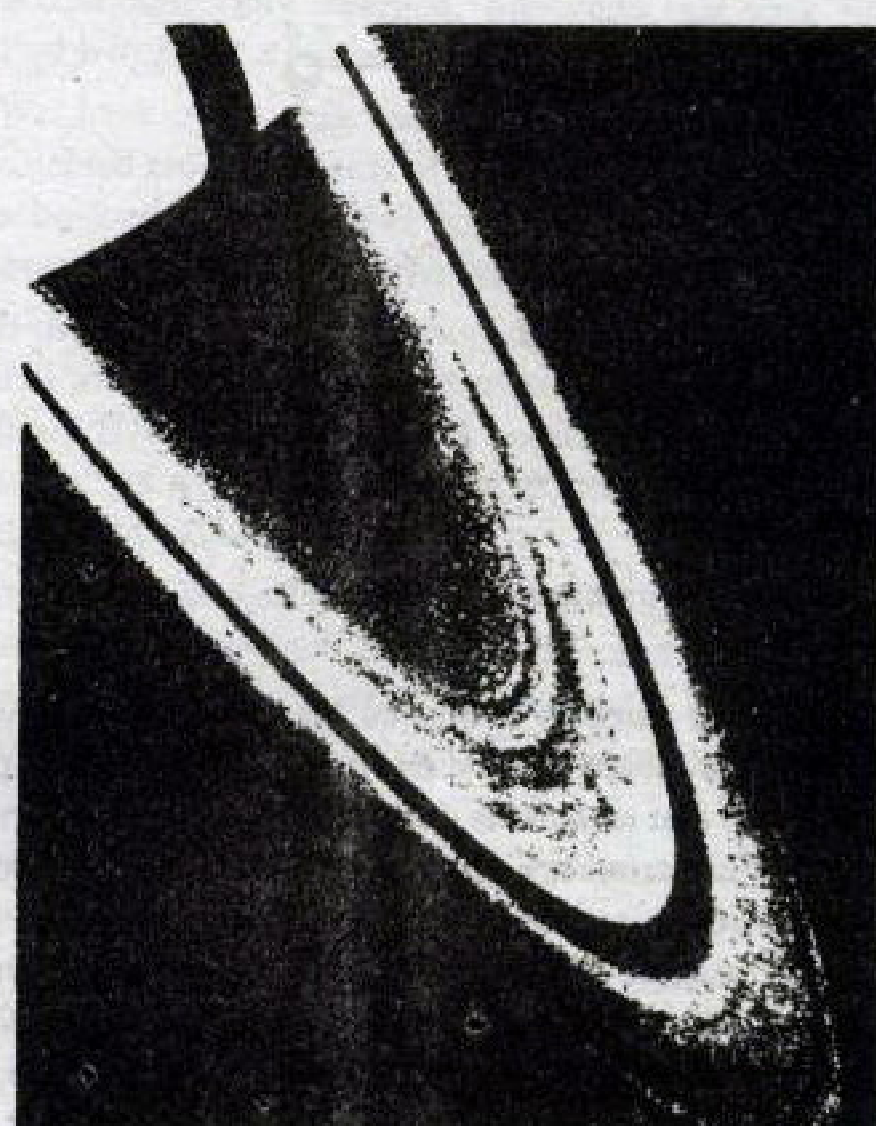


FIGURE 28-33. Exercise 20.

21. As a space shuttle moves through the dilute ionized gas of the Earth's ionosphere, its potential is typically changed by -1.0 V before it completes one revolution. By assuming that the shuttle is a sphere of radius 10 m, estimate the amount of charge it collects.

22. Suppose that the negative charge in a copper one-cent coin were removed to a very large distance from the Earth—perhaps to a distant galaxy—and that the positive charge were distributed uniformly over the Earth's surface. By how much would the electric potential at the surface of the Earth change? (See Sample Problem 25-1.)

23. An electric field of approximately 100 V/m is often observed near the surface of the Earth. If this field were the same over the entire surface, what would be the electric potential of a point on the surface? Assume that $V = 0$ at infinity.

24. The ammonia molecule NH_3 has a permanent electric dipole moment equal to 1.47 D, where D is the debye unit with a value of 3.34×10^{-30} C·m. Calculate the electric potential due to an ammonia molecule at a point 52.0 nm away along the axis of the dipole. Assume $V = 0$ at infinity.

25. (*a*) For Fig. 28-34, derive an expression for $V_A - V_B$. (*b*) Does your result reduce to the expected answer when $d = 0$? When $a = 0$? When $q = 0$?

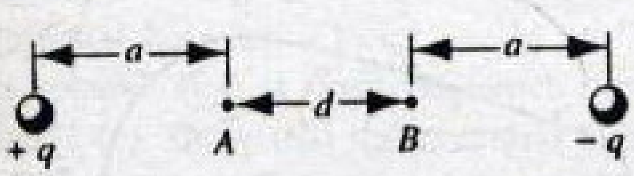


FIGURE 28-34. Exercise 25.

26. In Fig. 28-35, locate the points, if any, (*a*) where $V = 0$ and (*b*) where $E = 0$. Consider only points on the axis and take $V = 0$ at infinity.

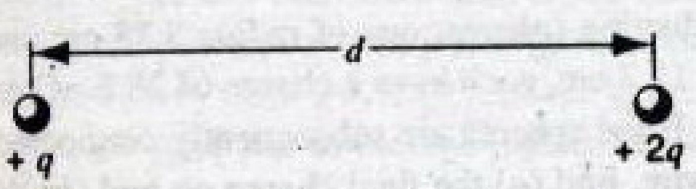


FIGURE 28-35. Exercise 26.

27. Two charges $q = +2.13\ \mu C$ are fixed in space a distance $d = 1.96$ cm apart, as shown in Fig. 28-36. (*a*) What is the electric potential at point *C*? Take $V = 0$ at infinity. (*b*) You bring a third charge $Q = +1.91\ \mu C$ slowly from infinity to *C*. How much work must you do? (*c*) What is the potential energy *U* of the configuration when the third charge is in place?

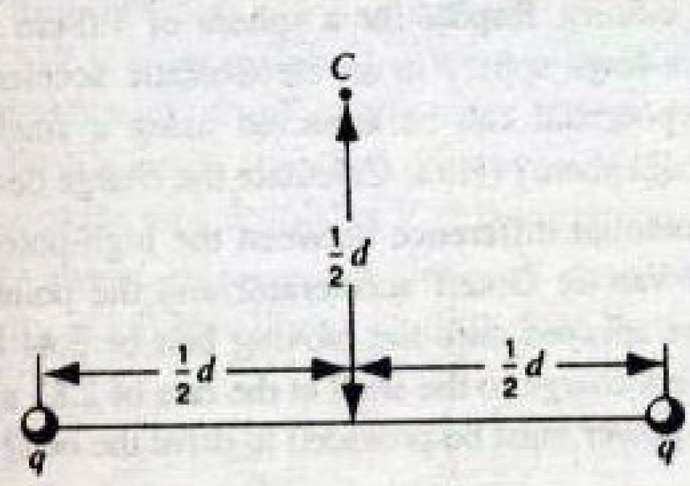


FIGURE 28-36. Exercise 27.

28-6 Electric Potential of Continuous Charge Distributions

28. At what distance along the axis of a uniformly charged disk of radius *R* is the electric potential equal to one-half the value of the potential at the surface of the disk at the center?

29. An electric charge of −9.12 nC is uniformly distributed around a ring of radius 1.48 m that lies in the *yz* plane with its center at the origin. A particle carrying a charge of −5.93 pC is located on the *x* axis at $x = 3.07$ m. Calculate the work done by an external agent in moving the point charge to the origin.

28-7 Calculating the Field from the Potential

30. Suppose that the electric potential varies along the *x* axis as shown in the graph of Fig. 28-37. Of the intervals shown (ignore the behavior at the endpoints of the intervals), determine the intervals in which E_x has (*a*) its greatest absolute value and (*b*) its least. (*c*) Plot E_x versus *x*.

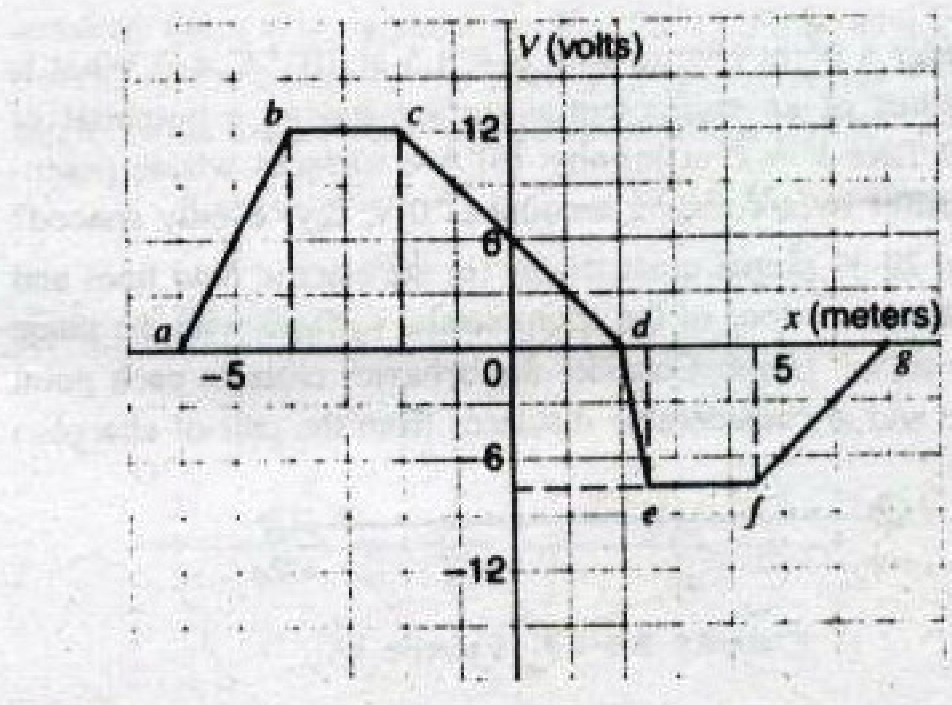


FIGURE 28-37. Exercise 30.

31. Two large, parallel, metal plates are 1.48 cm apart and carry equal but opposite charges on their facing surfaces. The negative plate is grounded and its potential is taken to be zero. If the potential halfway between the plates is +5.52 V, what is the electric field in this region?

32. From Eq. 28-30 derive an expression for *E* at axial points of a uniformly charged ring.

33. Calculate the radial potential gradient, $\partial V/\partial r$, at the surface of a gold nucleus. See Sample Problem 28-7.

34. Exercise 39 of Chapter 26 deals with Rutherford's calculation of the electric field a distance *r* from the center of an atom. He also gave the electric potential as

$$V = \frac{Ze}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{3}{2R} + \frac{r^2}{2R^3}\right).$$

(*a*) Show how the expression for the electric field given in Exercise 39 of Chapter 26 follows from this expression for *V*. (*b*) Why does this expression for *V* not go to zero as $r \to \infty$?

35. The electric potential *V* in the space between the plates of a particular, and now obsolete, vacuum tube is given by $V = (1530\ \text{V/m}^2)x^2$, where *x* is the distance from one of the plates. Calculate the magnitude and direction of the electric field at $x = 1.28$ cm.

28-8 Equipotential Surfaces

36. Two line charges are parallel to the *z* axis. One, of charge per unit length $+\lambda$, is a distance *a* to the right of this axis. The other, of charge per unit length $-\lambda$, is a distance *a* to the left of this axis (the lines and the *z* axis being in the same plane). Sketch some of the equipotential surfaces.

37. In moving from *A* to *B* along an electric field line, the electric field does 3.94×10^{-19} J of work on an electron in the field

illustrated in Fig. 28-38. What are the differences in the electric potential (a) $V_B - V_A$, (b) $V_C - V_A$, and (c) $V_C - V_B$?

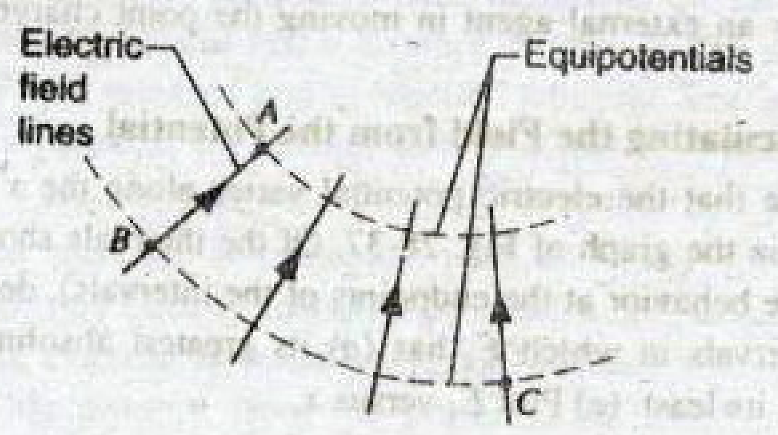


FIGURE 28-38. Exercise 37.

38. Consider a point charge with $q = 1.5 \times 10^{-8}$ C. (a) What is the radius of an equipotential surface having a potential of 30 V? Take $V = 0$ at infinity. (b) Are surfaces whose potentials differ by a constant amount (1.0 V, say) evenly spaced?

39. In Fig. 28-39 sketch qualitatively (a) the electric field lines and (b) the intersections of the equipotential surfaces with the plane of the figure. (Hint: Consider the behavior close to each point charge and at considerable distances from the pair of charges.)

FIGURE 28-39. Exercise 39.

40. Three long parallel lines of charge have the relative linear charge densities shown in Fig. 28-40. Sketch some electric field lines and the intersections of some equipotential surfaces with the plane of this figure.

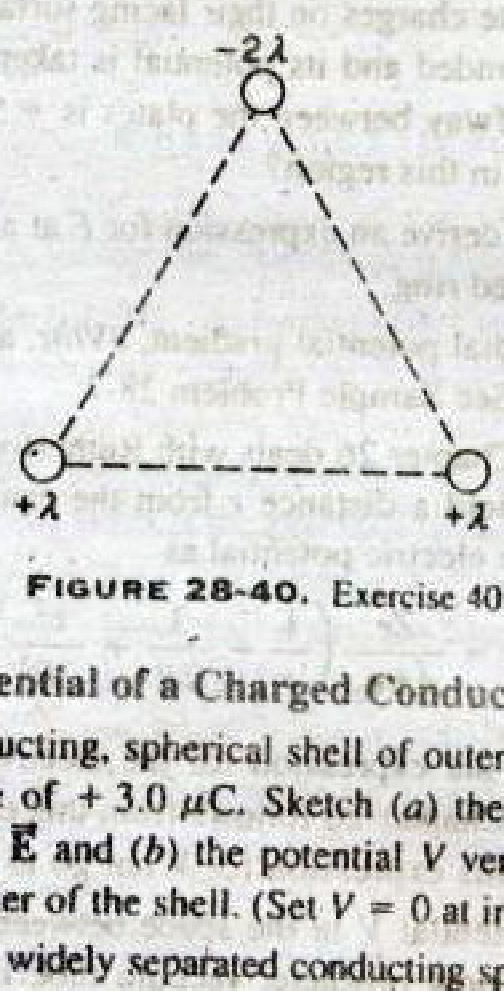


FIGURE 28-40. Exercise 40.

28-9 The Potential of a Charged Conductor

41. A thin, conducting, spherical shell of outer radius 20 cm carries a charge of $+3.0\ \mu$C. Sketch (a) the magnitude of the electric field $\vec{E}$ and (b) the potential V versus the distance r from the center of the shell. (Set $V = 0$ at infinity.)

42. Consider two widely separated conducting spheres, 1 and 2, the second having twice the diameter of the first. The smaller sphere initially has a positive charge q and the larger one is initially uncharged. You now connect the spheres with a long thin wire. (a) How are the final potentials V_1 and V_2 of the spheres related? (b) Find the final charges q_1 and q_2 on the spheres in terms of q.

43. (a) If the Earth had a net charge equivalent to 1 electron/m^2 of surface area (a very artificial assumption), what would be the Earth's potential? (Set $V = 0$ at infinity.) (b) What would be the electric field due to the Earth just outside its surface?

44. A charge of 15 nC can be produced by simple rubbing. To what potential (relative to $V = 0$ at infinity) would such a charge raise an isolated conducting sphere of 16 cm radius?

45. Find (a) the charge and (b) the charge density on the surface of a conducting sphere of radius 15.2 cm whose potential is 215 V. Take $V = 0$ at infinity.

46. The metal object in Fig. 28-41 is a figure of revolution about the horizontal axis. If it is charged negatively, sketch roughly a few equipotentials and electric field lines. Use physical reasoning rather than mathematical analysis.

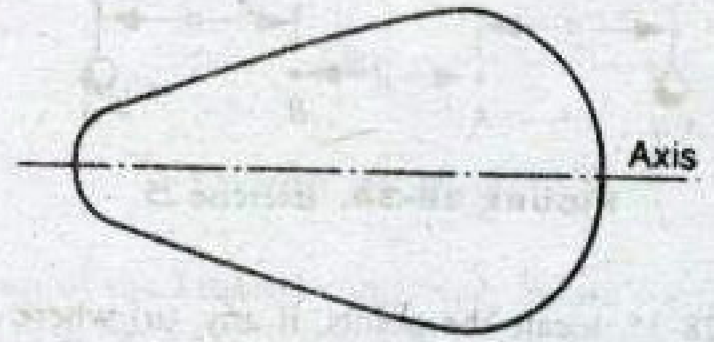


FIGURE 28-41. Exercise 46.

47. Two conducting spheres, one of radius 5.88 cm and the other of radius 12.2 cm, each have a charge of 28.6 nC and are very far apart. If the spheres are subsequently connected by a conducting wire, find (a) the final charge on and (b) the potential of each sphere, assuming $V = 0$ at infinity.

48. A charged metal sphere of radius 16.2 cm has a net charge of 31.5 nC. (a) Find the electric potential at the sphere's surface, if $V = 0$ at infinity. (b) At what distance from the sphere's surface has the electric potential decreased by 550 V?

28-10 The Electrostatic Accelerator

49. (a) How much charge is required to raise an isolated metallic sphere of 1.0-m radius to a potential of 1.0 MV? Assume $V = 0$ at infinity. Repeat for a sphere of 1.0-cm radius. (b) Why use a large sphere in an electrostatic accelerator when the same potential can be achieved using a smaller charge with a small sphere? (Hint: Calculate the charge densities.)

50. Let the potential difference between the high-potential inner shell of a Van de Graaff accelerator and the point at which charges are sprayed onto the moving belt be 3.41 MV. If the belt transfers charge to the shell at the rate of 2.83 mC/s, what minimum power must be provided to drive the belt?

PROBLEMS

1. (a) Through what potential difference must an electron fall, according to Newtonian mechanics, to acquire a speed v equal to the speed c of light? (b) Newtonian mechanics fails as $v \to c$. Therefore, using the correct relativistic expression for the kinetic energy (see Eq. 20-27)

$$K = mc^2\left[\frac{1}{\sqrt{1-(v/c)^2}} - 1\right]$$

in place of the Newtonian expression $K = \frac{1}{2}mv^2$, determine the actual electron speed acquired in falling through the potential difference computed in (*a*). Express this speed as an appropriate fraction of the speed of light.

2. Repeat Exercise 3, assuming that the electron is a hollow shell of radius 2.82×10^{-15} m with charge e evenly distributed over the surface.

3. A particle of (positive) charge Q is assumed to have a fixed position at P. A second particle of mass m and (negative) charge $-q$ moves at constant speed in a circle of radius r_1 centered at P. Derive an expression for the work W that must be done by an external agent on the second particle in order to increase the radius of the circle of motion, centered at P, to r_2.

4. The electric field inside a nonconducting sphere of radius R, containing uniform charge density, is radially directed and has magnitude

$$E = \frac{qr}{4\pi\epsilon_0 R^3},$$

where q is the total charge in the sphere and r is the distance from the center of the sphere. (*a*) Find the potential V inside the sphere, taking $V = 0$ at $r = 0$. (*b*) What is the difference in electric potential between a point on the surface and the center of the sphere? If q is positive, which point is at the higher potential? (*c*) Show that the potential at a distance r from the center, where $r < R$, is given by

$$V = \frac{q(3R^2 - r^2)}{8\pi\epsilon_0 R^3},$$

where the zero of potential is taken at $r = \infty$. Why does this result differ from that of part (*a*)?

5. Three charges of +122 mC each are placed on the corners of an equilateral triangle, 1.72 m on a side. If energy is supplied at the rate of 831 W, how many days would be required to move one of the charges onto the midpoint of the line joining the other two?

6. A particle of mass m, charge $q > 0$, and initial kinetic energy K is projected (from an infinite separation) toward a heavy nucleus of charge Q, assumed to have a fixed position in our reference frame. (*a*) If the aim is "perfect," how close to the center of the nucleus is the particle when it comes instantaneously to rest? (*b*) With a particular imperfect aim, the particle's closest approach to the nucleus is twice the distance determined in part (*a*). Determine the speed of the particle at this closest distance of approach. Assume that the particle does not reach the surface of the nucleus.

7. A spherical drop of water carrying a charge of 32.0 pC has a potential of 512 V at its surface. (*a*) What is the radius of the drop? (*b*) If two such drops of the same charge and radius combine to form a single spherical drop, what is the potential at the surface of the new drop? Set $V = 0$ at infinity.

8. Figure 28-42 shows, edge-on, an "infinite" sheet of positive charge density σ. (*a*) How much work is done by the electric field of the sheet as a small positive test charge q_0 is moved from an initial position on the sheet to a final position located a perpendicular distance z from the sheet? (*b*) Use the result from (*a*) to show that the electric potential of an infinite sheet of charge can be written

$$V = V_0 - (\sigma/2\epsilon_0)z,$$

where V_0 is the potential at the surface of the sheet.

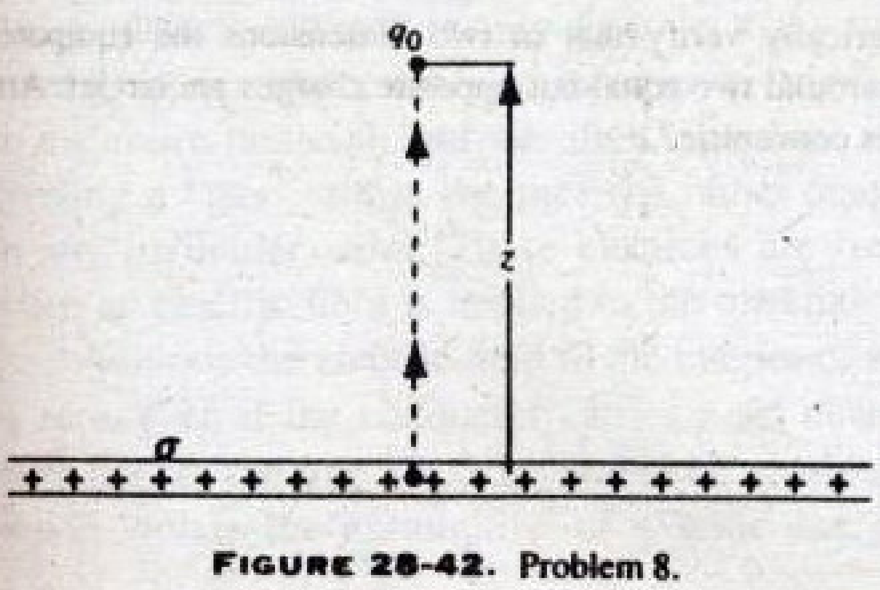


FIGURE 28-42. Problem 8.

9. A point charge $q_1 = +6e$ is fixed at the origin of a rectangular coordinate system, and a second point charge $q_2 = -10e$ is fixed at $x = 9.60$ nm, $y = 0$. With $V = 0$ at infinity, the locus of all points in the xy plane with $V = 0$ is a circle centered on the x axis, as shown in Fig. 28-43. Find (*a*) the location x_c of the center of the circle and (*b*) the radius R of the circle. (*c*) Is the $V = 5$ V equipotential also a circle?

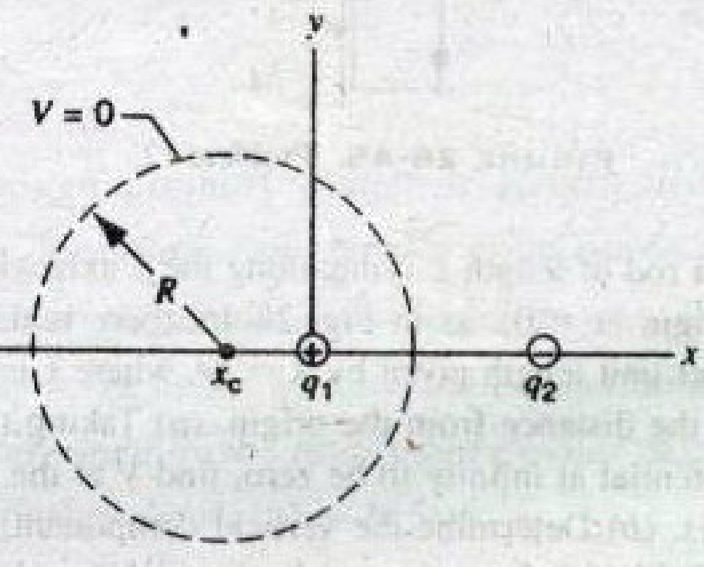


FIGURE 28-43. Problem 9.

10. A total amount of positive charge Q is spread onto a nonconducting, flat, circular annulus of inner radius a and outer radius b. The charge is distributed so that the charge density (charge per unit area) is given by $\sigma = k/r^3$, where r is the distance from the center of the annulus to any point on it. Show that (with $V = 0$ at infinity) the potential at the center of the annulus is given by

$$V = \frac{Q}{8\pi\epsilon_0}\left(\frac{a+b}{ab}\right).$$

11. For the charge configuration of Fig. 28-44, show that $V(r)$ for points on the vertical axis, assuming $r \gg d$, is given by

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}\left(1 + \frac{2d}{r}\right).$$

(Hint: The charge configuration can be viewed as the sum of an isolated charge and a dipole.) Set $V = 0$ at infinity.

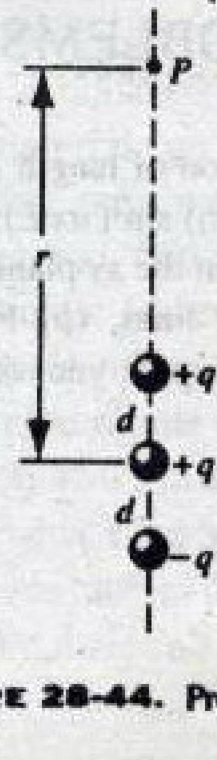


FIGURE 28-44. Problem 11.

12. A charge per unit length λ is distributed uniformly along a thin rod of length L. (*a*) Determine the potential (chosen to be zero at infinity) at a point P a distance y from one end of the rod and in line with it (see Fig. 28-45). (*b*) Use the result of (*a*) to compute the component of the electric field at P in the y direction (along the rod). (*c*) Determine the component of the electric field at P in a direction perpendicular to the rod.

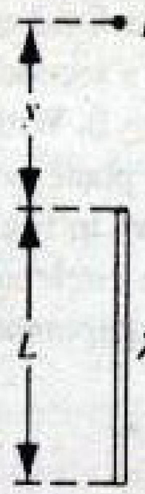


FIGURE 28-45. Problem 12.

13. On a thin rod of length L lying along the x axis with one end at the origin ($x = 0$), as in Fig. 28-46, there is distributed a charge per unit length given by $\lambda = kr$, where k is a constant and r is the distance from the origin. (*a*) Taking the electrostatic potential at infinity to be zero, find V at the point P on the y axis. (*b*) Determine the vertical component, E_y, of the electric field at P from the result of part (*a*) and also by direct calculation. (*c*) Why cannot E_x, the horizontal component of the electric field at P, be found using the result of part (*a*)? (*d*) At what distance from the rod along the y axis is the potential equal to one-half the value at the left end of the rod?

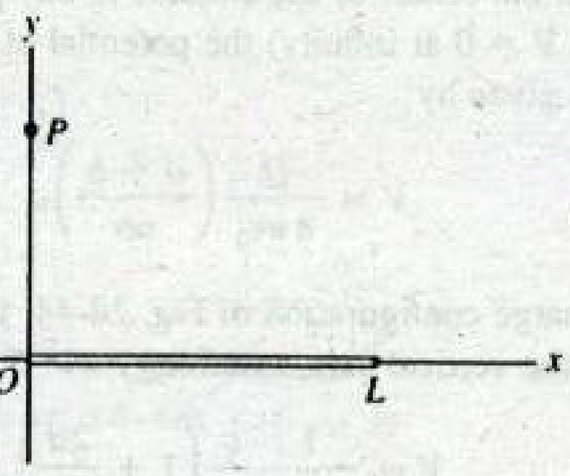


FIGURE 28-46. Problem 13.

14. Two identical conducting spheres of radius 15.0 cm are separated by a distance of 10.0 m. What is the charge on each sphere if the potential of one is +1500 V and the other is −1500 V? What assumptions have you made? Take $V = 0$ at infinity.

15. Consider the Earth to be a spherical conductor of radius 6370 km and to be initially uncharged. A metal sphere, having a radius of 13 cm and carrying a charge of −6.2 nC is earthed—that is, put into electrical contact with the Earth. Show that this process effectively discharges the sphere, by calculating the fraction of the excess electrons originally present on the sphere that remain after the sphere is earthed.

16. A copper sphere whose radius is 1.08 cm has a very thin surface coating of nickel. Some of the nickel atoms are radioactive, each atom emitting an electron as it decays. Half of these electrons enter the copper sphere, each depositing 100 keV of energy there. The other half of the electrons escape, each carrying away a charge of $-e$. The nickel coating has an activity of 10.0 mCi (= 10.0 millicuries = 3.70×10^8 radioactive decays per second). The sphere is hung from a long, nonconducting string and insulated from its surroundings. How long will it take for the potential of the sphere to increase by 1000 V?

17. Consider a thin, isolated, conducting, spherical shell that is uniformly charged to a constant charge density σ. How much work does it take to move a small positive test charge q_0 (*a*) from the surface of the shell to the interior, through a small hole; (*b*) from one point on the surface to another, regardless of path; (*c*) from point to point inside the shell; and (*d*) from any point P outside the shell over any path, whether or not it pierces the shell, back to P? (*e*) For the conditions given, does it matter whether or not the shell is conducting?

18. The high-voltage electrode of an electrostatic accelerator is a charged spherical metal shell having a potential $V = +9.15$ MV (relative to $V = 0$ at infinity). (*a*) Electrical breakdown occurs in the gas in this machine at a field $E = 100$ MV/m. To prevent such breakdown, what restriction must be made on the radius r of the shell? (*b*) A long, moving, rubber belt transfers charge to the shell at 320 μC/S, the potential of the shell remaining constant because of leakage. What minimum power is required to transfer the charge? (*c*) The belt is of width $w = 48.5$ cm and travels at speed $v = 33.0$ m/s. What is the surface charge density on the belt?

COMPUTER PROBLEMS

1. The charge density on a rod of length L centered on the x axis is given by $\lambda = (1.0\ \mu\text{C/m}) \sin^2(\pi x/L)$. (*a*) Numerically generate a plot of the potential in the xy plane, and then use your plot to generate equipotential lines. (*b*) From the plot, generate electric field lines and compare your result to Computer Problem 2 in Chapter 26.

2. Numerically verify that in two dimensions the equipotential lines around two equal but opposite charges are circles. Are the circles concentric?

CHAPTER 29

THE ELECTRICAL PROPERTIES OF MATERIALS

Although ordinary matter is electrically neutral, containing equal numbers of positive and negative charges, materials can reveal a great range of different behaviors when they are placed in electric fields. Some materials can conduct electricity even in very small fields, whereas others remain nonconducting in enormously large fields. In some materials that do not permit the movement of charge, the electrical properties are determined by the rotation of dipoles in an applied field, yet in others the applied field can create dipoles where they did not exist before.

In this chapter we shall consider the basic behavior of two kinds of materials: conductors and insulators. We will show how we can understand their behavior in applied fields based on simple models of forces and movement of charges. Even though a detailed understanding of the electrical properties of materials requires the methods of quantum mechanics, we can learn a great deal about materials from classical models that ignore the quantum behavior.

29-1 TYPES OF MATERIALS

Natural and artificially made materials show a wide range of electrical properties. These properties are determined partly by the behavior of individual atoms or molecules and partly by the interactions of atoms or molecules in the bulk material. The ability of a material to conduct electricity may also depend on the conditions of the material, such as its temperature and pressure.

Conductors (for example, most metals) are materials in which electric charges readily flow. In many metals, each atom gives up one or more of its outer or valence electrons to the entire material, and we often regard the electrons as forming a "gas" within the material rather than belonging to any particular atom. These electrons are free to move when an electric field is applied to the material. Under static conditions the electric field in the interior of a conductor is zero, even if the conductor carries a net charge. (If this were not so, the free electrons would be accelerated, which would violate the assumption of a static charge distribution.) In Section 29-2, we will discuss the effect of an external electric field applied to a conductor under static conditions.

In an *insulator,* on the other hand, the electrons are bound rather tightly to the atoms and are not free to move under the electric fields that might be applied under ordinary circumstances. An insulator can carry any distribution of electric charges on its surface or in its interior, and (in contrast to a conductor) the electric field in the interior of an insulator can have nonzero values.

An insulating material can often be regarded as a collection of molecules that are not easily ionized. In this case the electrical properties may depend on the electric dipole moment of the molecules. Materials in which the molecules have permanent dipole moments are called *polar,* and electric fields can align the dipole moments of molecules, as we discussed in Section 26-7. In some materials, the alignment of the dipoles remains even when we remove the applied field; these materials are called *ferroelectric* (in analogy with ferromagnetic materials, in which *magnetic* dipole moments remain aligned even when an external *magnetic* field is removed). Even nonpolar materials can show these effects, because the applied electric field can induce a dipole moment in the molecules. These effects are discussed in Section 29-5.

Ordinary matter is usually electrically neutral. In the absence of an external electric field, this neutrality applies to individual atoms as well as to the entire material. The application of an electric field can remove one or more electrons from atoms of the material. This process is called *ionization*, and the resulting positively charged atoms with a deficiency of electrons are called *ions*. In an insulator, a sufficiently large electric field can ionize the atoms, and as a result there are electrons available to move through the material. Under these circumstances an insulator can behave more like a conductor. This situation is called *breakdown* and requires fields typically in the range of 10^6 V/m in air to 10^7 V/m in plastics and ceramics.

Intermediate between insulators and conductors, are *semiconductors*. In a semiconductor, perhaps one atom in 10^{10} to 10^{12} might contribute an electron to the flow of electricity in the material (in contrast to a conductor, in which *every* atom typically contributes an electron to the flow of electricity). Commonly used semiconductors include silicon and germanium, as well as many compounds.

Even the best conductors (copper, silver, and gold) show a small but definitely nonzero resistance to the flow of electricity. Under certain conditions, often involving cooling to very low temperatures, electric charge can flow through some materials with no resistance at all. This property of materials is called *superconductivity*, and the materials under these conditions are called *superconductors*. Some materials can be relatively poor conductors at room temperature but can be superconductors at low temperatures.

In this chapter we study ways in which conductors and insulators respond to applied electric fields. Understanding the behavior of semiconductors and superconductors requires the methods of quantum mechanics, which are discussed in Chapter 49.

29-2 A CONDUCTOR IN AN ELECTRIC FIELD: STATIC CONDITIONS

Suppose we place a large rectangular slab of a conductor such as copper in a uniform electric field, as shown in Fig. 29-1*a*. We can regard the copper as a "gas" of electrons that are free to move in a lattice of copper ions in fixed locations. The electric field $\vec{E}_0$ exerts a force $\vec{F} = -e\vec{E}_0$ on the electrons, which causes the electrons to move in a direction opposite to the field. The electrons quickly move to the top surface of the copper, leaving a deficiency of electrons (a positive charge) on the bottom surface. When we place a conductor in an external field, the charges redistribute themselves almost instantaneously, after which electrostatic conditions apply.

The two surfaces of the conductor can be considered as sheets of charge, which set up an electric field $\vec{E}'$ as shown in Fig. 29-1*b*. Inside the copper, the net electric field $\vec{E}$ is the vector sum of the two fields: $\vec{E} = \vec{E}_0 + \vec{E}'$. In terms of magnitudes, the sum becomes a difference, because the two fields are in opposite directions: $E = E_0 - E'$. In the interior of the copper under static conditions, the net electric field E must be zero, as we discussed in Section 27-6. (In Section 27-6, we did not consider the presence of an externally applied electric field; however, the conclusion remains the same—the electric field inside the conductor must be zero, because otherwise the free electrons in the conductor would be accelerated, thus violating our assumption of a static situation.) The applied electric field E_0 must move just enough electrons to the surface to set up an electric field E' that has the same magnitude as E_0, giving a net field of zero inside the copper (Fig. 29-1*c*). Outside the slab, the sheets of charge on the two surfaces give electric fields that cancel, leaving the net field unchanged in those regions.

Figure 29-2 shows an uncharged conductor of irregular shape in an originally uniform electric field. Once again, free electrons in the conductor quickly move to the surface,

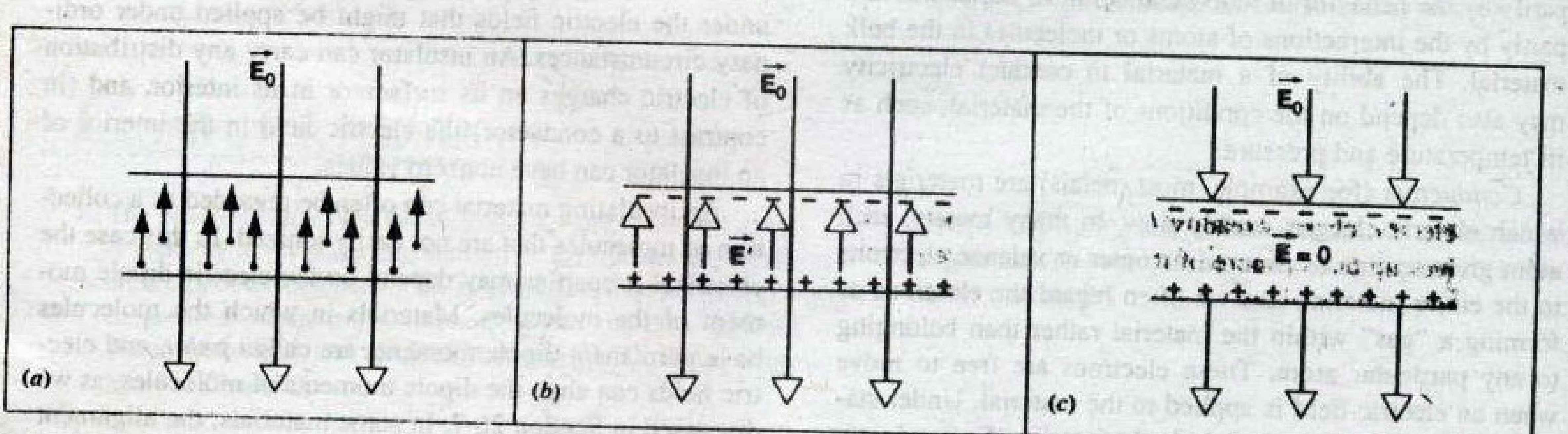


FIGURE 29-1. (*a*) A large slab of conductor is placed in a uniform electric field. Electrons in the material move upward in response to the field. (*b*) Electrons accumulate on the top surface, leaving positive ions on the bottom. These charges set up a field $\vec{E}'$. (*c*) Inside the slab, the net field is zero.

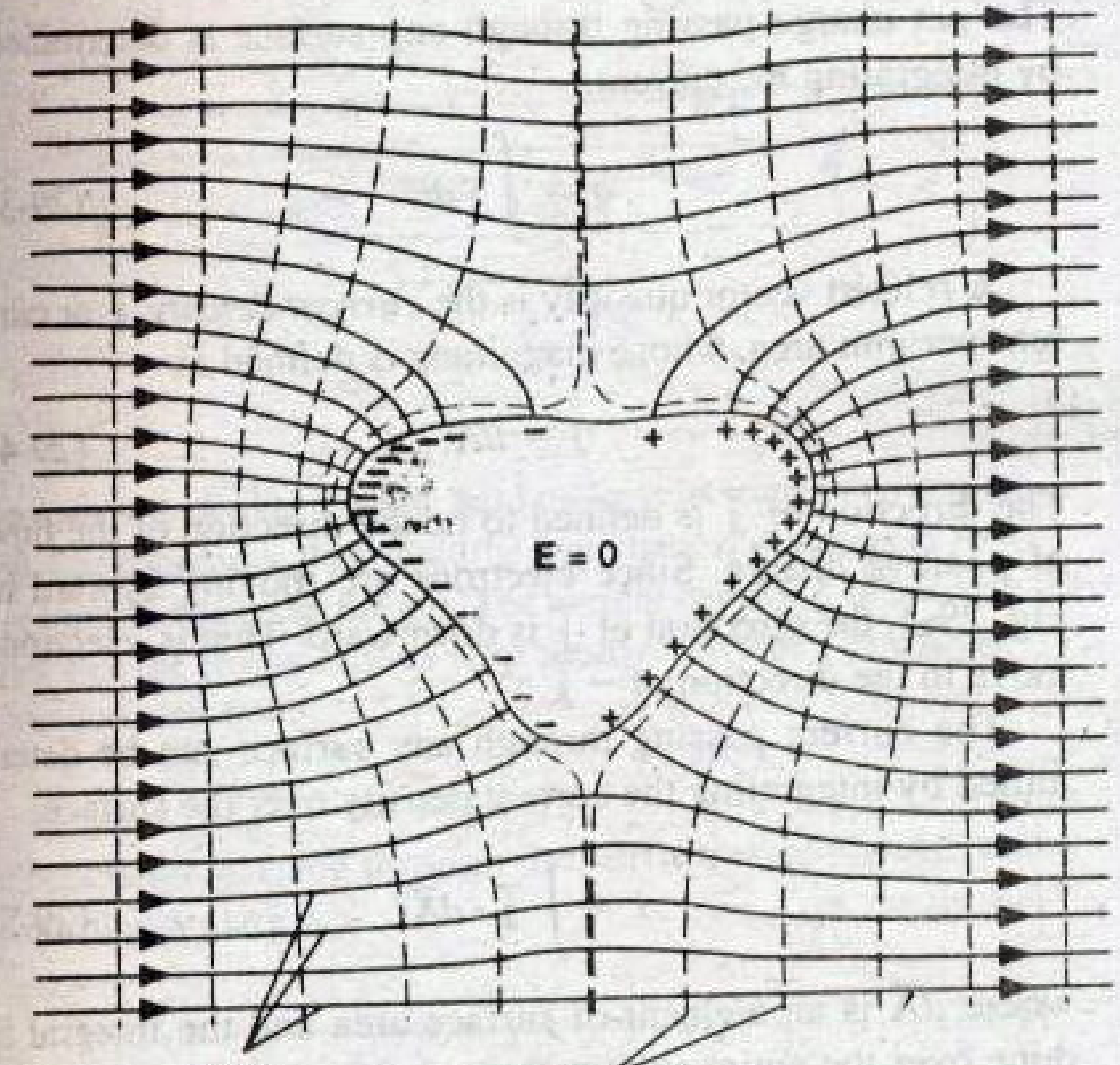


FIGURE 29-2. An uncharged conductor is placed in an external electric field. The conduction electrons distribute themselves on the surface to produce a charge distribution as shown, reducing the field inside the conductor to zero. Note the distortion of the lines of force (solid lines) and the equipotentials (dashed lines) when the conductor is placed in the previously uniform field. Note also how the electric field lines originate on positive charges and terminate on negative charges.

establishing a distribution of positive and negative charges that gives an electric field exactly canceling the applied field in the interior of the conductor. Outside the conductor, the field is the (vector) sum of the original uniform field and the field due to the charges on the surface of the conductor. Note that the field lines originate on positive charges and terminate on negative charges. Note also that the charge density is large on parts of the surface where the radius of curvature is small, as we discussed in Section 28-9, and that the field is large (the field lines are close together) where the charge density is large.

At the surface of the conductor in Fig. 29-2, the electric field lines are perpendicular to the surface. If this were not true, then there would be a component of the electric field parallel to the surface, which would cause charges to move. Since this would violate our assumption of a static situation, this component of the electric field cannot exist and the field must be perpendicular to the surface.

The figure also shows the equipotentials for this situation. Far from the conductor, where the field is uniform, the equipotentials are flat planes. As we move close to the conductor, the equipotentials are distorted, until at the surface the equipotential follows the surface exactly; as we discussed in Section 28-9, the surface of a conductor is an equipotential.

SAMPLE PROBLEM 29-1. A large, thin plate of copper is placed in a uniform electric field of magnitude E_0 = 450 N/C that is perpendicular to the plate (as in Fig. 29-1). Find the resulting surface charge density on the copper.

Solution The electric field causes a positive charge density σ on the lower surface of the plate and a negative charge density of equal magnitude on the upper surface. The field in the interior of the plate must be zero, which means that the two charge distributions must combine to give an electric field inside the plate of magnitude E_0 and direction opposite to the applied field. If we regard the plate as of very large dimensions, the field due to the positive charge distribution is, according to Eq. 26-20, $E_+ = \sigma/2\epsilon_0$, and the magnitude of the field due to the negative charge is $E_- = \sigma/2\epsilon_0$. These two fields are in the same direction and must add to give a total field of E_0:

$$E_0 = \sigma/2\epsilon_0 + \sigma/2\epsilon_0 = \sigma/\epsilon_0$$

and the charge density on each surface is

$$\sigma = \epsilon_0 E_0 = (8.85 \times 10^{-12}\ \mathrm{C^2/N\cdot m^2})(450\ \mathrm{N/C}) = 3.98 \times 10^{-9}\ \mathrm{C/m^2}.$$

Note that outside the copper plate, the fields due to the two sheets of charge cancel one another, so that the resulting field remains E_0. This is true only for the flat geometry of this problem and is not true in general; see for example Fig. 29-2.

29-3 A CONDUCTOR IN AN ELECTRIC FIELD: DYNAMIC CONDITIONS

In Fig. 29-1*a*, electrons move from the bottom of the slab of copper to the top under the action of the applied electric field, until the concentration of electrons at the top of the slab (and of positive ions at the bottom) creates a field that cancels the applied field in the interior of the copper and prevents the flow of additional electrons. Suppose there were a mechanism to remove electrons from the top of the slab, carry them around an external path, and re-inject them at the bottom of the slab (shown schematically in Fig. 29-3). In this case, there would be no build-up of charge on the top and bottom of the slab, and the electrostatic conditions of the previous section cannot be applied to the copper. In particular, the conclusion drawn in the last section is no longer valid—the electric field inside the copper will in general be nonzero when charges are flowing.

The continuous loop of flowing electrons is a simple representation of an electric circuit, and the flow of electrons (or other charged particles) is called an *electric current*.

Let us examine the flow of electric charge past a particular point in the interior of the material (Fig. 29-4). A quantity of charge dq will pass through a small surface of area A in a time dt. For example, the area A might be the cross-sectional area of a wire through which the charge

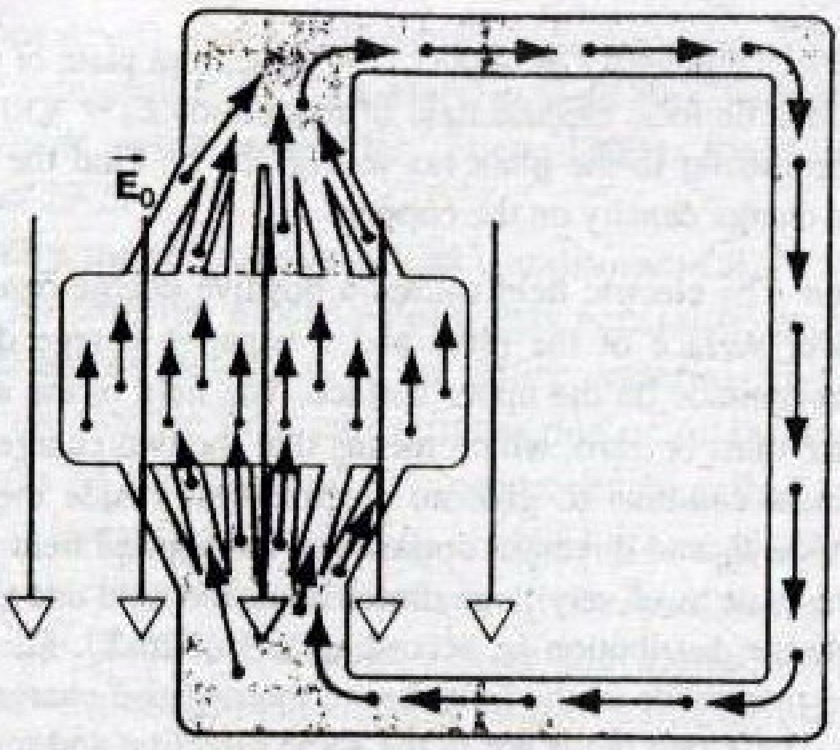


FIGURE 29-3. The electric field $\vec{E}_0$ moves electrons through the slab of copper. The electrons can be collected at the top of the slab and transported through an external path to the bottom of the copper slab.

is flowing. The *electric current i* is defined as the net charge that flows through the surface per unit time interval:

$$i = dq/dt. \tag{29-1}$$

For electric current to exist, there must be a net flow of charge across the surface. If neutral atoms travel across the surface, no current is flowing even though charges travel across the surface, because equal numbers of positive and negative charges cross the surface. If electrons are traveling randomly through the material, with equal numbers crossing the surface in either direction, no current flows because the net charge crossing the surface is zero.

The electric current has a direction, which is defined to be the direction of the flow of positive charge. Even though the current has a direction, current is a scalar and not a vector, because currents do not satisfy the laws for vector addition.

The SI unit of current is the *ampere* (A), defined as

1 ampere = 1 coulomb/second.

If the current is constant, then Eq. 29-1 becomes

$$i = q/t. \tag{29-2}$$

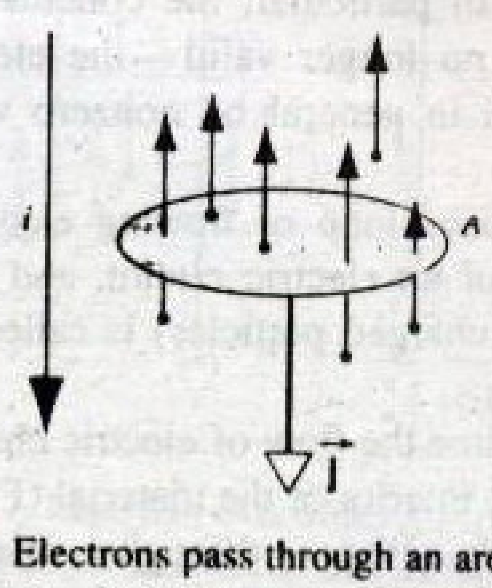


FIGURE 29-4. Electrons pass through an area *A*. The directions of the current *i* and of the vector current density $\vec{j}$ are opposite to the motion of the electrons.

The net charge passing through any surface is determined by integrating the current:

$$q = \int i\, dt. \tag{29-3}$$

A related vector quantity is the *current density* $\vec{j}$ or current per unit area, whose magnitude is defined as

$$j = i/A. \tag{29-4}$$

The direction of $\vec{j}$ is defined to be the direction of the flow of positive charge. Since electrons are moving upward in Fig. 29-4, the direction of $\vec{j}$ is downward. That is, electrons move in the direction of $-\vec{j}$.

The current passing through any surface can be determined by integrating the current density over the surface:

$$i = \int \vec{j} \cdot d\vec{A}, \tag{29-5}$$

where $d\vec{A}$ is an element of surface area and the integral is done over the entire surface through which we want to find the current. The vector $d\vec{A}$ is taken to be perpendicular to the surface element such that $\vec{j} \cdot d\vec{A}$ is positive, corresponding to a positive current *i*.

Current Density and Drift Speed

As electrons make their way through the copper, they are accelerated by an electric field, which exerts a force $-e\vec{E}$ on the electrons. In Section 29-2 we considered static conditions, in which the electric field is always zero inside a conductor. Here we consider charges in motion, so that static conditions do not apply and $\vec{E}$ can be nonzero inside a conductor.

The electrons collide with the ions of the lattice and transfer energy to them. The motion of individual electrons is therefore very irregular, consisting of a short interval of acceleration in a direction opposite to the electric field, followed by a collision with an ion that might send the electron into motion in any direction, followed by another acceleration, and so on. The net effect is a drift of electrons in a direction opposite to the field. There is no net acceleration of electrons, because they continually lose energy in collisions with the lattice of copper ions. In effect, energy is transferred from the applied field to the lattice (in the form of internal energy of the conductor, often observed as a temperature increase). On the average, electrons can be described as moving with a constant *drift velocity* $\vec{v}_d$ in a direction opposite to the field, as indicated in Fig. 29-5.

Consider the motion of electrons in a portion of the conductor of length *L*. The electrons are moving with drift speed v_d, so they travel the length *L* in a time $t = L/v_d$. The conductor has a cross-sectional area *A*, so in the time *t* all of the electrons in the volume *AL* will travel through a surface at the right end of the conductor. If the density of electrons (number per unit volume) is *n*, then the magnitude of

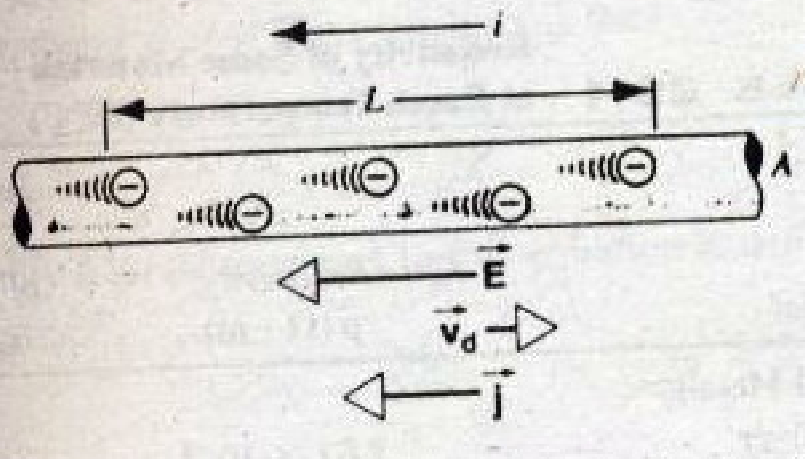


FIGURE 29-5. The electric field causes electrons to drift to the right. The conventional current (the hypothetical direction of flow of positive charge) is to the left. The current density $\vec{j}$ is likewise drawn as if the charge carriers were positive, so that $\vec{j}$ and $\vec{E}$ are in the same direction.

the net charge passing through the surface is $q = enAL$, and the current density is

$$j = \frac{q}{At} = \frac{enAL}{AL/v_d} = env_d. \qquad (29\text{-}6)$$

In vector notation, this is

$$\vec{j} = -en\vec{v}_d. \qquad (29\text{-}7)$$

The negative sign again reminds us that the direction of the current density is opposite to the motion of the electrons.

As the following sample problems illustrate, the drift speeds of electrons in typical materials are very small compared with the speed of the random thermal motions of electrons (typically 10^6 m/s).

SAMPLE PROBLEM 29-2. One end of an aluminum wire whose diameter is 2.5 mm is welded to one end of a copper wire whose diameter is 1.8 mm. The composite wire carries a steady current i of 1.3 A. What is the current density in each wire?

Solution We may take the current density as (a different) constant within each wire except for points near the junction. The current density is given by Eq. 29-4, $j = i/A$. The cross-sectional area A of the aluminum wire is

$$A_{Al} = \tfrac{1}{4}\pi d^2 = (\pi/4)(2.5 \times 10^{-3}\ \text{m})^2 = 4.91 \times 10^{-6}\ \text{m}^2,$$

so that

$$j_{Al} = \frac{1.3\ \text{A}}{4.91 \times 10^{-6}\ \text{m}^2} = 2.6 \times 10^5\ \text{A/m}^2 = 26\ \text{A/cm}^2.$$

As you can verify, the cross-sectional area of the copper wire is 2.54×10^{-6} m², so that

$$j_{Cu} = \frac{1.3\ \text{A}}{2.54 \times \text{j}10^{-6}\ \text{m}^2} = 5.1 \times 10^5\ \text{A/m}^2 = 51\ \text{A/cm}^2.$$

The fact that the wires are of different materials does not enter here.

SAMPLE PROBLEM 29-3. What is the drift speed of the conduction electrons in the copper wire of Sample Problem 29-2?

Solution In copper, there is very nearly one conduction electron per atom on the average. The number n of electrons per unit volume is therefore the same as the number of atoms per unit volume and is determined from

$$\frac{n}{N_A} = \frac{\rho_m}{M} \quad \text{or} \quad \frac{\text{atoms/m}^3}{\text{atoms/mol}} = \frac{\text{mass/m}^3}{\text{mass/mol}}.$$

Here ρ_m is the (mass) density of copper, N_A is the Avogadro constant, and M is the molar mass of copper.* Thus

$$n = \frac{N_A\rho_m}{M} = \frac{(6.02 \times 10^{23}\ \text{electrons/mol})(8.96 \times 10^3\ \text{kg/m}^3)}{63.5 \times 10^{-3}\ \text{kg/mol}}$$

$$= 8.49 \times 10^{28}\ \text{electrons/m}^3.$$

We then have, using Eq. 29-6 ($v_d = j/ne$),

$$v_d = \frac{5.1 \times 10^5\ \text{A/m}^2}{(8.49 \times 10^{28}\ \text{electrons/m}^3)(1.60 \times 10^{-19}\ \text{C/electron})}$$

$$= 3.8 \times 10^{-5}\ \text{m/s} = 14\ \text{cm/h}.$$

You should be able to show that for the aluminum wire, $v_d = 2.7 \times 10^{-5}$ m/s = 9.7 cm/h. Can you explain, in physical terms, why in this example the drift speed is smaller in aluminum than in copper, even though the two wires carry the same current?

If the electrons drift at such a low speed, why do electrical effects seem to occur immediately after a switch is thrown, such as when you turn on the room lights? Confusion on this point results from not distinguishing between the drift speed of the electrons and the speed at which *changes* in the electric field configuration travel along wires. This latter speed approaches that of light. Similarly, when you turn the valve on your garden hose, with the hose full of water, a pressure wave travels along the hose at the speed of sound in water. The speed at which the water moves through the hose—measured perhaps with a dye marker—is much lower.

SAMPLE PROBLEM 29-4. A strip of silicon, of cross-sectional width w = 3.2 mm and thickness d = 250 μm, carries a current i of 190 mA. The silicon is an *n-type semiconductor,* having been "doped" with a controlled amount of phosphorus impurity. The doping has the effect of greatly increasing n, the number of charge carriers (electrons, in this case) per unit volume, as compared with the value for pure silicon. In this case, $n = 8.0 \times 10^{21}\ \text{m}^{-3}$. (*a*) What is the current density in the strip? (*b*) What is the drift speed?

Solution (*a*) From Eq. 29-4,

$$j = \frac{i}{wd} = \frac{190 \times 10^{-3}\ \text{A}}{(3.2 \times 10^{-3}\ \text{m})(250 \times 10^{-6}\ \text{m})}$$

$$= 2.4 \times 10^5\ \text{A/m}^2.$$

(*b*) From Eq. 29-6,

$$v_d = \frac{j}{ne} = \frac{2.4 \times 10^5\ \text{A/m}^2}{(8.0 \times 10^{21}\ \text{m}^{-3})(1.60 \times 10^{-19}\ \text{C})} = 190\ \text{m/s}.$$

The drift speed (190 m/s) of the electrons in this silicon semiconductor is much greater than the drift speed (3.8×10^{-5} m/s) of the conduction electrons in the metallic copper conductor of Sample Problem 29-3, even though the current densities are similar. The number of charge carriers in this semiconductor (8.0 ×

*We use the subscript m to make it clear that the density referred to here is a mass density (kg/m³), not a charge density (C/m³).

$10^{21}\ m^{-3}$) is much smaller than the number of charge carriers in the copper conductor ($8.49 \times 10^{28}\ m^{-3}$). The smaller number of charge carriers must drift faster in the semiconductor if they are to establish the same current density that the greater number of charge carriers establish in copper.

29-4 OHMIC MATERIALS

Between collisions with the lattice ions, the electrons in a conducting material are accelerated by the electric field $\vec{E}$, and so their drift velocity is proportional to $\vec{E}$. The current density $\vec{j}$ is also proportional to $\vec{v}_d$, so it is reasonable that $\vec{j}$ should be proportional to $\vec{E}$. In fact, we observe this type of behavior for a wide class of materials. The proportionality constant between the current density and electric field is the *electrical conductivity* σ of the material:

$$\vec{j} = \sigma \vec{E}. \tag{29-8}$$

A large value of σ indicates that the material is a good conductor of electric current. The conductivity is a property of the material, not of any particular sample of the material. The SI unit for conductivity is the *siemens per meter* (S/m), where the siemens is defined as

$$1 \text{ siemens} = 1 \text{ ampere/volt}.$$

It is more common to find materials characterized by their *resistivity*, which is the inverse of the conductivity:

$$\rho = 1/\sigma, \tag{29-9}$$

in which case Eq. 29-8 becomes

$$\vec{E} = \rho \vec{j}. \tag{29-10}$$

Units of resistivity are *ohm · meter*, where the ohm (symbol Ω) is defined as

$$1 \text{ ohm} = 1 \text{ volt/ampere}.$$

Note that 1 ohm = (1 siemens)$^{-1}$.

Equations 29-8 and 29-10 are valid only for isotropic materials, whose electrical properties are the same in all directions. In these materials, $\vec{j}$ will always be in the same direction as $\vec{E}$.

Table 29-1 gives some values of the resistivity for various materials. A perfect insulator would have $\rho = \infty$ (or $\sigma = 0$). Note that even good insulators are weakly conducting.

We can use Eq. 29-10 to determine the resistivity of any material by applying an electric field and measuring the resulting current density. For some materials, we find that the resistivity is not a constant but depends on the strength of the electric field. That is, if we double the electric field the current density does not double. For other materials, we find that the resistivity does not depend on the strength of the applied field for a wide range of applied fields. For such materials, a plot of E against j gives a straight line, whose slope is the resistivity ρ. These materials are known as *ohmic* materials. Equivalently, such materials are said to satisfy *Ohm's law:*

> ***The resistivity (or conductivity) of a material is independent of the magnitude and direction of the applied electric field.***

Many homogeneous materials, including conducting metals such as copper, obey Ohm's law for a certain range of values of the applied electric field. If the field is sufficiently large, all materials will behave in violation of Ohm's law.

The resistivity values in Table 29-1 are properties of the materials listed. We might also want to know the *resistance* of a particular object, such as a block of copper of certain dimensions. Figure 29-6 illustrates the situation for a homogeneous, isotropic conductor of length L and uniform cross-sectional area A, to which we have applied a potential difference ΔV. Inside the object, there is a uniform electric

TABLE 29-1 Resistivity of Some Materials at Room Temperature (20°C)

Material	Resistivity, ρ ($\Omega \cdot$ m)	Temperatur[e] Coefficient of Resistivi[ty] α_{av} (per C°)
Typical Metals		
Silver	1.62×10^{-8}	$4.1 \times 10^{[illegible]}$
Copper	1.69×10^{-8}	$4.3 \times 10^{[illegible]}$
Aluminum	2.75×10^{-8}	$4.4 \times 10^{[illegible]}$
Tungsten	5.25×10^{-8}	$4.5 \times 10^{[illegible]}$
Iron	9.68×10^{-8}	$6.5 \times 10^{[illegible]}$
Platinum	10.6×10^{-8}	$3.9 \times 10^{[illegible]}$
Manganin[a]	48.2×10^{-8}	$0.002 \times 10^{[illegible]}$
Typical Semiconductors		
Silicon pure	2.5×10^{3}	$-70 \times 10^{[illegible]}$
Silicon n-type[b]	8.7×10^{-4}	
Silicon p-type[c]	2.8×10^{-3}	
Typical Insulators		
Pure water	2.5×10^{5}	
Glass	$10^{10} - 10^{14}$	
Polystyrene	$> 10^{14}$	
Fused quartz	$\sim 10^{16}$	

[a] An alloy specifically designed to have a small value of α.
[b] Pure silicon "doped" with phosphorus impurities to a charge carrie[r] density of $10^{23}\ m^{-3}$.
[c] Pure silicon "doped" with aluminum impurities to a charge carrie[r] density of $10^{23}\ m^{-3}$.

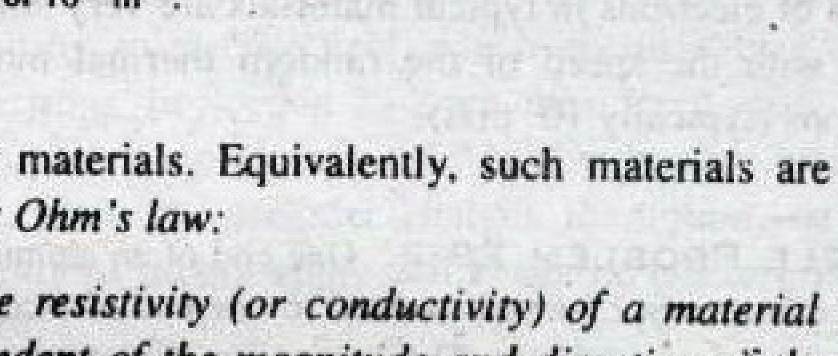


FIGURE 29-6. A potential difference ΔV is applied across a cylindrical conductor of length L and cross-sectional area A, establishing a current i.

field $E = \Delta V/L$. If the current density is also uniform over the area A, then $j = i/A$. The resistivity is then

$$\rho = \frac{E}{j} = \frac{\Delta V/L}{i/A}. \qquad (29\text{-}11)$$

The quantity $\Delta V/i$ that appears in this equation is defined as the resistance R:

$$R = \frac{\Delta V}{i}. \qquad (29\text{-}12)$$

Combining Eqs. 29-11 and 29-12, we obtain an expression for the resistance R:

$$R = \rho\frac{L}{A}. \qquad (29\text{-}13)$$

The resistance R is characteristic of a particular object and depends on the material of which it is made as well as on its length and cross-sectional area; the resistivity ρ is characteristic of the material in general. The units of resistance are ohms (Ω).

Equation 29-12 gives us another basis for stating Ohm's law. For a particular object, we can measure the current i for various applied potential differences and plot i as a function of ΔV. If this plot gives a straight line, then the object is ohmic and obeys Ohm's law. An equivalent statement of Ohm's law is:

The resistance of an object is independent of the magnitude or sign of the applied potential difference.

Ordinary resistors that are found in electric circuits are ohmic for the range of potential differences that are normally used in circuits. Semiconducting devices, such as diodes and transistors, usually are nonohmic. Figure 29-7 compares the current–voltage plots for ohmic and non-ohmic devices.

Keep in mind that the relationship $\Delta V = iR$ is *not* a statement of Ohm's law. This equation defines the resistance and is true for both ohmic and nonohmic objects. Even for nonohmic devices, we can find a value of the resistance R for a particular value of ΔV; for a different ΔV, a different value of R will be obtained. For ohmic devices, we get the same value of R for any value of ΔV.

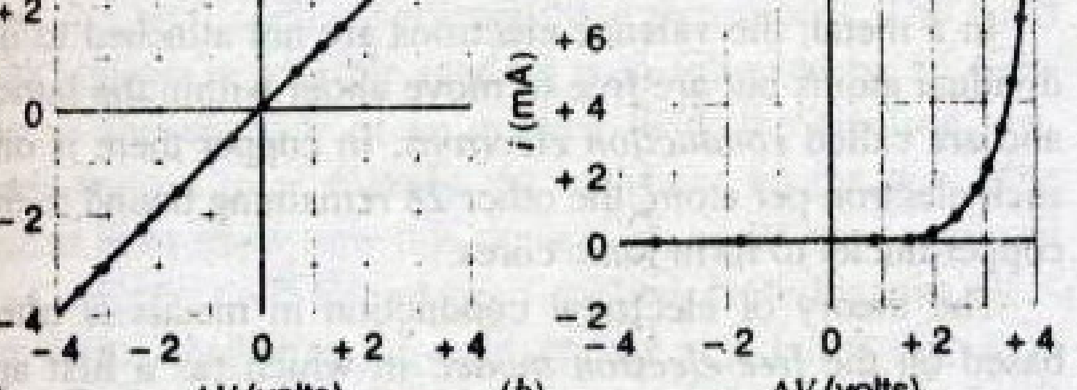


FIGURE 29-7. (*a*) A current–voltage plot for a material that obeys Ohm's law, in this case a 1000-Ω resistor. (*b*) A current–voltage plot for a material that does not obey Ohm's law, in this case a *pn* junction diode.

ΔV, i, and R are *macroscopic* quantities, applying to a particular body or extended region. The corresponding *microscopic* quantities are $\vec{\mathbf{E}}$, $\vec{\mathbf{j}}$, and ρ (or σ); they have values at every point in a body. The macroscopic quantities are related by Eq. 29-12 ($\Delta V = iR$) and the microscopic quantities by Eq. 29-10 ($\vec{\mathbf{E}} = \rho\vec{\mathbf{j}}$).

The macroscopic quantities ΔV, i, and R are of primary interest when we are making electrical measurements on real conducting objects. They are the quantities whose values are indicated on meters. The microscopic quantities $\vec{\mathbf{E}}$, $\vec{\mathbf{j}}$, and ρ are of primary importance when we are concerned with the fundamental behavior of matter (rather than of specimens of matter), as we usually are in the research area of *solid state* (or *condensed matter*) physics. Section 29-5 accordingly deals with an atomic view of the *resistivity* of a metal and not of the *resistance* of a metallic specimen.

SAMPLE PROBLEM 29-5. A rectangular block of iron has dimensions 1.2 cm × 1.2 cm × 15 cm. (*a*) What is the resistance of the block measured between the two square ends? (*b*) What is the resistance between two opposing rectangular faces? The resistivity of iron at room temperature is $9.68 \times 10^{-8}\ \Omega\cdot\text{m}$.

Solution (*a*) The area of a square end is $(1.2 \times 10^{-2}\ \text{m})^2$ or $1.44 \times 10^{-4}\ \text{m}^2$. From Eq. 29-13,

$$R = \frac{\rho L}{A} = \frac{(9.68 \times 10^{-8}\ \Omega\cdot\text{m})(0.15\ \text{m})}{1.44 \times 10^{-4}\ \text{m}^2} = 1.0 \times 10^{-4}\ \Omega = 100\ \mu\Omega.$$

(*b*) The area of a rectangular face is $(1.2 \times 10^{-2}\ \text{m})(0.15\ \text{m})$ or $1.80 \times 10^{-3}\ \text{m}^2$. From Eq. 29-13,

$$R = \frac{\rho L}{A} = \frac{(9.68 \times 10^{-8}\ \Omega\cdot\text{m})(1.2 \times 10^{-2}\ \text{m})}{1.80 \times 10^{-3}\ \text{m}^2} = 6.5 \times 10^{-7}\ \Omega = 0.65\ \mu\Omega.$$

We assume in each case that the potential difference is applied to the block in such a way that the surfaces between which the resistance is desired are equipotentials. Then the electric field is uniform between the surfaces, and as a result the current density is also uniform. Otherwise, Eq. 29-13 would not apply.

Analogy between Current and Heat Flow (Optional)

A close analogy exists between the flow of charge established by a potential difference and the flow of heat established by a temperature difference. Consider a thin electrically conducting slab of thickness Δx and area A. Let a potential difference ΔV be maintained between opposing faces. The current i is given by Eqs. 29-12 ($i = \Delta V/R$) and 29-13 ($R = \rho L/A$), or

$$i = \frac{\Delta V}{R} = \frac{\Delta V}{\rho\Delta x/A} = \sigma A\frac{\Delta V}{\Delta x}$$

using $\sigma = \rho^{-1}$. In the limiting case of a slab of thickness dx this becomes

$$\frac{dq}{dt} = -\sigma A \frac{dV}{dx}. \qquad (29\text{-}14)$$

The negative sign in Eq. 29-14 indicates that positive charge flows in the direction of decreasing V; that is, dq/dt is positive when dV/dx is negative.

The analogous heat flow equation (see Section 23-2) is

$$\frac{dQ}{dt} = -kA \frac{dT}{dx}, \qquad (29\text{-}15)$$

which shows that k, the thermal conductivity, corresponds to σ, and dT/dx, the temperature gradient, corresponds to dV/dx, the potential gradient. For pure metals there is more than a formal mathematical analogy between Eqs. 29-14 and 29-15. Both heat energy and charge are carried by the free electrons in such metals; empirically, a good electrical conductor (silver, say) is also a good heat conductor, and the electrical conductivity σ is directly related to the thermal conductivity k. ■

Temperature Variation of Resistivity (Optional)

Figure 29-8 shows a summary of some experimental measurements of the resistivity of copper at different temperatures. For practical use of this information, it would be helpful to express it in the form of an equation. Over a limited range of temperature, the relationship between resistivity and temperature is nearly linear. We can fit a straight line to any selected region of Fig. 29-8, using two points to determine the slope of the line. Choosing a reference point, such as that labeled T_0, ρ_0 in the figure, we can express the resistivity ρ at an arbitrary temperature T from the empirical equation of the straight line in Fig. 29-8, which is

$$\rho - \rho_0 = \rho_0 \alpha_{av}(T - T_0). \qquad (29\text{-}16)$$

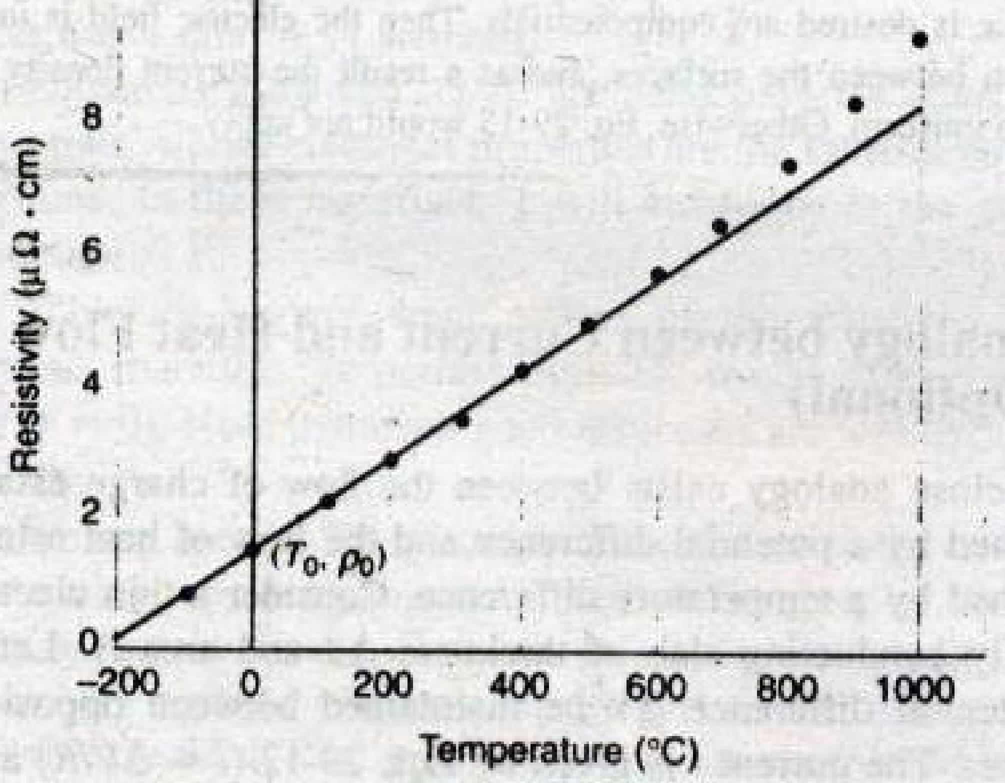


FIGURE 29-8. The dots show selected measurements of the resistivity of copper at different temperatures. Over any given range of temperature, the variation in the resistivity with T can be approximated by a straight line; for example, the line shown fits the data from about −100°C to 400°C.

(This expression is very similar to that for linear thermal expansion, $\Delta L = \alpha L\, \Delta T$, which we introduced in Section 21-4.) We have written the slope of this line as $\rho_0 \alpha_{av}$. If we solve Eq. 29-16 for α_{av}, we obtain

$$\alpha_{av} = \frac{1}{\rho_0} \frac{\rho - \rho_0}{T - T_0}. \qquad (29\text{-}17)$$

The quantity α_{av} is the *mean (or average) temperature coefficient of resistivity* over the region of temperature between the two points used to determine the slope of the line. We can define a more general temperature coefficient of resistivity as

$$\alpha = \frac{1}{\rho} \frac{d\rho}{dT}, \qquad (29\text{-}18)$$

which is the fractional change in resistivity $d\rho/\rho$ per change in temperature dT. That is, α gives the dependence of resistivity on temperature *at a particular temperature*, whereas α_{av} gives the average dependence *over a particular interval*. The coefficient α is in general dependent on temperature.

For most practical purposes, Eq. 29-16 gives results that are within the acceptable range of accuracy. Typical values of α_{av} are given in Table 29-1. For more precise work, such as the use of the platinum resistance thermometer to measure temperature (see Section 21-3), the linear approximation is not sufficient. In this case we can add terms in $(T - T_0)^2$ and $(T - T_0)^3$ to the right side of Eq. 29-16 to improve the precision. The coefficients of these additional terms must be determined empirically, in analogy with the coefficient α_{av} in Eq. 29-16. ■

29-5 OHM'S LAW: A MICROSCOPIC VIEW

As we discussed previously, Ohm's law is not a fundamental law of electromagnetism because it depends on the properties of the conducting medium. The law is very simple in form, and it is curious that many materials obey it so well, whereas other materials do not obey it at all. Let us see if we can understand why metals obey Ohm's law—that is, why their resistivities ρ are constant (and not, for example, dependent on the applied electric field).

In a metal, the valence electrons are not attached to individual atoms but are free to move about within the lattice and are called *conduction electrons*. In copper there is one such electron per atom, the other 28 remaining bound to the copper nuclei to form ionic cores.

The theory of electrical conduction in metals is often based on the *free-electron model*, in which (as a first approximation) the conduction electrons are assumed to move throughout the conducting material, somewhat like molecules of gas in a container. In fact, the assembly of conduction electrons is sometimes called an *electron gas*. As we shall see, however, we cannot neglect the effect of the ion cores on this "gas."

The classical Maxwellian velocity distribution (see Section 22-4) for the electron gas would suggest that the conduction electrons have a broad distribution of velocities from zero to infinity, with a well-defined average. However, in considering the electrons we cannot ignore quantum mechanics, which gives a very different view. In the quantum distribution (see Section 49-4), the electrons that readily contribute to electrical conduction are concentrated in a very narrow interval of kinetic energies and therefore of speeds. To a very good approximation, we can assume that the electrons move with a uniform average speed. In the case of copper, this speed is about $v_{av} = 1.6 \times 10^6$ m/s. Furthermore, whereas the Maxwellian average speed depends strongly on the temperature, the effective speed obtained from the quantum distribution is nearly independent of temperature.

In the absence of an electric field, the electrons move randomly, again like the molecules of gas in a container. Occasionally, an electron collides with an ionic core of the lattice, suffering a sudden change in direction in the process. As we did in the case of collisions of gas molecules, we can associate a mean free path λ and a mean free time τ to the average distance and time between collisions. (Collisions between the electrons themselves are rare and do not affect the electrical properties of the conductor.)

In an ideal metallic crystal (containing no defects or impurities) at 0 K, electron–lattice collisions would not occur, according to the predictions of quantum physics; that is, $\lambda \rightarrow \infty$ as $T \rightarrow 0$ K for ideal crystals. Collisions take place in actual crystals because (1) the ionic cores at any temperature T are vibrating about their equilibrium positions in a random way; (2) impurities—that is, foreign atoms—may be present; and (3) the crystal may contain lattice imperfections, such as missing atoms and displaced atoms. Consequently, the resistivity of a metal can be increased by (1) raising its temperature, (2) adding small amounts of impurities, and (3) straining it severely, as by drawing it through a die, to increase the number of lattice imperfections.

When we apply an electric field to a metal, the electrons modify their random motion in such a way that they drift slowly, in the opposite direction to that of the field, with an average drift speed v_d. This drift speed is very much less (by a factor of something like 10^{10}; see Sample Problem 29-3) than the effective average speed v_{av}. Figure 29-9 suggests the relationship between these two speeds. The solid lines suggest a possible random path followed by an electron in the absence of an applied field; the electron proceeds from x to y, making six collisions on the way. The dashed lines show how this same event *might* have occurred if an electric field $\vec{E}$ had been applied. Note that the electron drifts steadily to the right, ending at y' rather than at y. In preparing Fig. 29-9, it has been assumed that the drift speed v_d is $0.02v_{av}$; actually, it is more like $10^{-10}v_{av}$, so that the "drift" exhibited in the figure is greatly exaggerated.

We can calculate the drift speed v_d in terms of the applied electric field E and of v_{av} and λ. When a field is applied to an electron in the metal, it experiences a force eE,

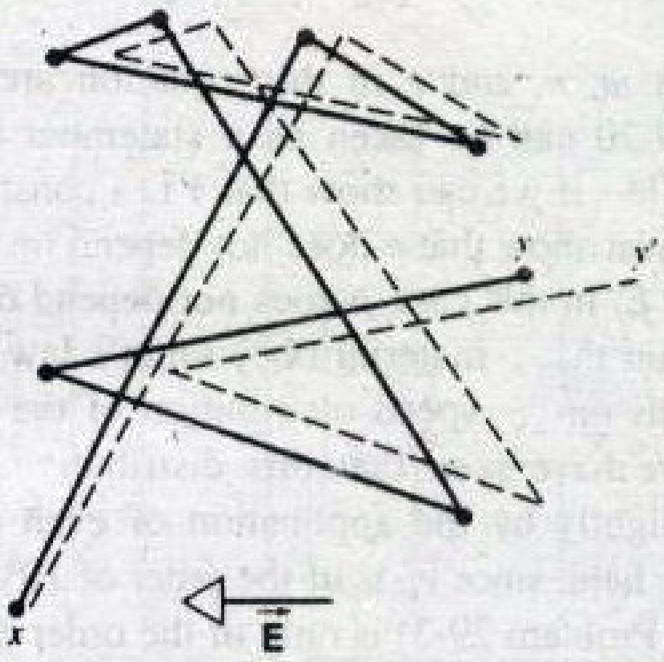


FIGURE 29-9. The solid line segments show an electron moving from x to y, making six collisions en route. The dashed lines show what its path *might* have been in the presence of an applied electric field $\vec{E}$. Note the gradual but steady drift in the direction of $-\vec{E}$. (Actually, the dashed lines should be slightly curved to represent the parabolic paths followed by the electrons between collisions.)

which imparts to it an acceleration a given by Newton's second law,

$$a = \frac{eE}{m}.$$

Consider an electron that has just collided with an ion core. The collision, in general, momentarily destroys the tendency to drift, and the electron has a truly random direction after the collision. During the time interval to the next collision, the electron's speed changes, on the average, by an amount $a(\lambda/v_{av})$ or $a\tau$, where τ is the mean time between collisions. We identify this with the drift speed v_d, or*

$$v_d = a\tau = \frac{eE\tau}{m}. \qquad (29\text{-}19)$$

We may also express v_d in terms of the current density (Eq. 29-6), which gives

$$v_d = \frac{j}{ne} = \frac{eE\tau}{m}.$$

Combining this with Eq. 29-10 ($\rho = E/j$), we finally obtain

$$\rho = \frac{m}{ne^2\tau}. \qquad (29\text{-}20)$$

*It may be tempting to write Eq. 29-19 as $v_d = \frac{1}{2}a\tau$, reasoning that $a\tau$ is the electron's *final* velocity, and thus that its *average* velocity is half that value. The extra factor of $\frac{1}{2}$ would be correct if we followed a typical electron, taking its drift speed to be the average of its velocity over its mean time τ between collisions. However, the drift speed is proportional to the current density j and must be calculated from the average velocity of *all* the electrons taken at one instant of time. For each electron, the velocity at any time is at, where t is the time since the last collision for that electron. Since the acceleration a is the same for all electrons, the average value of at at a given instant is $a\tau$, where τ is the average time since the last collision, which is the same as the mean time between collisions. For a discussion of this point, see *Electricity and Magnetism*, 2nd ed., by Edward Purcell (McGraw-Hill, 1985), Section 4.4. See also "Drift Speed and Collision Time," by Donald E. Tilley, *American Journal of Physics*, June 1976, p. 597.

Note that m, n, and e in this equation are constants. Thus Eq. 29-20 can be taken as a statement that metals obey Ohm's law if we can show that τ is a constant. In particular, we must show that τ does not depend on the applied electric field E. In this case ρ does not depend on E, which is the criterion that a material obey Ohm's law. The quantity τ depends on the speed distribution of the conduction electrons. We have seen that this distribution is affected only very slightly by the application of even a relatively large electric field, since v_{av} is of the order of 10^6 m/s, and v_d (see Sample Problem 29-3) is only of the order of 10^{-4} m/s, a ratio of 10^{10}. Whatever the value of τ is (for copper at 20°C, say) in the absence of a field, it remains essentially unchanged when the field is applied. Thus the right side of Eq. 29-20 is independent of E (which means that ρ is independent of E), and the material obeys Ohm's law.

SAMPLE PROBLEM 29-6. (*a*) What is the mean free time τ between collisions for the conduction electrons in copper? (*b*) What is the mean free path λ for these collisions? Assume an effective speed v_{av} of 1.6×10^6 m/s.

Solution (*a*) From Eq. 29-20 we have

$$\tau = \frac{m}{ne^2\rho}$$

$$= \frac{9.11 \times 10^{-31}\ \text{kg}}{(8.49 \times 10^{28}\ \text{m}^{-3})(1.60 \times 10^{-19}\ \text{C})^2(1.69 \times 10^{-8}\ \Omega\cdot\text{m})}$$

$$= 2.48 \times 10^{-14}\ \text{s}.$$

The value of n, the number of conduction electrons per unit volume in copper, was obtained from Sample Problem 29-3; the value of ρ comes from Table 29-1.

(*b*) We define the mean free path from

$$\lambda = \tau v_{av} = (2.48 \times 10^{-14}\ \text{s})(1.6 \times 10^6\ \text{m/s})$$
$$= 4.0 \times 10^{-8}\ \text{m} = 40\ \text{nm}.$$

This is about 150 times the distance between nearest-neighbor ions in a copper lattice. A full treatment based on quantum physics reveals that we cannot view a "collision" as a direct interaction between an electron and an ion. Rather, it is an interaction between an electron and the thermal vibrations of the lattice, lattice imperfections, or lattice impurity atoms. An electron can pass very freely through an "ideal" lattice—that is, a geometrically "perfect" lattice close to the absolute zero of temperature. Mean free paths as large as 10 cm have been observed under such conditions.

29-6 AN INSULATOR IN AN ELECTRIC FIELD

So far we have been talking only about the behavior of conducting materials in electric fields. We now consider what happens when we apply an external electric field to an insulating material. That is, we shall repeat the experiment of Fig. 29-1 with the conducting material replaced by an insulating material.

In an insulator, the electric charges are not free to move. No current results when an insulator is placed in an electric field. The electrons remain firmly locked to their atoms or molecules. Instead of moving charges through the material, all the electric field can do in an insulator is to produce a slight rearrangement of the electric charges within the atoms. However, this small effect can have a substantial influence on the electric field in an insulator.

We begin by considering an insulator such as pure water. The water molecule has a permanent electric dipole moment, as we illustrated in Fig. 26-20. When a water molecule, with its electric dipole moment, is placed in an electric field, as in Fig. 26-19, the field exerts a torque on the dipole that tries to align it with the field. Figure 29-10 shows a collection of dipoles, which have been rotated into alignment by an external field.

To an external observer, the collection of dipoles in Fig. 29-10*b* appears to show negative charges on its upper surface and positive charges on its lower surface. In this respect the insulator resembles the conductor of Fig. 29-1, but the explanation is very different—there is no movement of electrons through the insulating material. In an insulator, an external electric field causes charges to move only over distances that are less than an atomic diameter.

Figure 29-11*a* shows a slab of insulating material that has been placed in an externally applied electric field $\vec{E}_0$. As a result of the rotation of the dipole moments, there is an apparent sheet of positive charge on the lower surface of the material and a sheet of negative charge on the upper surface. These two sheets of *induced surface charge* establish an electric field $\vec{E}'$ in the insulator that opposes the applied field, as shown in Fig. 29-11*b*. The effect of aligning the dipoles in the insulator is called *polarization*, and the field $\vec{E}'$ is known as the polarization field.

The net field $\vec{E}$ inside the insulator is the vector sum of the applied field $\vec{E}_0$ and the polarization field $\vec{E}'$:

$$\vec{E} = \vec{E}_0 + \vec{E}'. \qquad (29\text{-}21)$$

Because $\vec{E}_0$ and $\vec{E}'$ are in opposite directions, we can write the vector sum as a difference of magnitudes:

$$E = E_0 - E'. \qquad (29\text{-}22)$$

Figure 29-11*c* shows the net field inside the insulator, which is smaller than the applied field. ***When an insulator is placed in an electric field, induced surface charges appear that tend to weaken the original field within the material.***

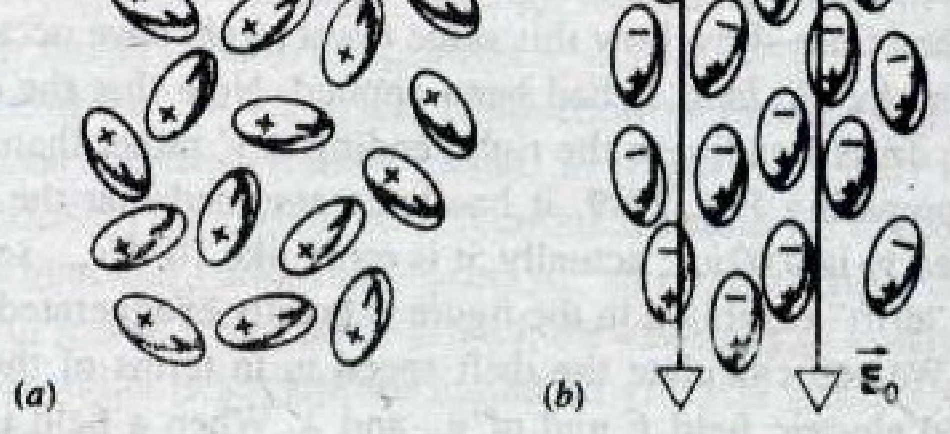


FIGURE 29-10. (*a*) A collection of randomly oriented dipoles. (*b*) An external electric field aligns the dipoles.

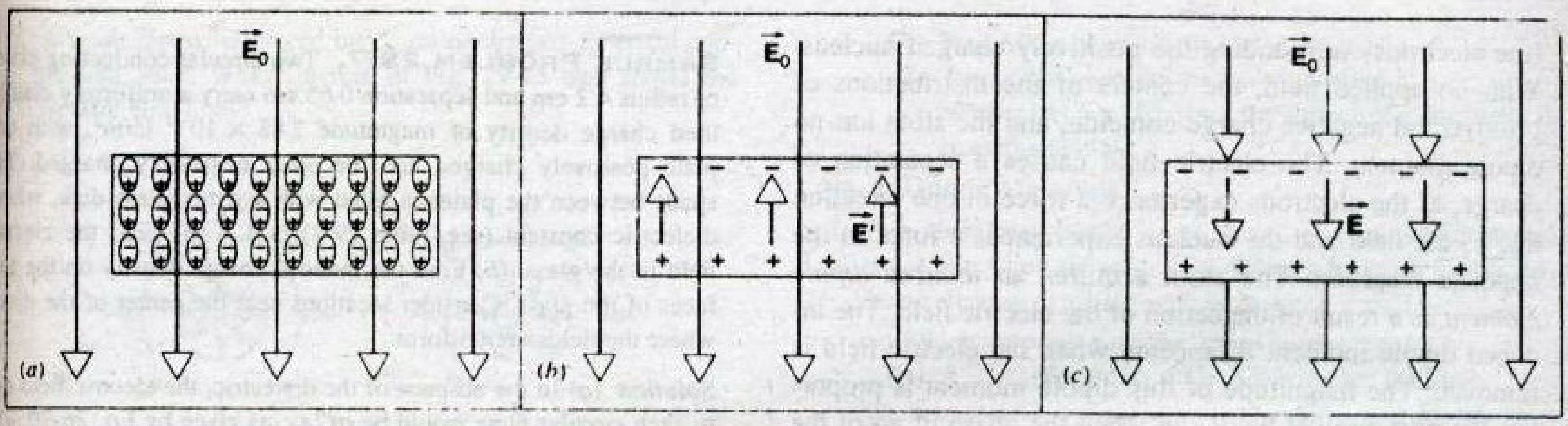


FIGURE 29-11. (*a*) When an insulator is placed in an external field, the dipoles become aligned. (*b*) Induced surface charges on the insulator establish a polarization field $\vec{E}'$ in its interior. (*c*) The net field $\vec{E}$ in the insulator is the vector sum of $\vec{E}_0$ and $\vec{E}'$.

As we increase the applied field $\vec{E}_0$, the polarization field will generally increase. The dipoles in the insulator are in random thermal motion, which tends to destroy their alignment. The greater the applied field, the greater is the torque on the dipoles, the greater is their degree of alignment, and the greater is the polarization field. For many materials, which are called *linear* materials, the polarization field increases in direct proportion to the applied field: $E' \propto E_0$. Using Eq. 29-22, we can also write this proportionality as $E \propto E_0$ and, introducing a constant of proportionality, we have

$$E = \frac{1}{\kappa_e} E_0, \tag{29-23}$$

where the dimensionless constant κ_e is called the *dielectric constant* of the material. The dielectric constant is greater than 1, and so the net field E in the insulator is smaller than the applied field. Like the conductivity or the resistivity, the dielectric constant is characteristic of the type of material (and its temperature), and is independent of the size or shape of any particular object made from the material.

Insulating materials are also known as *dielectric materials,* and we shall use the two terms interchangeably. Table 29-2 shows values of the dielectric constants for various materials at room temperature. Materials with large dielectric constants have large polarization fields, and therefore the fields in their interiors are considerably reduced compared with the applied field.

If we apply a large enough electric field to an insulating material, we can ionize atoms or molecules of the insulator and thus create a condition for electric charge to flow, as in a conductor. The fields necessary for the *breakdown* of various insulators, called the *dielectric strengths,* are given in Table 29-2.

Water is an example of a *polar dielectric* material, because its molecules have permanent electric dipole moments. Effects similar to those described in this section will also occur for *nonpolar* dielectrics, whose molecules do not have permanent dipole moments. Figure 29-12 shows the effect of an electric field on an atom. The atom can be considered as a spherically symmetric cloud of negative charge

TABLE 29-2 Some Properties of Dielectrics[a]

Material	*Dielectric Constant* κ_e	*Dielectric Strength* (kV/mm)
Vacuum	1 (exact)	∞
Air (1 atm)	1.00059	3
Polystyrene	2.6	24
Paper	3.5	16
Transformer oil	4.5	12
Pyrex	4.7	14
Mica	5.4	160
Porcelain	6.5	4
Silicon	12	
Water (25°C)	78.5	
Water (20°C)	80.4	
Titania ceramic	130	
Strontium titanate	310	8

[a] Measured at room temperature.

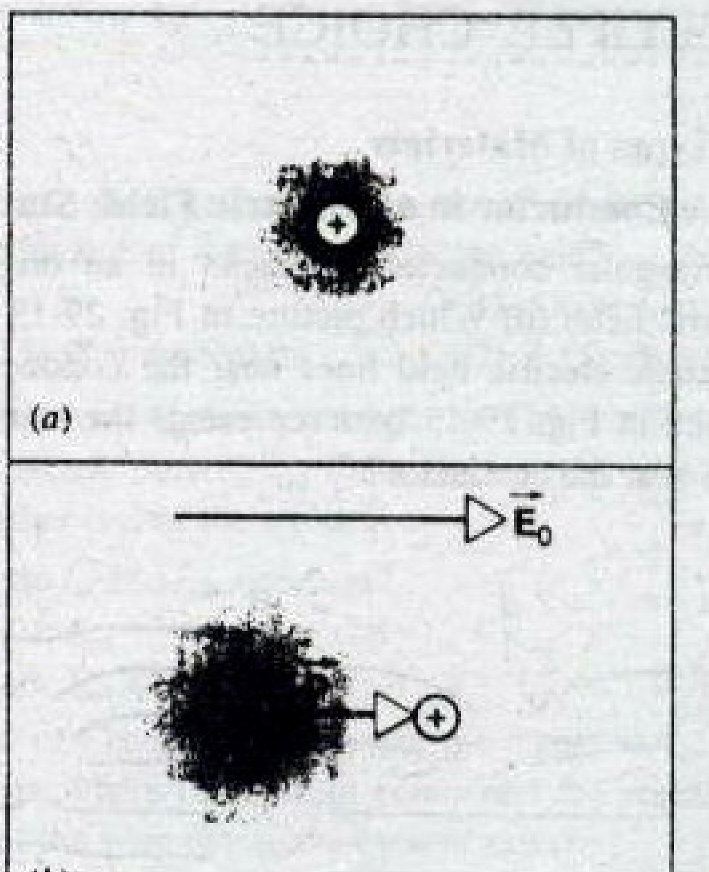


FIGURE 29-12. (*a*) An atom is represented by its positively charged nucleus and its diffuse, negatively charged electron cloud. The centers of positive and negative charge coincide. (*b*) When the atom is placed in an external electric field, the positive and negative charges experience forces in opposite directions, and the centers of the positive and negative charges no longer coincide. The atom acquires an induced dipole moment.

(the electrons) surrounding the positively charged nucleus. With no applied field, the centers of the distributions of positive and negative charge coincide, and the atom has no dipole moment. The electric field causes a separation of charge, as the electrons experience a force in one direction due to the field and the nucleus experiences a force in the opposite direction. The atom acquires an *induced dipole moment* as a result of the action of the electric field. The induced dipole moment disappears when the electric field is removed. The magnitude of this dipole moment is proportional to the applied field, and when the effect of all of the induced dipoles in the material is taken into account, we have again a polarization field E' that is proportional to the applied field, for ordinary field strengths. The induced dipole moment is often responsible for the attraction of a charged object for an uncharged insulator, such as the charged comb and the bits of paper shown in Fig. 25-5.

Because all expressions for electric fields in empty space due to various charge distributions include a factor of $1/\epsilon_0$, Eq. 29-23 suggests that expressions for electric fields in matter will include the factor $1/\kappa_e\epsilon_0$. Since this factor occurs frequently, it is designated by the symbol ϵ:

$$\epsilon = \kappa_e\epsilon_0. \qquad (29\text{-}24)$$

ϵ is called the *permittivity* of the material (recall that the electric constant ϵ_0 is also known as the permittivity of free space). We can often change equations for electric fields in empty space to apply to electric fields in matter by replacing ϵ_0 with ϵ.

SAMPLE PROBLEM 29-7. Two circular conducting pla of radius 4.2 cm and separation 0.65 cm carry a uniformly distr uted charge density of magnitude 2.88×10^{-7} C/m², with o plate positively charged and the other negatively charged. T space between the plates is filled with a pyrex glass disk, who dielectric constant (see Table 29-2) is 4.7. (*a*) Find the elect field in the glass. (*b*) Find the induced charge density on the su faces of the glass. Consider locations near the center of the disk where the fields are uniform.

Solution (*a*) In the absence of the dielectric, the electric field d to each circular plate would be $\sigma/2\epsilon_0$, as given by Eq. 26-20. T fields due to the two plates are in the same direction, so they a to give a net field of

$$E_0 = \frac{\sigma}{\epsilon_0} = \frac{2.88 \times 10^{-7}\ \text{C/m}^2}{8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2} = 3.25 \times 10^4\ \text{N/C}.$$

With the dielectric present, the net field is

$$E = \frac{E_0}{\kappa_e} = \frac{3.25 \times 10^4\ \text{N/C}}{4.7} = 6.9 \times 10^3\ \text{N/C}.$$

(*b*) The polarization field due to the induced surface charge is

$$E' = E_0 - E = 3.25 \times 10^4\ \text{N/C} - 6.9 \times 10^3\ \text{N/C} = 2.56 \times 10^4\ \text{N/C}.$$

The two sheets of induced charges set up the electric field E' ju as the two sheets of free charges set up the field E_0. With E' = σ_{ind}/ϵ_0, we have

$$\sigma_{ind} = \epsilon_0 E' = (8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(2.56 \times 10^4\ \text{N/C}) = 2.27 \times 10^{-7}\ \text{C/m}^2.$$

MULTIPLE CHOICE

29-1 Types of Materials

29-2 A Conductor in an Electric Field: Static Conditions

1. A triangular conductor is placed in an originally uniform electric field. (*a*) Which picture in Fig. 29-13 best represents the static electric field lines near the conductor? (*b*) Which picture in Fig. 29-13 best represents the static equipotential lines near the conductor?

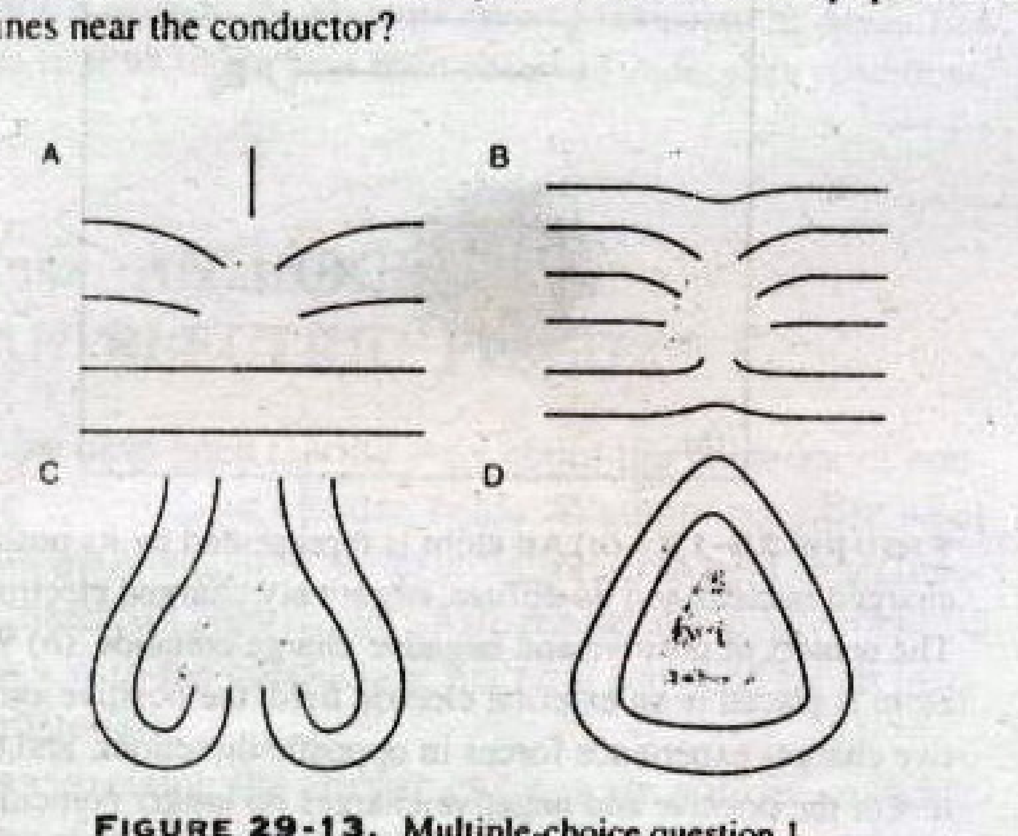


FIGURE 29-13. Multiple-choice question 1.

2. A point charge is placed inside an uncharged spherical conducting shell. Which picture in Fig. 29-14 best shows the electric field lines?

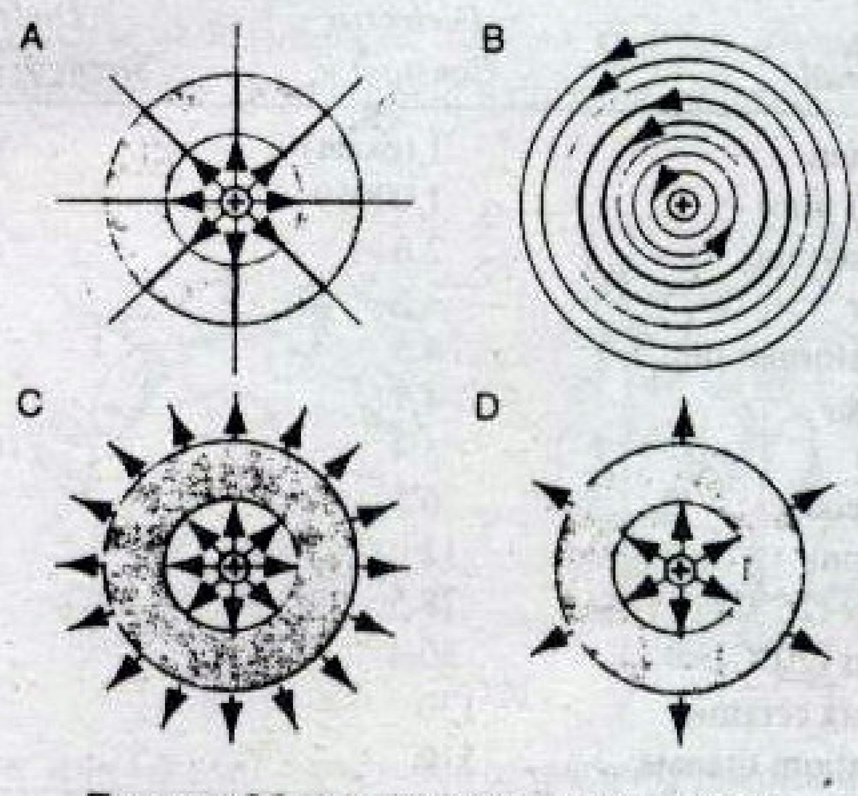


FIGURE 29-14. Multiple-choice question 2.

3. A point charge is placed inside an uncharged spherical conducting shell. Which picture in Fig. 29-15 best shows the electric field lines?

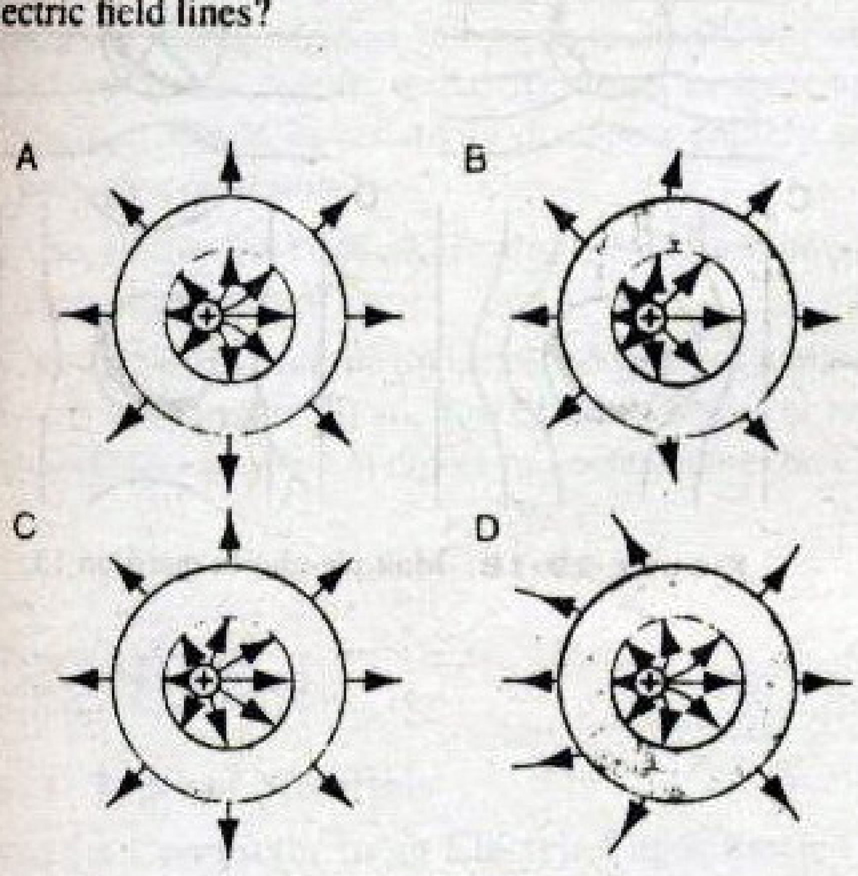


FIGURE 29-15. Multiple-choice question 3.

29-3 A Conductor in an Electric Field: Dynamic Conditions

4. Both current and current density have directions associated with them. Are they vectors?
   (A) Only current is a vector.
   (B) Only current density is a vector.
   (C) Both current and current density are vectors.
   (D) Neither current nor current density is a vector.

5. A constant current flows through a conical conductor as shown in Fig. 29-16. End surfaces S_1 and S_2 are two different equipotential surfaces.
   (*a*) Through which plane does the greatest current flow?
   (A) 1 (B) 2 (C) 3 (D) 4
   (E) The current is the same through all.
   (*b*) Through which plane is the greatest electric flux?
   (A) 1 (B) 2 (C) 3 (D) 4
   (E) The electric flux is the same through all.
   (*c*) How does the magnitude of the electric field E vary along the central axis moving from S_1 to S_2?
   (A) E is constant. (B) E increases.
   (C) E decreases.

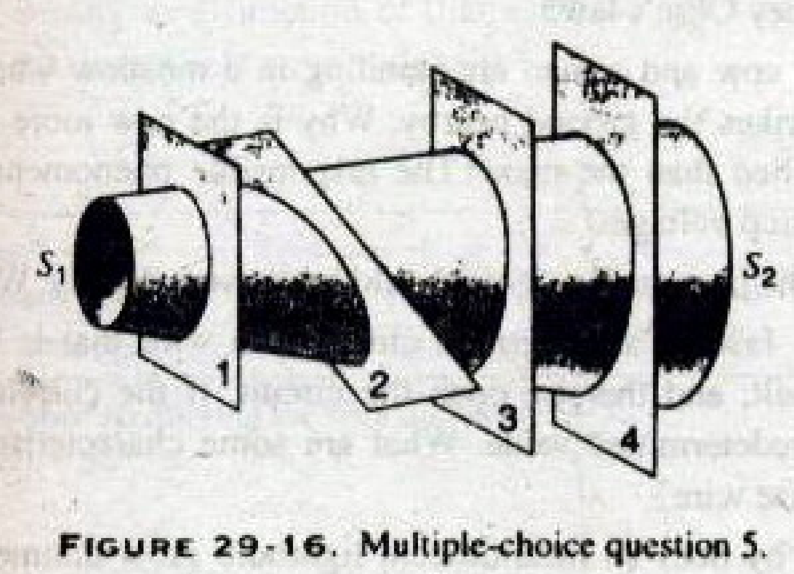


FIGURE 29-16. Multiple-choice question 5.

6. A current flows through of a long cylindrical conductor. In which direction does the current flow?
   (A) Toward the end with the higher potential
   (B) Toward the end with the lower potential
   (C) Neither (A) nor (B), since the surface of a conductor is an equipotential

29-4 Ohmic Materials

7. Two identically shaped wires, A and B, carry identical currents. The wires are made of different substances having differing electron densities, with $n_A > n_B$.
   (*a*) Which wire will have the largest current density?
   (A) A (B) B (C) The wires are the same.
   (*b*) Which wire will have the larger drift speed for the electrons?
   (A) A (B) B (C) The wires are the same.
   (*c*) Which wire will have the larger electric field E in its interior?
   (A) A (B) B (C) The wires are the same.

8. The current–voltage relationship for a certain substance is shown in Fig. 29-17. This substance is ohmic for
   (A) all values of ΔV. (B) ΔV between 0 and 3 V.
   (C) ΔV greater than 3 V. (D) no values of ΔV.

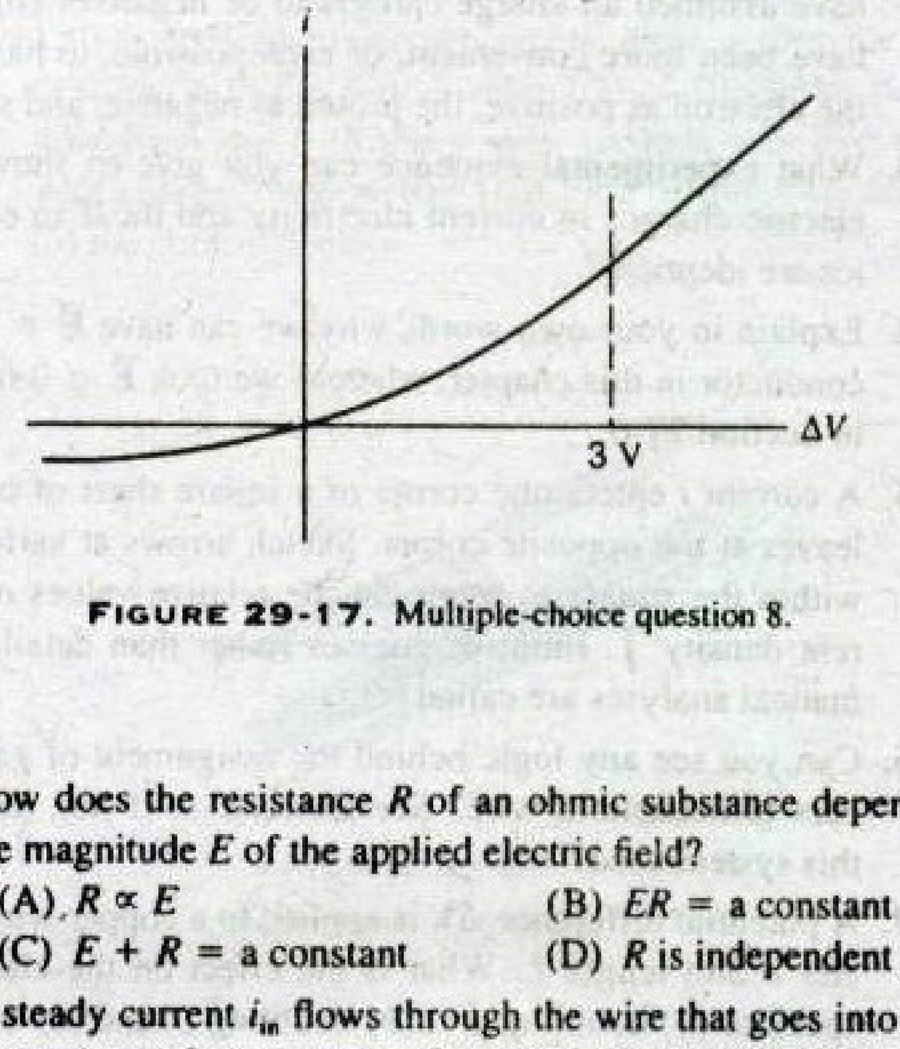


FIGURE 29-17. Multiple-choice question 8.

9. How does the resistance R of an ohmic substance depend on the magnitude E of the applied electric field?
   (A) $R \propto E$ (B) ER = a constant
   (C) $E + R$ = a constant (D) R is independent of E.

10. A steady current i_{in} flows through the wire that goes into a resistor. A steady current i_{out} flows through the wire that comes out the other end of the resistor.
   (*a*) How do i_{in} and i_{out} compare?
   (A) $i_{in} > i_{out}$ (B) $i_{in} < i_{out}$
   (C) $i_{in} = i_{out}$ always (D) $i_{in} = i_{out}$ only if $R = 0$
   (*b*) What can be concluded about the potential V_{in} on the end of the wire where the current enters and the potential V_{out} on the end of the wire where the current exits?
   (A) $V_{in} > V_{out}$ (B) $V_{in} < V_{out}$
   (C) $V_{in} = V_{out}$ always
   (D) Nothing, unless more information is given

29-5 Ohm's Law: A Microscopic View

11. How does the drift speed of electrons change as they move through a resistor?
   (A) It increases. (B) It decreases.
   (C) It remains the same.

12. The resistivity of most conductors increases with temperature. A plausible reason is that, in a conductor,
(A) the electron density changes with temperature.
(B) the charge on each electron changes with temperature.
(C) the time between collisions changes with temperature.
(D) the mass of the electron changes with temperature.

29-6 An Insulator in an Electric Field

13. A spherical insulator is placed in an originally uniform electric field.

(*a*) Which picture in Fig. 29-18 best represents the static electric field lines near and inside the insulator?

(*b*) Which picture in Fig. 29-18 best represents the static equipotential lines near and inside the insulator?

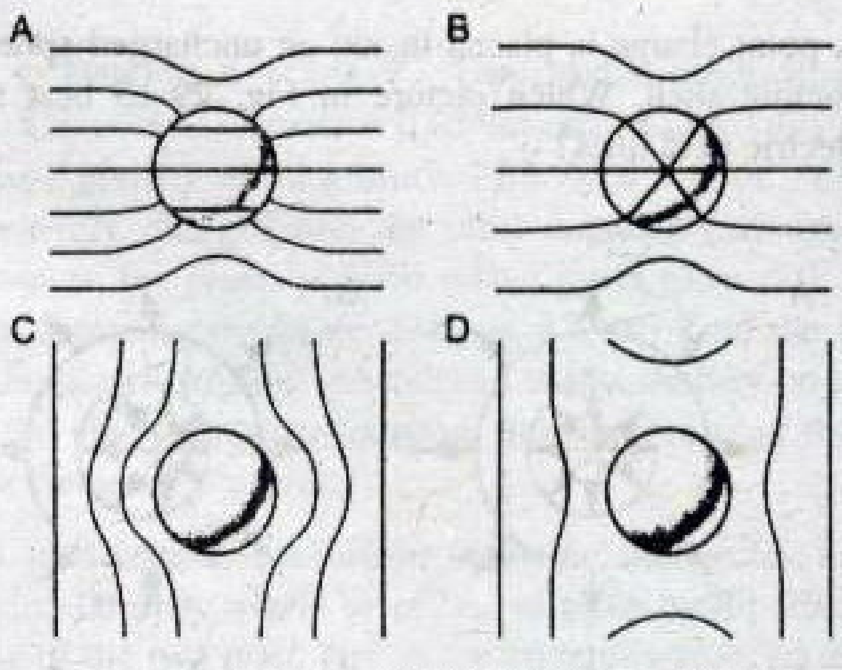


FIGURE 29-18. Multiple-choice question 13.

QUESTIONS

1. Name other physical quantities that, like current, are scalars having a sense represented by an arrow in a diagram.

2. In our convention for the direction of current arrows (*a*) would it have been more convenient, or even possible, to have assumed all charge carriers to be negative? (*b*) Would it have been more convenient, or even possible, to have labeled the electron as positive, the proton as negative, and so on?

3. What experimental evidence can you give to show that the electric charges in current electricity and those in electrostatics are identical?

4. Explain in your own words why we can have $\vec{E} \neq 0$ inside a conductor in this chapter, whereas we took $\vec{E} = 0$ for granted in Section 27-6.

5. A current i enters one corner of a square sheet of copper and leaves at the opposite corner. Sketch arrows at various points within the square to represent the relative values of the current density $\vec{j}$. Intuitive guesses rather than detailed mathematical analyses are called for.

6. Can you see any logic behind the assignment of gauge numbers to household wire? See Exercise 6. If not, then why is this system used?

7. A potential difference ΔV is applied to a copper wire of diameter d and length L. What is the effect on the electron drift speed of (*a*) doubling ΔV, (*b*) doubling L, and (*c*) doubling d?

8. Why is it not possible to measure the drift speed for electrons by timing their travel along a conductor?

9. Describe briefly some possible designs of variable resistors.

10. A potential difference ΔV is applied to a circular cylinder of carbon by clamping it between circular copper electrodes, as

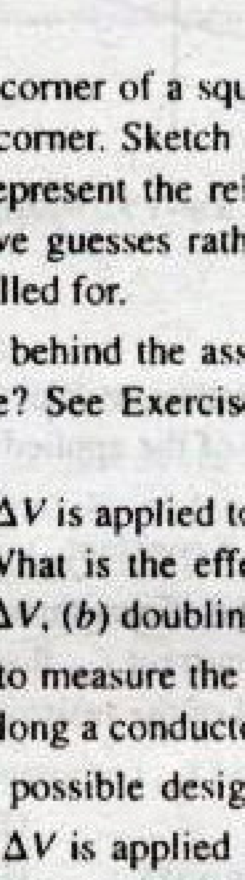


FIGURE 29-19. Question 10.

in Fig. 29-19. Discuss the difficulty of calculating the resi tance of the carbon cylinder using the relation $R = \rho L/A$.

11. You are given a cube of aluminum and access to two batte terminals. How would you connect the terminals to the cub to ensure (*a*) a maximum and (*b*) a minimum resistance?

12. How would you measure the resistance of a pretzel-shape block of metal? Give specific details to clarify the concept.

13. Sliding across the seat of an automobile can generate poten tials of several thousand volts. Why is the slider not electro cuted?

14. Discuss the difficulties of testing whether the filament of lightbulb obeys Ohm's law.

15. Will the drift velocity of electrons in a current-carrying meta conductor change when the temperature of the conductor i increased? Explain.

16. Explain why the momentum that conduction electrons trans fer to the ions in a metal conductor does not give rise to a re sultant force on the conductor.

17. List in tabular form similarities and differences between th flow of charge along a conductor, the flow of water through horizontal pipe, and the conduction of heat through a slab Consider such ideas as what causes the flow, what opposes i what particles (if any) participate, and the units in which th flow may be measured.

18. How does the relation $\Delta V = iR$ apply to resistors that do no obey Ohm's law?

19. A cow and a man are standing in a meadow when lightnin strikes the ground nearby. Why is the cow more likely to b killed than the man? The responsible phenomenon is calle "step voltage."

20. The lines in Fig. 29-9 should be curved slightly. Why?

21. A fuse in an electrical circuit is a wire that is designed t melt, and thereby open the circuit, if the current exceeds predetermined value. What are some characteristics of idea fuse wire?

22. Why does an incandescent lightbulb grow dimmer with use

23. The character and quality of our daily lives are influence greatly by devices that do not obey Ohm's law. What can yo say in support of this claim?

24. From a student's paper: "The relationship $R = \Delta V/i$ tells u that the resistance of a conductor is directly proportional to the potential difference applied to it." What do you think of this proposition?

25. Carbon has a negative temperature coefficient of resistivity, which means that its resistivity drops as its temperature increases. Would its resistivity disappear entirely at some high enough temperature?

26. Can a dielectric conduct electricity? Can a conductor have dielectric properties?

27. Would you expect the dielectric constant of a material to vary with temperature? If so, how? Does whether or not the molecules have permanent dipole moments matter here?

28. Show that the dielectric constant of a conductor can be taken to be infinitely great.

29. An electric field can polarize gases in several ways: by distorting the electron clouds of molecules, by orienting pola molecules, by bending or stretching the bonds in polar molecules. How does this differ from polarization of molecules ir liquids and solids?

30. A dielectric object in a nonuniform electric field experiences a net force. Why is there no net force if the field is uniform?

31. A stream of tap water can be deflected if a charged rod is brought close to the stream. Explain carefully how this happens.

EXERCISES

29-1 Types of Materials

29-2 A Conductor in an Electric Field: Static Conditions

29-3 A Conductor in an Electric Field: Dynamic Conditions

1. A current of 4.82 A exists in a 12.4 Ω resistor for 4.60 min. (*a*) How much charge and (*b*) how many electrons pass through any cross section of the resistor in this time?

2. The current in the electron beam of a typical video display terminal is 200 μA. How many electrons strike the screen each minute?

3. Suppose that we have 2.10×10^8 doubly charged positive ions per cubic centimeter, all moving north with a speed of 1.40×10^5 m/s. (*a*) Calculate the current density, in magnitude and direction. (*b*) Can you calculate the total current in this ion beam? If not, what additional information is needed?

4. A small but measurable current of 123 pA exists in a copper wire whose diameter is 2.46 mm. Calculate (*a*) the current density and (*b*) the electron drift speed. See Sample Problem 29-3.

5. Suppose that the material composing a fuse (see Question 21) melts once the current density rises to 440 A/cm². What diameter of cylindrical wire should be used for the fuse to limit the current to 0.552 A?

6. The (United States) National Electric Code, which sets maximum safe currents for rubber-insulated copper wires of various diameters, is given (in part) below. Plot the safe current density as a function of diameter. Which wire gauge has the maximum safe current density?

Gauge[a]	4	6	8	10	12	14	16	18
Diameter (mils)[b]	204	162	129	102	81	64	51	40
Safe current (A)	70	50	35	25	20	15	6	3

[a] A way of identifying the wire diameter.
[b] 1 mil = 10^{-3} in.

7. A current is established in a gas discharge tube when a sufficiently high potential difference is applied across the two electrodes in the tube. The gas ionizes; electrons move toward the positive terminal and singly charged positive ions toward the negative terminal. What are the magnitude and direction of the current in a hydrogen discharge tube in which 3.1×10^{18} electrons and 1.1×10^{18} protons move past a cross sectional area of the tube each second?

8. A *pn* junction is formed from two different semiconducting materials in the form of identical cylinders with radiu 0.165 mm, as depicted in Fig. 29-20. In one application 3.50×10^{15} electrons per second flow across the junction from the *n* to the *p* side while 2.25×10^{15} holes per second flow from the *p* to the *n* side. (A hole acts like a particle with charge $+1.6 \times 10^{-19}$ C.) Find (*a*) the total current and (*b*) the current density.

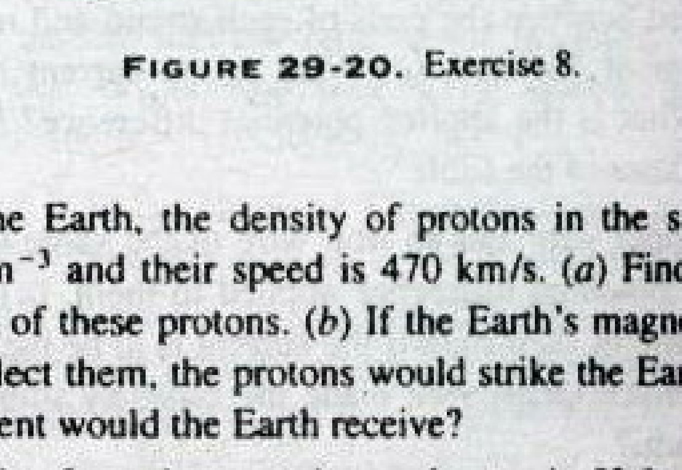


FIGURE 29-20. Exercise 8.

9. Near the Earth, the density of protons in the solar wind i 8.70 cm^{-3} and their speed is 470 km/s. (*a*) Find the current density of these protons. (*b*) If the Earth's magnetic field did not deflect them, the protons would strike the Earth. What total current would the Earth receive?

10. The belt of an electrostatic accelerator is 52.0 cm wide and travels at 28.0 m/s. The belt carries charge into the sphere at rate corresponding to 95.0 μA. Compute the surface charge density on the belt. See Section 28-10.

11. How long does it take electrons to get from a car battery to the starting motor? Assume that the current is 115 A and the electrons travel through copper wire with cross-sectional area 31.2 mm² and length 85.5 cm. See Sample Problem 29-3.

29-4 Ohmic Materials

12. A human being can be electrocuted if a current as small a 50 mA passes near the heart. An electrician working wit sweaty hands makes good contact with two conductor

being held one in each hand. If the electrician's resistance is 1800 Ω, what might be the fatal voltage difference? (Electricians often work with "live" wires.)

13. A steel trolley-car rail has a cross-sectional area of 56 cm^2. What is the resistance of 11 km of rail? The resistivity of the steel is $3.0 \times 10^{-7}\ \Omega \cdot m$.

14. From the slope of the line in Fig. 29-8, estimate the average temperature coefficient of resistivity for copper at room temperature and compare with the value given in Table 29-1.

15. A wire 4.0 m long and 6.0 mm in diameter has a resistance of 15 mΩ. A potential difference of 23 V is applied between the ends. (*a*) What is the current in the wire? (*b*) Calculate the current density. (*c*) Calculate the resistivity of the wire material. Can you identify the material? See Table 29-1.

16. The copper windings of a motor have a resistance of 50 Ω at 20°C when the motor is idle. After running for several hours the resistance rises to 58 Ω. What is the temperature of the windings? Ignore changes in the dimensions of the windings. See Table 29-1.

17. Show that if changes in the dimensions of a conductor with changing temperature can be ignored, then the resistance varies with temperature according to $R - R_0 = \alpha_{av}R_0(T - T_0)$.

18. A coil is formed by winding 250 turns of insulated gauge 8 copper wire (see Exercise 6) in a single layer on a cylindrical form whose radius is 12.2 cm. Find the resistance of the coil. Neglect the thickness of the insulation. See Table 29-1.

19. Two conductors are made of the same material and have the same length. Conductor *A* is a solid wire of diameter *D*. Conductor *B* is a hollow tube of outside diameter 2*D* and inside diameter *D*. Find the resistance ratio, R_A/R_B, measured between their ends.

20. What must be the diameter of an iron wire if it is to have the same resistance as a copper wire 1.19 mm in diameter, both wires being the same length?

21. An electrical cable consists of 125 strands of fine wire, each having 2.65-μΩ resistance. The same potential difference is applied between the ends of each strand and results in a total current of 750 mA. (*a*) What is the current in each strand? (*b*) What is the applied potential difference? (*c*) What is the resistance of the cable?

22. A copper wire and an iron wire of the same length have the same potential difference applied to them. (*a*) What must be the ratio of their radii if the current is to be the same? (*b*) Can the current density be made the same by suitable choices of the radii?

23. When a potential difference of 115 V is applied between the ends of a 9.66-m-long wire, the current density is 1.42 A/cm^2. Calculate the conductivity of the wire material.

24. In the lower atmosphere of the Earth there are negative and positive ions, created by radioactive elements in the soil and cosmic rays from space. In a certain region, the atmospheric electric field strength is 120 V/m, directed vertically down. Due to this field, singly charged positive ions, 620 per cm^3, drift downward, and singly charged negative ions, 550 per cm^3, drift upward; see Fig. 29-21. The measured conductivity is $2.70 \times 10^{-14}/\Omega \cdot m$. Calculate (*a*) the ion drift speed, assumed to be the same for positive and negative ions; and (*b*) the current density.

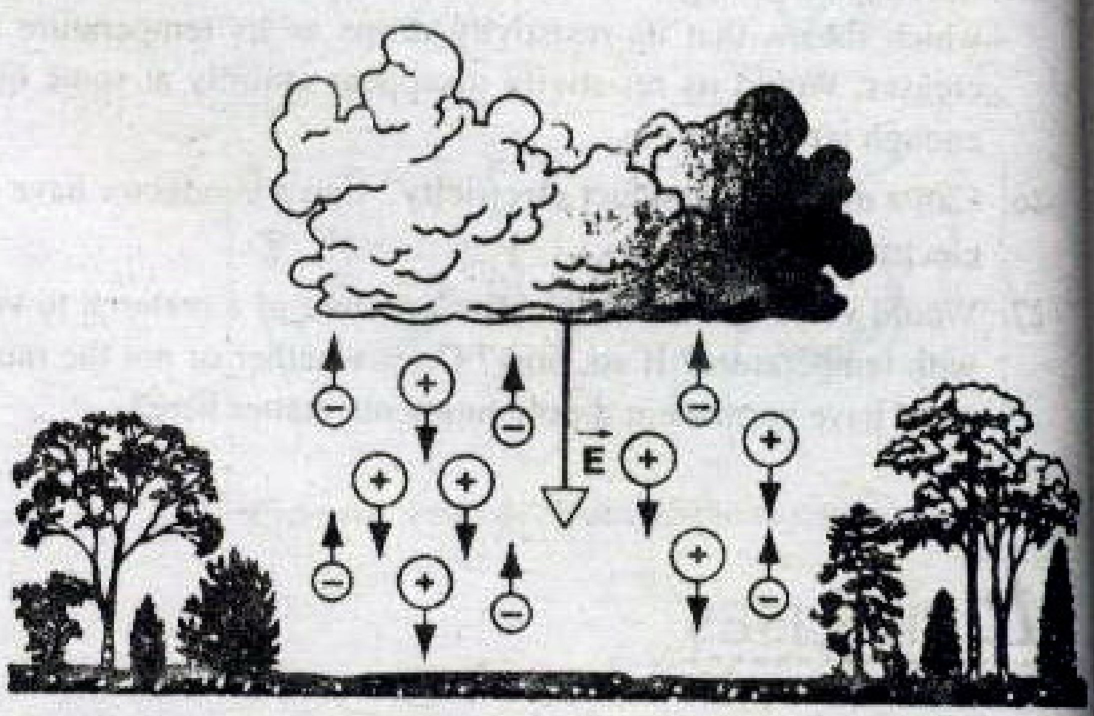


FIGURE 29-21. Exercise 24.

25. Copper and aluminum are being considered for a high-voltage transmission line that must carry a current of 62.3 A. The resistance per unit length is to be 0.152 Ω/km. Compute for each choice of cable material (*a*) the current density and (*b*) the mass of 1.00 m of the cable. The densities of copper and aluminum are 8960 and 2700 kg/m^3, respectively.

26. Using data from Fig. 29-7*b*, plot the resistance of the *pn* junction diode as a function of applied potential difference.

27. For a hypothetical electronic device, the potential difference ΔV, measured across the device, is related to the current *i* by $\Delta V = (3.55 \times 10^6\ V/A^2)i^2$. (*a*) Find the resistance when the current is 2.40 mA. (*b*) At what value of the current is the resistance equal to 16.0 Ω?

29-5 Ohm's Law: A Microscopic View

28. Calculate the mean free time between collisions for conduction electrons in aluminum at 20°C. Each atom of aluminum contributes three conduction electrons. Take needed data from Table 29-1 and Appendix D. See also Sample Problem 29-3.

29-6 An Insulator in an Electric Field

29. A 1-μC point charge is embedded in the center of a solid Pyrex sphere of radius *R* = 10 cm. (*a*) Calculate the electric field strength *E* just beneath the surface of the sphere. (*b*) Assuming that there are no other *free* charges, calculate the strength of the electric field just outside the surface of the sphere. (*c*) What is the induced surface charge density σ_{ind} on the surface of the Pyrex sphere?

30. Two equal but opposite point charges +*q* and −*q* are separated by a distance of 10 cm in air. What value of *q* will provide for an electric field strength midway between the charges that will exceed the dielectric strength of air?

31. A spherical conductor of radius *R* is at a potential *V*; assume *V* = 0 at infinity. (*a*) What is the minimum value of *V* that will result in an electric field strength just above the surface of the sphere that will exceed the dielectric strength of air? (*b*) Is it easier to get a "spark" from a ball at a given potential with a larger or a smaller radius? (*c*) Use your answer to explain why lightning rods are pointed.

PROBLEMS

1. You are given an isolated conducting sphere of 13-cm radius. One wire carries a current of 1.0000020 A into it. Another wire carries a current of 1.0000000 A out of it. How long would it take for the sphere to increase in potential by 980 V?

2. In a hypothetical fusion research lab, high-temperature helium gas is completely ionized, each helium atom being separated into two free electrons and the remaining positively charged nucleus (alpha particle). An applied electric field causes the alpha particles to drift to the east at 25 m/s while the electrons drift to the west at 88 m/s. The alpha particle density is 2.8×10^{15} cm^{-3}. Calculate the net current density; specify the current direction.

3. A 4.0-cm-long caterpillar crawls in the direction of electron drift along a 5.2-mm-diameter bare copper wire that carries a current of 12 A. (*a*) Find the potential difference between the two ends of the caterpillar. (*b*) Is its tail positive or negative compared to its head? (*c*) How much time could it take the caterpillar to crawl 1.0 cm and still keep up with the drifting electrons in the wire?

4. A steady beam of alpha particles ($q = 2e$) traveling with kinetic energy 22.4 MeV carries a current of 250 nA. (*a*) If the beam is directed perpendicular to a plane surface, how many alpha particles strike the surface in 2.90 s? (*b*) At any instant, how many alpha particles are there in a given 18.0-cm length of the beam? (*c*) Through what potential difference was it necessary to accelerate each alpha particle from rest to bring it to an energy of 22.4 MeV?

5. In the two intersecting storage rings of circumference 950 m at CERN, protons of kinetic energy 28.0 GeV formed beams of current 30.0 A each. (*a*) Find the total charge carried by the protons in each ring. Assume that the protons travel at very nearly the speed of light. (*b*) A beam is deflected out of a ring onto a 43.5-kg copper block. By how much does the temperature of the block rise?

6. (*a*) The current density across a cylindrical conductor of radius R varies according to the equation

$$j = j_0(1 - r/R),$$

where r is the distance from the axis. Thus the current density is a maximum j_0 at the axis $r = 0$ and decreases linearly to zero at the surface $r = R$. Calculate the current in terms of j_0 and the conductor's cross-sectional area $A = \pi R^2$. (*b*) Suppose that, instead, the current density is a maximum j_0 at the surface and decreases linearly to zero at the axis, so that

$$j = j_0 r/R.$$

Calculate the current. Why is the result different from (*a*)?

7. (*a*) At what temperature would the resistance of a copper conductor be double its resistance at 20°C? (Use 20°C as the reference point in Eq. 29-16; compare your answer with Fig. 29-8.) (*b*) Does this same temperature hold for all copper conductors, regardless of shape or size?

8. A common flashlight bulb is rated at 310 mA and 2.90 V, the respective values of the current and voltage under operating conditions. If the resistance of the bulb filament when cold (T_0 = 20°C) is 1.12 Ω, calculate the temperature of the filament when the bulb is on. The filament is made of tungsten. Assume that Eq. 29-16 holds over the temperature range encountered.

9. A wire with a resistance of 6.0 Ω is drawn out through a die so that its new length is three times its original length. Find the resistance of the longer wire, assuming that the resistivity and density of the material are not changed during the drawing process.

10. A block in the shape of a rectangular solid has a cross-sectional area of 3.50 cm^2, a length of 15.8 cm, and a resistance of 935 Ω. The material of which the block is made has 5.33×10^{22} conduction electrons/m^3. A potential difference of 35.8 V is maintained between its ends. (*a*) Find the current in the block. (*b*) Assuming that the current density is uniform, what is its value? Calculate (*c*) the drift velocity of the conduction electrons and (*d*) the electric field in the block.

11. A rod of a certain metal is 1.6 m long and 5.5 mm in diameter. The resistance between its ends (at 20°C) is 1.09×10^{-3} Ω. A round disk is formed of this same material, 2.14 cm in diameter and 1.35 mm thick. (*a*) What is the material? (*b*) What is the resistance between the opposing round faces, assuming equipotential surfaces?

12. When a metal rod is heated, not only its resistance but also its length and its cross-sectional area change. The relation $R = \rho L/A$ suggests that all three factors should be taken into account in measuring ρ at various temperatures. (*a*) If the temperature changes by 1.0 C°, what fractional changes in R, L, and A occur for a copper conductor? (*b*) What conclusion do you draw? The coefficient of linear expansion is 1.7×10^{-5}/C°.

13. It is desired to make a long cylindrical conductor whose temperature coefficient of resistivity at 20°C will be close to zero. If such a conductor is made by assembling alternate disks of iron and carbon, find the ratio of the thickness of a carbon disk to that of an iron disk. (For carbon, $\rho = 3500 \times 10^{-8}$ Ω·m and $\alpha = -0.50 \times 10^{-3}$/C°.)

14. A resistor is in the shape of a truncated right circular cone (Fig. 29-22). The end radii are a and b, and the length is L. If the taper is small, we may assume that the current density is uniform across any cross section. (*a*) Calculate the resistance of this object. (*b*) Show that your answer reduces to $\rho L/A$ for the special case of zero taper ($a = b$).

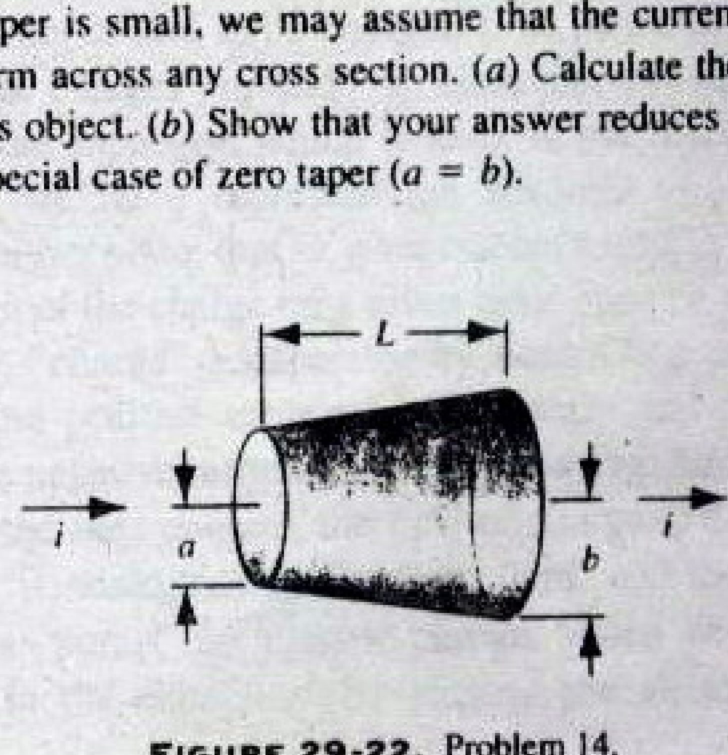


FIGURE 29-22. Problem 14.

15. A resistor is in the shape of a spherical shell, with an inside surface of radius of a covered with a conducting material and an outside surface of radius b covered with a conducting material. Assuming a uniform resistivity ρ, calculate the resistance between the conducting surfaces.

16. Show that, according to the free-electron model of electrical conduction in metals and classical physics, the resistivity of metals should be proportional to $\sqrt{T}$, where T is absolute temperature. (Hint: Treat the electrons as an ideal gas.)

COMPUTER PROBLEM

1. Assume a lightbulb has a tungsten filament that radiates energy at a rate proportional to the temperature difference between the filament and room temperature. Call the constant of proportionality C. Estimate C for a 120-watt bulb in a 120-volt circuit, assuming that all the energy transferred to the filament is radiated as heat transferred to the environment; assume the temperature of the filament is 2500 °C. (*a*) Numerically generate a graph that shows the equilibrium temperature of the bulb as a function of the applied potential difference, remembering that the resistivity of tungsten changes with temperature. (*b*) At what applied voltage must the bulb "burn-out"? (Hint: The filament will melt if it gets too hot.) (*c*) Repeat the above procedure, except now assume that energy is radiated from the bulb according to $k(T^4 - T_0^4)$, where k is a constant you must determine, T_0 is room temperature in kelvin, and T is the filament temperature in kelvin. Compare your results.

CHAPTER 30

CAPACITANCE

In many applications of electric circuits, the goal is to store electrical charge or energy in an electrostatic field. A device that stores charge is called a capacitor, *and the property that determines how much charge it can store is its* capacitance. *We shall see that the capacitance depends on the geometrical properties of the device and not on the electric field or the potential.*

In this chapter we define capacitance and show how to calculate the capacitance of a few simple devices and of combinations of capacitors. We study the energy stored in capacitors and show how it is related to the strength of the electric field. Finally, we investigate how the presence of a dielectric in a capacitor enhances its ability to store electric charge.

30-1 CAPACITORS

A *capacitor** is a device that stores energy in an electrostatic field. A flashbulb, for example, requires a short burst of electric energy that exceeds what a battery can generally provide. A capacitor can draw energy relatively slowly (over several seconds) from the battery, and it then can release the energy rapidly (within milliseconds) through the bulb. Much larger capacitors are used to produce short laser pulses in attempts to induce thermonuclear fusion in tiny pellets of hydrogen. In this case the power level during the pulse is about 10^{14} W, about 200 times the entire electrical generating capacity of the United States, but the pulses typically last only for 10^{-9} s.

Capacitors are also used to produce electric fields, such as the parallel-plate device that gives the very nearly uniform electric field that deflects beams of electrons in a TV or oscilloscope tube.

In circuits, capacitors are often used to smooth out the sudden variations in line voltage that can damage computer memories. In another application, the tuning of a radio or TV receiver is usually done by varying the capacitance of the circuit.

*See "Capacitors," by Donald M. Trotter, Jr., *Scientific American*, July 1988, p. 86.

30-2 CAPACITANCE

Figure 30-1 shows a generalized capacitor, consisting of two conductors *a* and *b* of arbitrary shape. No matter what their geometry, these conductors are called *plates*. We assume that they are totally isolated from their surroundings. We further assume, for the time being, that the conductors exist in a vacuum.

A capacitor is said to be *charged* if its plates carry equal and opposite charges $+q$ and $-q$. Note that q is *not* the net charge on the capacitor, which is zero. In our discussion of capacitors, we let q represent the absolute value of the charge on either plate; that is, q represents a magnitude only, and the sign of the charge on a given plate must be specified.

We can "charge" a capacitor by connecting one of its plates to the positive terminal of a battery and the other plate to the negative terminal, as shown in Fig. 30-2. As we discuss in the next chapter, the flow of charge in an electrical circuit is analogous to the flow fluid, and the battery serves as a "pump" for electric charge. When we connect a battery to the capacitor (by closing the switch in the

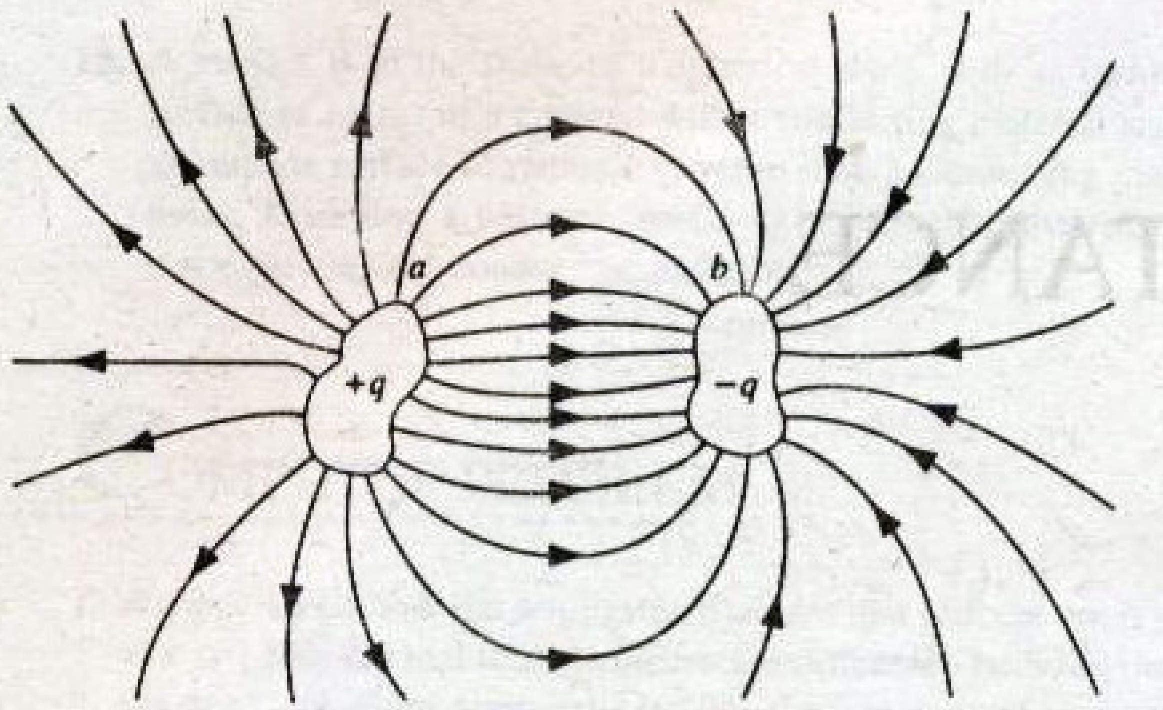


FIGURE 30-1. Two conductors, isolated from one another and from their surroundings, form a capacitor. When the capacitor is charged, the conductors carry equal but opposite charges of magnitude q. The two conductors are called *plates* no matter what their shape.

circuit), the battery "pumps" electrons from the (previously uncharged) positive plate of the capacitor to the negative plate. After the battery moves a quantity of charge of magnitude q, the charge on the positive plate is $+q$ and the charge on the negative plate is $-q$.

An ideal battery maintains a constant potential difference between its terminals. The positive plate and the wire connecting it to the positive terminal of the battery are conductors, and so (under electrostatic conditions) they must be at the same potential V_+ as the positive terminal of the battery. The negative plate and the wire connecting it to the negative terminal of the battery are also conductors, and so (when the switch is closed) they must be at the same potential V_- as the negative terminal of the battery. The potential difference $\Delta V = V_+ - V_-$ between the battery terminals is the same potential difference that appears between the capacitor plates when the switch is closed. We usually describe this as the potential difference "across" the capacitor, meaning the potential difference between its plates.

Figure 30-3 shows the circuit for charging a capacitor by a battery that maintains a constant potential difference

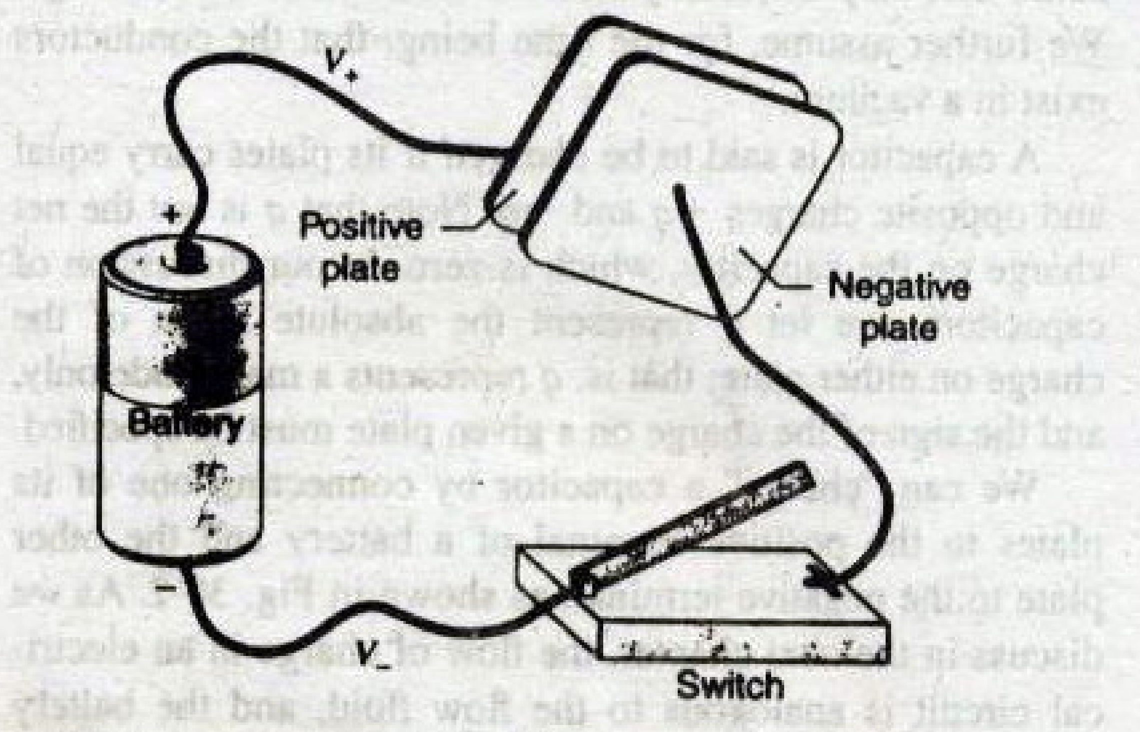


FIGURE 30-2. When the switch is closed, the capacitor becomes charged as the battery moves electrons from the positive plate to the negative plate.

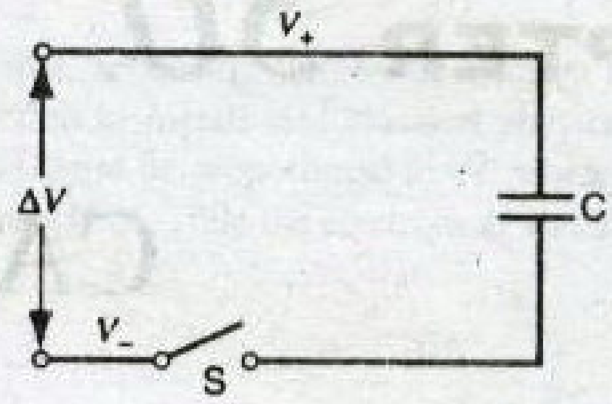


FIGURE 30-3. A schematic circuit diagram equivalent to Fig. 30-2, showing the capacitor C, switch S, and constant potential difference ΔV (supplied by a battery that is not shown in the diagram).

$\Delta V = V_+ - V_-$ between its terminals. In a circuit, a capacitor is represented by the symbol ⊣⊢, in which the two parallel lines suggest the two plates of the capacitor.

When we charge a capacitor, we find that the charge q that appears on the capacitor plates is always directly proportional to the potential difference ΔV between the plates: $q \propto \Delta V$. The *capacitance* C is the constant of proportionality necessary to make this relationship into an equation:

$$q = C\,\Delta V. \tag{30-1}$$

The capacitance is a geometrical factor that depends on the size, shape, and separation of the plates and on the material that occupies the space between the plates (which for now we assume is a vacuum). The capacitance of a capacitor does *not* depend on ΔV or q.

The SI unit of capacitance that follows from Eq. 30-1 is the coulomb/volt, which is given the name *farad* (abbreviation F):

$$1 \text{ farad} = 1 \text{ coulomb/volt}.$$

The unit is named in honor of Michael Faraday who, among his other contributions, developed the concept of capacitance. The submultiples of the farad, the *microfarad* ($1\ \mu F = 10^{-6}$ F) and the *picofarad* ($1\ pF = 10^{-12}$ F), are more convenient units in practice. Figure 30-4 shows some capacitors in the microfarad or picofarad range that might be found in electronic or computing equipment.

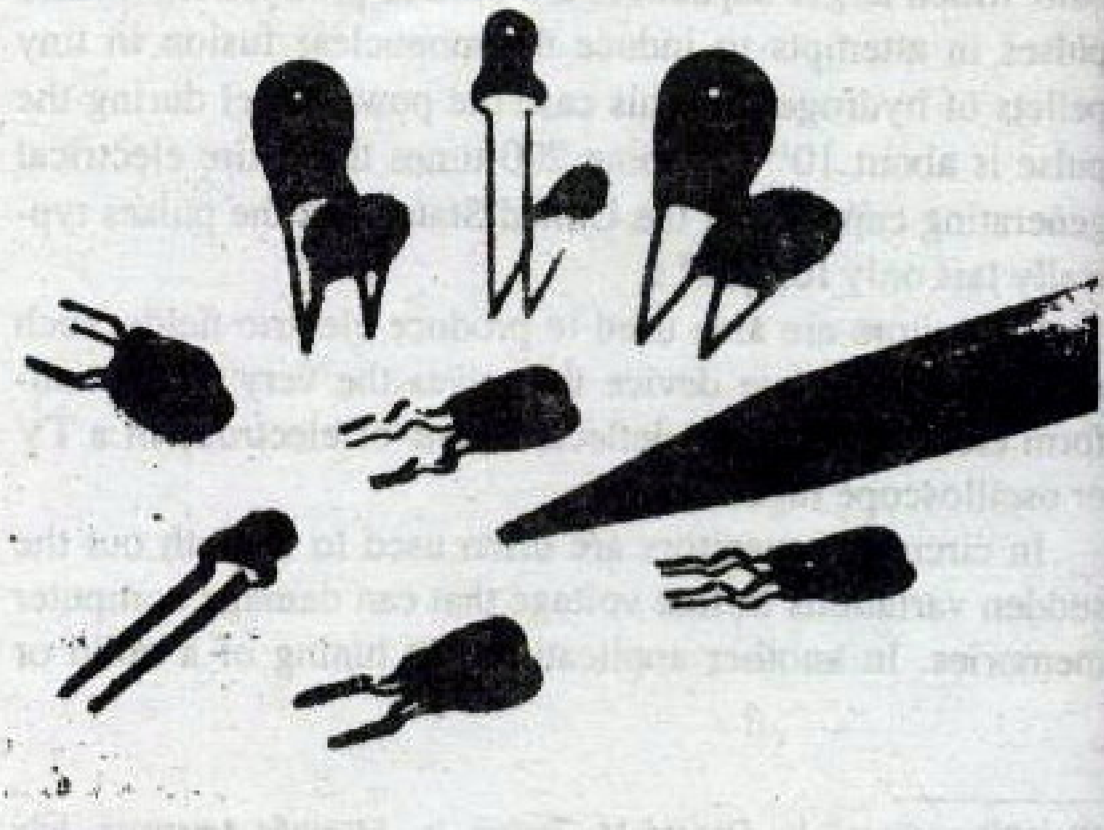


FIGURE 30-4. An assortment of capacitors that might be found in electronic circuits.

SAMPLE PROBLEM 30-1. A storage capacitor on a random access memory (RAM) chip has a capacitance of 0.055 pF. If it is charged to 5.3 V, how many excess electrons are there on its negative plate?

Solution If the negative plate has N excess electrons, it carries a net charge of magnitude $q = Ne$. Using Eq. 30-1, we obtain

$$N = \frac{q}{e} = \frac{C\,\Delta V}{e} = \frac{(0.055 \times 10^{-12}\ \mathrm{F})(5.3\ \mathrm{V})}{1.60 \times 10^{-19}\ \mathrm{C}}$$

$$= 1.8 \times 10^{6}\ \text{electrons}.$$

For electrons, this is a very small number. A speck of household dust, so tiny that it essentially never settles, contains about 10^{17} electrons (and the same number of protons).

Analogy with Fluid Flow (Optional)

In situations involving electric circuits, it is often useful to draw analogies between the movement of electric charge and the movement of material particles such as occurs in fluid flow. In the case of a capacitor, an analogy can be made between a capacitor carrying a charge q and a rigid container of volume v (we use v rather than V for volume so as not to confuse it with potential difference) containing n moles of an ideal gas. The gas pressure p is directly proportional to n for a fixed temperature, according to the ideal gas law (Eq. 21-13)

$$n = \left(\frac{v}{RT}\right)p.$$

For the capacitor (Eq. 30-1)

$$q = C\,\Delta V.$$

Comparison shows that the capacitance C of the capacitor is analogous to the volume v of the container, assuming a fixed temperature for the gas. In fact, the word "capacitor" brings to mind the word "capacity," in the same sense that the volume of a container for gas has a certain "capacity."

We can force more gas into the container by imposing a higher pressure, just as we can force more charge into the capacitor by imposing a higher voltage. Note that any amount of charge can be put on the capacitor, and any mass of gas can be put in the container, up to certain limits. These correspond to electrical breakdown ("arcing over") for the capacitor and to rupture of the walls for the container. ■

30-3 CALCULATING THE CAPACITANCE

Our goal in this section is to calculate the capacitance of a capacitor from its geometry. We do this using the following procedure. (1) We first find the electric field in the region between the plates, using methods such as those described in Section 26-4. (2) We then use Eq. 28-15 to find the potential difference between the positive and negative plates by integrating the electric field along any convenient path connecting the plates:

$$\Delta V = V_+ - V_- = -\int_-^+ \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}} = \int_+^- \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}}. \quad (30\text{-}2)$$

(3) The outcome of Eq. 30-2 will involve the magnitude of the charge q on the right-hand side. Using Eq. 30-1, we can then find $C = q/\Delta V$.

As we have defined it, ΔV is a positive number. Since q is an absolute magnitude, the capacitance C will always be positive.

We now illustrate this method with several examples.

A Parallel-Plate Capacitor

Figure 30-5 shows a capacitor in which the two flat plates are very large and very close together; that is, the separation d is much smaller than the length or width of the plates. We can neglect the "fringing" of the electric field that occurs near the edges of the plates and assume that the electric field has the same magnitude and direction everywhere in the volume between the plates.

We obtained the electric field for a single large uniformly charged disk at points near its center in Section 26-4: $E = \sigma/2\epsilon_0$. If the capacitor plates are very large, their shape is not important, and we can assume that the electric field due to each plate has this magnitude. The net electric field is the sum of the fields due to the two plates: $\vec{\mathbf{E}} = \vec{\mathbf{E}}_+ + \vec{\mathbf{E}}_-$. As Fig. 30-5 shows, the fields due to the positive and negative plates have the same direction, so we can write

$$E = E_+ + E_- = \sigma/2\epsilon_0 + \sigma/2\epsilon_0 = \sigma/\epsilon_0. \quad (30\text{-}3)$$

Using $\sigma = q/A$, where A is the surface area of each plate, and substituting Eq. 30-3 into Eq. 30-2, we obtain

$$\Delta V = \int_+^- E\,ds = \frac{q}{\epsilon_0 A}\int_+^- ds = \frac{qd}{\epsilon_0 A}, \quad (30\text{-}4)$$

where we have chosen an integration path along one of the lines of the electric field, so that $\vec{\mathbf{E}}$ and $d\vec{\mathbf{s}}$ are parallel (see Fig. 30-5).

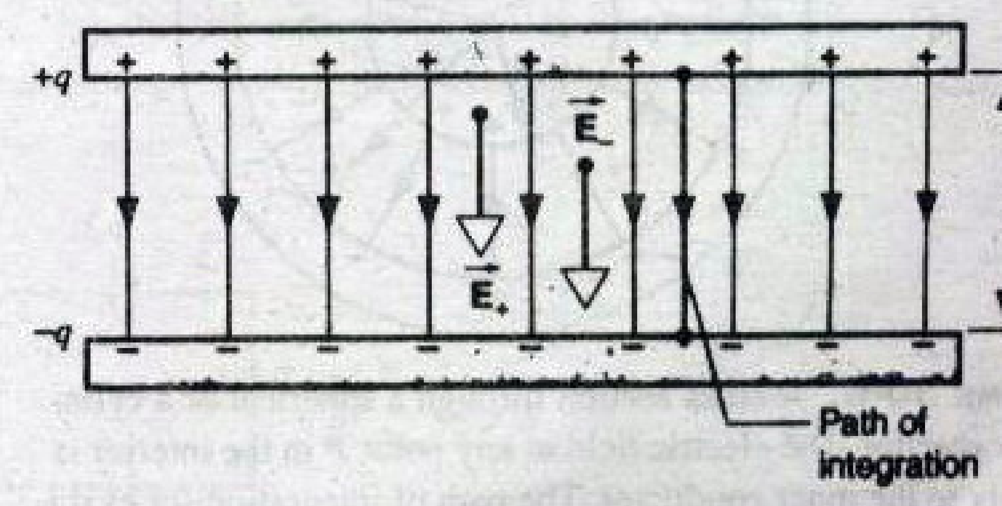


FIGURE 30-5. A parallel-plate capacitor. The path of integration for evaluating Eq. 30-4 is shown.

The capacitance is then obtained from Eq. 30-1: $C = q/\Delta V$, or

$$C = \frac{\epsilon_0 A}{d} \qquad \text{(parallel-plate capacitor).} \qquad (30\text{-}5)$$

You can see from this equation why we say that the capacitance depends on geometrical factors, in this case the plate separation d and area A. The capacitance does not depend on the voltage difference between the plates or the charge carried by the plates.

Note that the right-hand side of Eq. 30-5 has the form of ϵ_0 times a quantity with the dimension of length (A/d). We will find that all expressions for capacitance have essentially this same form, which suggests that the units of ϵ_0 can be expressed as capacitance divided by length:

$$\epsilon_0 = 8.85 \times 10^{-12}\ \text{F/m} = 8.85\ \text{pF/m}.$$

These units for ϵ_0 are often more useful for calculations of capacitance than our previous (and equivalent) units of $C^2/N \cdot m^2$.

A Spherical Capacitor

Figure 30-6 shows a cross section of a spherical capacitor, in which the inner conductor is a solid sphere of radius a, and the outer conductor is a hollow spherical shell of inner radius b. We assume that the inner sphere carries a charge $+q$ and that the outer sphere has a charge $-q$. From our analysis of conductors using Gauss' law (see Section 27-6), we know that the charge on the inner conductor resides on its surface and that the charge on the outer conductor resides on its inner surface. (Draw a spherical Gaussian surface of radius slightly larger than b; the surface lies entirely within the outer conductor, so $E = 0$ everywhere on the surface and the flux through the surface is zero. Therefore the surface encloses no net charge, as Fig. 30-6 shows.)

In the region $a < r < b$, we can use Gauss' law to determine that, in the region between the conductors, the electric field depends only on the charge on the inner sphere,

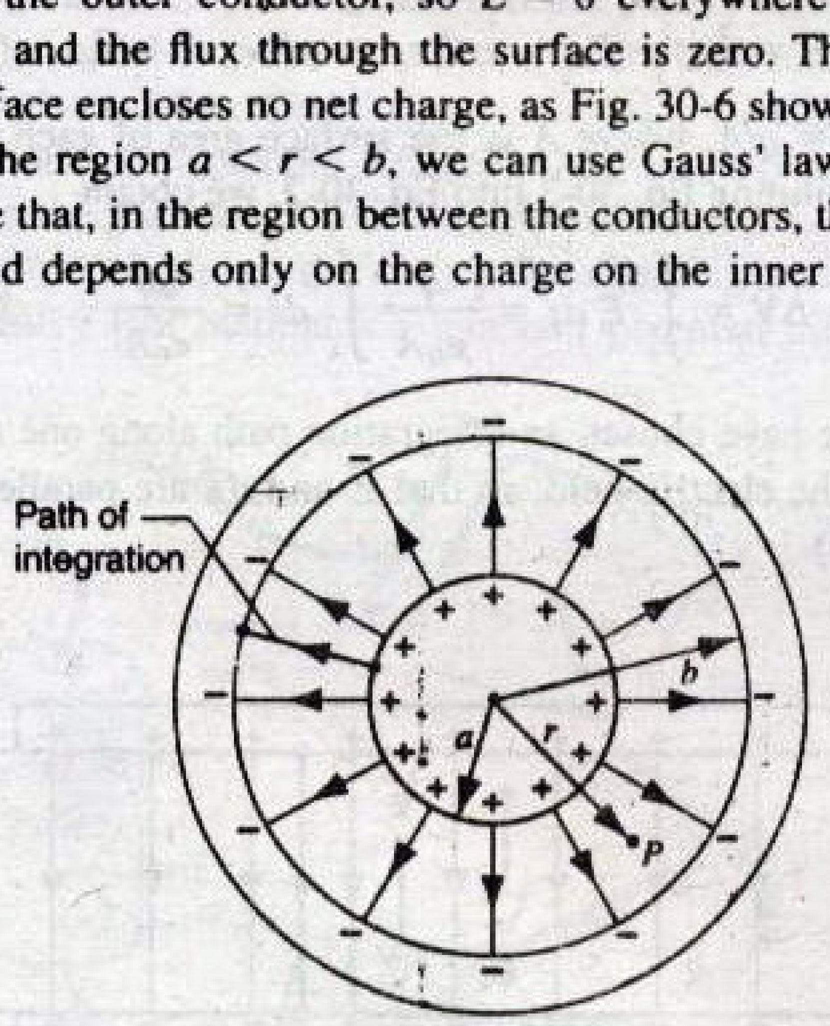


FIGURE 30-6. A cross section through a spherical or a cylindrical capacitor. The electric field at any point P in the interior is due only to the inner conductor. The path of integration for evaluating Eq. 30-7 or Eq. 30-10 is shown.

and that this field is the same as that of a point charge at its center (recall the shell theorems discussed in Section 27-5). We therefore have

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \qquad a < r < b. \qquad (30\text{-}6)$$

Substituting this expression for the electric field into Eq. 30-2 and integrating along the path shown in Fig. 30-6 from the positive plate to the negative plate, we obtain

$$\Delta V = \int_+^- E\,ds = \int_a^b \frac{q}{4\pi\epsilon_0}\frac{dr}{r^2} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$
$$= \frac{q}{4\pi\epsilon_0}\frac{b-a}{ab}. \qquad (30\text{-}7)$$

Because the path of integration is in the radial direction, we have $\vec{E} \cdot d\vec{s} = E\,ds$ and $ds = dr$.

Using $C = q/\Delta V$, we now find

$$C = 4\pi\epsilon_0 \frac{ab}{b-a} \qquad \text{(spherical capacitor).} \qquad (30\text{-}8)$$

Note that the capacitance again has the form of ϵ_0 times a quantity with the dimension of length.

A Cylindrical Capacitor

Figure 30-6 can also represent the cross section of a cylindrical capacitor, in which the inner conductor is a solid rod of radius a carrying a charge $+q$ uniformly distributed over its surface, and the outer conductor is a coaxial cylindrical shell of inner radius b carrying a charge of $-q$ uniformly distributed over its inner surface. The capacitor has length L, and we assume $L \gg b$ so that, as was the case with the parallel-plate capacitor, we can neglect the "fringing" field at the ends of the capacitor.

Just as we used Gauss' law in the spherical geometry to obtain the two shell theorems, we can obtain two similar results in the cylindrical geometry. If only the uniformly charged outer cylindrical conductor were present, we could construct a Gaussian surface in the shape of a long cylinder of radius $r < b$ having the same axis as the outer cylinder. This surface encloses no net charge, so we conclude that $E = 0$ everywhere on the Gaussian surface. As in the case of the spherical shell, a uniformly charged cylindrical shell produces no electric field in its interior. Using a cylindrical Gaussian surface with $r > a$, we can deduce that the inner cylinder behaves just like a uniform line of charge, for which the field points radially outward from the axis and has a magnitude that we calculated in Section 26-4 (Eq. 26-17):

$$E = \frac{1}{2\pi\epsilon_0}\frac{q}{Lr} \qquad a < r < b, \qquad (30\text{-}9)$$

where we have replaced the linear charge density λ with q/L and the distance y with the radial coordinate r. Equation 30-2 now gives

$$\Delta V = \int_{+}^{-} E\,ds = \frac{q}{2\pi\epsilon_0 L}\int_a^b \frac{dr}{r}$$

$$= \frac{q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right). \quad (30\text{-}10)$$

As we did for the spherical capacitor, we have chosen an integrating path from the positive plate to the negative plate in the radial direction, so $\vec{\mathbf{E}}\cdot d\vec{s} = E\,ds$ and $ds = dr$.

Equation 30-1 now gives the capacitance:

$$C = 2\pi\epsilon_0 \frac{L}{\ln(b/a)} \quad \text{(cylindrical capacitor).} \quad (30\text{-}11)$$

Note once again that only geometrical factors appear in this equation and that the capacitance has the form of ϵ_0 multiplied by a quantity with the dimension of length.

SAMPLE PROBLEM 30-2. The plates of a parallel-plate capacitor are separated by a distance $d = 1.0$ mm. What must be the plate area if the capacitance is to be 1.0 F?

Solution From Eq. 30-5 we have

$$A = \frac{Cd}{\epsilon_0} = \frac{(1.0\ \text{F})(1.0\times10^{-3}\ \text{m})}{8.85\times10^{-12}\ \text{F/m}} = 1.1\times10^8\ \text{m}^2.$$

This is the area of a square more than 10 km on edge. The farad is indeed a large unit. Modern technology, however, has permitted the construction of 1-F capacitors of very modest size. These "Supercaps" are used as backup voltage sources for computers; they can maintain the computer memory for up to 30 days in case of power failure.

SAMPLE PROBLEM 30-3. The space between the conductors of a long coaxial cable, used to transmit TV signals, has an inner radius $a = 0.15$ mm and an outer radius $b = 2.1$ mm. What is the capacitance per unit length of this cable?

Solution From Eq. 30-11 we have

$$\frac{C}{L} = \frac{2\pi\epsilon_0}{\ln(b/a)} = \frac{(2\pi)(8.85\ \text{pF/m})}{\ln(2.1\ \text{mm}/0.15\ \text{mm})} = 21\ \text{pF/m}.$$

SAMPLE PROBLEM 30-4. What is the capacitance of the Earth, viewed as an isolated conducting sphere of radius $R = 6370$ km?

Solution We can assign a capacitance to a single isolated spherical conductor by assuming that the "missing plate" is a conducting sphere of infinite radius.

If we let $b \to \infty$ in Eq. 30-8 and substitute R for a, we find

$$C = 4\pi\epsilon_0 R \quad \text{(isolated sphere).} \quad (30\text{-}12)$$

Substituting, we obtain

$$C = (4\pi)(8.85\times10^{-12}\ \text{F/m})(6.37\times10^6\ \text{m})$$
$$= 7.1\times10^{-4}\ \text{F} = 710\ \mu\text{F}.$$

A tiny 1-F Supercap has a capacitance that is about 1400 times larger than that of the Earth.

30-4 CAPACITORS IN SERIES AND PARALLEL

In analyzing electric circuits, it is often desirable to know the *equivalent capacitance* of two or more capacitors that are connected in a certain way. By "equivalent capacitance" we mean the capacitance of a single capacitor that can be substituted for the combination with no change in the operation of the rest of the circuit.

Capacitors Connected in Parallel

Figure 30-7*a* shows two capacitors connected *in parallel*. There are three properties that characterize a parallel connection of circuit elements. (1) In traveling from *a* to *b*, we can take any of several (two, in this case) *parallel* paths, each of which goes through *only one* of the parallel elements. (2) When a battery of potential difference ΔV is connected across the combination (that is, one terminal of the battery is connected to point *a* in Fig. 30-7*a* and the other terminal to point *b*), the same potential difference ΔV appears across each element of the parallel connection. The wires and capacitor plates are conductors and therefore equipotentials under electrostatic conditions. The potential at *a* appears on the wires connected to *a* and on the two left-hand capacitor plates; similarly, the potential at *b* appears on all the wires connected to *b* and on the two right-hand capacitor plates. (3) The total charge that is delivered by the battery to the combination is shared among the elements; some charge "pumped" by the battery ends up on C_1 and some on C_2.

With these principles in mind, we can now find the equivalent capacitance C_{eq} that gives the same total capacitance between points *a* and *b*, as indicated in Fig. 30-7*b*. We assume a battery of potential difference ΔV to be connected between points *a* and *b*. For each capacitor, we can write (using Eq. 30-1)

$$q_1 = C_1\Delta V \quad \text{and} \quad q_2 = C_2\Delta V. \quad (30\text{-}13)$$

In writing these equations, we have used the same value of the potential difference across the capacitors, in accordance with the second characteristic of a parallel connection stated previously. The battery extracts charge *q* from one side of the circuit and moves it to the other side. This charge is shared among the two elements according to the

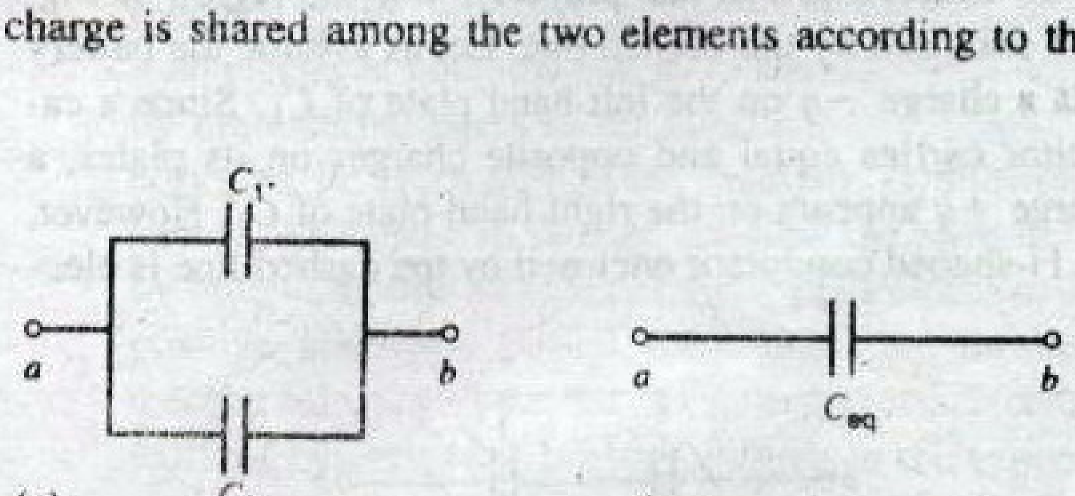


FIGURE 30-7. (*a*) Two capacitors in parallel. (*b*) The equivalent capacitance that can replace the parallel combination.

third characteristic, such that the sum of the charges on the two capacitors equals the total charge:

$$q = q_1 + q_2. \tag{30-14}$$

If the parallel combination were replaced with a single capacitor C_{eq} and connected to the same battery, the requirement that the circuit operate in identical fashion means that the same charge q must be transferred by the battery. That is, for the equivalent capacitor,

$$q = C_{eq}\Delta V. \tag{30-15}$$

Substituting Eq. 30-14 into Eq. 30-15, and then putting Eqs. 30-13 into the result, we obtain

$$C_{eq}\Delta V = C_1\Delta V + C_2\Delta V$$

or

$$C_{eq} = C_1 + C_2. \tag{30-16}$$

If we have more than two capacitors in parallel, we can first replace C_1 and C_2 with their equivalent C_{12}, determined according to Eq. 30-16. We then find the equivalent capacitance of C_{12} and the next parallel capacitor C_3. Continuing this process, we can extend Eq. 30-16 to any number of capacitors connected in parallel:

$$C_{eq} = \sum_n C_n \quad \text{(parallel combination)}. \tag{30-17}$$

That is, to find the equivalent capacitance of a parallel combination, simply add the individual capacitances. Note that the equivalent capacitance is always larger than the largest capacitance in the parallel combination. The parallel combination can store more charge than any one of the individual capacitors.

Capacitors Connected in Series

Figure 30-8 shows two capacitors connected *in series*. There are three properties that distinguish a series connection of circuit elements. (1) If we attempt to travel from a to b, we must pass through *all* the circuit elements *in succession*. (2) When a battery is connected across the combination, the potential difference ΔV of the battery equals the sum of the potential differences across each of the elements. (3) The charge q delivered to each element of the series combination has the same value.

To understand this last property, note the region of Fig. 30-8 enclosed by the dashed line. Let us assume the battery puts a charge $-q$ on the left-hand plate of C_1. Since a capacitor carries equal and opposite charges on its plates, a charge $+q$ appears on the right-hand plate of C_1. However, the H-shaped conductor enclosed by the dashed line is electrically isolated from the rest of the circuit; initially it carries no net charge, and no charge can be transferred to it. If a charge $+q$ appears on the right-hand plate of C_1, the charge $-q$ must appear on the left-hand plate of C_2. That is, n $(=q/e)$ electrons move from the right-hand plate of C_1 to the left-hand plate of C_2. If there were more than two capacitors in series, a similar argument could be made across the entire line of capacitors, the result being that the left-hand plate of *every* capacitor in the series connection carries a charge q of one sign, and the right-hand plate of every capacitor in the series connection carries a charge of equal magnitude q and opposite sign.

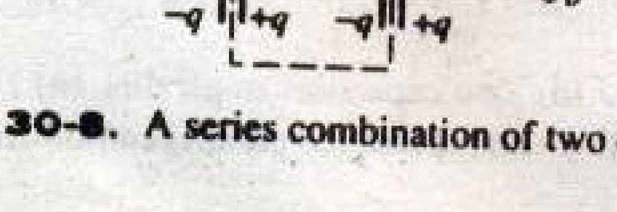


FIGURE 30-8. A series combination of two capacitors.

For the individual capacitors we can write, using Eq. 30-1,

$$\Delta V_1 = \frac{q}{C_1} \quad \text{and} \quad \Delta V_2 = \frac{q}{C_2}, \tag{30-18}$$

with the same charge q on each capacitor but different potential differences across each. According to the second property of a series connection, we have

$$\Delta V = \Delta V_1 + \Delta V_2. \tag{30-19}$$

We seek the equivalent capacitance C_{eq} that can replace the combination, such that the battery would move the same amount of charge:

$$\Delta V = \frac{q}{C_{eq}}. \tag{30-20}$$

Substituting Eq. 30-19 into Eq. 30-20 and then using Eqs. 30-18, we obtain

$$\frac{q}{C_{eq}} = \frac{q}{C_1} + \frac{q}{C_2},$$

or

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}. \tag{30-21}$$

If we have several capacitors in series, we can use Eq. 30-21 to find the equivalent capacitance C_{12} of the first two. We then find the equivalent capacitance of C_{12} and the next capacitor in series, C_3. Continuing in this way, we find the equivalent capacitance of any number of capacitors in series,

$$\frac{1}{C_{eq}} = \sum_n \frac{1}{C_n} \quad \text{(series combination)}. \tag{30-22}$$

That is, to find the equivalent capacitance of a series combination, take the reciprocal of the sum of the reciprocals of the individual capacitances. Note that the equivalent capacitance of the series combination is always smaller than the smallest individual capacitance in the series.

Occasionally, capacitors are connected in ways that are not immediately identifiable as series or parallel combinations. As Sample Problem 30-5 shows, such combinations can often (but not always) be broken down into smaller units that can be analyzed as series or parallel connections.

SAMPLE PROBLEM 30-5. ***(a)*** ***Find the equivalent capacitance of the combination shown in Fig. 30-9a, with $C_1 = 12.0\ \mu F$,***

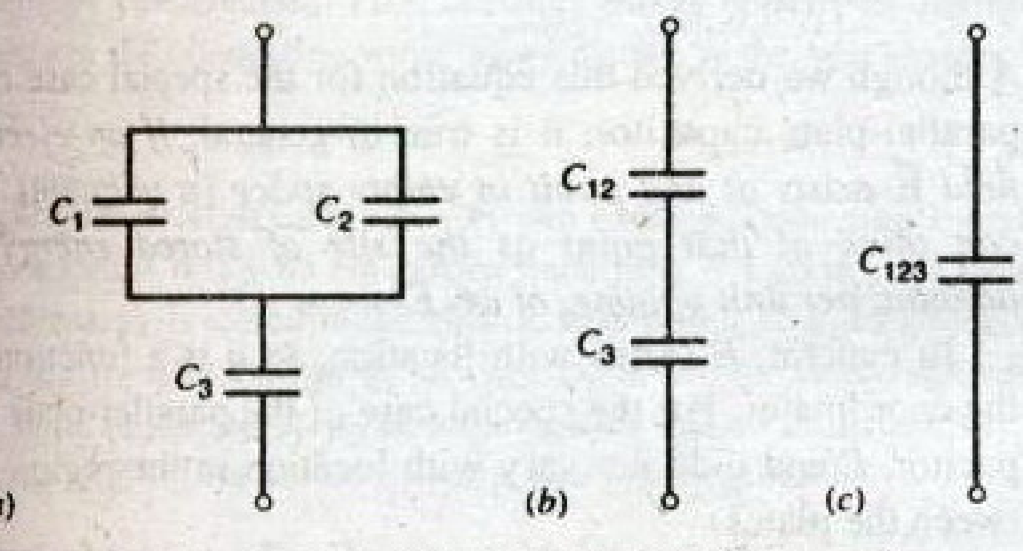


FIGURE 30-9. Sample Problem 30-5. (*a*) A combination of three capacitors. (*b*) The parallel combination of C_1 and C_2 has been replaced by its equivalent, C_{12}. (*c*) The series combination of C_{12} and C_3 has been replaced by its equivalent, C_{123}.

$C_2 = 5.3\ \mu F$, and $C_3 = 4.5\ \mu F$. (*b*) A potential difference $\Delta V =$ 12.5 V is applied to the terminals in Fig. 30-9*a*. What is the charge on C_1?

Solution (*a*) Capacitors C_1 and C_2 are in parallel. From Eq. 30-16, their equivalent capacitance is

$$C_{12} = C_1 + C_2 = 12.0\ \mu\text{F} + 5.3\ \mu\text{F} = 17.3\ \mu\text{F}.$$

In Fig. 30-9*b*, C_1 and C_2 have been replaced by their parallel combination, C_{12}. As the figure shows, C_{12} and C_3 are in series. From Eq. 30-21, the final equivalent combination (see Fig. 30-9*c*) is found from

$$\frac{1}{C_{123}} = \frac{1}{C_{12}} + \frac{1}{C_3} = \frac{1}{17.3\ \mu\text{F}} + \frac{1}{4.5\ \mu\text{F}} = 0.280\ \mu\text{F}^{-1},$$

or

$$C_{123} = \frac{1}{0.280\ \mu\text{F}^{-1}} = 3.57\ \mu\text{F}.$$

(*b*) We treat the equivalent capacitors C_{12} and C_{123} exactly as we would real capacitors having that capacitance. The charge on C_{123} in Fig. 30-9*c* is then

$$q_{123} = C_{123}\,\Delta V = (3.57\ \mu\text{F})(12.5\ \text{V}) = 44.6\ \mu\text{C}.$$

This same charge exists on each capacitor in the series combination of Fig. 30-9*b*. The potential difference across C_{12} in that figure is then

$$\Delta V_{12} = \frac{q_{12}}{C_{12}} = \frac{44.6\ \mu\text{C}}{17.3\ \mu\text{F}} = 2.58\ V.$$

This same potential difference appears across C_1 in Fig. 30-9*a*, so that

$$q_1 = C_1\,\Delta V_1 = (12\ \mu\text{F})(2.58\ \text{V}) = 31\ \mu\text{C}.$$

30-5 ENERGY STORAGE IN AN ELECTRIC FIELD

An important use of capacitors is to store electrostatic energy in applications ranging from flash lamps to laser systems (see Fig. 30-10), both of which depend for their operation on the charging and discharging of capacitors.

In Section 28-2 we showed that any charge configuration has a certain *electric potential energy U*, equal to the work W (which may be positive or negative) that is done by an external agent that assembles the charge configuration from its individual components, originally assumed to be infinitely far apart and at rest. This potential energy is similar to that of mechanical systems, such as a compressed spring or the Earth–Moon system.

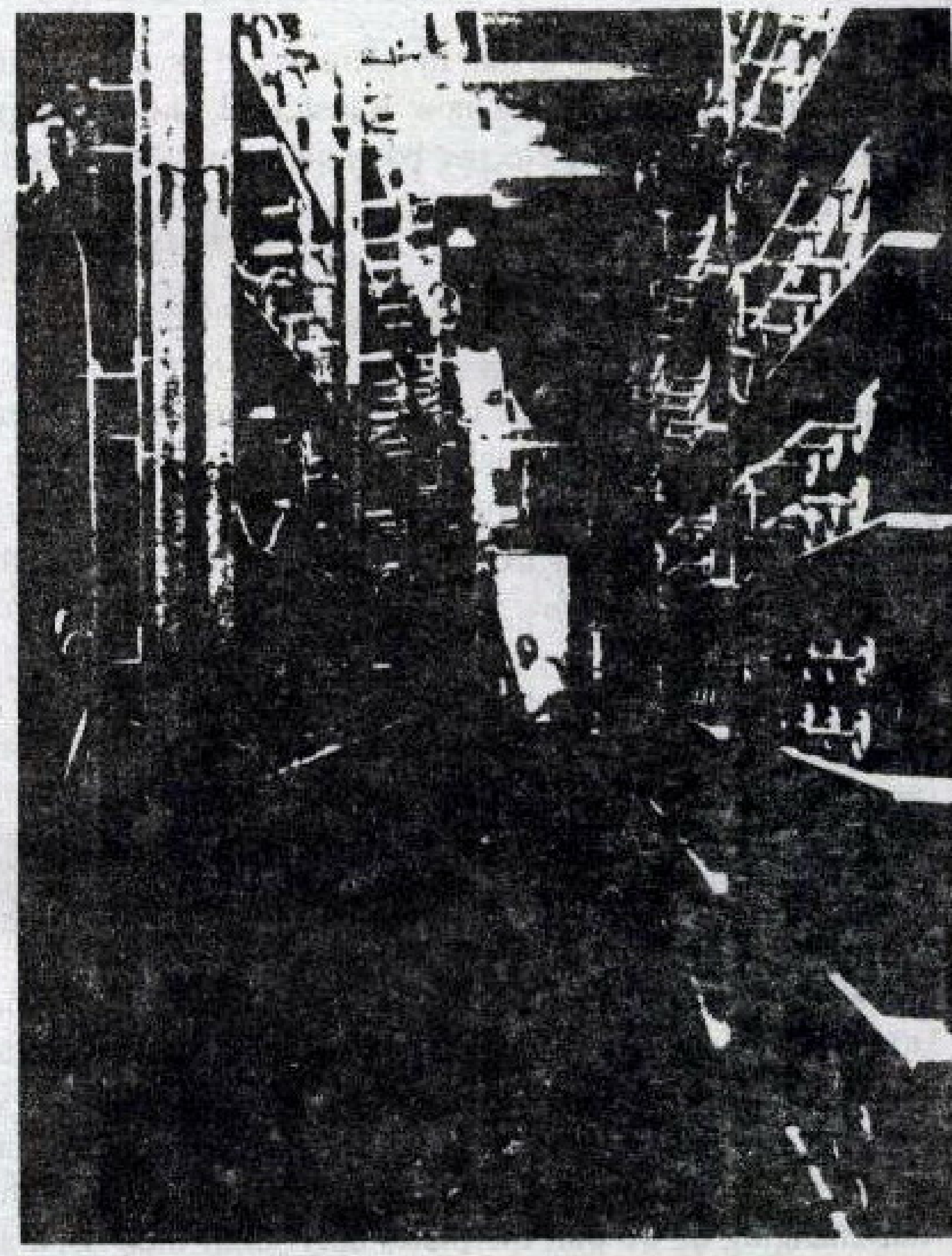


FIGURE 30-10. This bank of 10,000 capacitors at the Lawrence Livermore National Laboratory stores 60 MJ of electric energy and releases it in 1 ms to flashlamps that drive a system of lasers. The installation is part of the Nova project, which is attempting to produce sustained nuclear fusion reactions.

For a simple example, work is done when two equal and opposite charges are separated. This energy is stored as electric potential energy in the system, and it can be recovered as kinetic energy if the charges are allowed to come together again. Similarly, a charged capacitor has stored in it an electrical potential energy U equal to the work W done by the external agent as the capacitor is charged. This energy can be recovered if the capacitor is allowed to discharge. Alternatively, we can visualize the work of charging by imagining that an external agent pulls electrons from the positive plate and pushes them onto the negative plate, thereby bringing about the charge separation. Normally, the work of charging is done by a battery, at the expense of its store of chemical energy.

Suppose that at a time t a charge q' has already been transferred from one plate to the other. The potential difference

$\Delta V'$ between the plates at that moment is $\Delta V' = q'/C$. If an increment of charge dq' is now transferred, the resulting small change dU in the electric potential energy is, according to Eq. 28-9 ($\Delta V = \Delta U/q_0$),

$$dU = \Delta V' dq' = \frac{q'}{C} dq'.$$

If this process is continued until a total charge q has been transferred, the total potential energy is

$$U = \int dU = \int_0^q \frac{q'}{C} dq' \qquad (30\text{-}23)$$

or

$$U = \frac{q^2}{2C}. \qquad (30\text{-}24)$$

From the relation $q = C\,\Delta V$ we can also write this as

$$U = \tfrac{1}{2}C(\Delta V)^2 \qquad (30\text{-}25)$$

Where does this energy reside? Equations 30-24 and 30-25 do not give us a direct answer, but we can determine the location of the stored energy by reasoning as follows. Suppose we have an isolated parallel-plate capacitor (that is, not connected to a battery) that carries a charge q. Without changing q, we pull the plates apart until their separation is twice as large as it was initially. According to Eq. 30-5, if the plate separation d becomes twice as large, the capacitance becomes only half as large. Equation 30-24 shows that if C becomes half as large, the stored energy doubles. Now in pulling the plates apart we have not changed the capacitor plates, so it would not be reasonable to conclude that the extra energy is stored there. What we have done is to double the volume of the space between the plates, and since the energy has also doubled it seems reasonable to conclude that this electric potential energy resides in the volume between the plates. More specifically, *the energy is stored in the electric field that is present in this region.*

In a parallel-plate capacitor, neglecting fringing, the electric field has the same value for all points between the plates. Based on our conclusion that the energy resides in the field, it follows that the *energy density* u, which is the stored energy per unit volume, should also be the same everywhere between the plates. u is given by the stored energy U divided by the volume Ad, or

$$u = \frac{U}{Ad} = \frac{\frac{1}{2}C(\Delta V)^2}{Ad} \qquad (30\text{-}26)$$

Substituting the relation $C = \epsilon_0 A/d$ (Eq. 30-5) leads to

$$u = \frac{\epsilon_0}{2}\left(\frac{\Delta V}{d}\right)^2 \qquad (30\text{-}27)$$

However, $\Delta V/d$ is the electric field E, so that

$$u = \tfrac{1}{2}\epsilon_0 E^2. \qquad (30\text{-}28)$$

Although we derived this equation for the special case of a parallel-plate capacitor, it is true in general. *If an electric field* $\vec{\mathbf{E}}$ *exists at any point in empty space (a vacuum), we can think of that point as the site of stored energy in amount, per unit volume, of* $\frac{1}{2}\epsilon_0 E^2$.

In general, E varies with location, so u is a function of the coordinates. For the special case of the parallel-plate capacitor, E and u do not vary with location in the region between the plates.

SAMPLE PROBLEM 30-6. A 3.55-μF capacitor C_1 is charged to a potential difference $\Delta V_0 = 6.30$ V, using a battery. The charging battery is then removed, and the capacitor is connected as in Fig. 30-11 to an uncharged 8.95-μF capacitor C_2. After the switch S is closed, charge flows from C_1 to C_2 until an equilibrium is established, with both capacitors at the same potential difference ΔV. (*a*) What is this common potential difference? (*b*) What is the energy stored in the electric field before and after the switch S in Fig. 30-11 is closed?

Solution (*a*) Electric charge must be conserved, so the original charge q_0 is shared by two capacitors, or

$$q_0 = q_1 + q_2.$$

Applying the relation $q = C\,\Delta V$ to each term yields

$$C_1\,\Delta V_0 = C_1\,\Delta V + C_2\,\Delta V.$$

or

$$\Delta V = \Delta V_0 \frac{C_1}{C_1 + C_2} = \frac{(6.30\text{ V})(3.55\ \mu\text{F})}{3.55\ \mu\text{F} + 8.95\ \mu\text{F}} = 1.79\text{ V}.$$

If we know the battery voltage ΔV_0 and the value of C_1, we can determine an unknown capacitance C_2 by measuring the value of ΔV in an arrangement similar to that of Fig. 30-11.

(*b*) The initial stored energy is

$$U_i = \tfrac{1}{2}C_1(\Delta V_0)^2 = \tfrac{1}{2}(3.55 \times 10^{-6}\text{ F})(6.30\text{ V})^2$$
$$= 7.05 \times 10^{-5}\text{ J} = 70.5\ \mu\text{J}.$$

The final energy is

$$U_f = \tfrac{1}{2}C_1(\Delta V)^2 + \tfrac{1}{2}C_2(\Delta V)^2 = \tfrac{1}{2}(C_1 + C_2)(\Delta V)^2$$
$$= \tfrac{1}{2}(3.55 \times 10^{-6}\text{ F} + 8.95 \times 10^{-6}\text{ F})(1.79\text{ V})^2$$
$$= 2.00 \times 10^{-5}\text{ J} = 20.0\ \mu\text{J}.$$

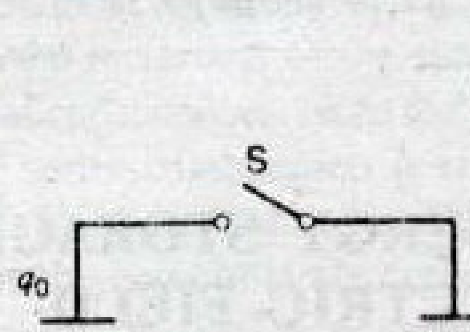


FIGURE 30-11. Sample Problem 30-6. Capacitor C_1 has previously been charged to a potential difference ΔV_0 by a battery that has been removed. When the switch S is closed, the initial charge q_0 on C_1 is shared with C_2.

We conclude that $U_f < U_i$, by about 72%. This is not a violation of energy conservation. The "missing" energy appears as thermal energy in the connecting wires, as we discuss in the next chapter.*

SAMPLE PROBLEM 30-7. An isolated conducting sphere whose radius R is 6.85 cm carries a charge q = 1.25 nC. (*a*) How much energy is stored in the electric field of this charged conductor? (*b*) What is the energy density at the surface of the sphere? (*c*) What is the radius R_0 of an imaginary spherical surface such that one-half of the stored potential energy lies within it?

Solution (*a*) From Eqs. 30-24 and 30-12 we have

$$U = \frac{q^2}{2C} = \frac{q^2}{8\pi\epsilon_0 R} = \frac{(1.25 \times 10^{-9}\ \text{C})^2}{(8\pi)(8.85 \times 10^{-12}\ \text{F/m})(0.0685\ \text{m})}$$
$$= 1.03 \times 10^{-7}\ \text{J} = 103\ \text{nJ}.$$

(*b*) To find the energy density, we must first find E at the surface of the sphere. This is given by

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}.$$

The energy density is then, using Eq. 30-28,

$$u = \tfrac{1}{2}\epsilon_0 E^2 = \frac{q^2}{32\pi^2\epsilon_0 R^4}$$
$$= \frac{(1.25 \times 10^{-9}\ \text{C})^2}{(32\pi^2)(8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(0.0685\ \text{m})^4}$$
$$= 2.54 \times 10^{-5}\ \text{J/m}^3 = 25.4\ \mu\text{J/m}^3.$$

(*c*) The energy that lies in a spherical shell between radii r and $r + dr$ is

$$dU = (u)(4\pi r^2)(dr),$$

where $(4\pi r^2)(dr)$ is the volume of the spherical shell. Using the result of part (*b*) for the energy density evaluated at a radius r, we obtain

$$dU = \frac{q^2}{32\pi^2\epsilon_0 r^4}4\pi r^2\,dr = \frac{q^2}{8\pi\epsilon_0}\frac{dr}{r^2}.$$

The condition given for this problem is

$$\int_R^{R_0} dU = \frac{1}{2}\int_R^{\infty} dU$$

or, using the result obtained above for dU and canceling constant factors from both sides,

$$\int_R^{R_0}\frac{dr}{r^2} = \frac{1}{2}\int_R^{\infty}\frac{dr}{r^2},$$

which becomes

$$\frac{1}{R} - \frac{1}{R_0} = \frac{1}{2R}.$$

Solving for R_0 yields

$$R_0 = 2R = (2)(6.85\ \text{cm}) = 13.7\ \text{cm}.$$

Half the stored energy is contained within a spherical surface whose radius is twice the radius of the conducting sphere.

30-6 CAPACITOR WITH DIELECTRIC

In Section 29-6 we discussed the effect of applying an electric field to an insulating material (a dielectric). We showed that the effect of the dielectric is to reduce the strength of the electric field in its interior from its initial value E_0 in vacuum to $E = E_0/\kappa_e$ inside the dielectric. The parameter κ_e, the dielectric constant, has values greater than 1 for all materials, so that the electric field in the dielectric is smaller than the field in vacuum.

In this section, we consider the effect of filling the interior of a capacitor with a dielectric material. This effect was first investigated in 1837 by Michael Faraday. Faraday constructed two identical capacitors, filling one with a dielectric material and leaving the other with air between its plates. When both capacitors were connected to batteries with the *same potential difference,* Faraday found that the charge on the capacitor filled with the dielectric was greater than the charge on the capacitor with air between its plates. That is, the presence of the dielectric enables the capacitor to store more charge. Since storage of charge for later discharge is one of the purposes for which we use capacitors, the presence of a dielectric can enhance the performance of a capacitor.

The effect of filling a capacitor with dielectric depends on whether we do so with the battery connected (as in Faraday's experiment) or disconnected. First we consider the situation as in Faraday's experiment (Fig. 30-12). A capacitor with capacitance C is connected to a battery of potential difference ΔV and allowed to become fully charged, such that the plates carry a charge q, as in Fig. 30-12*a*. With the battery remaining connected, we then fill the interior of the capacitor with a material of dielectric constant κ_e, as in Fig. 30-12*b*. The battery maintains the same potential difference ΔV across the plates.

Equation 30-2 shows that, if the potential differences in Figs. 30-12*a* and 30-12*b* are the same, then the electric fields inside the capacitor must be the same. However, we

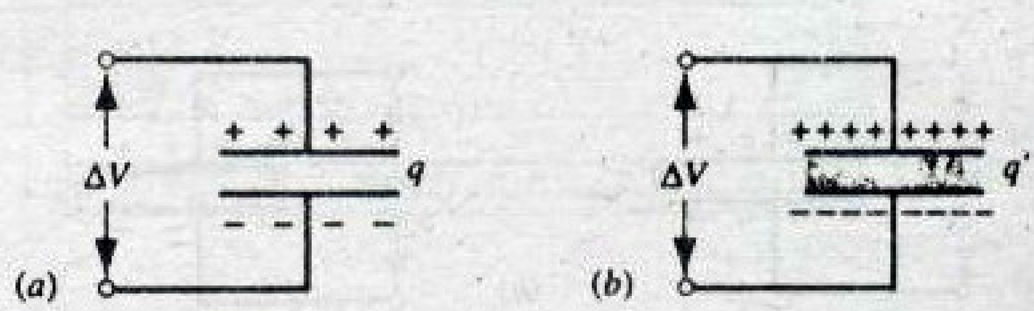


FIGURE 30-12. (*a*) An empty capacitor is charged by connecting it to a battery that establishes a potential difference ΔV. (*b*) The battery remains connected as the capacitor is filled with a dielectric. In this case, the potential difference ΔV remains constant, but q increases.

*Some slight amount of energy is also radiated away. For a critical discussion, see "Two-Capacitor Problem: A More Realistic View," by R. A. Powell, *American Journal of Physics*, May 1979, p. 460.

would expect the presence of the dielectric to reduce the strength of the electric field. As Faraday concluded, the tendency of the dielectric to reduce the field is exactly balanced by the additional charge that the battery delivers to the plates as the dielectric is inserted.

Let us assume that we are using a parallel-plate capacitor. With the capacitor empty, the electric field is given by Eq. 30-3: $E = \sigma/\epsilon_0 = q/\epsilon_0 A$. When the dielectric is present, the electric field is reduced by the factor $1/\kappa_e$ due to the presence of the dielectric, but the field is also changed because the plates now carry a charge q', so the field is $E' = q'/\kappa_e\epsilon_0 A$. Since the fields must be equal, we can set $E' = E$ and conclude that

$$q' = \kappa_e q. \tag{30-29}$$

The dielectric constant is greater than 1, so the capacitor can store more charge with the dielectric present than it can when it is empty. As the dielectric material is inserted into the already-charged capacitor, the battery moves additional charge $q' - q = q(\kappa_e - 1)$ from the negative to the positive plate.

The capacitance with the dielectric present is $C' = q'/\Delta V'$. Using $q' = \kappa_e q$ and $\Delta V' = \Delta V$, we obtain

$$C' = \kappa_e C. \tag{30-30}$$

The presence of the dielectric increases the capacitance by the factor κ_e. For a parallel-plate capacitor with dielectric, the capacitance can be found by combining Eqs. 30-5 and 30-30:

$$C' = \frac{\kappa_e \epsilon_0 A}{d}. \tag{30-31}$$

The capacitance of *any* capacitor is increased by the same factor κ_e when a dielectric substance completely fills the space between the plates. Equations 30-8 and 30-11 can be similarly modified to account for the presence of a dielectric filling the capacitor.

Although the effect on the capacitance is the same, the derivation is very different if the dielectric material is inserted with the battery *not* connected. We first connect the capacitor to a battery, so that the plates acquire a potential difference ΔV and charge q, after which we disconnect the battery, as in Fig. 30-13*a*. Now we fill the capacitor with dielectric, as in Fig. 30-13*b*. In this case, *the charge must remain constant,* since there is no battery present to move charge from one plate to another. With the charge remaining constant, the electric field is changed only by the presence of the dielectric, so $E' = E/\kappa_e$. Using this electric field in Eq. 30-2 to find the potential difference, we obtain $\Delta V' = \Delta V/\kappa_e$. That is, the potential difference in this case decreases by the factor $1/\kappa_e$. With $\Delta V' = q'/C'$ and $q' = q$, once again we obtain $C' = \kappa_e C$, as in Eq. 30-30. The capacitance does not depend on *how* we charge the capacitor or insert the dielectric; it depends only on the geometry of the capacitor and the material with which it is filled.

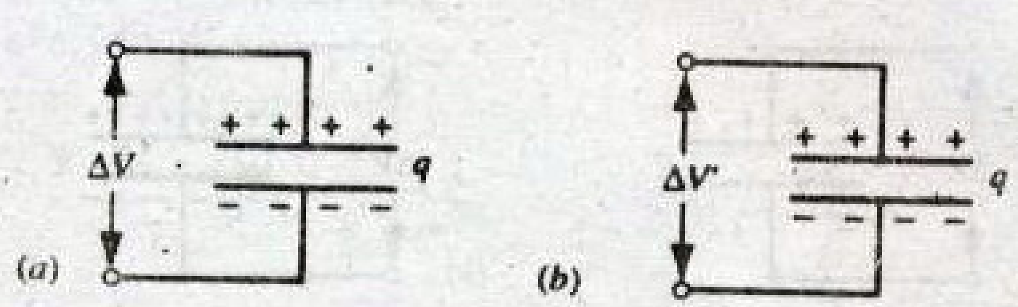


FIGURE 30-13. (*a*) An empty capacitor is charged and then disconnected from a battery. (*b*) The capacitor is now filled with a dielectric. The charge remains constant, but the potential difference decreases from ΔV to $\Delta V'$.

SAMPLE PROBLEM 30-8. A parallel-plate capacitor whose capacitance C is 13.5 pF has a potential difference ΔV = 12.5 V across its plates. The charging battery is now disconnected and a porcelain slab ($\kappa_e = 6.5$) is slipped between the plates as in Fig. 30-13*b*. What is the stored energy of the unit, both before and after the slab is introduced?

Solution The initial stored energy is given by Eq. 30-25 as

$$U_i = \tfrac{1}{2}C\,\Delta V^2 = \tfrac{1}{2}(13.5 \times 10^{-12}\ \text{F})(12.5\ \text{V})^2$$
$$= 1.055 \times 10^{-9}\ \text{J} = 1055\ \text{pJ}.$$

We can write the final energy from Eq. 30-24 in the form $U_f = q^2/2C'$ because, from the conditions of the problem statement, q (but not ΔV) remains constant as the slab is introduced. After the slab is in place, the capacitance increases to $C' = \kappa_e C$ so that

$$U_f = \frac{q^2}{2\kappa_e C} = \frac{U_i}{\kappa_e} = \frac{1055\ \text{pJ}}{6.5} = 162\ \text{pJ}.$$

The energy after the slab is introduced is smaller by a factor of $1/\kappa_e$.

The "missing" energy, in principle, would be apparent to the person who introduced the slab. The capacitor would exert a force on the slab and would do work on it, in the amount

$$W = U_i - U_f = 1055\ \text{pJ} - 162\ \text{pJ} = 893\ \text{pJ}.$$

If the slab were introduced with no restraint and if there were no friction, the slab would oscillate into and out of the region between the plates. The system consisting of capacitor + slab has a constant energy of 1055 pJ; the energy shuttles back and forth between kinetic energy of the moving slab and stored energy of the electric field. At the instant the oscillating slab filled the space between the plates, its kinetic energy would be 893 pJ.

Dielectrics and Gauss' Law

So far our use of Gauss' law has been confined to situations in which no dielectric was present. Now let us apply this law to a parallel-plate capacitor filled with a material of dielectric constant κ_e.

Figure 30-14 shows the capacitor both with and without the dielectric. We assume that the charge q on the plates is the same in each case. Gaussian surfaces have been drawn partly through the upper plate and partly through the region between the plates.

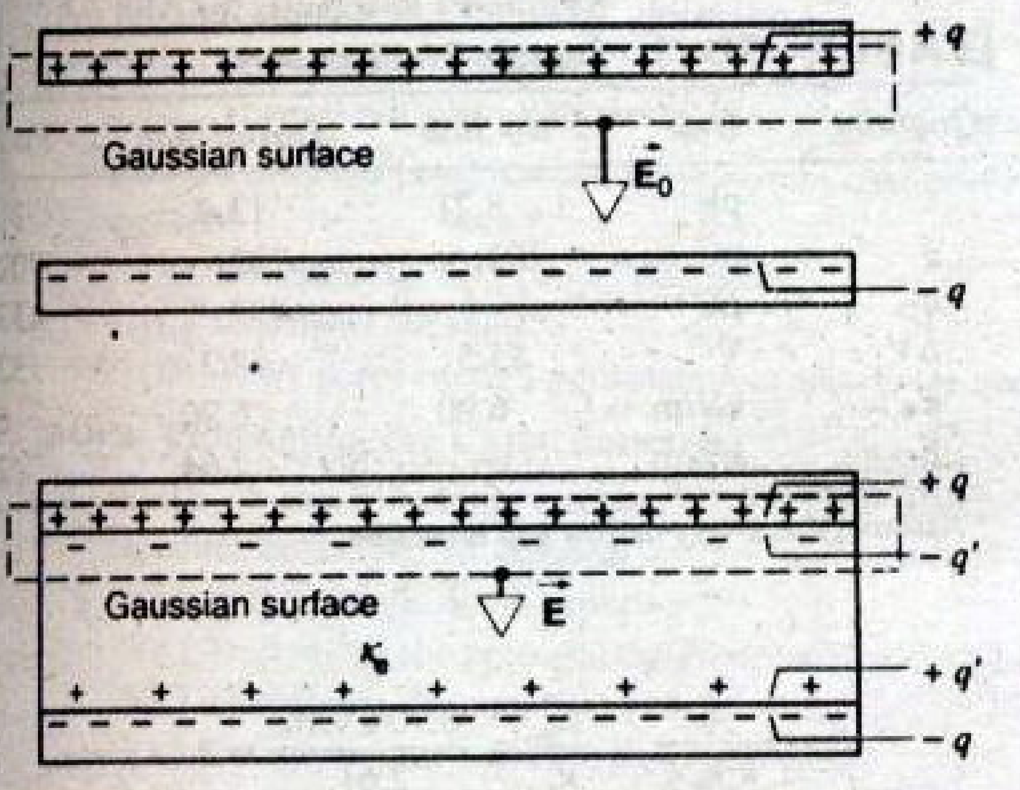


FIGURE 30-14. (*a*) A parallel-plate capacitor. (*b*) A dielectric slab is inserted, while the charge q on the plates remains constant. Induced charged q' appears on the surface of the dielectric slab.

If no dielectric is present (Fig. 30-14*a*), Gauss' law gives

$$\epsilon_0 \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \epsilon_0 E_0 A = q,$$

because the electric field exists only on that portion of the Gaussian surface between the plates. Thus,

$$E_0 = \frac{q}{\epsilon_0 A}. \qquad (30\text{-}32)$$

If the dielectric is present (Fig. 30-14*b*), Gauss' law gives

$$\epsilon_0 \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \epsilon_0 E A = q - q'$$

or

$$E = \frac{q}{\epsilon_0 A} - \frac{q'}{\epsilon_0 A}, \qquad (30\text{-}33)$$

in which $-q'$, the *induced surface charge*, must be distinguished from q, the *free charge* on the plates. These two charges $+q$ and $-q'$, both of which lie within the Gaussian surface, are opposite in sign; the *net* charge within the Gaussian surface is $q + (-q') = q - q'$.

The dielectric reduces the electric field by the factor κ_e, and so

$$E = \frac{E_0}{\kappa_e} = \frac{q}{\kappa_e \epsilon_0 A}. \qquad (30\text{-}34)$$

Inserting this in Eq. 30-33 yields

$$\frac{q}{\kappa_e \epsilon_0 A} = \frac{q}{\epsilon_0 A} - \frac{q'}{\epsilon_0 A}$$

or

$$q' = q\left(1 - \frac{1}{\kappa_e}\right). \qquad (30\text{-}35)$$

This shows that the induced surface charge q' is always less in magnitude than the free charge q and is equal to zero if no dielectric is present—that is, if $\kappa_e = 1$.

Now we write Gauss' law for the case of Fig. 30-14*b* in the form

$$\epsilon_0 \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = q - q', \qquad (30\text{-}36)$$

$q - q'$ again being the net charge within the Gaussian surface. Substituting from Eq. 30-35 for q' leads, after some rearrangement, to

$$\epsilon_0 \oint \kappa_e \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = q. \qquad (30\text{-}37)$$

This important relation, although derived for a parallel-plate capacitor, is true generally and is the form in which Gauss' law is usually written when dielectrics are present. Note the following:

1. The flux integral now deals with $\kappa_e \vec{\mathbf{E}}$ instead of $\vec{\mathbf{E}}$. This is consistent with the *reduction* of E in a dielectric by the factor κ_e, because $\kappa_e \vec{\mathbf{E}}$ (dielectric present) equals $\vec{\mathbf{E}}_0$ (no dielectric). For generality, we allow for the possibility that the dielectric is not uniform by putting κ_e inside the integral.

2. The charge q contained within the Gaussian surface is taken to be the *free charge only*. Induced surface charge is deliberately omitted on the right side of Eq. 30-37, having been taken into account by the introduction of κ_e on the left side. Equations 30-36 and 30-37 are completely equivalent formulations.

SAMPLE PROBLEM 30-9. Figure 30-15 shows a parallel-plate capacitor of plate area A and plate separation d. A potential difference ΔV is applied across the plates. The battery is then disconnected, and a dielectric slab of thickness b and dielectric constant κ_e is placed between the plates as shown. Assume that

$$A = 115\text{ cm}^2, \quad d = 1.24\text{ cm}, \quad b = 0.78\text{ cm},$$
$$\kappa_e = 2.61, \quad \Delta V = 85.5\text{ V}.$$

(*a*) What is the capacitance C before the slab is inserted? (*b*) What free charge appears on the plates? (*c*) What is the electric

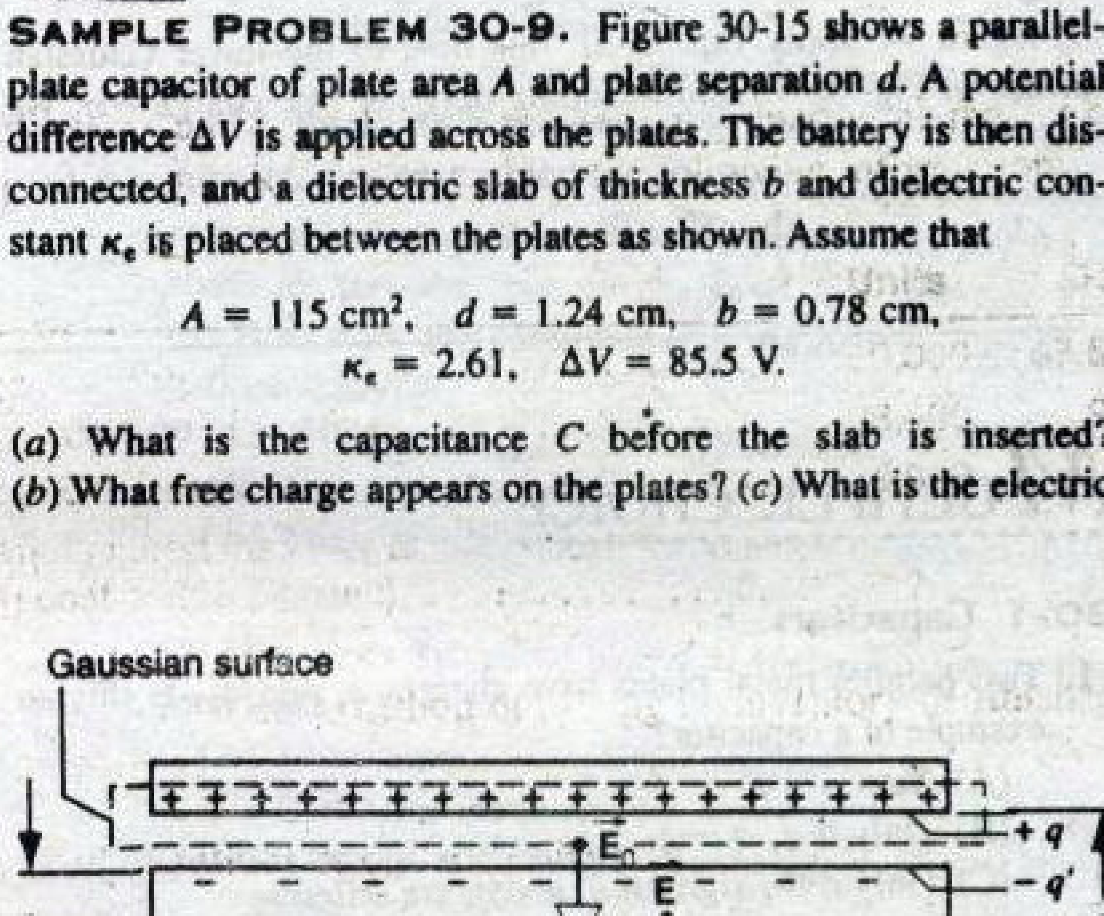


FIGURE 30-15. Sample Problem 30-9. A parallel-plate capacitor contains a dielectric that only partially fills the space between the plates.

field E_0 in the gaps between the plates and the dielectric slab? (*d*) Calculate the electric field E in the dielectric slab. (*e*) What is the potential difference $\Delta V'$ between the plates after the slab has been introduced? (*f*) What is the capacitance C' with the slab in place?

Solution (*a*) From Eq. 30-5 we have

$$C = \frac{\epsilon_0 A}{d} = \frac{(8.85 \times 10^{-12}\ \text{F/m})(115 \times 10^{-4}\ \text{m}^2)}{1.24 \times 10^{-2}\ \text{m}}$$
$$= 8.21 \times 10^{-12}\ \text{F} = 8.21\ \text{pF}.$$

(*b*) The free charge on the plates can be found from Eq. 30-1,

$$q = C\Delta V = (8.21 \times 10^{-12}\ \text{F})(85.5\ \text{V})$$
$$= 7.02 \times 10^{-10}\ \text{C} = 702\ \text{pC}.$$

Because the charging battery was disconnected before the slab was introduced, the free charge remains unchanged as the slab is put into place.

(*c*) Let us apply Gauss' law in the form given in Eq. 30-37 to the upper Gaussian surface in Fig. 30-15, which encloses only the free charge on the upper capacitor plate. We have

$$\epsilon_0 \oint \kappa_e \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \epsilon_0 (1) E_0 A = q$$

or

$$E_0 = \frac{q}{\epsilon_0 A} = \frac{7.02 \times 10^{-10}\ \text{C}}{(8.85 \times 10^{-12}\ \text{F/m})(115 \times 10^{-4}\ \text{m}^2)}$$
$$= 6900\ \text{V/m} = 6.90\ \text{kV/m}.$$

Note that we put $\kappa_e = 1$ in this equation because the Gaussian surface over which Gauss' law was integrated does not pass through any dielectric. Note too that the value of E_0 remains unchanged as the slab is introduced. It depends only on the free charge on the plates.

(*d*) Again we apply Eq. 30-37, this time to the lower Gaussian surface in Fig. 30-15 and including only the free charge $-q$. We find

$$\epsilon_0 \oint \kappa_e \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = -\epsilon_0 \kappa_e EA = -q$$

TABLE 30-1 Summary of Results for Sample Problem 30-9

Quantity	*Unit*	*No Slab*	*Partial Slab*	*Full Slab*^{a}
C	pF	8.21	13.4	21.4
q	pC	702	702	702
q'	pC	—	433	433
ΔV	V	85.5	52.3	32.8
E_0	kV/m	6.90	6.90	6.90
E	kV/m	—	2.64	2.64

^{a} Assumes that a very narrow gap is present.

or

$$E = \frac{q}{\kappa_e \epsilon_0 A} = \frac{E_0}{\kappa_e} = \frac{6.90\ \text{kV/m}}{2.61} = 2.64\ \text{kV/m}.$$

The negative sign appears when we evaluate the dot product $\vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$ because $\vec{\mathbf{E}}$ and $d\vec{\mathbf{A}}$ are in opposite directions, $d\vec{\mathbf{A}}$ always being in the direction of the *outward* normal to the closed Gaussian surface.

(*e*) To find the potential difference $\Delta V'$, we use Eq. 30-2

$$\Delta V' = \int_+^- E\,ds = E_0(d - b) + Eb$$
$$= (6900\ \text{V/m})(0.0124\ \text{m} - 0.0078\ \text{m}) + (2640\ \text{V/m})(0.0078\ \text{m})$$
$$= 52.3\ \text{V}.$$

This contrasts with the original applied potential difference of 85.5 V.

(*f*) From Eq. 30-1, the capacitance with the slab in place is

$$C' = \frac{q}{\Delta V'} = \frac{7.02 \times 10^{-10}\ \text{C}}{52.3\ \text{V}}$$
$$= 1.34 \times 10^{-11}\ \text{F} = 13.4\ \text{pF}.$$

Table 30-1 summarizes the results of this sample problem and also includes the results that would have followed if the dielectric slab had completely filled the space between the plates.

MULTIPLE CHOICE

30-1 Capacitors

1. Two parallel metal plates have charges q_1 and q_2. Is this an example of a capacitor?
   (A) Yes
   (B) Only if $q_1 = -q_2$
   (C) Only if the signs of q_1 and q_2 are different
   (D) No

30-2 Capacitance

2. The centers of two identical conducting spheres of radius r are separated by a distance $d > 2r$. A charge $+q$ is on one sphere, and a charge $-q$ is on the other. The capacitance of the system is C_0. Additional charge is now transferred so that the charge on each sphere is doubled.

   (*a*) What is the new capacitance C' now that the charges have changed?
   (A) $C' = 4C_0$ (B) $C' = 2C_0$
   (C) $C' = C_0$ (D) $C' = C_0/2$
   (E) There is not enough information to answer the question.

   (*b*) What is the new potential difference $\Delta V'$ between the spheres?
   (A) $\Delta V' = 4q/C_0$ (B) $\Delta V' = 2q/C_0$
   (C) $\Delta V' = q/C_0$ (D) $\Delta V' = q/2C_0$
   (E) There is not enough information to answer the question.

3. The centers of two identical conducting spheres of radius r are separated by a distance $d > 2r$.

   (*a*) How does the capacitance of this system change if the separation of the two spheres is decreased?

(A) C increases. (B) C decreases.
(C) C remains the same.
(D) There is not enough information to answer the question.

(*b*) How does the capacitance of this system change if r is decreased?
(A) C increases. (B) C decreases.
(C) C remains the same.
(D) There is not enough information to answer the question.

30-3 Calculating the Capacitance

4. Which of the following changes to an ideal parallel-plate capacitor connected to an ideal battery will result in an increase in the charge on the capacitor?
(A) Decreasing the potential difference across the plates
(B) Decreasing the area of the plates
(C) Decreasing the separation of the plates
(D) None of the above

5. Equation 30-5 neglects fringing effects near the edge of the plates. Does this cause Eq. 30-5 to under- or overestimate the capacitance of a real parallel-plate capacitor?
(A) Overestimate (B) Underestimate
(C) Neither, the expression is exact

6. What is the capacitance of a *single* spherical conductor of radius r?
(A) $4\pi\epsilon_0$ (B) $4\pi\epsilon_0 r$ (C) $4\pi\epsilon_0/r$
(D) Capacitance is undefined for a single object.

30-4 Capacitors in Series and Parallel

7. Two capacitors C_1 and C_2 are connected in series; assume that $C_1 < C_2$. The equivalent capacitance of this arrangement is C, where
(A) $C < C_1/2$. (B) $C_1/2 < C < C_1$.
(C) $C_1 < C < C_2$. (D) $C_2 < C < 2C_2$.
(E) $2C_2 < C$.

8. Two capacitors C_1 and C_2 are connected in parallel; assume that $C_1 < C_2$. The equivalent capacitance of this arrangement is C, where
(A) $C < C_1/2$. (B) $C_1/2 < C < C_1$.
(C) $C_1 < C < C_2$. (D) $C_2 < C < 2C_2$.
(E) $2C_2 < C$.

9. Four possible capacitor arrangements are shown in Fig. 30-16 for three identical capacitors.

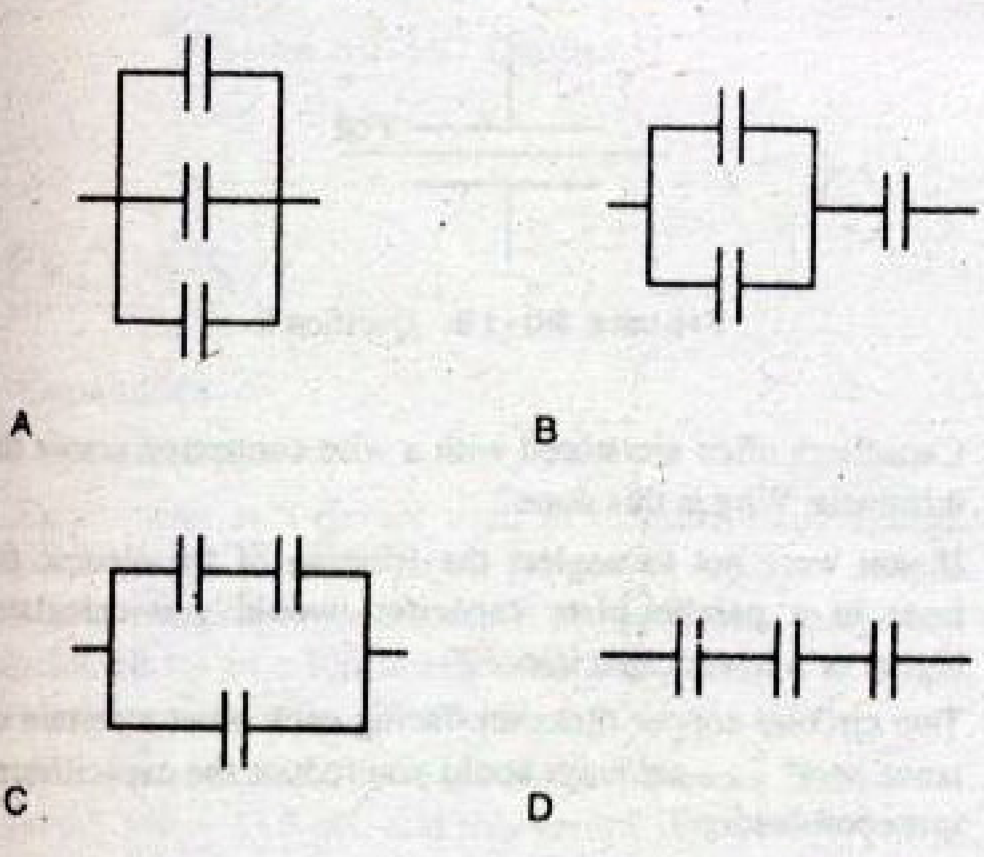


FIGURE 30-16. Multiple-choice question 9.

(*a*) Which arrangement would have the largest equivalent capacitance?

(*b*) If each arrangement were connected to a 12-V potential difference, in which case would the largest amount of charge be transferred?

(*c*) Each arrangement is connected to a potential difference so that the same amount of charge is transferred. Which requires the largest potential difference?

30-5 Energy Storage in an Electric Field

10. A parallel-plate capacitor is connected to an ideal battery, which provides a fixed potential difference. Originally the energy stored in the capacitor is U_0. If the distance between the plates is doubled, then the new energy stored in the capacitor will be
(A) $4U_0$. (B) $2U_0$. (C) U_0.
(D) $U_0/2$. (E) $U_0/4$.

11. A parallel-plate capacitor is charged by connecting it to an ideal battery; the capacitor is then disconnected. Originally the energy stored in the capacitor is U_0. If the distance between the plates is doubled, then the new energy stored in the capacitor will be
(A) $4U_0$. (B) $2U_0$. (C) U_0.
(D) $U_0/2$. (E) $U_0/4$.

12. A student originally charges a fixed capacitor to have a potential energy of 1 J. If the student wishes to give the capacitor a potential energy of 4 J, then the student should
(A) quadruple the potential difference across the capacitor but leave the charge unchanged.
(B) double the potential difference across the capacitor but leave the charge unchanged.
(C) double both the potential difference across the capacitor and the charge.
(D) leave the potential difference across the capacitor unchanged while doubling the charge.

13. An inflated balloon is covered with a conducting surface that carries a charge q. The balloon develops a leak and the radius starts to decrease, but no charge is lost from the surface.
(*a*) How does the capacitance of the balloon change as the balloon leaks?
(A) C increases. (B) C decreases.
(C) C remains the same.
(D) There is not enough information to answer the question.

(*b*) How does the stored electrical energy change as the balloon leaks?
(A) U increases. (B) U decreases.
(C) U remains the same.
(D) There is not enough information to answer the question.

30-6 Capacitor with Dielectric

14. Consider a parallel-plate capacitor originally with a charge q_0, capacitance C_0, and potential difference ΔV_0. There is an electrostatic force of magnitude F_0 between the plates, and the capacitor has a stored energy U_0. The terminals of the capacitor are *not* connected to anything.
(*a*) A dielectric slab with $\kappa_e > 1$ is inserted between the plates. Which quantities increase? (Choose all that apply.)
(A) q (B) C (C) ΔV (D) F (E) U

(*b*) What is the direction of the electrostatic force on the dielectric slab while it is being inserted?

(A) The force pulls the slab into the capacitor.
(B) The force pushes the slab out of the capacitor.
(C) There is no electrostatic force on the slab.

(*c*) Later the dielectric slab is removed. What is the direction of the electrostatic force on the dielectric slab while it is being removed?

(A) The force pulls the slab into the capacitor.
(B) The force pushes the slab out of the capacitor.
(C) There is no electrostatic force on the slab.

15. Consider a parallel-plate capacitor originally with charge q_0 and capacitance C_0. There is an electrostatic force of magnitude F_0 between the plates, and the capacitor has a stored energy U_0. The terminals of the capacitor are connected to an ideal battery, which supplies a potential difference ΔV_0.

(*a*) A dielectric slab with $\kappa_e > 1$ is inserted between the plates. Which quantities increase? (Choose all that apply.)

(A) q (B) C (C) ΔV
(D) F (E) U

(*b*) What is the direction of the electrostatic force on the dielectric slab while it is being inserted?

(A) The force pulls the slab into the capacitor.
(B) The force pushes the slab out of the capacitor.
(C) There is no electrostatic force on the slab.

(*c*) Later the dielectric slab is removed. What is the direction of the electrostatic force on the dielectric slab while it is being removed?

(A) The force pulls the slab into the capacitor.
(B) The force pushes the slab out of the capacitor.
(C) There is no electrostatic force on the slab.

QUESTIONS

1. A capacitor is connected across a battery. (*a*) Why does each plate receive a charge of exactly the same magnitude? (*b*) Is this true even if the plates are of different sizes?

2. You are given two capacitors, C_1 and C_2, in which $C_1 > C_2$. How could things be arranged so that C_2 could hold more charge than C_1?

3. The relation $\sigma \propto 1/R$, in which σ is the surface charge density and R is the radius of curvature (see Eq. 28-42), suggests that the charge placed on an isolated conductor concentrates on points and avoids flat surfaces, where $R = \infty$. How do we reconcile this with Fig. 30-5, in which the charge is definitely on the flat surface of either plate?

4. In connection with Eq. 30-1 ($q = C\,\Delta V$) we said that C is a constant. Yet we pointed out (see Eq. 30-5) that it depends on the geometry (and also, as we saw later, on the medium). If C is indeed a constant, with respect to what variables does it remain constant?

5. In Fig. 30-1, suppose that a and b are nonconductors, the charge being distributed arbitrarily over their surfaces. (*a*) Would Eq. 30-1 ($q = C\,\Delta V$) hold, with C independent of the charge arrangements? (*b*) How would you define ΔV in this case?

6. You are given a parallel-plate capacitor with square plates of area A and separation d, in a vacuum. What is the qualitative effect of each of the following on its capacitance? (*a*) Reduce d. (*b*) Put a slab of copper between the plates, touching neither plate. (*c*) Double the area of both plates. (*d*) Double the area of one plate only. (*e*) Slide the plates parallel to each other so that the area of overlap is 50%. (*f*) Double the potential difference between the plates. (*g*) Tilt one plate so that the separation remains d at one end but is $\frac{1}{2}d$ at the other.

7. You have two isolated conductors, each of which has a certain capacitance; see Fig. 30-17. If you join these conductors by a fine wire, how do you calculate the capacitance of the combination? In joining them with the wire, have you connected them in series or parallel?

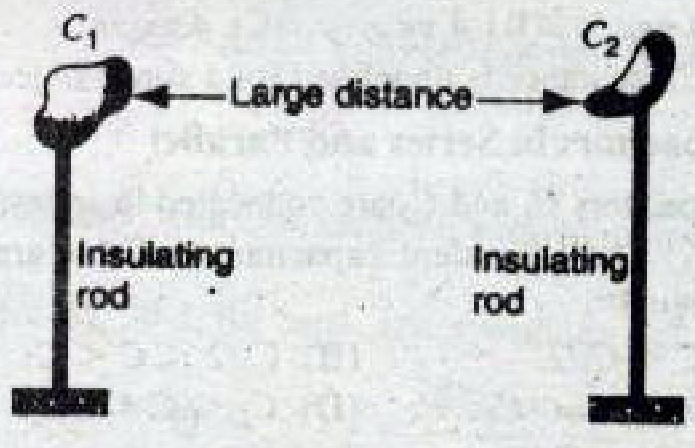


FIGURE 30-17. Question 7.

8. The capacitance of a conductor is affected by the presence of a second conductor that is uncharged and isolated electrically. Why?

9. A sheet of aluminum foil of negligible thickness is placed between the plates of a capacitor as in Fig. 30-18. What effect has it on the capacitance if (*a*) the foil is electrically insulated and (*b*) the foil is connected to the upper plate?

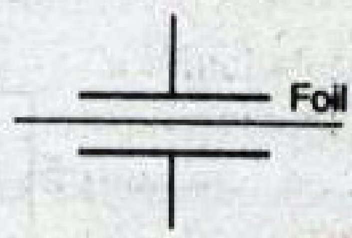


FIGURE 30-18. Question 9.

10. Capacitors often are stored with a wire connected across their terminals. Why is this done?

11. If you were not to neglect the fringing of the electric field lines in a parallel-plate capacitor, would you calculate a higher or a lower capacitance?

12. Two circular copper disks are facing each other a certain distance apart. In what ways could you reduce the capacitance of this combination?

13. Discuss similarities and differences when (*a*) a dielectric slab and (*b*) a conducting slab are inserted between the plates of a parallel-plate capacitor. Assume the slab thicknesses to be one-half the plate separation.

14. An oil-filled, parallel-plate capacitor has been designed to have a capacitance C and to operate safely at or below a certain maximum potential difference ΔV_m without arcing over. However, the designer did not do a good job and the capacitor occasionally arcs over. What can be done to redesign the capacitor, keeping C and ΔV_m unchanged and using the same dielectric?

15. For a given potential difference, does a capacitor store more or less charge with a dielectric than it does without a dielectric (vacuum)? Explain in terms of the microscopic picture of the situation.

16. Water has a high dielectric constant (see Table 29-2). Why is it not used ordinarily as a dielectric material in capacitors?

17. Figure 30-19 shows an actual 1-F capacitor available for use in student laboratories. It is only a few centimeters in diameter. Considering the result of Sample Problem 30-2, how can such a capacitor be constructed?

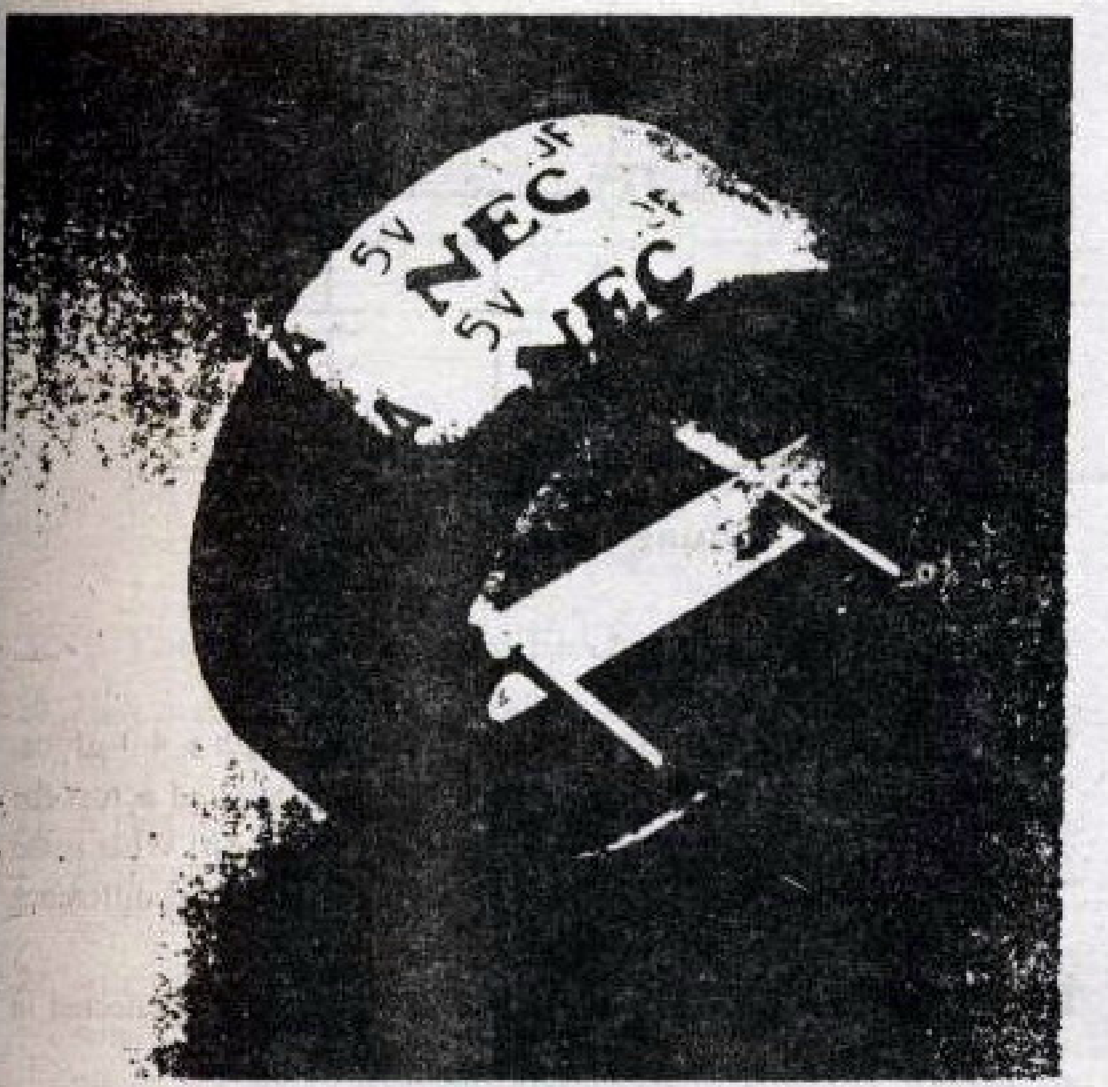


FIGURE 30-19. Question 17.

18. A dielectric slab is inserted in one end of a charged parallel-plate capacitor (the plates being horizontal and the charging battery having been disconnected) and then released. Describe what happens. Neglect friction.

19. A parallel-plate capacitor is charged by using a battery, which is then disconnected. A dielectric slab is then slipped between the plates. Describe qualitatively what happens to the charge, the capacitance, the potential difference, the electric field, and the stored energy.

20. While a parallel-plate capacitor remains connected to a battery, a dielectric slab is slipped between the plates. Describe qualitatively what happens to the charge, the capacitance, the potential difference, the electric field, and the stored energy. Is work required to insert the slab?

21. Imagine a dielectric slab, of width equal to the plate separation, inserted only halfway into a parallel-plate capacitor carrying a fixed charge q. Sketch qualitatively the distribution of the charge q on the plates and the induced charge q' on the slab.

22. Two identical capacitors are connected as shown in Fig. 30-20. A dielectric slab is slipped between the plates of one capacitor, the battery remaining connected so that a constant potential difference ΔV is maintained. Describe qualitatively what happens to the charge, the capacitance, the potential difference, the electric field, and the stored energy for each capacitor.

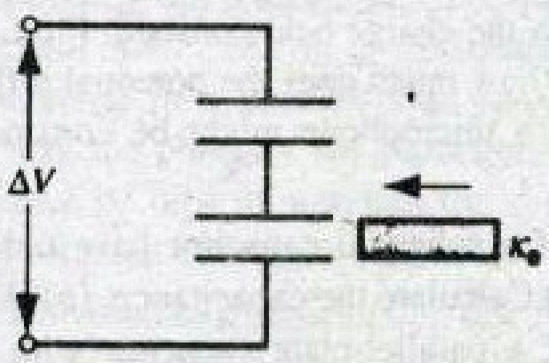


FIGURE 30-20. Question 22.

23. In this chapter we have assumed electrostatic conditions; that is, the potential difference ΔV between the capacitor plates remains constant. Suppose, however, that, as it often does in practice, ΔV varies sinusoidally with time with an angular frequency ω. Would you expect the dielectric constant κ_e to vary with ω?

EXERCISES

30-1 Capacitors

30-2 Capacitance

1. An electrometer is a device used to measure static charge. Unknown charge is placed on the plates of a capacitor and the potential difference is measured. What minimum charge can be measured by an electrometer with a capacitance of 50 pF and a voltage sensitivity of 0.15 V?

2. The two metal objects in Fig. 30-21 have net charges of +73.0 pC and −73.0 pC, and this results in a 19.2-V potential difference between them. (*a*) What is the capacitance of the system? (*b*) If the charges are changed to +210 pC and −210 pC, what does the capacitance become? (*c*) What does the potential difference become?

FIGURE 30-21. Exercise 2.

3. The capacitor in Fig. 30-22 has a capacitance of 26.0 μF and is initially uncharged. A battery supplies a potential difference ΔV of 125 V. After switch S has been closed for a long time, how much charge will have been moved by the battery?

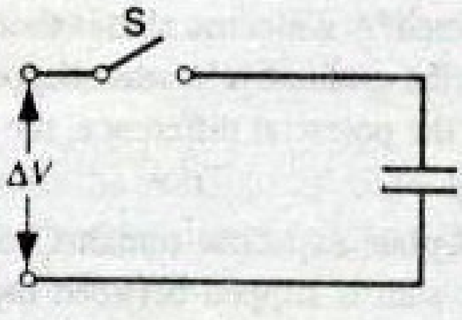


FIGURE 30-22. Exercise 3.

30-3 Calculating the Capacitance

4. A parallel-plate capacitor has circular plates of 8.22-cm radius and 1.31-mm separation. (*a*) Calculate the capacitance. (*b*) What charge will appear on the plates if a potential difference of 116 V is applied?

5. The plate and cathode of a vacuum tube diode are in the form of two concentric cylinders with the cathode as the central cylinder. The cathode diameter is 1.62 mm and the plate diameter is 18.3 mm, with both elements having a length of 2.38 cm. Calculate the capacitance of the diode.

6. Two sheets of aluminum foil have a separation of 1.20 mm, a capacitance of 9.70 pF, and are charged to 13.0 V. (*a*) Calculate the plate area. (*b*) The separation is now decreased by 0.10 mm with the charge held constant. Find the new capacitance. (*c*) By how much does the potential difference change? Explain how a microphone might be constructed using this principle.

7. The plates of a spherical capacitor have radii 38.0 mm and 40.0 mm. (*a*) Calculate the capacitance. (*b*) What must be the plate area of a parallel-plate capacitor with the same plate separation and capacitance?

8. Suppose that the two spherical shells of a spherical capacitor have their radii approximately equal. Under these conditions the device approximates a parallel-plate capacitor with $b - a = d$. Show that Eq. 30-8 for the spherical capacitor does indeed reduce to Eq. 30-5 for the parallel-plate capacitor in this case.

30-4 Capacitors in Series and Parallel

9. How many 1.00-μF capacitors must be connected in parallel to store a charge of 1.00 C with a potential difference of 110 V across the capacitors?

10. In Fig. 30-23, find the equivalent capacitance of the combination. Assume that C_1 = 10.3 μF, C_2 = 4.80 μF, and C_3 = 3.90 μF.

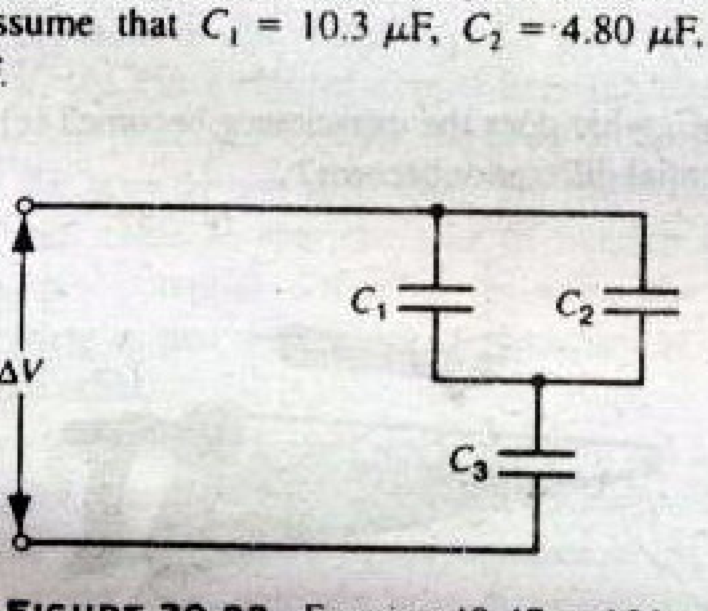


FIGURE 30-23. Exercises 10, 17, and 26.

11. In Fig. 30-24, find the equivalent capacitance of the combination. Assume that C_1 = 10.3 μF, C_2 = 4.80 μF, and C_3 = 3.90 μF.

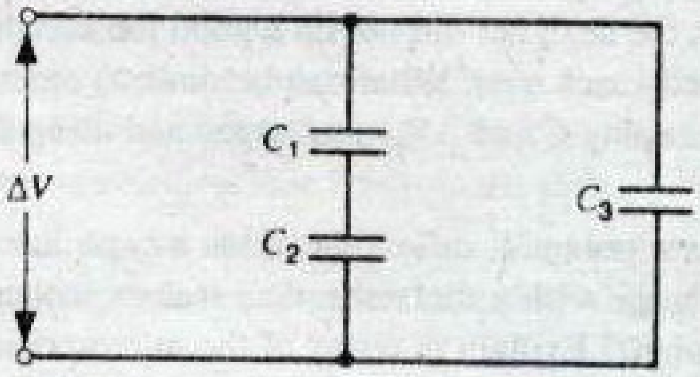


FIGURE 30-24. Exercise 11.

12. Each of the uncharged capacitors in Fig. 30-25 has a capacitance of 25.0 μF. A potential difference of 4200 V is established when the switch S is closed. How much charge then passes through the meter A?

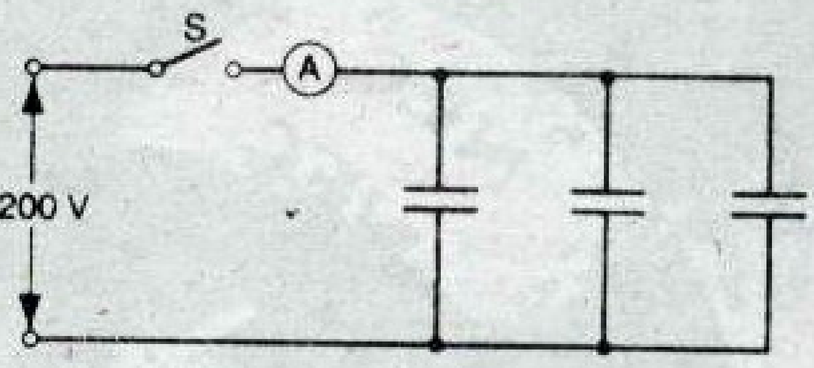


FIGURE 30-25. Exercise 12.

13. A 6.0-μF capacitor is connected in series with a 4.0-μF capacitor; a potential difference of 200 V is applied across the pair. (*a*) Calculate the equivalent capacitance. (*b*) What is the charge on each capacitor? (*c*) What is the potential difference across each capacitor?

14. Work Exercise 13 for the same two capacitors connected in parallel.

15. (*a*) Three capacitors are connected in parallel. Each has plate area *A* and plate spacing *d*. What must be the spacing of a single capacitor of plate area *A* if its capacitance equals that of the parallel combination? (*b*) What must be the spacing if the three capacitors are connected in series?

16. You have several 2.0-μF capacitors, each capable of withstanding 200 V without breakdown. How would you assemble a combination having an equivalent capacitance of (*a*) 0.40 μF or (*b*) 1.2 μF, each combination capable of withstanding 1000 V?

17. In Fig. 30-23, suppose that capacitor C_3 breaks down electrically, becoming equivalent to a conducting path. What changes in (*a*) the charge and (*b*) the potential difference occur for capacitor C_1? Assume that ΔV = 115 V.

18. A 108-μF capacitor is charged to a potential difference of 52.4 V, then the charging battery is disconnected. The capacitor is then connected in parallel with a second (initially uncharged) capacitor. The measured potential differ-

ence drops to 35.8 V. Find the capacitance of this second capacitor.

19. A portion of an infinite array of identical 1 μF-capacitors is shown in Fig. 30-26. The array is initially uncharged. A battery is then connected across two distant junctions. Show that the potential at any junction is the average of the potential at the four nearest junctions. You will use the result to solve Computer Problem 1. (Hint: What is the net charge on any junction?)

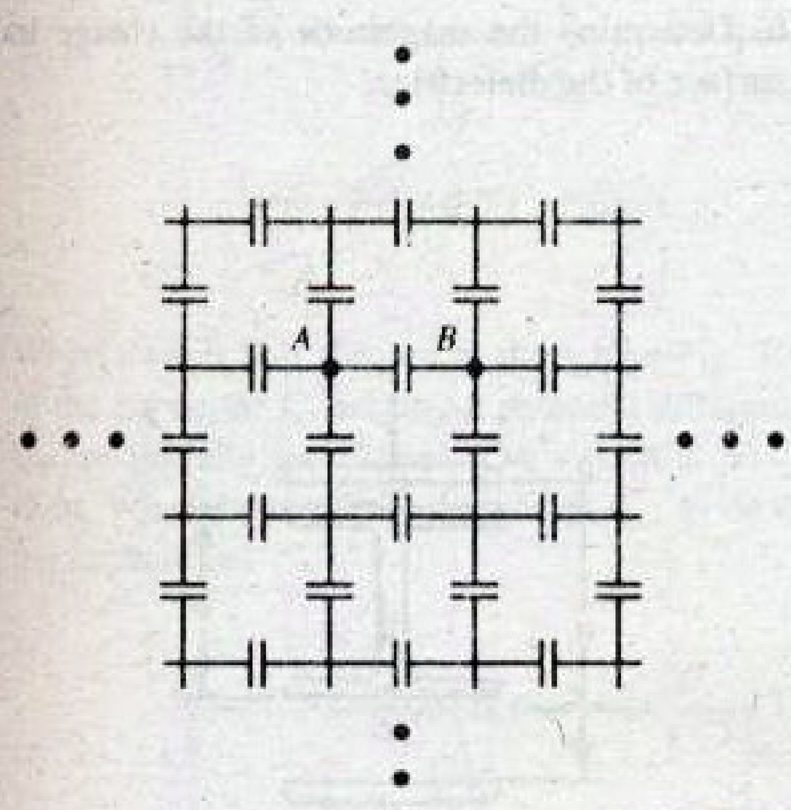


FIGURE 30-26. Exercise 19 and computer problem 1.

30-5 Energy Storage in an Electric Field

20. How much energy is stored in 2.0 m^3 of air due to the "fair weather" electric field of strength 150 V/m?

21. A parallel-connected bank of 2100 5.0-μF capacitors is used to store electric energy. What does it cost to charge this bank to 55 kV, assuming a rate of 3.0¢/kW · h?

22. Attempts to build a controlled thermonuclear fusion reactor, which, if successful, could provide the world with a vast supply of energy from heavy hydrogen in seawater, usually involve huge electric currents for short periods of time in magnetic field windings. These currents are often provided by discharging large banks of capacitors. One such capacitor bank provides 61.0 mF at 10.0 kV. Calculate the stored energy (*a*) in joules and (*b*) in kW · h.

23. A parallel-plate, air-filled capacitor having area 42.0 cm^2 and spacing of 1.30 mm is charged to a potential difference of 625 V. Find (*a*) the capacitance, (*b*) the magnitude of the charge on each plate, (*c*) the stored energy, (*d*) the electric field between the plates, and (*e*) the energy density between the plates.

24. Two capacitors, 2.12 μF and 3.88 μF, are connected in series across a 328-V potential difference. Calculate the total energy stored in the capacitors.

25. An isolated metal sphere whose diameter is 12.6 cm has a potential of 8150 V (where $V = 0$ at infinity). Calculate the energy density in the electric field near the surface of the sphere.

26. In Fig. 30-23, find (*a*) the charge, (*b*) the potential difference, and (*c*) the stored energy for each capacitor. Assume the numerical values of Exercise 10, with $\Delta V = 112$ V.

27. A cylindrical capacitor has radii a and b as in Fig. 30-6. Show that half the stored electric potential energy lies within a cylinder whose radius is $r = \sqrt{ab}$.

30-6 Capacitor with Dielectric

28. An air-filled, parallel-plate capacitor has a capacitance of 51.3 pF. (*a*) If its plates each have an area of 0.350 m^2, what is their separation? (*b*) If the region between the plates is now filled with material having a dielectric constant of 5.60, what is the capacitance?

29. An air-filled, parallel-plate capacitor has a capacitance of 1.32 pF. The separation of the plates is doubled and wax is inserted between them. The new capacitance is 2.57 pF. Find the dielectric constant of the wax.

30. Given a 7.40-pF air-filled capacitor, you are asked to design a capacitor to store up to 6.61 μJ with a maximum potential difference of 630 V. What dielectric in Table 29-2 will you use to fill the gap in the capacitor if you do not allow for a margin of error?

31. For making a parallel-plate capacitor you have available two plates of copper, a sheet of mica (thickness = 0.10 mm, κ_e = 5.4), a sheet of glass (thickness = 0.20 mm, κ_e = 7.0), and a slab of paraffin (thickness = 1.0 cm, κ_e = 2.0). To obtain the largest capacitance, which sheet should you place between the copper plates?

32. A certain substance has a dielectric constant of 2.80 and a dielectric strength of 18.2 MV/m. If the substance is used as the dielectric material in a parallel-plate capacitor, what minimum area may the plates of the capacitor have in order that the capacitance be 68.4 nF and that the capacitor be able to withstand a potential difference of 4.13 kV?

33. A coaxial cable used in a transmission line responds as a "distributed" capacitance to the circuit feeding it. Calculate the capacitance of 1.00 km for a cable having an inner radius of 0.110 mm and an outer radius of 0.588 mm. Assume that the space between the conductors is filled with polystyrene.

34. You have been assigned to design a transportable capacitor that can store 250 kJ of energy. You select a parallel-plate type with dielectric. (*a*) What is the minimum capacitor volume achievable using a dielectric selected from those listed in Table 29-2 that have values of dielectric strength? (*b*) Modern, high-performance capacitors that can store 250 kJ have volumes of 0.087 m^3. Assuming that the dielectric used has the same dielectric strength as in (*a*), what must be its dielectric constant?

35. You are asked to construct a capacitor having a capacitance near 1000 pF and a breakdown potential in excess of 10 kV. You think of using the sides of a tall drinking glass (Pyrex), lining the inside and outside with aluminum foil (neglect the ends). What are (*a*) the capacitance and (*b*) the breakdown potential? You use a glass 15 cm tall with an inner radius of 3.6 cm and an outer radius of 3.8 cm.

36. In Sample Problem 30-8, suppose that the battery remains connected during the time that the dielectric slab is being introduced. Calculate (*a*) the capacitance, (*b*) the charge on the capacitor plates, (*c*) the electric field in the gap, and (*d*) the electric field in the slab, after the slab is introduced.

37. A slab of copper of thickness b is thrust into a parallel-plate capacitor as shown in Fig. 30-27. (*a*) What is the capacitance after the slab is introduced? (*b*) If a charge q is maintained on the plates, find the ratio of the stored energy before to that after the slab is inserted. (c) How much work is done on the slab as it is inserted? Is the slab pulled in or do you have to push it in?

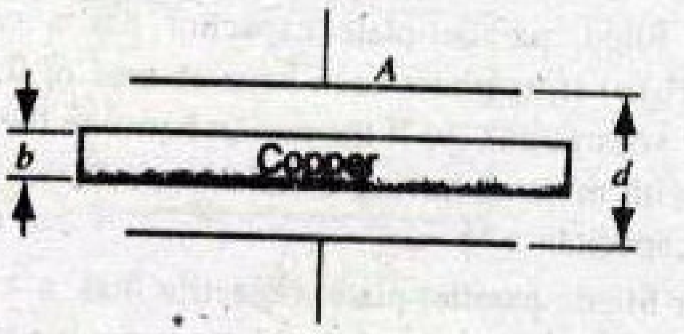


FIGURE 30-27. Exercise 37.

38. Reconsider Exercise 37, assuming that the potential di ence ΔV rather than the charge is held constant.

39. A parallel-plate capacitor has a capacitance of 112 pF, a p area of 96.5 cm^2, and a mica dielectric (κ_e = 5.40). A 55.0-V potential difference, calculate the magnitudes (a) the electric field in the mica, (b) the free charge on plates, and (c) the induced surface charge.

40. Two parallel plates of area 110 cm^2 are each given equal opposite charges of 890 nC. The electric field within dielectric material filling the space between the plates 1.40 MV/m. (a) Calculate the dielectric constant of the ma ial. (b) Determine the magnitude of the charge induced each surface of the dielectric.

PROBLEMS

1. In Section 30-2, the capacitance of a cylindrical capacitor was calculated. Using the approximation (see Appendix I) that $\ln(1 + x) \approx x$ when $x \ll 1$, show that the capacitance approaches that of a parallel-plate capacitor when the spacing between the two cylinders is small.

2. A capacitor is to be designed to operate, with constant capacitance, in an environment of fluctuating temperature. As shown in Fig. 30-28, the capacitor is a parallel-plate type with plastic "spacers" to keep the plates aligned. (a) Show that the rate of change of capacitance C with temperature T is given by

$$\frac{dC}{dT} = C\left(\frac{1}{A}\frac{dA}{dT} - \frac{1}{x}\frac{dx}{dT}\right),$$

where A is the plate area and x is the plate separation. (b) If the plates are aluminum, what should be the coefficient of thermal expansion of the spacers in order that the capacitance not vary with temperature? (Ignore the effect of the spacers on the capacitance.)

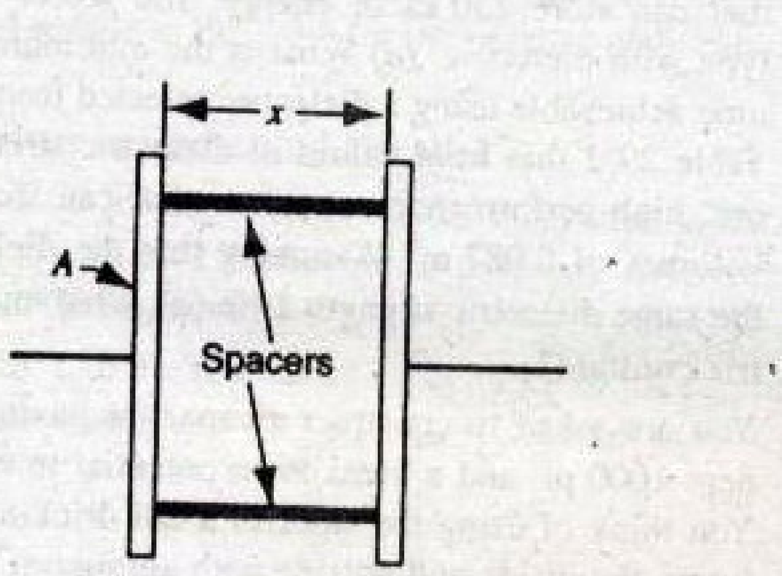


FIGURE 30-28. Problem 2.

3. Figure 30-29 shows two capacitors in series, the rigid center section of length b being movable vertically. Show that the equivalent capacitance of the series combination is independent of the position of the center section and is given by

$$C = \frac{\epsilon_0 A}{a - b}.$$

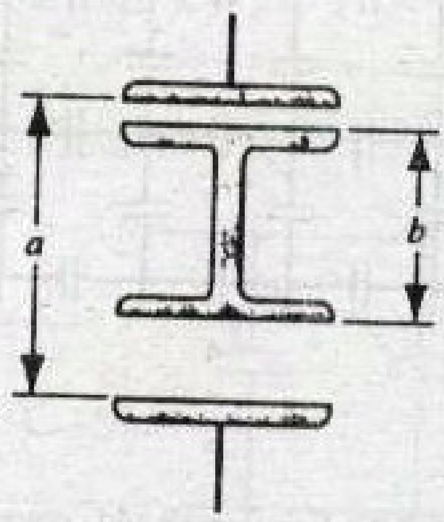


FIGURE 30-29. Problem 3.

4. In Fig. 30-30, a variable, air-filled capacitor of the type used in tuning radios is shown. Alternate plates are connected to gether, one group being fixed in position, the other group be ing capable of rotation. Consider a pile of n plates of alterna polarity, each having an area A and separated from adjacen plates by a distance d. Show that this capacitor has a maxi mum capacitance of

$$C = \frac{(n - 1)\epsilon_0 A}{d}.$$

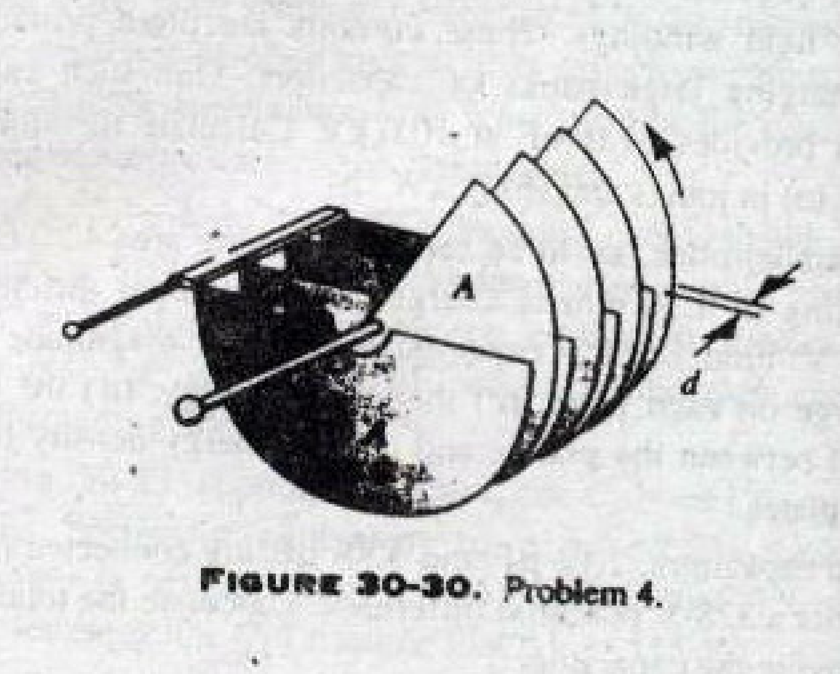


FIGURE 30-30. Problem 4.

5. In Fig. 30-31, capacitors C_1 = 1.16 μF and C_2 = 3.22 μF are each charged to a potential difference ΔV = 96.6 V but with opposite polarity, so that points a and c are on the side of the respective positive plates of C_1 and C_2, and points b and d are on the side of the respective negative plates. Switches S_1 and

S_2 are now closed. (*a*) What is the potential difference between points *e* and *f*? (*b*) What is the charge on C_1? (*c*) What is the charge on C_2?

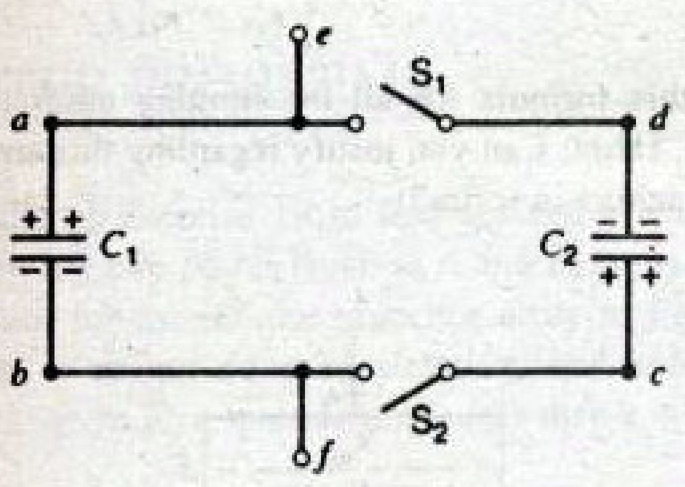


FIGURE 30-31. Problem 5.

6. When switch S is thrown to the left in Fig. 30-32, the plates of the capacitor C_1 acquire a potential difference ΔV_0. C_2 and C_3 are initially uncharged. The switch is now thrown to the right. What are the final charges q_1, q_2, q_3 on the corresponding capacitors?

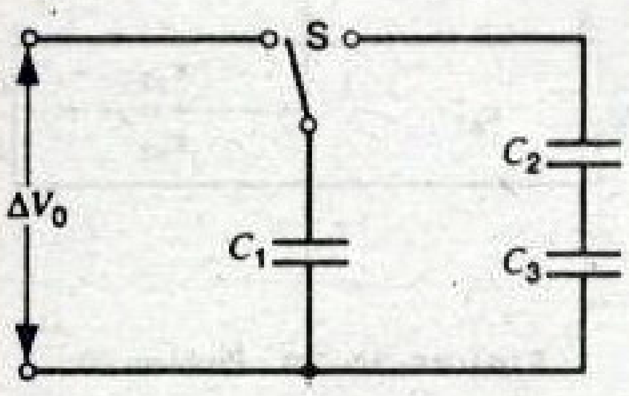


FIGURE 30-32. Problem 6.

7. Figure 30-33 shows two identical capacitors of capacitance *C* in a circuit with two (ideal) diodes D. A 100-V battery is connected to the input terminals, (*a*) first with terminal *a* positive and (*b*) later with terminal *b* positive. In each case, what is the potential difference across the output terminals? (An ideal diode has the property that positive charge flows through it only in the direction of the arrow and negative charge flows through it only in the opposite direction.)

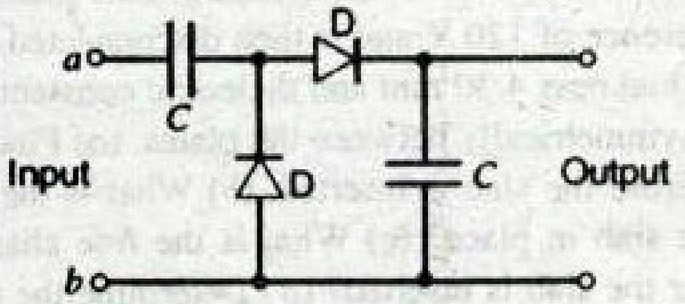


FIGURE 30-33. Problem 7.

8. A capacitor has square plates, each of side *a*, making an angle θ with each other as shown in Fig. 30-34. Show that for small θ the capacitance is given by

$$C = \frac{\epsilon_0 a^2}{d}\left(1 - \frac{a\theta}{2d}\right).$$

(Hint: The capacitor may be divided into differential strips that are effectively in parallel.)

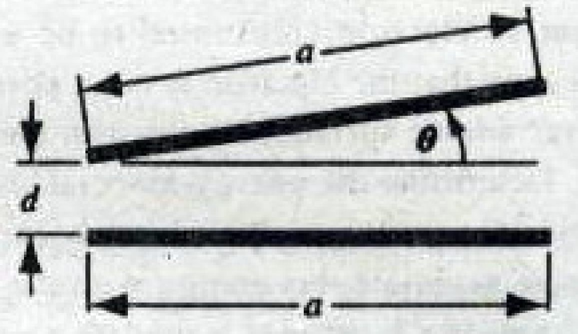


FIGURE 30-34. Problem 8.

9. In Fig. 30-35, a battery supplies a potential difference ΔV of 12 V. (*a*) Find the charge on each capacitor when switch S_1 is closed and (*b*) when (later) switch S_2 is also closed. Take $C_1 = 1.0\ \mu F$, $C_2 = 2.0\ \mu F$, $C_3 = 3.0\ \mu F$, and $C_4 = 4.0\ \mu F$.

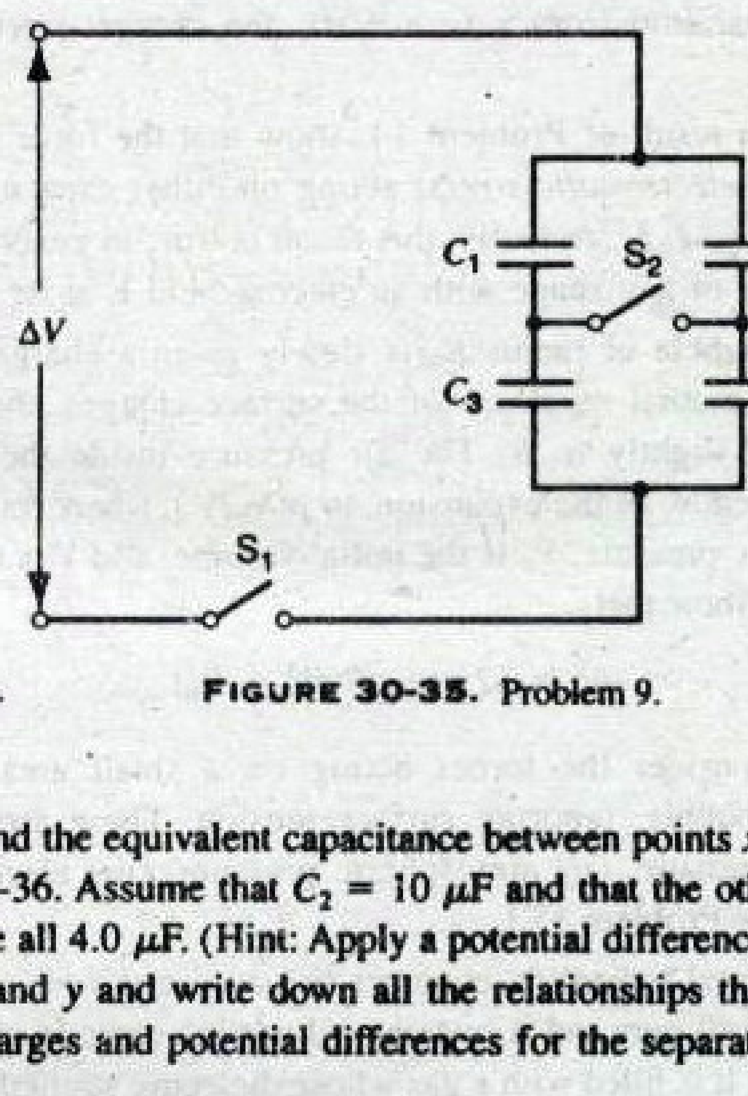


FIGURE 30-35. Problem 9.

10. Find the equivalent capacitance between points *x* and *y* in Fig. 30-36. Assume that $C_2 = 10\ \mu F$ and that the other capacitors are all $4.0\ \mu F$. (Hint: Apply a potential difference ΔV between *x* and *y* and write down all the relationships that involve the charges and potential differences for the separate capacitors.)

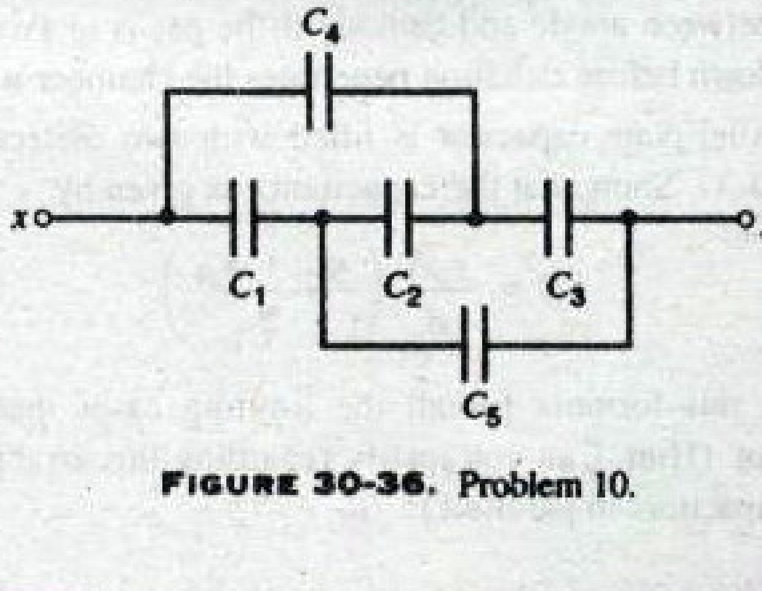


FIGURE 30-36. Problem 10.

11. One capacitor is charged until its stored energy is 4.0 J, and the charging battery is then removed. A second uncharged capacitor is then connected to it in parallel. (*a*) If the charge distributes equally, what is now the total energy stored in the electric fields? (*b*) Where did the excess energy go?

12. A fluid with resistivity 9.40 Ω·m seeps into the space between the plates of a 110-pF parallel-plate, air-filled capacitor. When the space is completely filled, what is the resistance between the plates?

13. (*a*) Calculate the energy density of the electric field at a distance r from an electron (presumed to be a particle) at rest. (*b*) Assume now that the electron is not a point but a sphere of radius R over whose surface the electron charge is uniformly distributed. Determine the energy associated with the external electric field in vacuum of the electron as a function of R. (*c*) If you now associate this energy with the mass of the electron, you can, using $E_0 = mc^2$, calculate a value for R. Evaluate this radius numerically; it is often called the classical radius of the electron.

14. Show that the plates of a parallel-plate capacitor attract each other with a force given by

$$F = \frac{q^2}{2\epsilon_0 A}.$$

Prove this by calculating the work necessary to increase the plate separation from x to $x + dx$, the charge q remaining constant.

15. Using the result of Problem 14, show that the force per unit area (the *electrostatic stress*) acting on either capacitor plate is given by $\frac{1}{2}\epsilon_0 E^2$. Actually, this result is true, in general, for a conductor of any shape with an electric field $\vec{E}$ at its surface.

16. A soap bubble of radius R_0 is slowly given a charge q. Because of mutual repulsion of the surface charges, the radius increases slightly to R. The air pressure inside the bubble drops, because of the expansion, to $p(V_0/V)$, where p is the atmospheric pressure, V_0 is the initial volume, and V is the final volume. Show that

$$q^2 = 32\pi^2\epsilon_0 pR(R^3 - R_0^3).$$

[Hint: Consider the forces acting on a small area of the charged bubble, ignoring surface tension. These are due to (i) gas pressure, (ii) atmospheric pressure, (iii) electrostatic stress. See Problem 15.]

17. A cylindrical ionization chamber has a central wire anode of radius 0.180 mm and a coaxial cylindrical cathode of radius 11.0 mm. It is filled with a gas whose dielectric strength is 2.20 MV/m. Find the largest potential difference that should be applied between anode and cathode if the gas is to avoid electric breakdown before radiation penetrates the chamber window.

18. A parallel-plate capacitor is filled with two dielectrics as in Fig. 30-37. Show that the capacitance is given by

$$C = \frac{\epsilon_0 A}{d}\left(\frac{\kappa_{e1} + \kappa_{e2}}{2}\right).$$

Check this formula for all the limiting cases that you can think of. (Hint: Can you justify regarding this arrangement as two capacitors in parallel?)

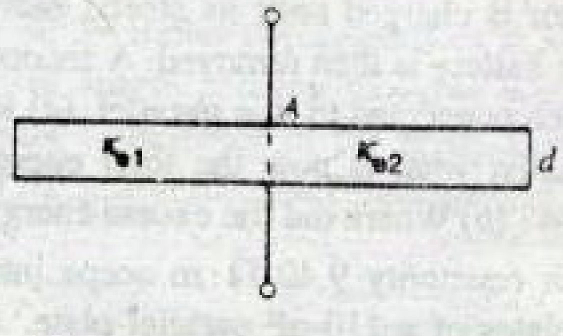


FIGURE 30-37. Problem 18

19. A parallel-plate capacitor is filled with two dielectrics as in Fig. 30-38. Show that the capacitance is given by

$$C = \frac{2\epsilon_0 A}{d}\left(\frac{\kappa_{e1}\kappa_{e2}}{\kappa_{e1} + \kappa_{e2}}\right).$$

Check this formula for all the limiting cases that you can think of. (Hint: Can you justify regarding this arrangement as two capacitors in series?)

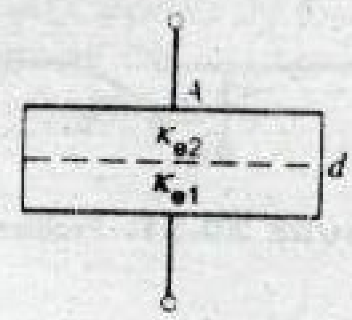


FIGURE 30-38. Problem 19.

20. What is the capacitance of the capacitor in Fig. 30-39?

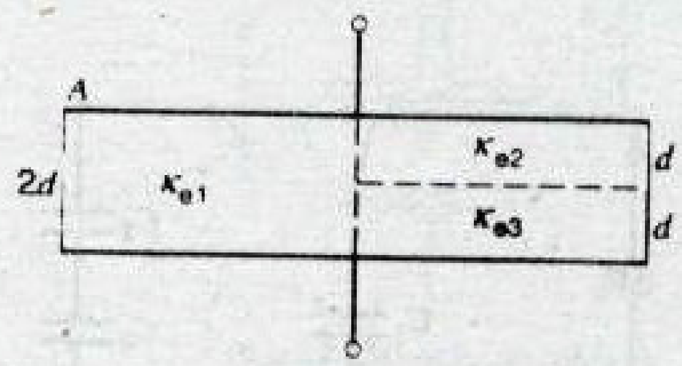


FIGURE 30-39. Problem 20.

21. A parallel-plate capacitor has plates of area A and separation d and is charged to a potential difference ΔV. The charging battery is then disconnected and the plates are pulled apart until their separation is $2d$. Derive expressions in terms of A, d, and ΔV for (*a*) the new potential difference, (*b*) the initial and the final stored energy, and (*c*) the work required to separate the plates.

22. In the capacitor of Sample Problem 30-9 (Fig. 30-15), (*a*) what fraction of the energy is stored in the air gaps? (*b*) What fraction is stored in the slab?

23. A parallel-plate capacitor has plates of area 0.118 m² and a separation of 1.22 cm. A battery charges the plates to a potential difference of 120 V and is then disconnected. A dielectric slab of thickness 4.30 mm and dielectric constant 4.80 is then placed symmetrically between the plates. (*a*) Find the capacitance before the slab is inserted. (*b*) What is the capacitance with the slab in place? (*c*) What is the free charge q before and after the slab is inserted? (*d*) Determine the electric field in the space between the plates and dielectric. (*e*) What is the electric field in the dielectric? (*f*) With the slab in place, what is the potential difference across the plates? (*g*) How much external work is involved in the process of inserting the slab?

24. A dielectric slab of thickness b is inserted between the plates of a parallel-plate capacitor of plate separation d. Show that the capacitance is given by

$$C = \frac{\kappa_e \epsilon_0 A}{\kappa_e d - b(\kappa_e - 1)}.$$

(Hint: Derive the formula following the pattern of Sample Problem 30-9.) Does this formula predict the correct numerical result of that sample problem? Verify that the formula gives reasonable results for the special cases of $b = 0$, $\kappa_e = 1$, and $b = d$.

COMPUTER PROBLEMS

1. Use the results of Exercise 19 to find the equivalent capacitance between any two points (such as A and B) separated by a single capacitor for the infinite capacitor array in Fig. 30-26. This problem is easiest done by iteration, and can be programmed and solved on a spreadsheet in less than a minute!

2. Repeat Computer Problem 1 for a torus instead of an infinite sheet. Start with a square grid of 10×10 capacitors. Join two opposite edges to make a cylinder, then join the two ends to make a torus (a "donut" shape). By how much does the answer change if the original grid is doubled in size?

CHAPTER 31

DC CIRCUITS

In Chapter 29 we discussed some general properties of current and resistance. In this chapter we begin to study the behavior of specific electric circuits that include resistive elements, which may be individual resistors or the internal resistances of circuit elements such as batteries or wires.

We confine our study in this chapter to direct current (DC) *circuits, in which the direction of the current does not change with time. In DC circuits that contain only batteries and resistors, the magnitude of the current does not vary with time, whereas in DC circuits containing capacitors, the magnitude of the current may be time dependent. Alternating current (AC) circuits, in which the current changes direction periodically, are considered in Chapter 37.*

31-1 ELECTRIC CURRENT

In Section 29-3, we discussed the flow of electric charge through conductors. The electric current i is the net amount of charge per unit time passing through an element of surface area at any particular location in the conductor. More often, we are concerned with the total current passing through a circuit, in which case the area is the entire cross section of the wires that connect the parts of the circuit.

Figure 31-1 illustrates the general problem we analyze in this chapter. A battery is connected to a "device." The device may be a single circuit element, such as a resistor or a capacitor, or it may be a combination of circuit elements. The battery maintains the upper terminal at a potential V_+ and the lower terminal at a potential V_-. For an ideal battery, the potential difference $V_+ - V_-$ between its terminals is independent of the amount of current that it is providing to the circuit. As we discuss later in this chapter, for real batteries the potential difference *does* depend on the current.

In the electrostatic case, in which conductors are equipotentials, the potential V_+ at the positive terminal of the battery would characterize the entire wire connecting the upper end of the device to the battery. In this case, the potential difference $V_+ - V_-$ between the battery terminals would also appear between the upper and lower terminals of the device. When currents are flowing in the wires, the conclusions of electrostatics are no longer valid; in particular, we know from our discussion in Section 29-4 (see Eq. 29-12) that when a current i flows in a conductor there is a potential difference $\Delta V = iR$ across the conductor. However, the

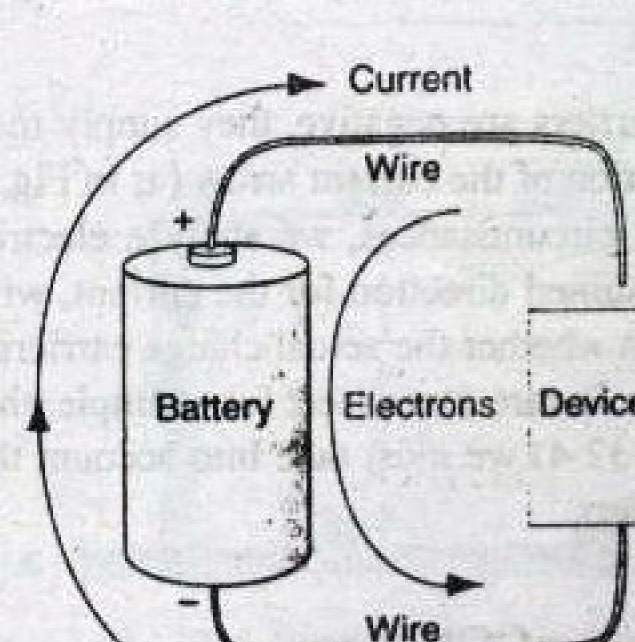


FIGURE 31-1. A battery is connected to an electrical device by two wires. The direction of the current is opposite to the direction of motion of the electrons.

resistance of the wires is usually very small compared with the resistance of the device in our circuit, so we are usually *justified in neglecting the effect of the wires; in particular,* we assume that there is no potential drop in the wires, and in this case the full voltage difference of the battery terminals does appear across the terminals of the device.

The battery can be considered a "pump" for charge, as if it were bringing positive charge through the battery from the negative terminal to the positive one. In actuality, it is usually the motion of the negatively charged electrons that is responsible for the current flow. Another way of interpreting the flow of charge in the circuit is to regard the positive charge as "falling" through the device from a region of high potential (the part of the device connected to the positive terminal of the battery) to a region of lower potential (the part of the device connected to the negative terminal of the battery).

The function of the battery in the circuit is to maintain the potential difference that enables the flow of charge. The battery is *not* a source of electrons. Electrons pass through the battery and have their energy raised as the move inside the battery from the positive to the negative terminal. When we say that a battery is "drained," we do not mean that it has "used up" its supply of electrons; instead, we mean that we have exhausted the source of energy (often a chemical reaction) that was responsible for raising the energies of the electrons. Note in Fig. 31-1 that the electrons move throughout the entire circuit; they do not "come from" the battery.

Although in metals the charge carriers are electrons, in electrolytes or in gaseous conductors (plasmas) they may also be positive or negative ions, or both. We need a convention for labeling the direction of current because charges of opposite sign move in opposite directions in a given field. A positive charge moving in one direction is equivalent in nearly all external effects to a negative charge moving in the opposite direction. Hence, for simplicity and algebraic consistency, we adopt the following convention:

> *The direction of current is the direction that positive charges would move, even if the actual charge carriers are negative.*

If the charge carriers are negative, they simply move opposite to the direction of the current arrow (as in Fig. 31-1).

Under most circumstances, we analyze electric circuits based on an assumed direction for the current, without taking into account whether the actual charge carriers are positive or negative. In rare cases (see, for example, the Hall effect in Section 32-4) we must take into account the sign of the charge carriers.

Conservation of Charge

When we first connect the battery to the device, the circuit behaves in an irregular manner. The situation is similar to what occurs when you first turn on a garden hose connected to a sprinkler. At first the water gushes through the hose, creating whirlpools and eddies. When it reaches the sprinkler it may at first by chance emerge from some of the holes and not from others. After a few seconds a steady *flow is established, and the water flows past any point at a* constant rate. In electrical circuits, we usually ignore the initial behavior (called the *transient behavior*) and consider only the steady situation, which is reached very rapidly (within nanoseconds).

We assume that, under steady conditions, charge does not collect at or drain away from any point in our idealized wire. In the language of fluid flow, there are no sources or sinks of charge in the wire. When we made this assumption in our study of incompressible fluids, we concluded that the rate at which the fluid flows past any cross section of a pipe is the same even if the cross section varies. The fluid flows faster where the pipe is smaller and slower where it is larger, but the volume rate of flow, measured perhaps in liters/second, remains constant. In the same way, *the electric current i is the same for all cross sections of a conductor, even though the cross-sectional area may be different at different points.* The current density $\vec{j}$ (current per unit area) will change as the cross-sectional area changes, but the current i remains the same.

Figure 31-2a shows the circuit of Fig. 31-1 in simpler notation. The device is indicated merely by a box. Note that the current going into the device is the same as the current coming out of the device. This is an example of conservation of charge; no net charge is retained by the device—for every electron that enters at one end, one electron leaves the other end.

Figure 31-2b shows another circuit in which the current travels in succession through three devices, labeled A, B, and C. The current i_A in device A is exactly equal to the current i_B in device B and also to the current i_C in device C; that is, $i_A = i_B = i_C$. No current is "used up" in traveling through any circuit element. Figure 31-2b is an example of circuit elements connected in *series*, in which the same current must travel in succession through each element of the circuit.

Figure 31-2c shows the current in a different combination of circuit elements. Here the current must divide when it reaches point a in the circuit, with an amount i_A passing through device A and an amount i_B passing through device B. (The relative amounts in A and B, which are not important for this discussion, depend on the properties of A and B.) At point b the currents must recombine. Because no charge gets trapped at point a, the current entering point a must be exactly the same as the current leaving it, or $i = i_A + i_B$. Similarly, the current entering point b must be the same as the current leaving point b, or $i_A + i_B = i$. This is often called the *junction rule* for analyzing circuits:

> *At any junction in an electric circuit, the total current entering the junction must be equal to the total current leaving the junction.*

In this rule the term "junction" means a point in a circuit where several wire segments meet, such as points a or b in Fig. 31-2c. The junction rule (sometimes known as *Kirch-*

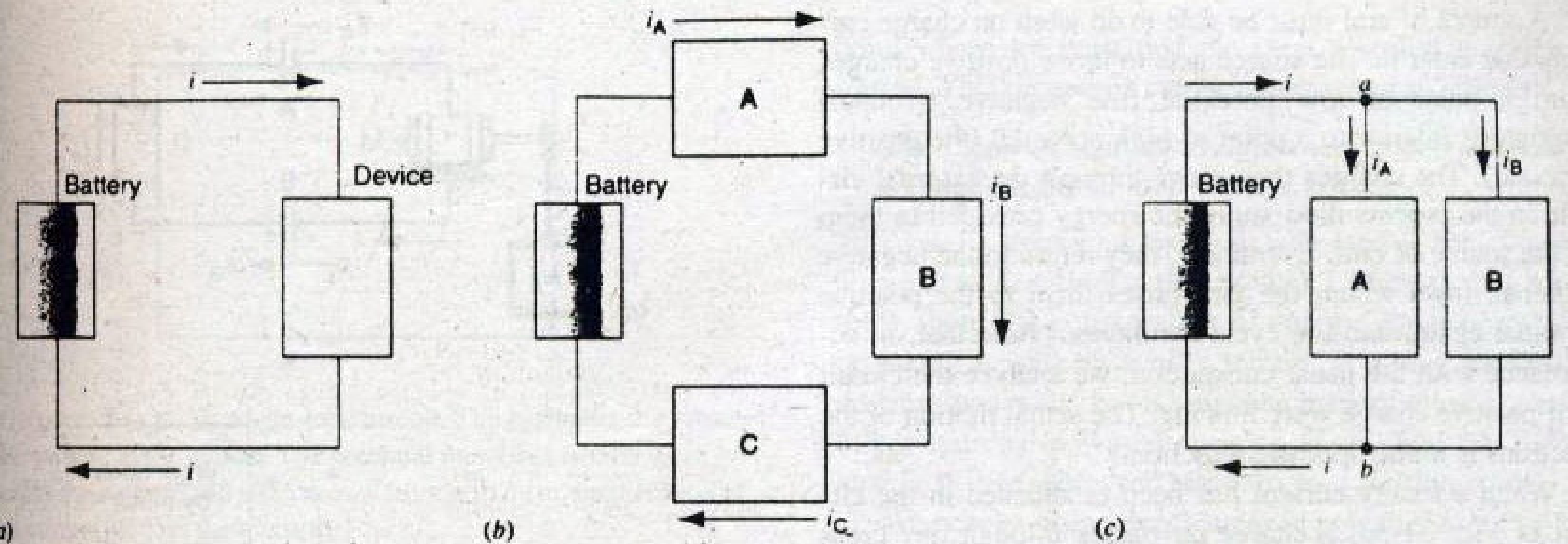


FIGURE 31-2. (*a*) The circuit of Fig. 31-1 in symbolic notation. (*b*) The same current flows in succession through devices A, B, and C. (*c*) The current divides at junction *a* and recombines at junction *b*.

hoff's first law) is really a statement about the conservation of electric charge.

Figure 31-2*c* is an example of a *parallel* connection of circuit elements. A characteristic of a parallel connection is that the current must divide to pass separately through individual elements and then recombine.

31-2 ELECTROMOTIVE FORCE

An external energy source is required by most electrical circuits to move charge through the circuit. The circuit therefore must include a device that maintains a potential difference between two points in the circuit, just as a circulating fluid requires an analogous device (a pump) that maintains a *pressure* difference between two points.

Any device that performs this task in an electrical circuit is called a source (or a seat) of *electromotive force* (symbol $\mathcal{E}$; abbreviation emf, which is pronounced "ee-em-eff"). It is sometimes useful to consider a source of emf as a mechanism that creates a "hill" of potential and moves charge "uphill," from which the charge flows "downhill" through the rest of the circuit. A common source of emf is the ordinary battery; another is the electric generator found in power plants. Solar cells are sources of emf used both in spacecraft and in pocket calculators. Other less commonly found sources of emf are fuel cells (used to power the space shuttle) and thermopiles. Biological systems, including the human heart, also function as sources of emf.

Figure 31-3*a* shows a source of emf connected to an electronic device, which might be a resistor, capacitor, or other circuit element. The emf is represented in the circuit by an arrow that is placed next to the source and points in the direction in which the emf, acting alone, would cause a positive charge carrier to move in the external circuit. We draw a small circle on the tail of the emf arrow so that it will not be mistaken for a current arrow. In the external circuit, positive charge carriers would be driven in the direction shown by the current arrows marked *i*. In other words, the source of emf sets up a clockwise current in the circuit of Fig. 31-3*a*.

The most common source of emf that we will use is the ordinary battery. As shown in Fig. 31-3*a*, a battery is represented in a circuit by two parallel lines of different length. The longer line always indicates the positive terminal of the battery. The source of emf (the battery) maintains its upper terminal at a high potential V_+ and its lower terminal at a low potential V_-. Common batteries (size AAA, AA, C, or D) that we use for flashlights or portable CD players have an emf of 1.5 volts. As we will see in the next section, the battery emf is equal to the terminal potential difference $V_+ - V_-$ *only* if there is no current in the circuit or if the battery has a negligible internal resistance.

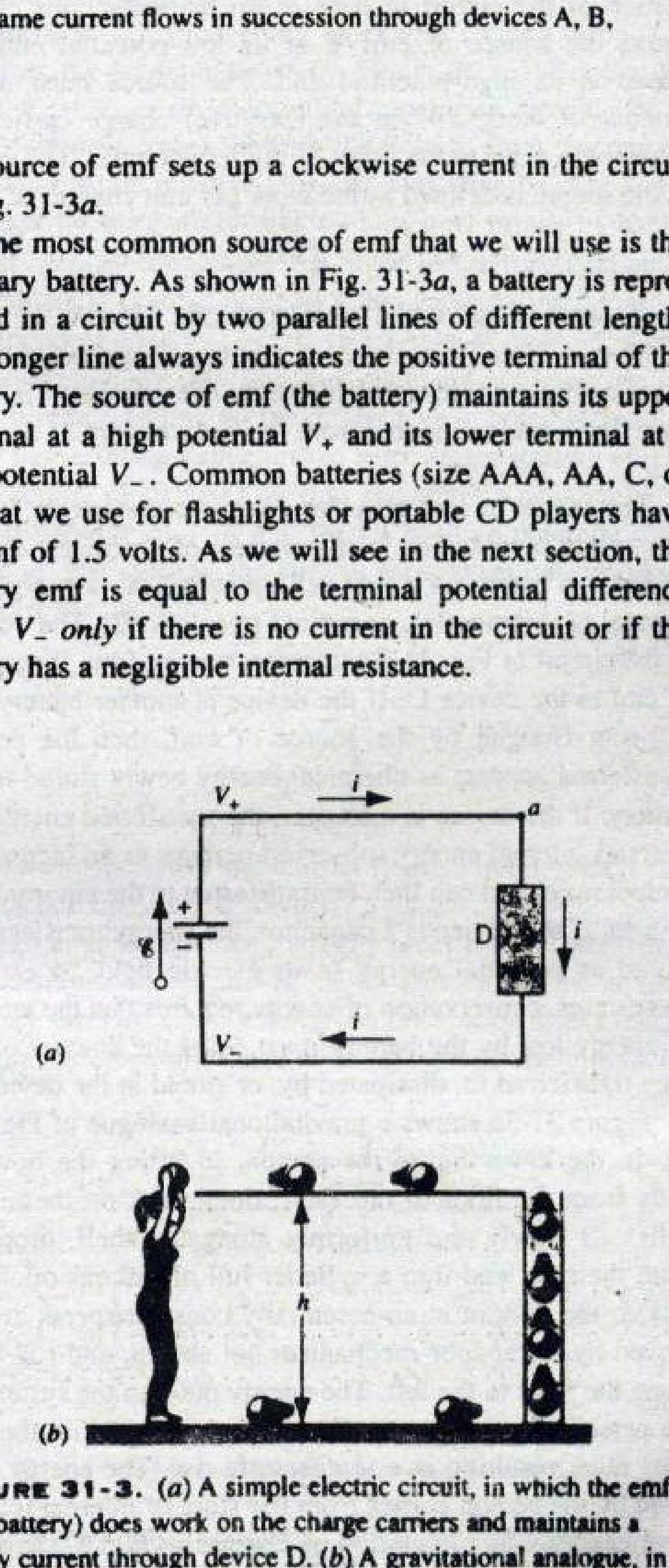


FIGURE 31-3. (*a*) A simple electric circuit, in which the emf $\mathcal{E}$ (a battery) does work on the charge carriers and maintains a steady current through device D. (*b*) A gravitational analogue, in which work done by the person maintains a steady flow of bowling balls through the viscous medium.

A source of emf must be able to do work on charge carriers that enter it. The source acts to move positive charges from a point of low potential (the negative terminal) through its interior to a point of high potential (the positive terminal). The charges then move through the external circuit, in the process dissipating the energy provided to them by the source of emf. Eventually, they return to the negative terminal, from which the emf raises them to the positive terminal again, and the cycle continues. (Note that, in accordance with the usual convention, we analyze the circuit as if positive charge were flowing. The actual motion of the electrons is in the opposite direction.)

When a steady current has been established in the circuit of Fig. 31-3*a*, a charge dq passes through *any* cross section of the circuit in time dt. In particular, this charge enters the source of emf $\mathscr{E}$ at its low-potential end and leaves at its high-potential end. The source must do an amount of work dW on the (positive) charge carriers to force them to go to the point of higher potential. The emf $\mathscr{E}$ of the source is defined as the work per unit charge, or

$$\mathscr{E} = dW/dq. \qquad (31\text{-}1)$$

The unit of emf is the joule/coulomb, which is the same as the volt. Note from Eq. 31-1 that the electromotive force is not actually a force; that is, we do not measure it in newtons. The name originates from the early history of the subject.

The source of emf provides energy to the circuit. Its energy might be obtained from a variety of processes: chemical (as in a battery or a fuel cell), mechanical (a generator), thermal (a thermopile), or radiant (a solar cell). The current in the circuit of Fig. 31-3*a* transfers energy from the source of emf to the device D. If the device is another battery that is being charged by the source of emf, then the energy transferred appears as chemical energy newly stored in the battery. If the device is a resistor, the transferred energy appears as internal energy (observed perhaps as an increase in temperature) and can then be transferred to the environment as heat. If the device is a capacitor, the energy transferred is stored as potential energy in its electric field. In each of these cases, conservation of energy requires that the amount of energy lost by the battery must equal the amount of energy transferred to, dissipated by, or stored in the device D.

Figure 31-3*b* shows a gravitational analogue of Fig. 31-3*a*. In the lower figure the person, in lifting the bowling balls from the floor to the shelf, does work on them. The balls roll slowly and uniformly along the shelf, dropping from the right end into a cylinder full of viscous oil. They sink to the bottom at an essentially constant speed, are removed by a trapdoor mechanism not shown, and roll back along the floor to the left. The energy put into the system by the person appears eventually as internal energy in the viscous fluid, resulting in a temperature rise. The energy supplied by the person comes from her store of internal (chemical) energy. The circulation of charges in Fig. 31-3*a* stops eventually if the source of emf runs out of energy; the circulation of bowling balls in Fig. 31-3*b* stops eventually if the person runs out of energy.

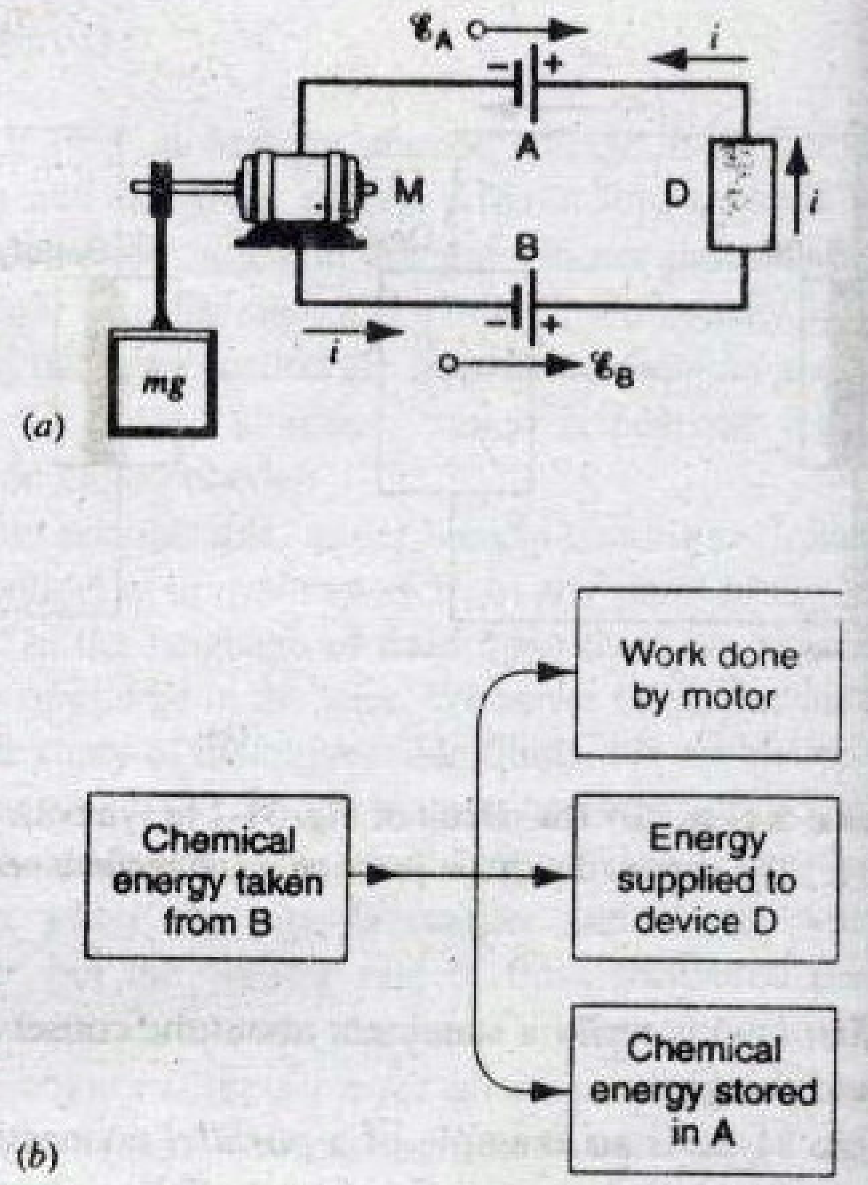


FIGURE 31-4. (*a*) $\mathscr{E}_B > \mathscr{E}_A$, so that battery B determines the direction of the current in this single-loop circuit. (*b*) Energy transfers in this circuit.

Figure 31-4*a* shows a circuit containing two sources of emf (batteries) A and B, a device D, and an ideal electric motor M employed in lifting a weight. The batteries are connected so that they tend to send charges around the circuit in opposite directions; the actual direction of the current is determined by battery B, which has the larger emf. Figure 31-4*b* shows the energy transfers in this circuit. The chemical energy in battery B is steadily depleted, the energy appearing in the three forms shown on the right. Battery A is being charged while battery B is being discharged.

Note that the circuit can transfer energy *from* a source of emf or *into* a source of emf. In the ideal case, the energy transfer process associated with a source of emf is *reversible* in the thermodynamic sense (see Section 24-1). A battery can either be *charged* (meaning an external source adds to the battery's supply of energy, *not* that we are forcing more charge into the battery) or *discharged* (meaning we take energy from the battery). Similarly, a generator can be driven mechanically to produce electrical energy, or it can use electrical energy to produce mechanical motion, as in a motor.

31-3 ANALYSIS OF CIRCUITS

The simplest electrical circuit consists of one source of emf (such as a battery) and one circuit device (such as a resistor). Examples of this kind of circuit might include a flashlight or an electric heater. Figure 31-5 shows a circuit con-

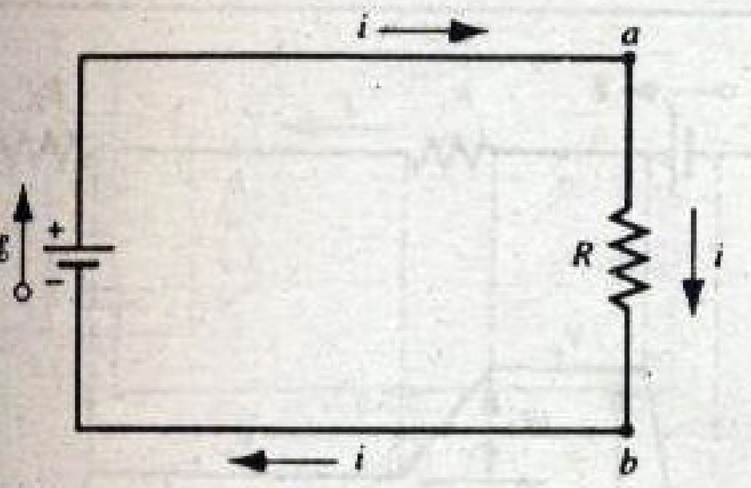


FIGURE 31-5. A single-loop circuit. The current is the same everywhere in the circuit. The potential *increases* from − to + across the battery, and it *decreases* from a to b (in the direction of the current) across the resistor.

sisting of a single ideal battery and a resistor R. The symbolic circuit notation for a resistor is -\/\/\/-.

Often our goal in analyzing circuits is to determine the magnitude and direction of the current, given the emfs and resistors in the circuit. We will analyze this circuit by considering the potential differences across each circuit element. Later in this chapter we consider another method based on the energy supplied or dissipated by each circuit element.

The first step in analyzing the circuit is to guess a direction for the current. Usually we try to make the best guess we can; if we choose the wrong direction the current will come out to be negative, indicating that our original guess for the direction was wrong, but the magnitude we calculate will still be correct. In the circuit of Fig. 31-5, we expect that the current is clockwise, determined by the emf of the battery. The battery maintains point a at a higher potential than point b, and so positive charges in the circuit would "fall" through the resistor from a to b before returning to the battery to be pumped back up to the higher potential of a.

When we analyze the circuit using the method of potential differences, we travel once around the loop and keep track of the differences in potential across each circuit element. It does not matter which way we travel around the loop in making this analysis. Let us try going around clockwise (in the same direction as the current), starting at point a.

Let ΔV_R be the potential difference across the resistor. That is, the potential at a is greater than the potential at b by an amount ΔV_R, which is equal to iR. How do we know that V_a is greater than V_b? We do not yet know this for certain, but it is consistent with the direction we have guessed for the current through R. If the current flows from a to b, then the positive charge carriers are "falling" through the resistor from the higher potential at a to the lower potential at b. If our initial guess for the direction of the current is wrong, our solution will show that both i and ΔV_R are negative.

We are now ready to analyze the circuit. Our procedure is to start at any point, travel once around the circuit adding all the potential differences, and then return to the starting point where we must find the same potential at which we started. This procedure can be summarized as follows:

The algebraic sum of all differences in potential around a complete circuit loop must be zero.

This rule is known as the *loop rule* (and is sometimes referred to as *Kirchhoff's second law*). Ultimately it is a statement about the conservation of energy. An analogy would be taking a walk in a hilly terrain; you may travel up or down as you walk, but if you keep track of all your changes in gravitational potential energy you will find that the total change is zero when you return to your starting point.

We now examine the changes in potential as we go once around the circuit of Fig. 31-5, starting at point a where the potential is V_a. Proceeding clockwise from a through the resistor, the potential drops by $\Delta V_R = iR$, so the potential at b is $V_b = V_a - iR$. Continuing clockwise around the circuit, we next travel through the battery from the negative terminal to the positive terminal, and thus the potential increases by the battery emf $\mathcal{E}$. This now brings us back to point a and the potential V_a. Since the starting and ending potentials at point a must be equal (potential being a path-independent quantity), we therefore have $V_a = V_a - iR + \mathcal{E}$. Equivalently, we can find this result by applying the loop rule directly, adding up the differences in potential, and setting the resulting sum equal to zero. Again starting at a and proceeding clockwise, we first encounter a negative potential difference of $-iR$ and then a positive potential difference of $+\mathcal{E}$. Setting the sum of these potential differences to zero, we obtain

$$-iR + \mathcal{E} = 0$$

or

$$i = \frac{\mathcal{E}}{R}. \qquad (31\text{-}2)$$

We have now found the current in this circuit, which completes our analysis.

Let us now consider a slightly more complicated single-loop circuit, shown in Fig. 31-6. This circuit has a single battery but two resistors. We can once again guess that the current is clockwise. Let us travel around the circuit in a

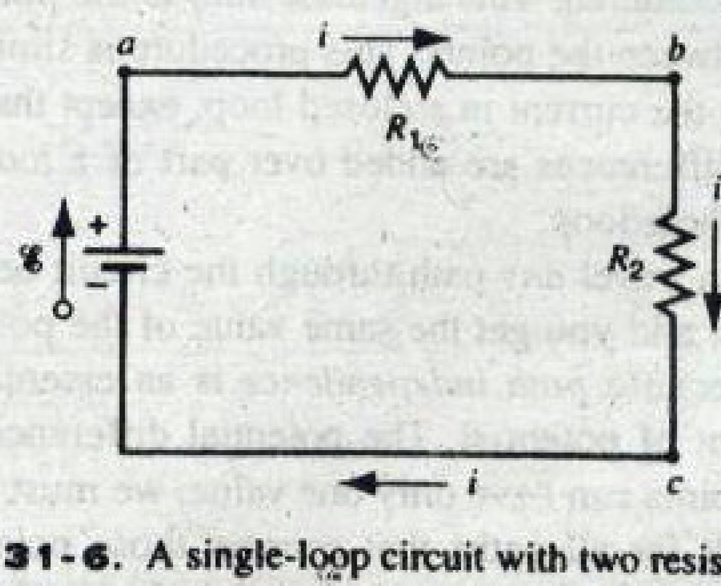


FIGURE 31-6. A single-loop circuit with two resistors. The current is the same everywhere; the potential decreases from a to b and also from b to c, in the direction of the current.

counterclockwise direction this time. Starting at point a, we first go through the battery and find a potential difference of $-\mathscr{E}$. Next we go through R_2 in a direction opposite to the current, so the potential *increases* and the potential difference is $+iR_2$. Similarly, the potential difference when we go through R_1 is $+iR_1$, after which we are back at our starting point. According to the loop rule, the total of these potential differences is zero:

$$-\mathscr{E} + iR_2 + iR_1 = 0$$

or

$$i = \frac{\mathscr{E}}{R_1 + R_2}. \tag{31-3}$$

Note that Eq. 31-3 reduces to Eq. 31-2 if either $R_1 = 0$ or $R_2 = 0$.

Potential Differences in a Circuit

We often want to find the potential difference between two points in a circuit. In Fig. 31-6, for example, how does the potential difference ΔV_{ab} ($= V_a - V_b$) between points b and a depend on the circuit elements $\mathscr{E}$, R_1 and R_2? To find their relationship, let us start at point b and move counterclockwise to point a, passing through resistor R_1. If V_a and V_b are the potentials at a and b, respectively, we have

$$V_b + iR_1 = V_a,$$

because we experience an increase in potential in traveling through a resistor in the direction opposite to the current. We rewrite this relation in terms of ΔV_{ab}, the potential difference between a and b, as

$$\Delta V_{ab} = V_a - V_b = +iR_1,$$

which tells us that ΔV_{ab} has magnitude iR_1 and that point a is at a higher potential than point b. Combining this last equation with Eq. 31-3 yields

$$\Delta V_{ab} = \mathscr{E}\frac{R_1}{R_1 + R_2}. \tag{31-4}$$

In summary, to find the potential difference between any two points in a circuit, start at one point, travel through the circuit to the other, and add algebraically the changes in potential encountered. This algebraic sum is the potential difference between the points. This procedure is similar to that for finding the current in a closed loop, except that here the potential differences are added over part of a loop and not over the whole loop.

You can travel *any* path through the circuit between the two points, and you get the same value of the potential difference, because *path independence* is an essential part of our concept of potential. The potential difference between any two points can have only one value; we must obtain the same result for all paths that connect those points. (Similarly, if we consider two points on the side of a hill, the measured difference in gravitational potential between them

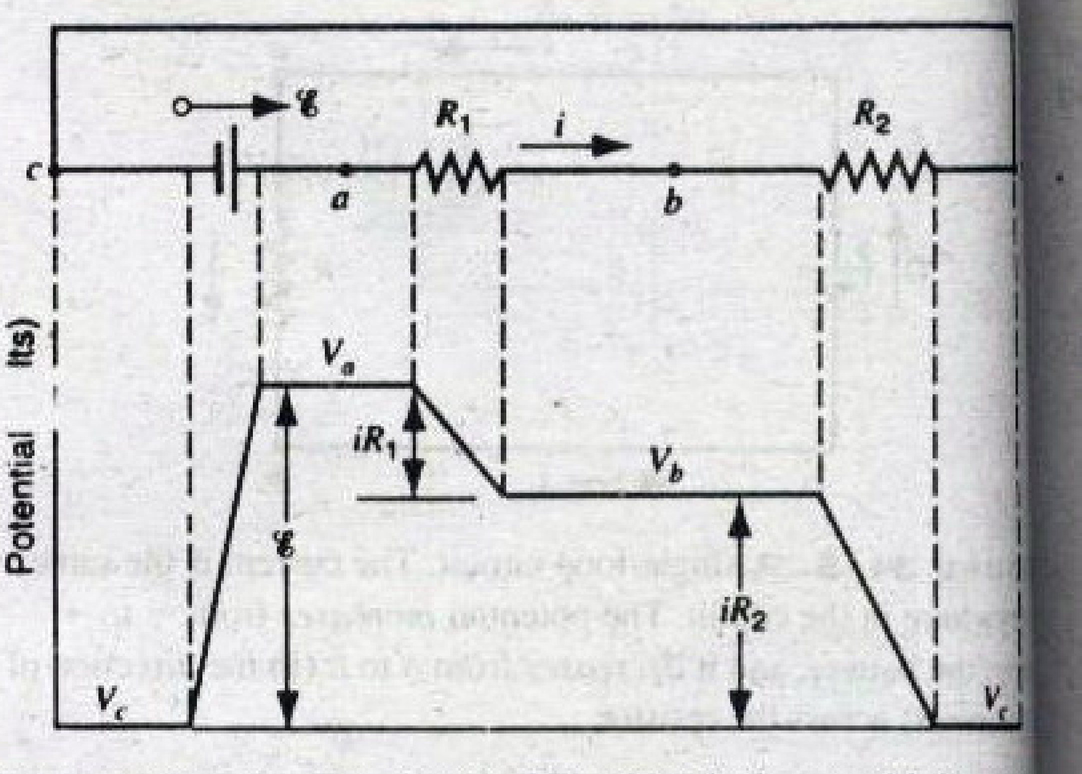


FIGURE 31-7. The circuit of Fig. 31-6 is drawn with its components along a straight line at the top. The potential difference across each of the elements is shown.

is the same no matter what path is followed in going from one to the other.) In Fig. 31-6 let us recalculate ΔV_{ab}, using a path starting at a and going counterclockwise through the source of emf to b. We have

$$V_a - \mathscr{E} + iR_2 = V_b$$

or

$$\Delta V_{ab} = V_a - V_b = +\mathscr{E} - iR_2.$$

Combining this result with Eq. 31-3 leads to Eq. 31-4.

Using similar methods, we can show that

$$\Delta V_{bc} = \mathscr{E}\frac{R_2}{R_1 + R_2}. \tag{31-5}$$

Note that, as we should expect, $\Delta V_{ab} + \Delta V_{bc} = \mathscr{E}$. The combination of resistors in the circuit of Fig. 31-6 is called a *voltage divider*. In effect, it divides the voltage difference of the battery into two pieces in proportion to the sizes of the two resistors.

Another way of illustrating the potential differences in this circuit is shown in Fig. 31-7. For convenience we have started at point c and traveled clockwise around the circuit. Here you can clearly see how the battery emf is "divided" into the potential differences across the two resistors.

Internal Resistance of a Source of EMF

In contrast to the ideal batteries we have been considering so far, real batteries have an internal resistance. This resistance is characteristic of the materials of which the battery is made. Because it is an inherent part of the battery, it cannot be removed; we would usually like to do so, for internal resistance has undesirable effects such as reducing the terminal voltage of the battery and limiting the current that can flow in the circuit.

Figure 31-8 shows the simple loop circuit of Fig. 31-5 with the internal resistance r of the battery taken into account. Even though they are part of the same device, we

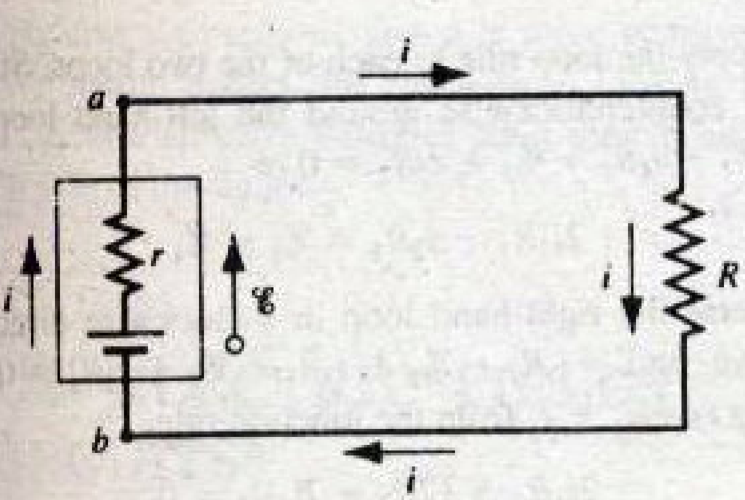


Figure 31-8. A battery is represented as a device containing a source of emf $\mathscr{E}$ and an internal resistance r.

show the source of emf and the internal resistance as separate elements.

The circuit of Fig. 31-8 is identical to the circuit of Fig. 31-6, and we can find the current by merely adapting Eq. 31-3 to the circuit elements of Fig. 31-8:

$$i = \frac{\mathscr{E}}{R + r}. \qquad (31\text{-}6)$$

The internal resistance *reduces* the current that the emf can supply to the external circuit.

The potential difference between the battery terminals is $\Delta V_{ab} = V_a - V_b = \mathscr{E} - iR$; using Eq. 31-6 we obtain

$$\Delta V_{ab} = \mathscr{E}\frac{R}{R + r}. \qquad (31\text{-}7)$$

From this expression we see that the potential difference between the battery terminals is not a constant, but now depends on the resistance R of the external circuit. As we make R smaller, thereby increasing the current, the potential difference between the battery terminals decreases. A 1.5-V battery has a terminal voltage difference of 1.5 V only when there is no current flowing through the battery. When the battery is connected to a circuit device such as a radio, the voltage difference between the terminals will be less than 1.5 V.

We see from Eq. 31-7 that ΔV_{ab} is equal to $\mathscr{E}$ only if either the battery has no internal resistance ($r = 0$) or the external circuit is open ($R = \infty$).

Sample Problem 31-1. What is the current in the circuit of Fig. 31-9*a*? The emfs and the resistors have the following values: $\mathscr{E}_1 = 2.1$ V, $\mathscr{E}_2 = 4.4$ V, $r_1 = 1.8\ \Omega$, $r_2 = 2.3\ \Omega$, $R = 5.5\ \Omega$.

Solution The two emfs are connected so that they oppose each other but $\mathscr{E}_2$, because it is larger than $\mathscr{E}_1$, controls the direction of the current in the circuit, which is counterclockwise. The loop rule, applied clockwise from point a, yields

$$-\mathscr{E}_2 + ir_2 + iR + ir_1 + \mathscr{E}_1 = 0.$$

Check that this same equation results by going around counterclockwise or by starting at some point other than a. Also, compare this equation term by term with Fig. 31-9*b*, which shows the potential changes graphically.

Solving for the current i, we obtain

$$i = \frac{\mathscr{E}_2 - \mathscr{E}_1}{R + r_1 + r_2} = \frac{4.4\text{ V} - 2.1\text{ V}}{5.5\ \Omega + 1.8\ \Omega + 2.3\ \Omega} = 0.24\text{ A}.$$

It is not necessary to know the direction of the current in advance. To show this, let us assume that the current in Fig. 31-9*a* is clockwise—that is, opposite to the direction of the current arrow in Fig. 31-9*a*. The loop rule would then yield (going clockwise from *a*)

$$-\mathscr{E}_2 - ir_2 - iR - ir_1 + \mathscr{E}_1 = 0$$

or

$$i = -\frac{\mathscr{E}_2 - \mathscr{E}_1}{R + r_1 + r_2}.$$

Substituting numerical values yields $i = -0.24$ A for the current. The negative sign is a signal that the current is in the opposite direction from that which we have assumed.

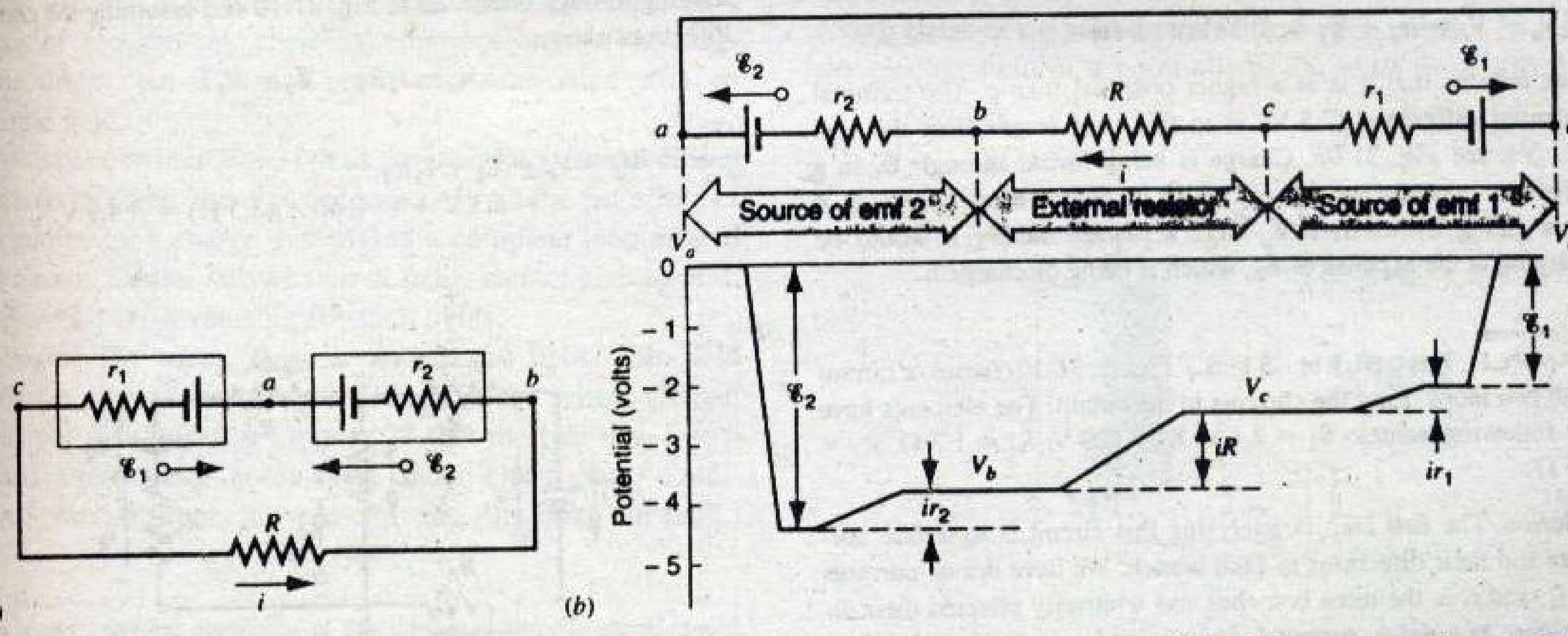


Figure 31-9. Sample Problems 31-1 and 31-2. (*a*) A single-loop circuit containing two sources of emf. (*b*) The changes in potential encountered in traveling clockwise around the circuit starting at point *a*.

In more complex circuits involving many loops and branches, it is often impossible to know in advance the actual directions for the currents in all parts of the circuit. However, the current directions for each branch may be chosen arbitrarily. If you get an answer with a positive sign for a particular current, you have chosen its direction correctly; if you get a negative sign, the current is opposite in direction to that chosen. In either case, the numerical value is correct.

SAMPLE PROBLEM 31-2. (*a*) What is the potential difference between points *a* and *b* in Fig. 31-9*a*? (*b*) What is the potential difference between points *a* and *c* in Fig. 31-9*a*?

Solution (*a*) This potential difference is the terminal potential difference of battery 2, which includes emf $\mathscr{E}_2$ and internal resistance r_2. Let us start at point *b* and traverse the circuit counterclockwise to point *a*, passing directly through the source of emf. We find

$$V_b - ir_2 + \mathscr{E}_2 = V_a$$

or

$$V_a - V_b = -ir_2 + \mathscr{E}_2 = -(0.24\text{ A})(2.3\ \Omega) + 4.4\text{ V} = +3.8\text{ V}.$$

We see that *a* is more positive than *b* and the potential difference between them (3.8 V) is *smaller* than the emf (4.4 V); see Fig. 31-9*b*.

We can verify this result by starting at point *b* in Fig. 31-9*a* and moving through the circuit clockwise to point *a*. For this different path we find

$$V_b + iR + ir_1 + \mathscr{E}_1 = V_a$$

or

$$\begin{aligned} V_a - V_b &= iR + ir_1 + \mathscr{E}_1 \\ &= (0.24\text{ A})(5.5\ \Omega + 1.8\ \Omega) + 2.1\text{ V} = +3.8\text{ V}, \end{aligned}$$

exactly as before. The potential difference between two points has the same value for all paths connecting those points.

(*b*) Note that the potential difference between *a* and *c* is the terminal potential difference of battery 1, consisting of emf $\mathscr{E}_1$ and internal resistance r_1. Let us start at *c* and traverse the circuit clockwise to point *a*. We find

$$V_c + ir_1 + \mathscr{E}_1 = V_a$$

or

$$V_a - V_c = ir_1 + \mathscr{E}_1 = (0.24\text{ A})(1.8\ \Omega) + 2.1\text{ V} = +2.5\text{ V}.$$

This tells us that *a* is at a higher potential than *c*. The terminal potential difference (2.5 V) is in this case *larger* than the emf (2.1 V); see Fig. 31-9*b*. Charge is being forced through $\mathscr{E}_1$ in a direction opposite to that in which it would send charge if it were acting by itself; if $\mathscr{E}_1$ were a storage battery it would be charging at the expense of $\mathscr{E}_2$, which is being discharged.

SAMPLE PROBLEM 31-3. Figure 31-10 shows a circuit with two loops. Find the currents in the circuit. The elements have the following values: $\mathscr{E}_1 = 2.1$ V, $\mathscr{E}_2 = 6.3$ V, $R_1 = 1.7\ \Omega$, $R_2 = 3.5\ \Omega$.

Solution The first step in analyzing this circuit is to define currents and their directions in each branch. We have drawn currents i_1, i_2, and i_3 in the three branches and arbitrarily selected their directions. At point *a*, current i_3 flows in and currents i_1 and i_2 flow out. Applying the junction rule (Section 31-1), we have

$$i_3 = i_1 + i_2.$$

We now apply the loop rule to each of the two loops. Starting at *a* and going counterclockwise around the left-hand loop, we find $-i_1R_1 - \mathscr{E}_1 - i_1R_1 + \mathscr{E}_2 + i_2R_2 = 0$ or

$$2i_1R_1 - i_2R_2 = \mathscr{E}_2 - \mathscr{E}_1.$$

If we traverse the right-hand loop in a clockwise direction from point *a*, we find $+i_3R_1 - \mathscr{E}_2 + i_3R_1 + \mathscr{E}_2 + i_2R_2 = 0$ or, after substituting $i_3 = i_1 + i_2$ from the junction rule,

$$2i_1R_1 + (2R_1 + R_2)i_2 = 0.$$

We now have two equations for the two currents i_1 and i_2. We can solve these equations for these variables, obtaining, after a little algebra,

$$\begin{aligned} i_1 &= \frac{(\mathscr{E}_2 - \mathscr{E}_1)(2R_1 + R_2)}{4R_1(R_1 + R_2)} \\ &= \frac{(6.3\text{ V} - 2.1\text{ V})(2 \times 1.7\ \Omega + 3.5\ \Omega)}{(4)(1.7\ \Omega)(1.7\ \Omega + 3.5\ \Omega)} = 0.82\text{ A}, \\ i_2 &= -\frac{\mathscr{E}_2 - \mathscr{E}_1}{2(R_1 + R_2)} \\ &= -\frac{6.3\text{ V} - 2.1\text{ V}}{(2)(1.7\ \Omega + 3.5\ \Omega)} = -0.40\text{ A}. \end{aligned}$$

The third current can be found from applying the junction rule:

$$i_3 = i_1 + i_2 = 0.82\text{ A} + (-0.40\text{ A}) = 0.42\text{ A}.$$

The signs of the currents tell us that we have guessed correctly about the directions of i_1 and i_3 but that we are wrong about the direction of i_2; it should point up—and not down—in the central branch of the circuit of Fig. 31-10.

Note that, having discovered that current i_2 is pointing in the wrong direction, we do not need to change it in Fig. 31-10. We can leave it in the figure as it is, as long as we remember to substitute a negative numerical value for i_2 in all subsequent calculations involving that current.

SAMPLE PROBLEM 31-4. What is the potential difference between points *a* and *b* in the circuit of Fig. 31-10?

Solution For the potential difference between *a* and *b*, we have, moving through branch *ab* in Fig. 31-10 and assuming the current directions shown,

$$V_a - i_2R_2 - \mathscr{E}_2 = V_b,$$

or

$$\begin{aligned} V_a - V_b &= \mathscr{E}_2 + i_2R_2 \\ &= 6.3\text{ V} + (-0.40\text{ A})(3.5\ \Omega) = +4.9\text{ V}. \end{aligned}$$

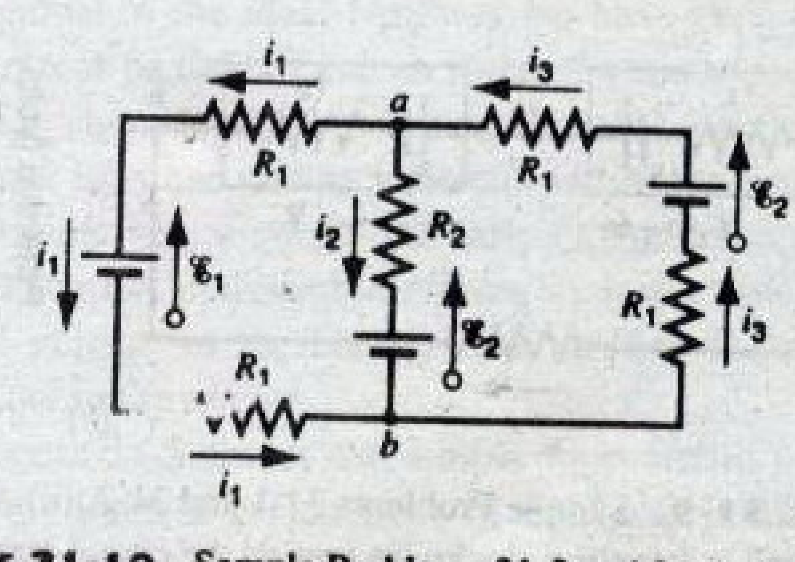


FIGURE 31-10. Sample Problems 31-3 and 31-4. A two-loop circuit.

The positive sign tells us that a is more positive in potential than b. We should expect this result from looking at the circuit diagram, because all three batteries have their positive terminals on the top side of the figure.

31-4 ELECTRIC FIELDS IN CIRCUITS*

So far we have been discussing circuits in a rather mysterious way. In Chapter 29 we considered the relationship between current and electric field in a conductor: $\vec{E} = \rho\vec{j}$ (Eq. 29-10), where ρ is the resistivity of the material and $\vec{j}$ is the current density (current per unit cross-sectional area). The wires in our circuits are conductors, and so there must be an electric field present to establish and sustain the current. Where does this electric field come from?

It is helpful at this point to return to the analogy between current and fluid flow in a pipe. Figure 31-11 shows a closed loop similar to the circuit of Fig. 31-5. The pump is analogous to the source of emf, and the constriction in the pipe is analogous to the resistor. In the steady state, the amount of fluid per unit time passing any particular point in the circuit must be the same as that passing any other point. In the constriction, the fluid must therefore flow more rapidly (assuming the fluid to be incompressible).

The battery provides the emf to the circuit; its role is to "pump" charges from low potential to high potential. The emf is defined as the work per unit charge done by the battery. With the conventional definition of work done by a force $\vec{F}$ as $W = \int \vec{F} \cdot d\vec{s}$, the work per unit charge W/q (the emf) must then be related to the force per unit charge $\vec{F}/q$:

$$\mathcal{E} = \oint (\vec{F}/q) \cdot d\vec{s} \qquad (31\text{-}8)$$

It is tempting to associate an electric field with the quantity $\vec{F}/q$, but it is generally not correct. The force $\vec{F}$ in this case is the one that acts inside the source of emf; it might be a force of mechanical, chemical, thermodynamic, or magnetic origin, but it is not necessarily associated with an electric field.

We have written Eq. 31-8 as the integral around a closed path. In this way, the emf depends only on the net effect of the source on a charge that makes a complete loop around the circuit. Conservative external fields cannot give an emf, because Eq. 31-8 vanishes for such fields.

Inside the wires, there *is* an electric field. This field must be present for charge to flow in the wires. (Recall from our discussion in Chapter 27 that the rule that $E = 0$ inside a conductor holds *only* under electrostatic conditions; when currents are present, this rule does not hold.)

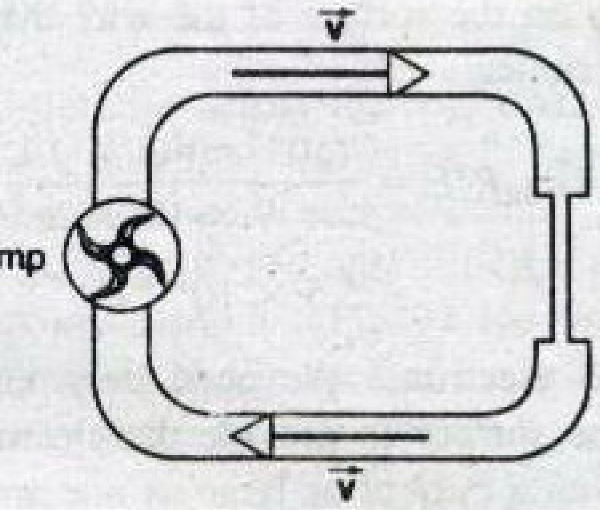


FIGURE 31-11. A fluid-mechanical analogy to the electrical circuit of Fig. 31-5. The constriction gives some resistance to the flow of fluid.

Figure 31-12 shows a representation of the electric fields in a conducting wire. How large must the electric fields be to maintain a typical current?

It is helpful to make some numerical estimates. Suppose a current $i = 1$ A flows in a wire of radius $R = 1$ mm. The current density is then

$$j = \frac{i}{A} = \frac{1\ \text{A}}{\pi(1 \times 10^{-3}\ \text{m})^2} \approx 3 \times 10^5\ \text{A/m}^2.$$

For copper wires, the resistivity ρ is $1.69 \times 10^{-8}\ \Omega\cdot\text{m}$, so the electric field associated with the current is

$$E = \rho j = (1.69 \times 10^{-8}\ \Omega\cdot\text{m})(3 \times 10^5\ \text{A/m}^2) \approx 5 \times 10^{-3}\ \text{V/m}.$$

This is a very small electric field.

When the battery is first connected to the circuit, initial transient currents are established. These currents distribute charges along the surfaces of the wires in just the precise way necessary to establish the electric field that maintains the steady current in the wires—and the entire process takes place in a time that is typically of the order of nanoseconds!

How much charge on the surface of the wire is needed to produce a field of 5×10^{-3} V/m in its interior? To find a rough order-of-magnitude estimate, we can use Eq. 26-6 for the electric field of a point charge, $E = q/4\pi\epsilon_0 R^2$, to find

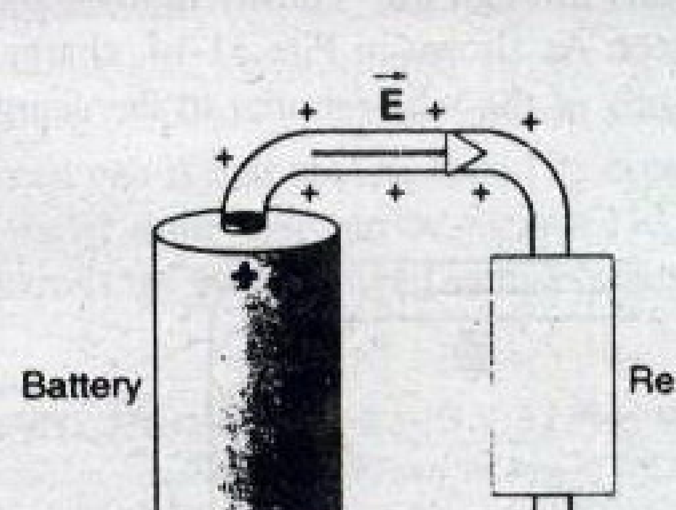


FIGURE 31-12. The electric fields in the connecting wires in a single-loop circuit. Charges on the surfaces of the wires are responsible for the fields in the wires.

*For a more detailed discussion of this topic, see *Electric and Magnetic Interactions* by R. Chabay and B. Sherwood (New York: Wiley, 1995), chapter 6. See also W. G. V. Rosser, *American Journal of Physics*, Vol. 31, 1963, p. 884 and Vol. 38, 1970, p. 265.

the charge q on the surface of the wire that would give a field E at its center:

$$q = 4\pi\epsilon_0 R^2 E = \frac{(10^{-3}\text{ m})^2(5 \times 10^{-3}\text{ V/m})}{9 \times 10^9\text{ N}\cdot\text{m}^2/\text{C}^2} = 5.5 \times 10^{-19}\text{ C}$$

or about 3.5 electrons! We need only tiny amounts of charge on the surface to provide the electric field needed to sustain even a current as large as one ampere in a conductor.

In a similar fashion, we can ask how the current "knows" to change direction when it encounters a bend in the wire. Once again, the initial transients must result in just enough charge on the surface to guide the current. Figure 31-13 shows a schematic view of a right-angle bend, for which the surface charges must be distributed roughly as shown. The negative charge sets up a field near the bend that opposes the motion of the oncoming current, and the positive charge provides an initial "push" in the new direction. The preceding calculation again gives an order-of-magnitude estimate of the necessary charges—a few electrons on the surface are sufficient to change the direction of a one-ampere current!

Consideration of surface charge can also help us understand the effect of a resistor in a circuit. Let us consider a carbon resistor, and for simplicity we will give it the same diameter as the wires in our circuit (Fig. 31-14). Carbon is a poor conductor, but not quite a good insulator—its resistivity is about $3 \times 10^{-5}\ \Omega\cdot\text{m}$, about 2000 times larger than that of copper, but not nearly so large as that of typical insulators (see Table 29-1). Because the resistor and the wires have the same cross-sectional area, the current density is the same in both. Using our previous current density for the 1-A current in the 1-mm-radius wire, we can find the electric field in the resistor:

$$E = \rho j = (3 \times 10^{-5}\ \Omega\cdot\text{m})(3 \times 10^5\text{ A/m}^2) \approx 10\text{ V/m},$$

giving a field about 2000 times larger than the field in the copper wires. (Can you now see why the potential drop in the wires is negligible compared with the potential drop in the resistor?) This large electric field is necessary to force the electrons through the "constriction" in the circuit due to the resistor. As shown in Fig. 31-14, charges building up on the ends of the wire (similar to the charges on a capacitor) are responsible for producing this large electric field. For a resistor of thickness 1 mm, you should be able to show that about 1000 electrons on each end can produce the required field.

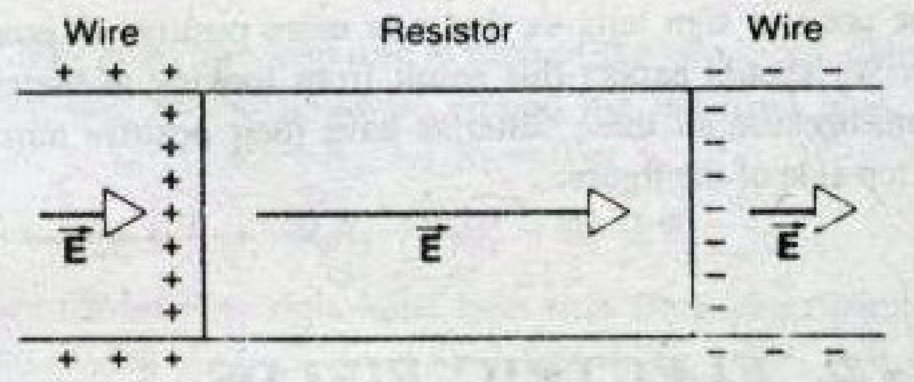


FIGURE 31-14. Details of surface charges near a resistor. The build-up of charge on the wire ends produces the large electric field in the resistor.

What we have described here is a wonderful, self-regulating system. The battery provides the initial "burst" of current to the circuit, and almost instantly the charge finds its way to the locations where it guides the steady current and prevents further build-up of charge on the surface of the wires. This equilibrium is maintained as long as the battery continues to pump charge around the circuit.

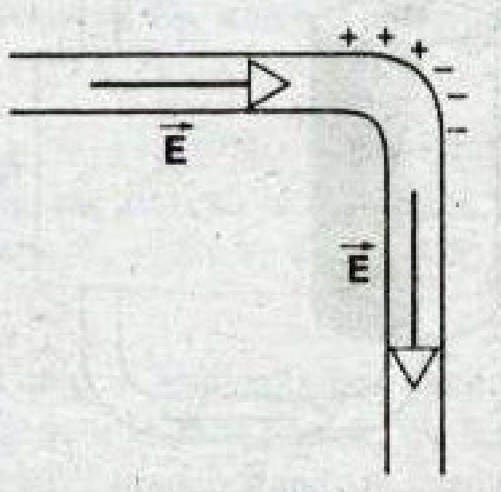


FIGURE 31-13. Detail of the surface charges near a right-angle bend.

31-5 RESISTORS IN SERIES AND PARALLEL

As was the case with capacitors (see Section 30-4), resistors often occur in circuits in various combinations. In analyzing such circuits, it is helpful to replace the combination of resistors with a single *equivalent resistance* R_{eq}, whose value is chosen such that the operation of the circuit is unchanged. We consider two ways that resistors can be combined.

Resistors Connected in Parallel

Recall our definition of a parallel combination of circuit elements in Section 30-4: We can travel through the combination by crossing *only one* of the elements, the same potential difference ΔV appears across each element, and the flow of charge is shared among the elements.

Figure 31-15 shows two resistors connected in parallel. We seek the equivalent resistance between points a and b. Let us assume we connect a battery (or other source of emf) that maintains a potential difference ΔV between points a and b. The potential difference across each resistor

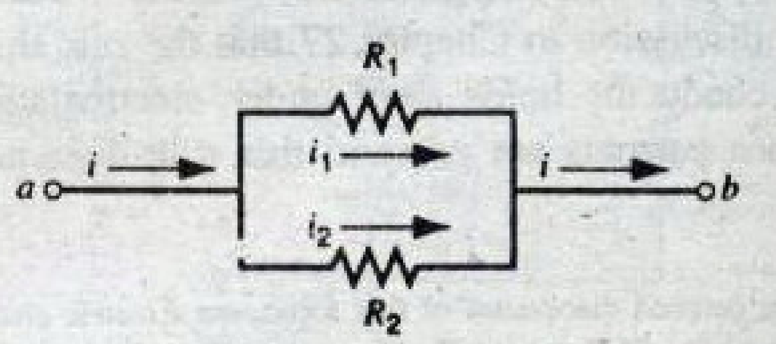


FIGURE 31-15. Two resistors in parallel.

is ΔV. The current through each of the resistors is, from Eq. 29-12,

$$i_1 = \Delta V/R_1 \quad \text{and} \quad i_2 = \Delta V/R_2. \qquad (31\text{-}9)$$

According to the properties of a parallel circuit, the total current i must be shared among the branches, so

$$i = i_1 + i_2. \qquad (31\text{-}10)$$

If we were to replace the parallel combination by a single equivalent resistance R_{eq}, the same total current i must flow (because the replacement must not change the operation of the circuit). The current is then

$$i = \Delta V/R_{eq}. \qquad (31\text{-}11)$$

Substituting Eqs. 31-9 and 31-11 into Eq. 31-10, we obtain

$$\frac{\Delta V}{R_{eq}} = \frac{\Delta V}{R_1} + \frac{\Delta V}{R_2}$$

or

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}. \qquad (31\text{-}12)$$

To find the equivalent resistance of a parallel combination of more than two resistors, we first find the equivalent resistance R_{12} of R_1 and R_2 using Eq. 31-12. We then find the equivalent resistance of R_{12} and the next parallel resistance, R_3, again using Eq. 31-12. Continuing in this way, we obtain a general expression for the equivalent resistance of a parallel combination of any number of resistors,

$$\frac{1}{R_{eq}} = \sum_n \frac{1}{R_n} \quad \text{(parallel combination).} \qquad (31\text{-}13)$$

That is, to find the equivalent resistance of a parallel combination, add the reciprocals of the individual resistances and take the reciprocal of the resulting sum. Note that R_{eq} is always *smaller than* the smallest resistance in the parallel combination—by adding more paths for the current, we get more current for the same potential difference.

In the special case of two resistors in parallel, Eq. 31-12 can be written

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}, \qquad (31\text{-}14)$$

or as the product of the two resistances divided by their sum.

Resistors Connected in Series

Figure 31-16 shows two resistors connected in series. Recall the properties of a series combination of circuit elements (see Section 30-4): to travel through the combination, we must travel through *all* the elements in succession, the potential difference across the combination is the sum of the potential differences across each element, and the same current is maintained in each element.

Suppose a battery of potential difference ΔV is connected across points a and b in Fig. 31-16. A current i is set up in the combination and in each of the resistors. The potential differences across the resistors are

$$\Delta V_1 = iR_1 \quad \text{and} \quad \Delta V_2 = iR_2. \qquad (31\text{-}15)$$

The sum of these potential differences must give the potential difference across points a and b maintained by the battery, or

$$\Delta V = \Delta V_1 + \Delta V_2. \qquad (31\text{-}16)$$

If we replaced the combination by its equivalent resistance R_{eq}, the same current i would be established, and so

$$\Delta V = iR_{eq}. \qquad (31\text{-}17)$$

Combining Eqs. 31-15, 31-16, and 31-17, we obtain

$$iR_{eq} = iR_1 + iR_2,$$

or

$$R_{eq} = R_1 + R_2. \qquad (31\text{-}18)$$

Extending this result to a series combination of any number of resistors, we obtain

$$R_{eq} = \sum_n R_n \quad \text{(series combination).} \qquad (31\text{-}19)$$

That is, to find the equivalent resistance of a series combination, find the sum of the individual resistors. Note that the equivalent resistance of a series combination is always *larger* than the largest resistance in the series—adding more resistors in series means we get less current for the same potential difference.

Comparing these results with Eqs. 30-17 and 30-22 for the series and parallel combinations of capacitors, we see that resistors in parallel add like capacitors in series, and resistors in series add like capacitors in parallel. This has to do with the different way the two quantities are defined, resistance being potential/current and capacitance being charge/potential.

Occasionally, resistors may appear in combinations that are neither parallel nor series. In such a case, the equivalent resistance can sometimes be found by breaking the problem into smaller units that can be regarded as series or parallel connections, as the following sample problems demonstrate.

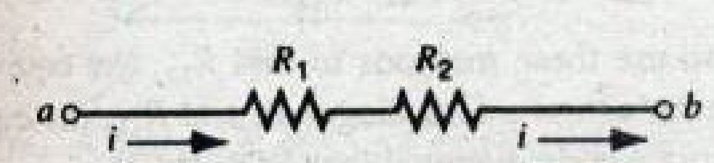


FIGURE 31-16. Two resistors in series.

SAMPLE PROBLEM 31-5. **(*a*) Find the equivalent resistance of the combination shown in Fig. 31-17*a*, using the values $R_1 = 4.6\ \Omega$, $R_2 = 3.5\ \Omega$, and $R_3 = 2.8\ \Omega$. (*b*) What is the value of the current through R_1 when a 12.0-V battery is connected across points *a* and *b*?**

Solution **(*a*) We first find the equivalent resistance R_{12} of the parallel combination of R_1 and R_2. Using Eq. 31-14 we obtain**

$$R_{12} = \frac{R_1 R_2}{R_1 + R_2} = \frac{(4.6\ \Omega)(3.5\ \Omega)}{4.6\ \Omega + 3.5\ \Omega} = 2.0\ \Omega.$$

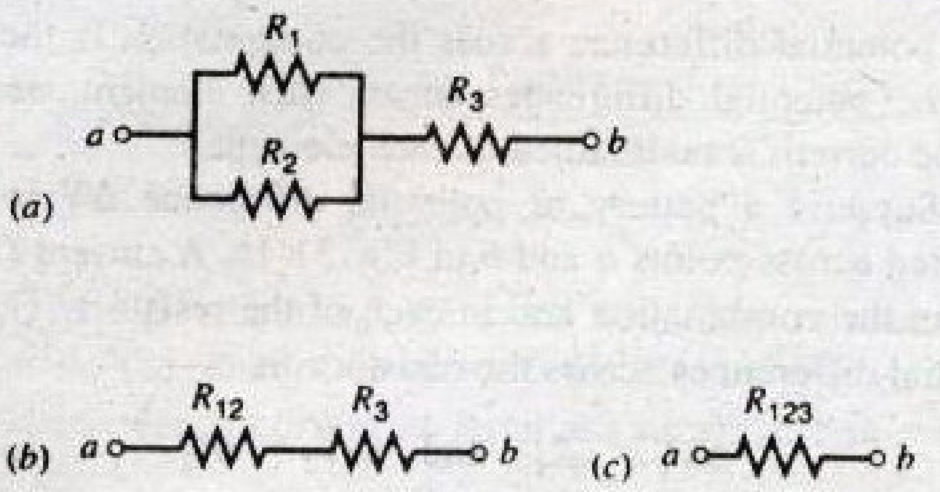


FIGURE 31-17. Sample Problem 31-5. (*a*) The parallel combination of R_1 and R_2 is in series with R_3. (*b*) The parallel combination of R_1 and R_2 has been replaced by its equivalent resistance, R_{12}. (*c*) The series combination of R_{12} and R_3 has been replaced by its equivalent resistance, R_{123}.

R_{12} and R_3 are in series, as shown in Fig. 31-17*b*. Using Eq. 31-18, we can find the equivalent resistance R_{123} of this series combination, which is the equivalent resistance of the entire original combination:

$$R_{123} = R_{12} + R_3 = 2.0\ \Omega + 2.8\ \Omega = 4.8\ \Omega.$$

(*b*) With a 12.0-V battery connected across points *a* and *b* in Fig. 31-17*c*, the resulting current is

$$i = \frac{\Delta V}{R_{123}} = \frac{12.0\ \text{V}}{4.8\ \Omega} = 2.5\ \text{A}.$$

With this current in the series combination in Fig. 31-17*b*, the potential difference across R_{12} is

$$\Delta V_{12} = iR_{12} = (2.5\ \text{A})(2.0\ \Omega) = 5.0\ \text{V}.$$

In a parallel combination, the same potential difference appears across each element (and across their combination). The potential difference across R_1 (and R_2) is therefore 5.0 V, and the current through R_1 is

$$i_1 = \frac{\Delta V_{12}}{R_1} = \frac{5.0\ \text{V}}{4.6\ \Omega} = 1.1\ \text{A}.$$

SAMPLE PROBLEM 31-6. Figure 31-18*a* shows a cube made of 12 resistors, each of resistance *R*. Find R_{12}, the equivalent resistance across a cube edge.

Solution Although this problem at first looks hopeless to divide into series and parallel subunits, the symmetry of the connections suggests a way to do so. The key is the realization that, from considerations of symmetry alone, points 3 and 6 must be at the same potential, and so must points 4 and 5.

If two points in a circuit have the same potential, the currents in the circuit do not change if you connect these points by a wire. There is no current in the wire because there is no potential difference between its ends. Points 3 and 6 may therefore be connected by a wire, and similarly points 4 and 5 may be connected.

This allows us to redraw the cube as in Fig. 31-18*b*. From this point, it is simply a matter of reducing the circuit between the input terminals to a single resistor, using the rules for resistors in series and in parallel. In Fig. 31-18*c*, we make a start by replacing five parallel combinations of two resistors by their equivalents, each of resistance $\frac{1}{2}R$.

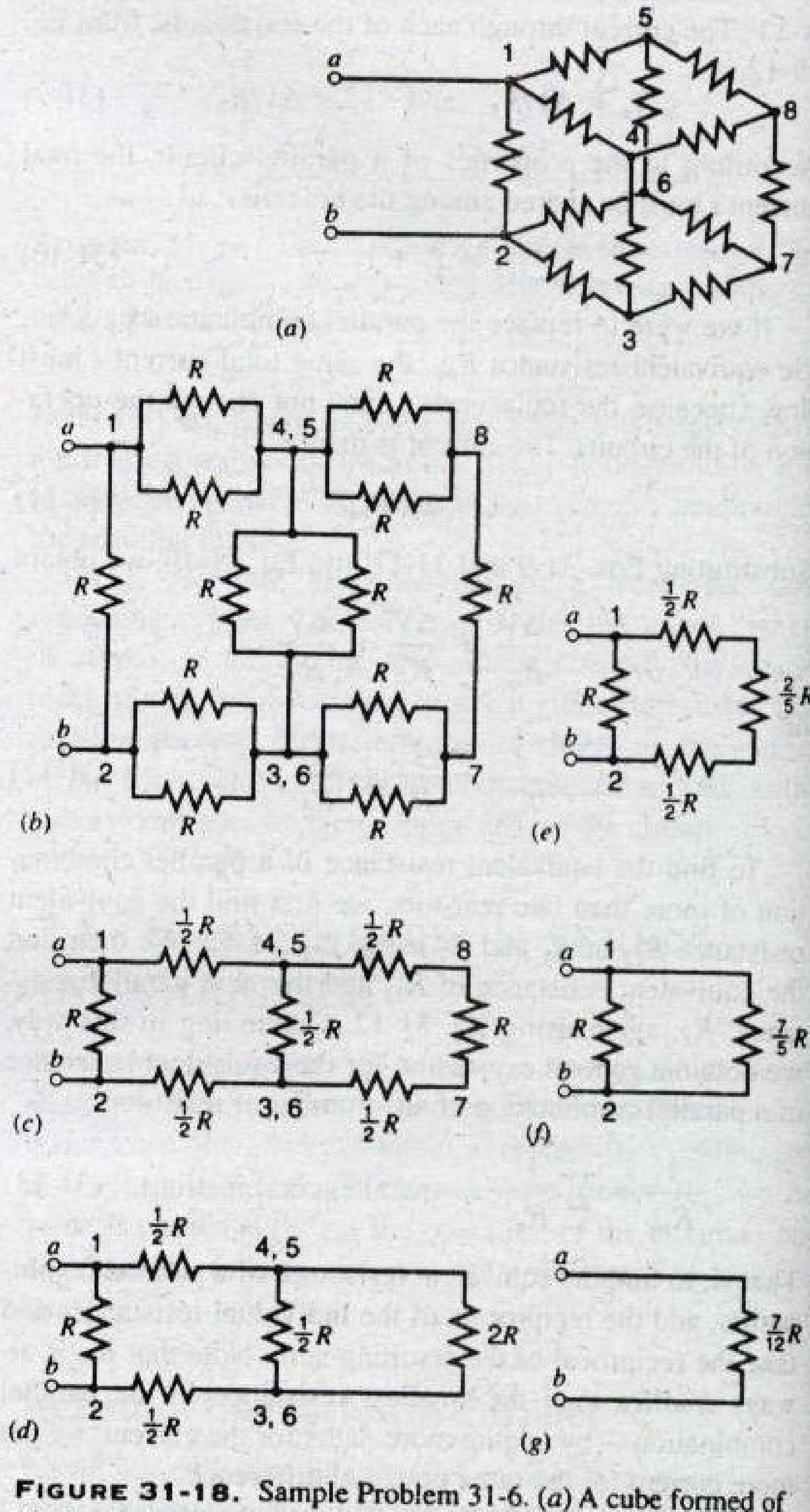


FIGURE 31-18. Sample Problem 31-6. (*a*) A cube formed of 12 identical resistors. (*b*)–(*g*) The step-by-step reduction of the cube to a single equivalent resistance.

In Fig. 31-18*d*, we have added the three resistors that are in series in the right-hand loop, obtaining a single equivalent resistance of 2*R*. In Fig. 31-18*e*, we have replaced the two resistors that now form the right-hand loop by a single equivalent resistor $\frac{2}{5}R$. In so doing, it is useful to recall that the equivalent resistance of two resistors in parallel is equal to their product divided by their sum (see Eq. 31-14).

In Fig. 31-18*f*, we have added the three series resistors of Fig. 31-18*e*, obtaining $\frac{7}{5}R$, and in Fig. 31-18*g* we have reduced this parallel combination to the single equivalent resistance that we seek—namely,

$$R_{12} = \tfrac{7}{12}R.$$

You can also use these methods to find R_{13}, the equivalent resistance of a cube across a face diagonal, and R_{17}, the equivalent resistance across a body diagonal (see Problem 7).

31-6 ENERGY TRANSFERS IN AN ELECTRIC CIRCUIT

Figure 31-19 shows a circuit consisting of a battery B connected to an electronic device, such as a resistor, a capacitor, a motor, or another battery. There is a current i in the wires and a potential difference ΔV_{ab} between the terminals of the device.

Let us first consider the operation of the battery, which we assume is an ideal (resistanceless) source of emf $\mathcal{E}$. As the battery moves a quantity of charge dq from its negative terminal to its positive terminal, it does work on the charge given by Eq. 31-1: $dW = \mathcal{E}\, dq$. The power delivered by the source of emf is determined by the rate at which work is being done; that is, $P_{\rm emf} = dW/dt = \mathcal{E}\, dq/dt$, or

$$P_{\rm emf} = \mathcal{E}i. \qquad (31\text{-}20)$$

This quantity gives the rate at which an ideal source of emf transfers energy to the rest of the circuit. As we discussed in Section 31-2, this energy might appear as the internal energy of a resistor, stored energy in the electric field of a capacitor, mechanical energy in a motor, or chemical energy in a battery that is being charged. If we consider the circuit to be an isolated system, then its total energy cannot change, and the decrease in energy of the source of emf must be balanced by an equivalent net increase in energy in the other parts of the circuit.

Suppose the circuit consists only of a source of emf and a resistor R. The potential difference between terminals a and b in Fig. 31-19 is $\Delta V_R = iR$. As a quantity of charge dq moves through the resistor from a to b, it experiences a potential energy change $dU = dq\, \Delta V_R$ (see Eq. 28-14). This energy must be transferred to the resistor, so the power transferred to the resistor (rate of energy transfer) is $P_R = dU/dt = (dq/dt)\, \Delta V_R = i\, \Delta V_R$ or

$$P_R = i^2R. \qquad (31\text{-}21)$$

With $i = \Delta V_R/R$, we can also write this result as

$$P_R = \frac{(\Delta V_R)^2}{R}. \qquad (31\text{-}22)$$

This energy transfer to a resistor in a circuit is often known as *Joule heating*.

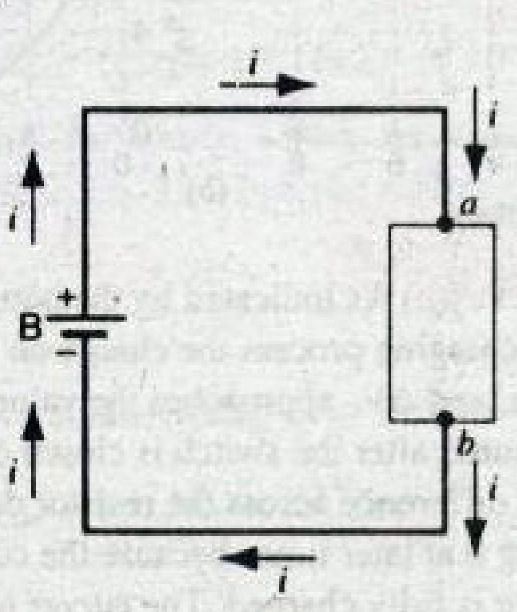


FIGURE 31-19. A battery B sets up a current i in a circuit containing an arbitrary electronic device.

As charge moves through the resistor from higher potential (terminal a) to lower potential (terminal b), it would tend to gain energy if it were not for the collisions with the atoms of the resistor. These collisions maintain the constant drift speed of the charge carriers, and the energy gained by the atoms in these collisions (which results in an increase in the amplitude of the vibration of the atoms about their equilibrium positions) can correspond to a temperature increase. This situation is analogous to the falling of a stone at its terminal velocity through a viscous medium such as air or water. As the stone falls in gravity, its decrease in potential energy is immediately transformed—not into an increase in its kinetic energy, but into an increase in the internal energy of the stone and the surrounding medium.

In a real battery with internal resistance r, the potential difference between the terminals is $\Delta V_{\rm batt} = \mathcal{E} - ir$, and the charge passing through the battery gains potential energy $dU = dq\, \Delta V_{\rm batt} = dq(\mathcal{E} - ir)$. The power delivered by this battery is $P_{\rm batt} = dU/dt$, or

$$P_{\rm batt} = \mathcal{E}i - i^2r = P_{\rm emf} - P_r. \qquad (31\text{-}23)$$

The energy available to the rest of the circuit is decreased by the Joule heating in the internal resistance.

The unit of power that follows from Eqs. 31-20 to 31-22 is the volt·ampere, which you can show to be equivalent to the watt by using the definitions of the volt (joule/coulomb) and ampere (coulomb/second).

SAMPLE PROBLEM 31-7. You are given a length of heating wire made of a nickel–chromium–iron alloy called Nichrome; it has a resistance R of 72 Ω. It is to be connected across a 120-V line. Under which circumstances will the wire dissipate more heat: (a) its entire length is connected across the line, or (b) the wire is cut in half and the two halves are connected in parallel across the line?

Solution (a) The power P_R dissipated by the entire wire is, from Eq. 31-22,

$$P_R = \frac{(\Delta V)^2}{R} = \frac{(120\ \text{V})^2}{72\ \Omega} = 200\ \text{W}.$$

(b) The power for a wire of half length (and thus half resistance) is

$$P_R' = \frac{(\Delta V)^2}{\frac{1}{2}R} = \frac{(120\ \text{V})^2}{36\ \Omega} = 400\ \text{W}.$$

There are two halves, so the power obtained from both of them is 800 W, or four times that for the single wire. This would seem to suggest that you could buy a heating wire, cut it in half, and reconnect it to obtain four times the heat output. Why is this not such a good idea?

31-7 *RC* CIRCUITS

The preceding sections dealt with circuits containing only resistors, in which the currents did not vary with time. Here we introduce the capacitor as a circuit element, which leads us to the study of time-varying currents.

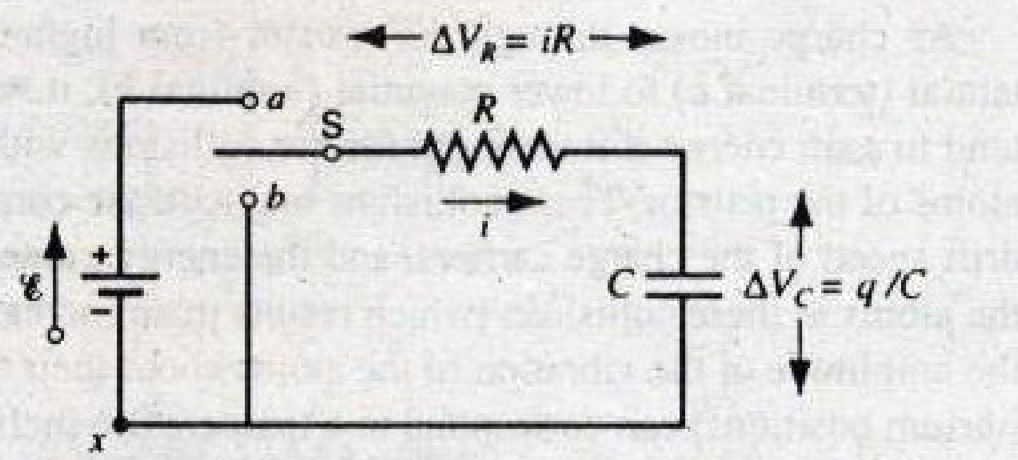


FIGURE 31-20. When switch S is connected to a, the capacitor C is charged by emf $\mathcal{E}$ through the resistor R. After the capacitor is charged, the switch is moved to b, and the capacitor discharges through R. We can easily measure the potential difference ΔV_R $(= iR)$ across the resistor to determine the current i and also measure the potential difference ΔV_C $(= q/C)$ across the capacitor to determine the charge q.

Suppose we charge the capacitor in Fig. 31-20 by throwing switch S to position a. (Later we consider the connection to position b.) What current is set up in the resulting single-loop circuit? Let us apply conservation of energy principles.

In time dt a charge dq $(= i\,dt)$ moves through any cross section of the circuit and is deposited on the positive plate of the capacitor. The work $(= \mathcal{E}\,dq$; see Eq. 31-1) done by the source of emf must equal the internal energy $(= i^2R\,dt)$ produced in the resistor during time dt, plus the increase dU in the amount of energy U $(= q^2/2C$; see Eq. 30-24) that is stored in the capacitor. Conservation of energy gives

$$\mathcal{E}\,dq = i^2R\,dt + d\left(\frac{q^2}{2C}\right)$$

or

$$\mathcal{E}\,dq = i^2R\,dt + \frac{q}{C}\,dq.$$

Dividing by dt yields

$$\mathcal{E}\frac{dq}{dt} = i^2R + \frac{q}{C}\frac{dq}{dt}.$$

Since q is the charge on the upper plate, positive i means positive dq/dt. With $i = dq/dt$, this equation becomes

$$\mathcal{E} = iR + \frac{q}{C}. \qquad (31\text{-}24)$$

Equation 31-24 also follows from the loop rule, as it must, since the loop rule was derived from the conservation of energy principle. Starting from point x and going around the circuit clockwise, we experience an increase in potential in going through the source of emf and decreases in potential in going through the resistor and the capacitor, or

$$\mathcal{E} - iR - \frac{q}{C} = 0,$$

which is identical to Eq. 31-24.

To solve Eq. 31-24, we first substitute dq/dt for i, which gives

$$\mathcal{E} = R\frac{dq}{dt} + \frac{q}{C}. \qquad (31\text{-}25)$$

We can rewrite Eq. 31-25 as

$$\frac{dq}{q - \mathcal{E}C} = -\frac{dt}{RC}. \qquad (31\text{-}26)$$

Integrating this result in the case that $q = 0$ at $t = 0$, we obtain (after solving for q),

$$q = C\mathcal{E}(1 - e^{-t/RC}). \qquad (31\text{-}27)$$

We can check that this function $q(t)$ is really a solution of Eq. 31-25 by substituting it and its first derivative into that equation and seeing whether an identity results. Differentiating Eq. 31-27 with respect to time yields

$$i = \frac{dq}{dt} = \frac{\mathcal{E}}{R}e^{-t/RC}. \qquad (31\text{-}28)$$

Substituting q (Eq. 31-27) and dq/dt (Eq. 31-28) into Eq. 31-25 yields an identity, as you should verify. Equation 31-27 is therefore a solution of Eq. 31-25.

In the laboratory we can determine i and q conveniently by measuring quantities that are proportional to them—namely, the potential difference ΔV_R $(= iR)$ across the resistor and the potential difference ΔV_C $(= q/C)$ across the capacitor. Such measurements can be obtained rather easily by connecting voltmeters (or oscilloscope probes) across the resistor and the capacitor. Figure 31-21 shows the resulting plots of ΔV_R and ΔV_C. Note the following: (1) At $t = 0$, $\Delta V_R = \mathcal{E}$ (the full potential difference appears across R), and $\Delta V_C = 0$ (the capacitor is not charged). (2) As $t \to \infty$, $\Delta V_C \to \mathcal{E}$ (the capacitor becomes fully charged), and $\Delta V_R \to 0$ (the current stops flowing). (3) At all times, $\Delta V_R + \Delta V_C = \mathcal{E}$, as Eq. 31-25 requires.

The quantity RC in Eqs. 31-27 and 31-28 has the dimensions of time and is called the *capacitive time constant*

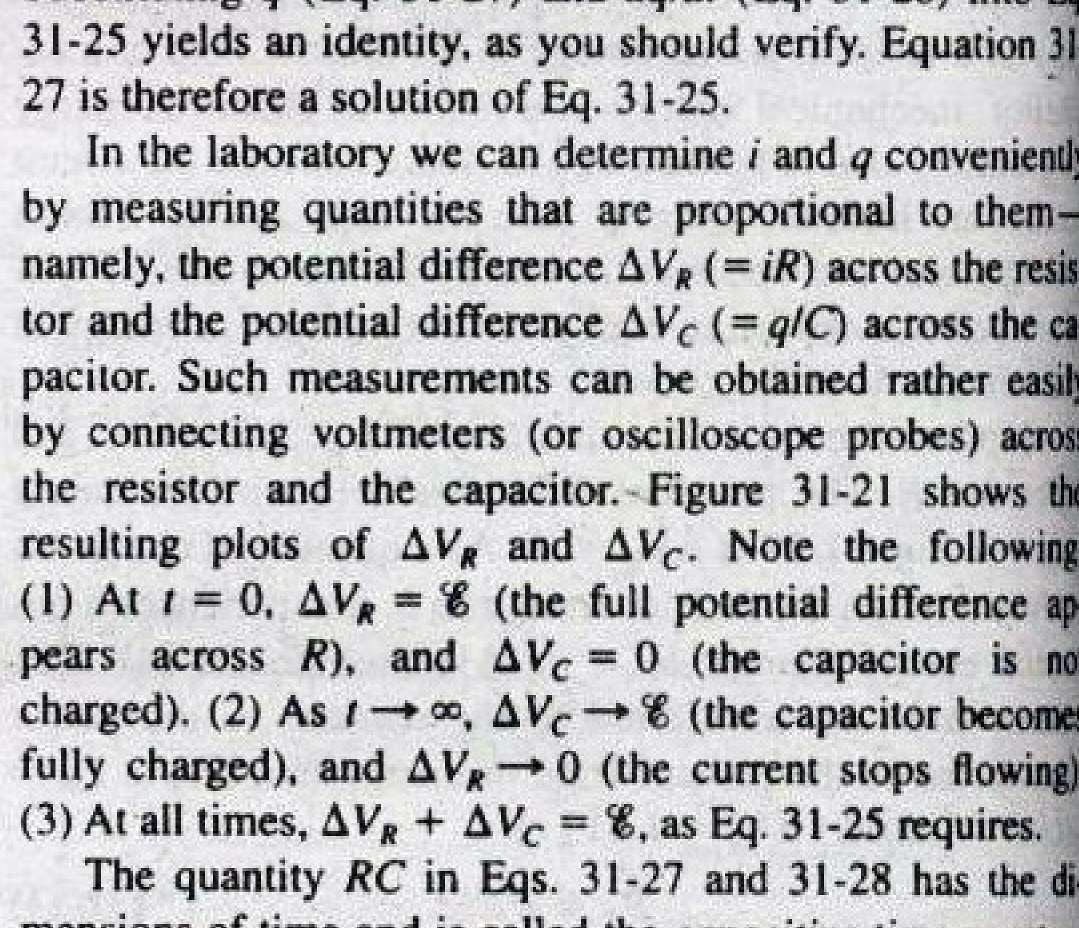


FIGURE 31-21. (*a*) As indicated by the potential difference ΔV_C, during the charging process the charge on the capacitor increases with time, and ΔV_C approaches the value of the emf $\mathcal{E}$. The time is measured after the switch is closed on a at $t = 0$. (*b*) The potential difference across the resistor decreases with time, approaching 0 at later times because the current falls to zero once the capacitor is fully charged. The curves have been drawn for $\mathcal{E} = 10$ V, $R = 2000\ \Omega$, and $C = 1\ \mu$F. The filled triangles represent successive RC time constants.

τ_C of the circuit:

$$\tau_C = RC. \quad (31\text{-}29)$$

It is the time at which the charge on the capacitor has increased to within a factor of $1 - e^{-1}$ ($\approx 63\%$) of its final value $C\mathscr{E}$. To show this, we put $t = \tau_C = RC$ in Eq. 31-27 to obtain

$$q = C\mathscr{E}(1 - e^{-1}) = 0.63C\mathscr{E}.$$

Figure 31-21*a* shows that if a resistance is included in a circuit with a charging capacitor, the increase of the charge of the capacitor toward its limiting value is *delayed* by a time characterized by the time constant *RC*. With no resistor present ($RC = 0$), the charge would rise immediately to its limiting value. Although we have shown that this time delay follows from an application of the loop rule to *RC* circuits, it is important to develop a physical understanding of the causes of the delay.

When switch S in Fig. 31-20 is closed on *a*, the charge on the capacitor is initially zero, so the potential difference across the capacitor is initially zero. At this time, Eq. 31-24 shows that $\mathscr{E} = iR$, and so $i = \mathscr{E}/R$ at $t = 0$. Because of this current, charge flows to the capacitor and the potential difference across the capacitor increases with time. Equation 31-24 now shows that, because the emf $\mathscr{E}$ is a constant, any increase in the potential difference across the capacitor must be balanced by a corresponding *decrease* in the potential difference across the resistor, with a similar decrease in the current. This decrease in the current means that the charge on the capacitor increases more slowly. This process continues until the current decreases to zero, at which time there is no potential drop across the resistor. The entire potential difference of the emf now appears across the capacitor, which is fully charged ($q = C\mathscr{E}$). Unless changes are made in the circuit, there is no further flow of charge, Review the derivations of Eqs. 31-27 and 31-28 and study Fig. 31-20 with the qualitative arguments of this paragraph in mind.

SAMPLE PROBLEM 31-8. A resistor R ($=6.2\ \mathrm{M\Omega}$) and a capacitor C ($= 2.4\ \mu\mathrm{F}$) are connected in series, and a 12-V battery of negligible internal resistance is connected across their combination. (*a*) What is the capacitive time constant of this circuit? (*b*) At what time after the battery is connected does the potential difference across the capacitor equal 5.6 V?

Solution (*a*) From Eq. 31-29,

$$\tau_C = RC = (6.2 \times 10^6\ \Omega)(2.4 \times 10^{-6}\ \mathrm{F}) = 15\ \mathrm{s}.$$

(*b*) The potential difference across the capacitor is $\Delta V_C = q/C$, which according to Eq. 31-27 can be written

$$\Delta V_C = \frac{q}{C} = \mathscr{E}(1 - e^{-t/RC}).$$

Solving for *t*, we obtain (using $\tau_C = RC$)

$$t = -\tau_C \ln\left(1 - \frac{\Delta V_C}{\mathscr{E}}\right)$$

$$= -(15\ \mathrm{s}) \ln\left(1 - \frac{5.6\ \mathrm{V}}{12\ \mathrm{V}}\right) = 9.4\ \mathrm{s}.$$

As we found above, after a time τ_C ($= 15$ s), the potential difference across the capacitor is $0.63\mathscr{E} = 7.6$ V. It is reasonable that in a shorter time of 9.4 s, the potential difference across the capacitor reaches only the smaller value of 5.6 V.

Discharging a Capacitor

Assume now that the switch S in Fig. 31-20 has been in position *a* for a time that is much greater than *RC*. For all practical purposes, the capacitor is fully charged, and no charge is flowing. The switch S is then thrown to position *b*. How do the charge of the capacitor and the current vary with time?

With the switch S closed on *b*, the capacitor discharges through the resistor. There is no emf in the circuit and Eq. 31-24 for the circuit, with $\mathscr{E} = 0$, becomes simply

$$iR + \frac{q}{C} = 0. \quad (31\text{-}30)$$

Putting $i = dq/dt$ allows us to write the equation of the circuit (compare Eq. 31-25) as

$$R\frac{dq}{dt} + \frac{q}{C} = 0. \quad (31\text{-}31)$$

The solution is, as you may readily derive by integration (after writing $dq/q = -dt/RC$) and verify by substitution,

$$q = q_0 e^{-t/\tau_C}, \quad (31\text{-}32)$$

q_0 being the initial charge on the capacitor ($=\mathscr{E}C$, in our case). The capacitive time constant τ_C ($= RC$) appears in this expression for a discharging capacitor as well as in that for a charging capacitor (Eq. 31-27). We see that at a time such that $t = \tau_C = RC$, the capacitor charge is reduced to $q_0 e^{-1}$, which is about 37% of the initial charge q_0.

Differentiating Eq. 31-32, we find the current during discharge,

$$i = \frac{dq}{dt} = -\frac{q_0}{RC} e^{-t/\tau_C}. \quad (31\text{-}33)$$

The negative sign shows that the current is in the direction opposite to that shown in Fig. 31-20. This is as it should be, since the capacitor is discharging rather than charging. If the capacitor is originally fully charged, then $q_0 = C\mathscr{E}$, and we can write Eq. 31-33 as

$$i = -\frac{\mathscr{E}}{R} e^{-t/\tau_C}. \quad (31\text{-}34)$$

The initial current, found by setting $t = 0$ in Eq. 31-34, is $-\mathscr{E}/R$. This is reasonable because the initial potential difference across the resistor is $\mathscr{E}$.

The potential differences across *R* and *C*, which are, respectively, proportional to *i* and *q*, can again be measured as indicated in Fig. 31-20. Typical results are shown in Fig. 31-22. Note that, as suggested by Eq. 31-32, ΔV_C ($=q/C$) falls exponentially from its maximum value, which occurs

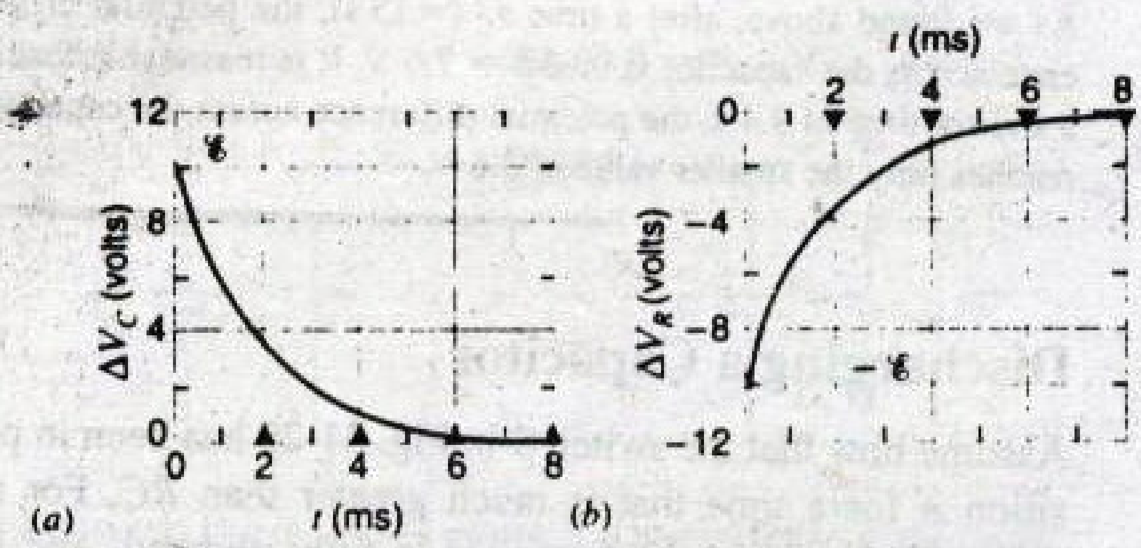


FIGURE 31-22. (*a*) After the capacitor has become fully charged, the switch in Fig. 31-20 is thrown from *a* to *b*, which we take to define a new $t = 0$. The potential difference across the capacitor decreases exponentially to zero as the capacitor discharges. (*b*) When the switch is initially moved to *b*, the potential difference across the resistor is negative compared with its value during the charging process shown in Fig. 31-21. As the capacitor discharges, the magnitude of the current falls exponentially to zero, and the potential drop across the resistor also approaches zero.

at $t = 0$, whereas $\Delta V_R\ (= iR)$ is negative and rises exponentially to zero. Note also that $\Delta V_C + \Delta V_R = 0$, as required by Eq. 31-30.

SAMPLE PROBLEM 31-9. A capacitor C discharges through a resistor R. (*a*) After how many time constants does its charge fall to one-half its initial value? (*b*) After how many time constants does the stored energy drop to half its initial value?

Solution (*a*) The charge on the capacitor varies according to Eq. 31-32,

$$q = q_0 e^{-t/\tau_C},$$

in which q_0 is the initial charge. We seek the time t at which $q = \frac{1}{2}q_0$, or

$$\tfrac{1}{2}q_0 = q_0 e^{-t/\tau_C}.$$

Canceling q_0 and taking the natural logarithm of each side, we find

$$-\ln 2 = -\frac{t}{\tau_C}$$

or

$$t = (\ln 2)\tau_C = 0.69\tau_C.$$

The charge drops to half its initial value after 0.69 time constants.

(*b*) The energy stored in the capacitor is

$$U = \frac{q^2}{2C} = \frac{q_0^2}{2C}e^{-2t/\tau_C} = U_0 e^{-2t/\tau_C},$$

in which U_0 is the initial stored energy. The time at which $U = \frac{1}{2}U_0$ is found from

$$\tfrac{1}{2}U_0 = U_0 e^{-2t/\tau_C}.$$

Canceling U_0 and taking the logarithm of each side, we obtain

$$-\ln 2 = -2t/\tau_C$$

or

$$t = \tau_C \frac{\ln 2}{2} = 0.35\tau_C.$$

The stored energy drops to half its initial value after 0.35 time constants have elapsed. This remains true no matter what the initial stored energy may be. The time ($0.69\tau_C$) needed for the *charge* to fall to half its initial value is greater than the time ($0.35\tau_C$) needed for the *energy* to fall to half its initial value. Why?

MULTIPLE CHOICE

31-1 Electric Current

1. The junction rule is a direct consequence of
   (A) Newton's second law.
   (B) conservation of momentum.
   (C) conservation of energy. (D) conservation of charge.
2. Figure 31-23 shows a network of wires carrying various currents. What is the current through A?
   (A) 1 A (B) 2 A (C) 3 A (D) 9 A
   (E) 11 A

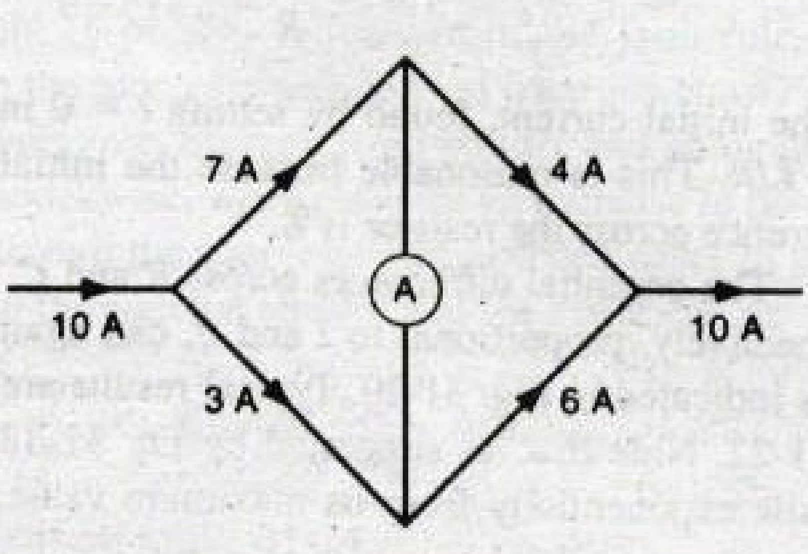


FIGURE 31-23. Multiple-choice question 2.

3. Figure 31-24 shows a network of wires carrying various currents. What is the current through A?
   (A) 1 A (B) 2 A (C) 6 A (D) 8 A
   (E) There is not enough information given.

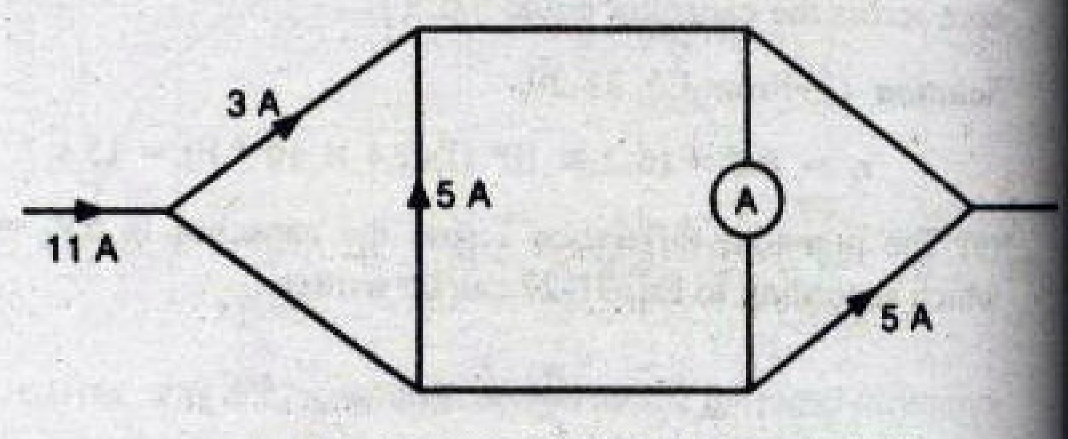


FIGURE 31-24. Multiple-choice question 3.

31-2 Electromotive Force

4. What are the units for ℰ, the electromotive force?
   (A) Emfs (B) Joules (C) Volts (D) Newtons
5. The function of the source of emf in a circuit is to
   (A) supply electrons to the circuit.
   (B) raise electrons to a higher potential.

(C) push electrons to a lower potential.
(D) accelerate electrons to greater speeds.

31-3 Analysis of Circuits

6. The loop rule is a direct consequence of
(A) Newton's second law.
(B) conservation of momentum.
(C) conservation of energy.
(D) conservation of charge.

7. A fixed resistor R is in series with a variable resistor and an ideal battery. Originally the resistances are the same.
(*a*) As the resistance of the variable resistor is decreased, the current through the variable resistor
(A) increases. (B) decreases.
(C) remains the same.
(D) cannot be determined without more information.
(*b*) As the resistance of the variable resistor is decreased, the potential difference across the variable resistor
(A) increases. (B) decreases.
(C) remains the same.
(D) cannot be determined without more information.

31-4 Electric Fields in Circuits

31-5 Resistors in Series and Parallel

8. Two resistors R_1 and R_2 are connected in series; assume that $R_1 < R_2$. The equivalent resistance of this arrangement is R, where
(A) $R < R_1/2$. (B) $R_1/2 < R < R_1$.
(C) $R_1 < R < R_2$. (D) $R_2 < R < 2R_2$.
(E) $2R_2 < R$.

9. Two resistors R_1 and R_2 are connected in parallel; assume that $R_1 < R_2$. The equivalent resistance of this argument is R, where
(A) $R < R_1/2$. (B) $R_1/2 < R < R_1$.
(C) $R_1 < R < R_2$. (D) $R_2 < R < 2R_2$.
(E) $2R_2 < R$.

10. What is the minimum number of resistors required to construct a network that *cannot* be analyzed to find the equivalent resistance by treating resistances in parallel or in series according to Eqs. 31-13 and 31-19?
(A) 3 (B) 4 (C) 5 (D) 6

31-6 Energy Transfers in an Electric Circuit

11. A standard lightbulb in the United States is the 60-watt bulb, which is designed to operate in a 120-volt circuit. During a brown-out it was noticed that the power output from the bulb decreased to 30 W. To what percentage of the original value did the voltage decrease?
(A) 75% (B) 70% (C) 50% (D) 33%

12. A fixed resistor R is in series with a variable resistor and a real battery (internal resistance is not negligible). Originally the fixed and variable resistors have the same resistance.
(*a*) As the resistance of the variable resistor is decreased, the rate at which energy is transferred to the fixed resistor
(A) increases. (B) decreases.
(C) remains the same.
(D) cannot be determined without more information.
(*b*) As the resistance of the variable resistor is decreased, the rate at which energy is transferred to the variable resistor
(A) increases. (B) decreases.
(C) remains the same.
(D) cannot be determined without more information.

13. A fixed resistor is in parallel with a variable resistor; both are connected to a real battery (internal resistance is not negligible). Originally the fixed and variable resistors have the same resistance.
(*a*) As the resistance of the variable resistor is decreased, the current through the fixed resistor
(A) increases. (B) decreases.
(C) remains the same.
(D) cannot be determined without more information.
(*b*) As the resistance of the variable resistor is decreased, the rate at which energy is transferred to the fixed resistor
(A) increases. (B) decreases.
(C) remains the same.
(D) cannot be determined without more information.
(*c*) As the resistance of the variable resistor is decreased, the rate at which energy is transferred to the variable resistor
(A) increases. (B) decreases.
(C) remains the same.
(D) cannot be determined without more information.

31-7 *RC* Circuits

14. A resistor, capacitor, switch, and ideal battery are in series. Originally the capacitor is uncharged. The switch is then closed, allowing current to flow.
(*a*) While the current is flowing, the potential difference across the resistor is
(A) increasing. (B) decreasing. (C) fixed.
(*b*) While the current is flowing, the potential difference across the capacitor is
(A) increasing. (B) decreasing. (C) fixed.

15. A capacitor is charged by connecting it in series with a resistor and an ideal battery. The battery supplies energy at a rate $P(t)$, the internal energy of the resistor increases at a rate $P_R(t)$, and the capacitor stores energy at a rate $P_C(t)$. What can be concluded about the relationship between $P_R(t)$ and $P_C(t)$?
(A) $P_R(t) > P_C(t)$ for all times t while charging.
(B) $P_R(t) = P_C(t)$ for all times t while charging.
(C) $P_R(t) < P_C(t)$ for all times t while charging.
(D) $P_R(t) > P_C(t)$ only at the beginning of the charging.
(E) $P_R(t) < P_C(t)$ only at the beginning of the charging.

QUESTIONS

1. Does the direction of the emf provided by a battery depend on the direction of current flow through the battery?

2. In Fig. 31-4, discuss what changes would occur if we increased the mass m by such an amount that the "motor" reversed direction and became a "generator"—that is, a source of emf.

3. Discuss in detail the statement that the energy method and the loop rule method for solving circuits are perfectly equivalent.

4. Devise a method for measuring the emf and the internal resistance of a battery.

5. What is the origin of the internal resistance of a battery? Does this depend on the age or size of the battery?

6. The current passing through a battery of emf $\mathscr{E}$ and internal resistance r is decreased by some external means. Does the potential difference between the terminals of the battery necessarily decrease or increase? Explain.

7. How could you calculate ΔV_{ab} in Fig. 31-8*a* by following a path from *a* to *b* that does not lie in the conducting circuit?

8. A 25-W, 120-V bulb glows at normal brightness when connected across a bank of batteries. A 500-W, 120-V bulb glows only dimly when connected across the same bank. How could this happen?

9. Under what circumstances can the terminal potential difference of a battery exceed its emf?

10. Automobiles generally use a 12-V electrical system. Years ago a 6-V system was used. Why the change? Why not 24 V?

11. The loop rule is based on the conservation of energy principle and the junction rule is based on the conservation of charge principle. Explain just how these rules are based on these principles.

12. Under what circumstances would you want to connect batteries in parallel? In series?

13. Compare and contrast the formulas for the equivalent values of series and parallel combinations of (*a*) capacitors and (*b*) resistors.

14. Under what circumstances would you want to connect resistors in parallel? In series?

15. What is the difference between an emf and a potential difference?

16. Referring to Fig. 31-10, use a qualitative argument to convince yourself that i_2 is drawn in the wrong direction.

17. Do the junction and loop rules apply to a circuit containing a capacitor?

18. Show that the product RC in Eq. 31-29 has the dimensions of time—that is, that 1 second = 1 ohm × 1 farad.

19. A capacitor, resistor, and battery are connected in series. The charge that the capacitor stores is unaffected by the resistance of the resistor. What purpose, then, is served by the resistor?

20. Explain why, in Sample Problem 31-9, the stored energy falls to half its initial value more rapidly than does the charge.

21. The light flash in a camera is produced by the discharge of a capacitor across the lamp. Why do we not just connect the photoflash lamp directly across the power supply used to charge the capacitor?

22. Does the time required for the charge on a capacitor in an RC circuit to build up to a given fraction of its final value depend on the value of the applied emf? Does the time required for the charge to change by a given amount depend on the applied emf?

23. A capacitor is connected across the terminals of a battery. Does the charge that eventually appears on the capacitor plates depend on the value of the internal resistance of the battery?

24. Devise a method whereby an RC circuit can be used to measure very high resistances.

25. In Fig. 31-20, suppose that switch S is closed on *a*. Explain why, in view of the fact that the negative terminal of the battery is not connected to resistance R, the current in R should be $\mathscr{E}/R$, as Eq. 31-28 predicts.

26. In Fig. 31-20, suppose that switch S is closed on *a*. Why does the charge on capacitor C not rise instantaneously to $q = C\mathscr{E}$? After all, the positive battery terminal is connected to one plate of the capacitor and the negative terminal to the other.

27. What special characteristics must heating wire have?

28. Equation 31-21 seems to suggest that the rate of increase of internal energy in a resistor is reduced if the resistance is made less; Eq. 31-22 seems to suggest just the opposite. How do you reconcile this apparent paradox?

29. Why do electric power companies reduce voltage during times of heavy demand? What is being saved?

30. Is the filament resistance lower or higher in a 500-W light-bulb than in a 100-W bulb? Both bulbs are designed to operate at 120 V.

31. Five wires of the same length and diameter are connected in turn between two points maintained at constant potential difference. Will internal energy be developed at a faster rate in the wire of (*a*) the smallest or (*b*) the largest resistance?

32. Why is it better to send 10 MW of electric power long distances at 10 kV rather than at 220 V?

EXERCISES

31-1 Electric Current

31-2 Electromotive Force

1. A 5.12-A current is set up in an external circuit by a 6.00-V storage battery for 5.75 min. By how much is the chemical energy of the battery reduced?

2. (*a*) How much work does a 12.0-V source of emf do on an electron as it passes through from the positive to the negative terminal? (*b*) If 3.40×10^{18} electrons pass through each second, what is the power output of the source?

3. A certain 12-V car battery can "pump" a total charge of 125 A · h before it runs down. Assuming that the potential difference across the terminals stays constant, for how long can it deliver energy at the rate of 110 W?

4. A standard flashlight battery can deliver about 2.0 W · h of energy before it runs down. (*a*) If a battery costs 80 cents, what is the cost of operating a 100-W lamp for 8.0 h using batteries? (*b*) What is the cost if power provided by an electric utility company, at 12 cents per kW · h, is used?

31-3 Analysis of Circuits

5. In Fig. 31-25, the potential at point P is 100 V. What is the potential at point Q?

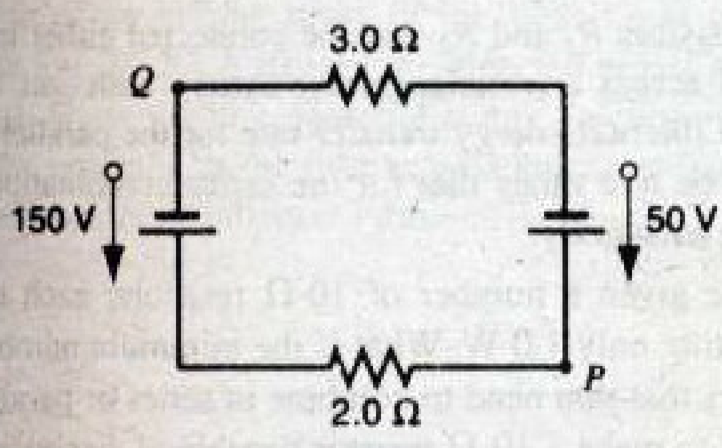


FIGURE 31-25. Exercise 5.

6. A gasoline gauge for an automobile is shown schematically in Fig. 31-26. The indicator (on the dashboard) has a resistance of 10 Ω. The tank unit is simply a float connected to a resistor that has a resistance of 140 Ω when the tank is empty, 20 Ω when it is full, and varies linearly with the volume of gasoline. Find the current in the circuit when the tank is (*a*) empty, (*b*) half full, and (*c*) full.

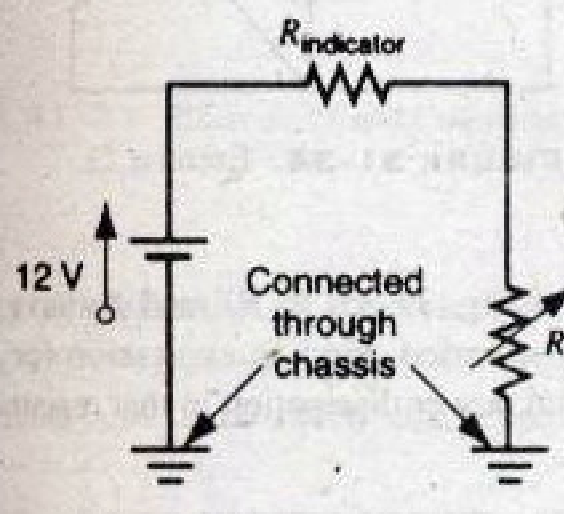


FIGURE 31-26. Exercise 6.

7. (*a*) In Fig. 31-27, what value must R have if the current in the circuit is to be 50 mA? Take $\mathscr{E}_1 = 2.0$ V, $\mathscr{E}_2 = 3.0$ V, and $r_1 = r_2 = 3.0\ \Omega$. (*b*) What is the rate at which internal energy appears in R?

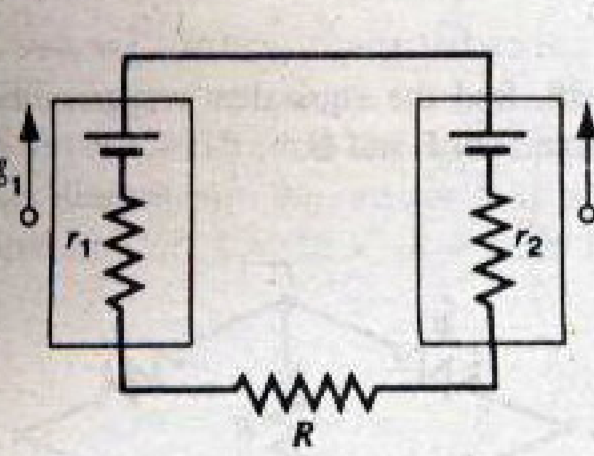


FIGURE 31-27. Exercise 7.

8. The current in a single-loop circuit is 5.0 A. When an additional resistance of 2.0 Ω is inserted in series, the current drops to 4.0 A. What was the resistance in the original circuit?

9. The section of circuit AB (see Fig. 31-28) absorbs 53.0 W of power when a current $i = 1.20$ A passes through it in the indicated direction. (*a*) Find the potential difference between A and B. (*b*) If the element C does not have an internal resistance, what is its emf? (*c*) Which terminal, left or right, is positive?

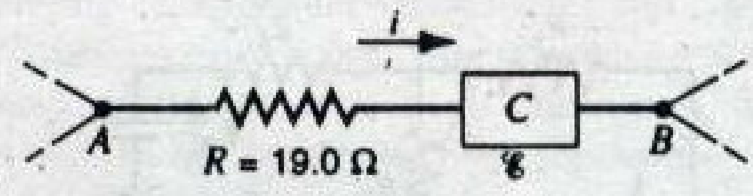


FIGURE 31-28. Exercise 9.

10. Internal energy is to be generated in a 108-mΩ resistor at the rate of 9.88 W by connecting it to a battery whose emf is 1.50 V. (*a*) What is the internal resistance of the battery? (*b*) What potential difference exists across the resistor?

11. What current, in terms of $\mathscr{E}$ and R, does the ammeter A in Fig. 31-29 read? Assume that A has zero resistance.

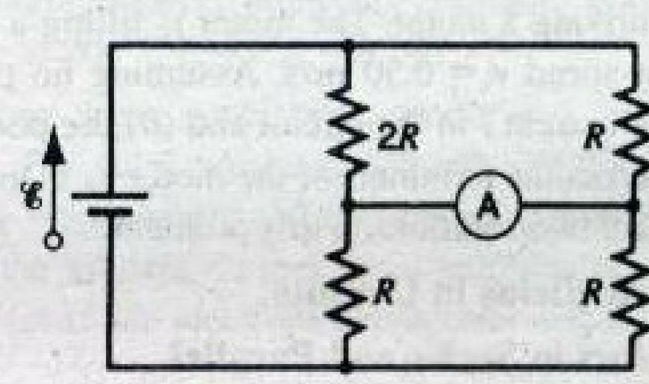


FIGURE 31-29. Exercise 11.

12. You are given two batteries of emf values $\mathscr{E}_1$ and $\mathscr{E}_2$ and internal resistances r_1 and r_2. They may be connected either in (*a*) parallel or (*b*) series and are used to establish a current in a resistor R, as shown in Fig. 31-30. Derive expressions for the current in R for both methods of connection.

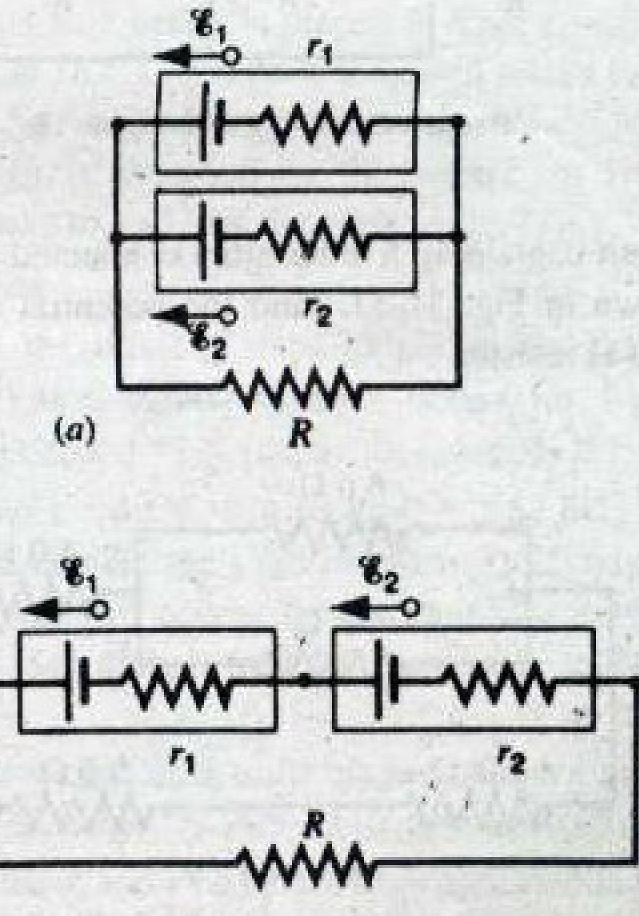


FIGURE 31-30. Exercise 12.

13. (*a*) Calculate the current through each source of emf in Fig. 31-31. (*b*) Calculate $V_b - V_a$. Assume that $R_1 = 1.20\ \Omega$, $R_2 = 2.30\ \Omega$, $\mathscr{E}_1 = 2.00$ V, $\mathscr{E}_2 = 3.80$ V, and $\mathscr{E}_3 = 5.00$ V.

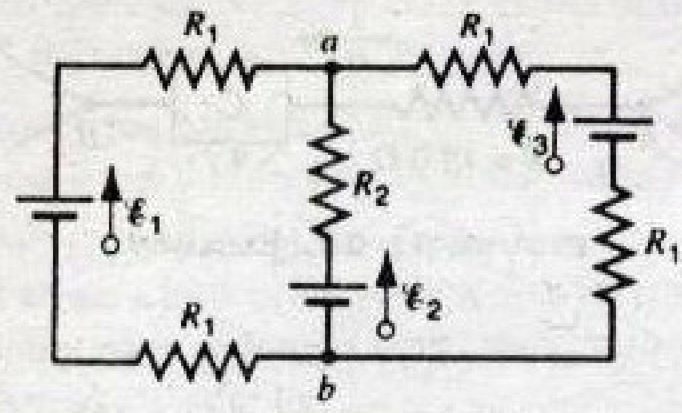


FIGURE 31-31. Exercise 13.

14. (*a*) In the circuit of Fig. 31-8, show that the power delivered to R as internal energy is a maximum when R is equal to the internal resistance r of the battery. (*b*) Show that this maximum power is $P = \mathscr{E}^2/4r$.

15. A battery of emf $\mathscr{E} = 2.0$ V and internal resistance $r = 0.50\ \Omega$ is driving a motor. The motor is lifting a 2.0-N object at constant speed $v = 0.50$ m/s. Assuming no power losses, find (*a*) the current i in the circuit and (*b*) the potential difference ΔV across the terminals of the motor. (*c*) Discuss the fact that there are two solutions to this problem.

31-4 Electric Fields in Circuits

31-5 Resistors in Series and Parallel

16. Four 18-Ω resistors are connected in parallel across a 27-V battery. What is the current through the battery?

17. By using only two resistors—singly, in series, or in parallel—you are able to obtain resistances of 3.0 Ω, 4.0 Ω, 12 Ω, and 16 Ω. What are the separate resistances of the resistors?

18. In Fig. 31-32, find the equivalent resistance between points (*a*) A and B, (*b*) A and C, and (*c*) B and C.

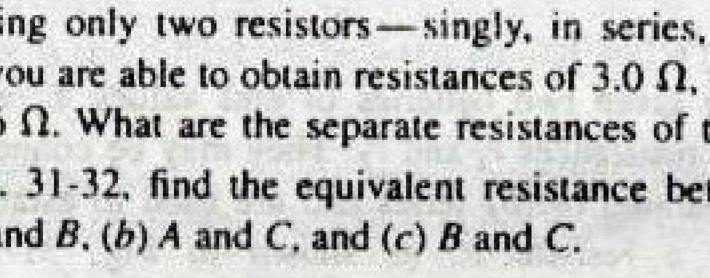


FIGURE 31-32. Exercise 18.

19. A circuit containing five resistors connected to a 12-V battery is shown in Fig. 31-33. Find the potential difference across the 5.0-Ω resistor.

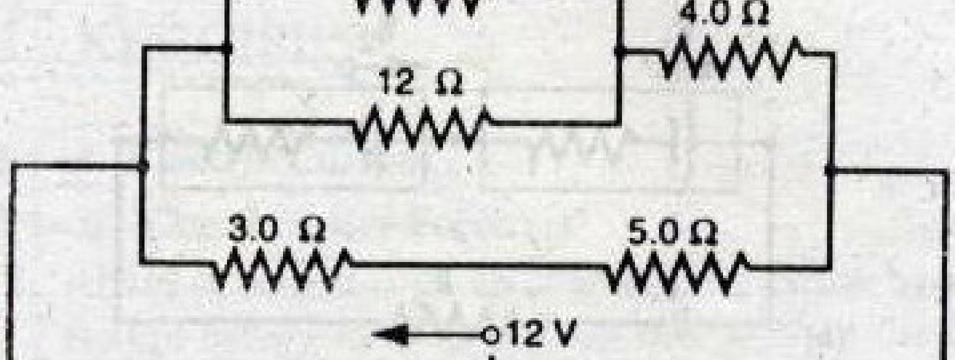


FIGURE 31-33. Exercise 19.

20. A 120-V power line is protected by a 15-A fuse. What is t maximum number of 500-W lamps that can be simulta ously operated in parallel on this line?

21. Two resistors R_1 and R_2 may be connected either in series parallel across a (resistanceless) battery with emf $\mathscr{E}$. We sire the internal energy transfer rate for the parallel combin tion to be five times that for the series combination. If R_1 100 Ω, what is R_2?

22. You are given a number of 10-Ω resistors, each capable dissipating only 1.0 W. What is the minimum number of su resistors that you need to combine in series or parallel comb nations to make a 10-Ω resistor capable of dissipating at lea 5.0 W?

23. (*a*) In Fig. 31-34, find the equivalent resistance of the netwo shown. (*b*) Calculate the current in each resistor. Put R_1 = 112 Ω, $R_2 = 42.0\ \Omega$, $R_3 = 61.6\ \Omega$, $R_4 = 75.0\ \Omega$, and $\mathscr{E}$ = 6.22 V.

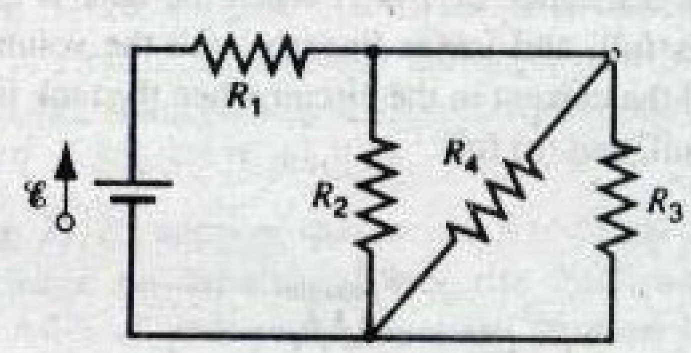


FIGURE 31-34. Exercise 23.

24. In the circuit of Fig. 31-35, $\mathscr{E}$, R_1, and R_2 have constant values but R can be varied. Find an expression for R that results in the maximum power dissipation in that resistor.

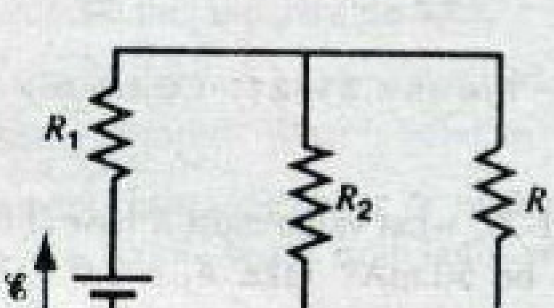


FIGURE 31-35. Exercise 24.

25. In Fig. 31-36, find the equivalent resistance between points (*a*) F and H and (*b*) F and G.

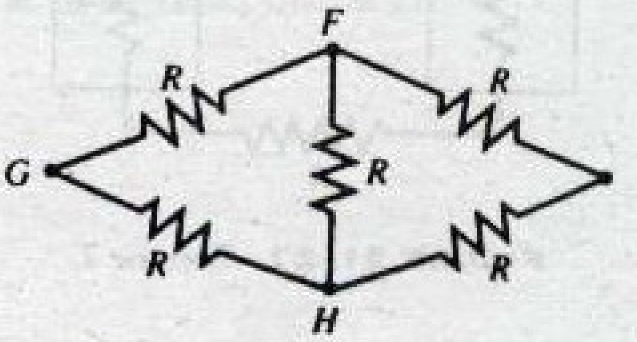


FIGURE 31-36. Exercise 25.

26. A voltage divider consists of two resistors in series. The applied potential difference across the resistors is 12 V, and the potential difference across the second resistor is 2.4 V. Find the resistances, assuming that the current through the two resistors is 1 mA.

27. Design a voltage divider that will input 1.50 V and output approximately 0.95 ± 0.01 V using only standard resistor values.

28. A portion of an infinite array of identical 1-$\mu\Omega$ resistors is shown in Fig. 31-37. A battery is connected across two distant junctions. Show that the potential at any junction is the average of the potential at the four nearest junctions. You will use this result to solve Computer Problem 1.

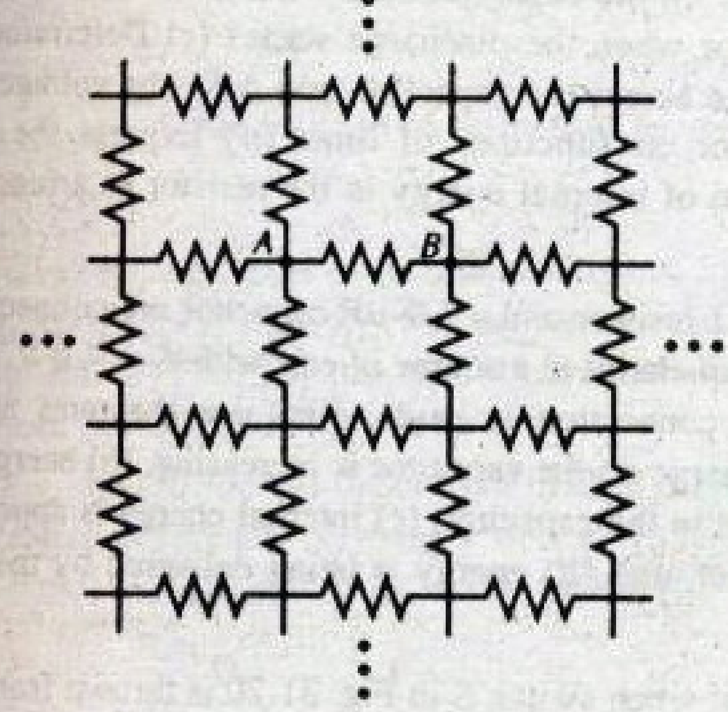


FIGURE 31-37. Exercise 28 and Computer Problem 1.

31-6 Energy Transfers in an Electric Circuit

29. A student's 9.0-V, 7.5-W portable radio was left on from 9:00 P.M. until 3:00 A.M. How much charge passed through the wires?

30. The headlights of a moving car draw 9.7 A from the 12-V alternator, which is driven by the engine. Assume that the alternator is 82% efficient, and calculate the horsepower the engine must supply to run the lights.

31. A space heater, operating from a 120-V line, has a hot resistance of 14.0 Ω. (*a*) At what rate is electrical energy transfered into internal energy? (*b*) At 5.22¢/kW · h, what does it cost to operate the device for 6 h 25 min?

32. Figure 31-38 shows a battery connected across a uniform resistor R_0. A sliding contact can move across the resistor from $x = 0$ at the left to $x = 10$ cm at the right. Find an expression for the power dissipated in the resistor R as a function of x. Plot the function for $\mathscr{E} = 50$ V, $R = 2000\ \Omega$, and $R_0 = 100\ \Omega$.

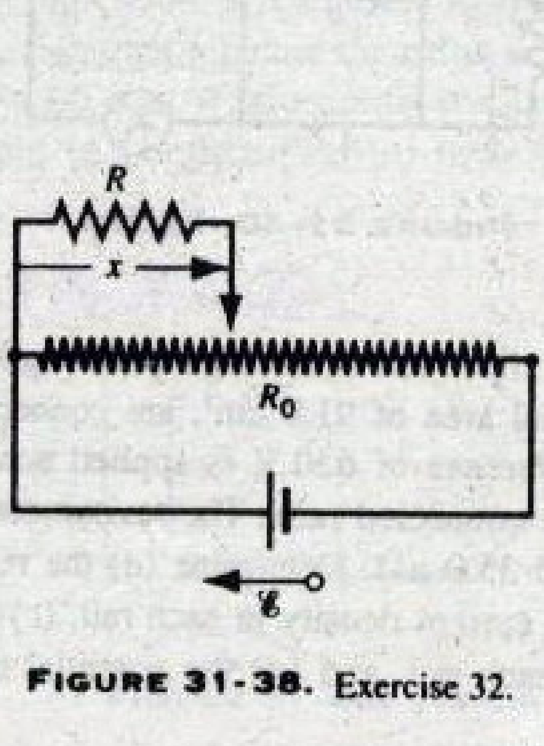


FIGURE 31-38. Exercise 32.

33. Two lightbulbs, one of resistance R_1 and the other of resistance R_2 ($< R_1$), are connected (*a*) in parallel and (*b*) in series. Which bulb is brighter in each case?

34. The National Board of Fire Underwriters has fixed safe current-carrying capacities for various sizes and types of wire. For #10 rubber-coated copper wire (diameter = 0.10 in.), the maximum safe current is 25 A. At this current, find (*a*) the current density, (*b*) the electric field, (*c*) the potential difference for 1000 ft of wire, and (*d*) the rate at which internal energy is developed for 1000 ft of wire.

35. A 100-W lightbulb is plugged into a standard 120-V outlet. (*a*) How much does it cost per month (31 days) to leave the light turned on? Assume electric energy costs 6¢/kW · h. (*b*) What is the resistance of the bulb? (*c*) What is the current in the bulb? (*d*) Is the resistance different when the bulb is turned off?

36. A Nichrome heater dissipates 500 W when the applied potential difference is 110 V and the wire temperature is 800°C. How much power would it dissipate if the wire temperature were held at 200°C by immersion in a bath of cooling oil? The applied potential difference remains the same; α for Nichrome at 800°C is 4.0×10^{-4}/C°.

37. An electron linear accelerator produces a pulsed beam of electrons. The pulse current is 485 mA and the pulse duration is 95.0 ns. (*a*) How many electrons are accelerated per pulse? (*b*) Find the average current for a machine operating at 520 pulses/s. (*c*) If the electrons are accelerated to an energy of 47.7 MeV, what are the values of average and peak power outputs of the accelerator?

38. A cylindrical resistor of radius 5.12 mm and length 1.96 cm is made of material that has a resistivity of 3.50×10^{-5} $\Omega \cdot$ m. What are (*a*) the current density and (*b*) the potential difference when the power dissipation is 1.55 W?

39. A heating element is made by maintaining a potential difference of 75 V along the length of a Nichrome wire with a 2.6-mm^2 cross section and a resistivity of $5.0 \times 10^{-7}\ \Omega \cdot$ m. (*a*) If the element dissipates 4.8 kW, what is its length? (*b*) If a potential difference of 110 V is used to obtain the same power output, what should the length be?

40. A 420-W immersion heater is placed in a pot containing 2.10 liters of water at 18.5°C. (*a*) How long will it take to bring the water to boiling temperature, assuming that 77.0% of the available energy is absorbed by the water? (*b*) How much longer will it take to boil half the water away?

31-7 *RC* Circuits

41. (*a*) Carry out the missing steps to obtain Eq. 31-27 from Eq. 31-26. (*b*) In a similar manner, obtain Eq. 31-32 from Eq. 31-31. Note that $q = q_0$ (capacitor charged) at $t = 0$.

42. In an *RC* series circuit $\mathscr{E} = 11.0$ V, $R = 1.42$ MΩ, and $C = 1.80\ \mu$F. (*a*) Calculate the time constant. (*b*) Find the maximum charge that will appear on the capacitor during charging. (*c*) How long does it take for the charge to build up to 15.5 μC?

43. How many time constants must elapse before a capacitor in an *RC* circuit is charged to 99% of its maximum charge?

44. A 15.2-kΩ resistor and a capacitor are connected in series and a 13.0-V potential is suddenly applied. The potential across the capacitor rises to 5.00 V in 1.28 μs. (*a*) Calculate the time constant. (*b*) Find the capacitance of the capacitor.

45. An *RC* circuit is discharged by closing a switch at time $t = 0$. The initial potential difference across the capacitor is 100 V. (*a*) If the potential difference has decreased to 1.06 V after 10.0 s, calculate the time constant of the circuit. (*b*) What will be the potential difference at $t = 17$ s?

46. A controller on an electronic arcade game consists of a variable resistor connected across the plates of a 220-nF capacitor. The capacitor is charged to 5.00 V, then discharged through the resistor. The time for the potential difference across the plates to decrease to 800 mV is measured by an internal clock. If the range of discharge times that can be handled is from 10.0 μs to 6.00 ms, what should be the range of the resistance of the resistor?

47. Figure 31-39 shows the circuit of a flashing lamp, like those attached to barrels at highway construction sites. The fluorescent lamp *L* is connected in parallel across the capacitor *C* of an *RC* circuit. Current passes through the lamp only when the potential across it reaches the breakdown voltage V_L; in this event, the capacitor discharges through the lamp and it flashes for a very short time. Suppose that two flashes per second are needed. Using a lamp with breakdown voltage $V_L = 72$ V, a 95-V battery, and a 0.15-μF capacitor, what should be the resistance *R* of the resistor?

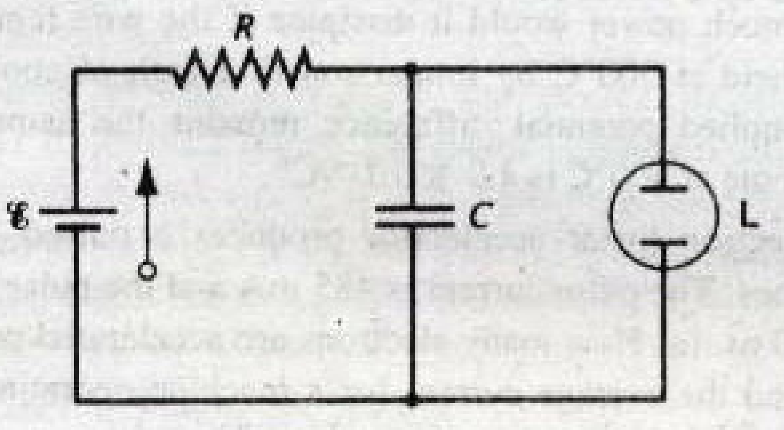


FIGURE 31-39. Exercise 47.

48. A 1.0-μF capacitor with an initial stored energy of 0.50 J is discharged through a 1.0-MΩ resistor. (*a*) What is the initial charge on the capacitor? (*b*) What is the current through the resistor when the discharge starts? (*c*) Determine ΔV_C, the voltage across the capacitor, and ΔV_R, the voltage across the resistor, as functions of time. (*d*) Express the rate of generation of internal energy in the resistor as a function of time.

49. A 3.0-MΩ resistor and a 1.0-μF capacitor are connected in a single-loop circuit to a source of emf with $\mathcal{E} = 4.0$ V. At 1.0 s after the connection is made, what are the rates at which (*a*) the charge on the capacitor is increasing, (*b*) energy is being stored in the capacitor, (*c*) internal energy is appearing in the resistor, and (*d*) energy is being delivered by the seat of emf?

50. Prove that when switch S in Fig. 31-20 is thrown from *a* to *b*, all the energy stored in the capacitor is transformed into internal energy in the resistor. Assume that the capacitor is fully charged before the switch is thrown.

PROBLEMS

1. The starting motor of an automobile is turning slowly and the mechanic has to decide whether to replace the motor, the cable, or the battery. The manufacturer's manual says that the 12-V battery can have no more than 0.020-Ω internal resistance, the motor no more than 0.200-Ω resistance, and the cable no more than 0.040-Ω resistance. The mechanic turns on the motor and measures 11.4 V across the battery, 3.0 V across the cable, and a current of 50 A. Which part is defective?

2. Two batteries having the same emf $\mathcal{E}$ but different internal resistances r_1 and r_2 ($r_1 > r_2$) are connected in series to an external resistance *R*. (*a*) Find the value of *R* that makes the potential difference zero between the terminals of one battery. (*b*) Which battery is it?

3. A solar cell generates a potential difference of 0.10 V when a 500-Ω resistor is connected across it and a potential difference of 0.16 V when a 1000-Ω resistor is substituted. What are (*a*) the internal resistance and (*b*) the emf of the solar cell? (*c*) The area of the cell is 5.0 cm^2 and the intensity of light striking it is 2.0 mW/cm^2. What is the efficiency of the cell for converting light energy to internal energy in the 1000-Ω external resistor?

4. When the lights of an automobile are switched on, an ammeter in series with them reads 10.0 A and a voltmeter connected across them reads 12.0 V. See Fig. 31-40. When the electric starting motor is turned on, the ammeter reading drops to 8.00 A and the lights dim somewhat. If the internal resistance of the battery is 50.0 mΩ and that of the ammeter is negligible, what are (*a*) the emf of the battery and (*b*) the current through the starting motor when the lights are on?

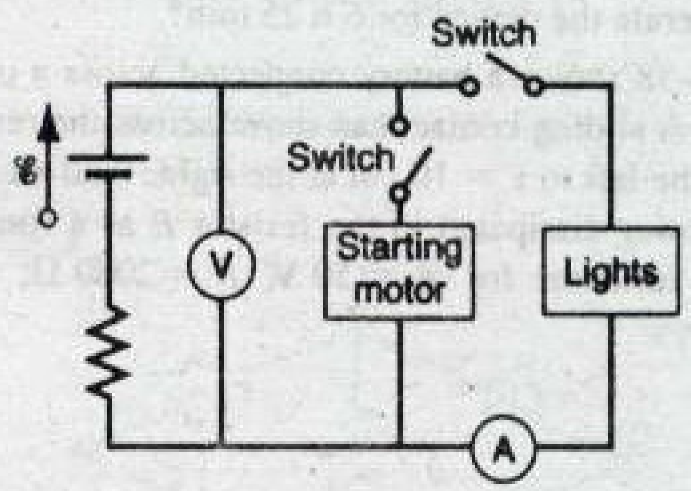


FIGURE 31-40. Problem 4.

5. Conducting rails *A* and *B*, having equal lengths of 42.6 m and cross-sectional area of 91.0 cm^2, are connected in series. A potential difference of 630 V is applied across the terminal points of the connected rails. The resistances of the rails are 76.2 $\mu\Omega$ and 35.0 $\mu\Omega$. Determine (*a*) the resistivities of the rails, (*b*) the current density in each rail, (*c*) the electric field strength in each rail, and (*d*) the potential difference across each rail.

6. Find the equivalent resistance between points x and y shown in Fig. 31-41. Four of the resistors have equal resistance R, as shown; the "middle" resistor has value $r \neq R$. (Compare with Problem 10 of Chapter 30.)

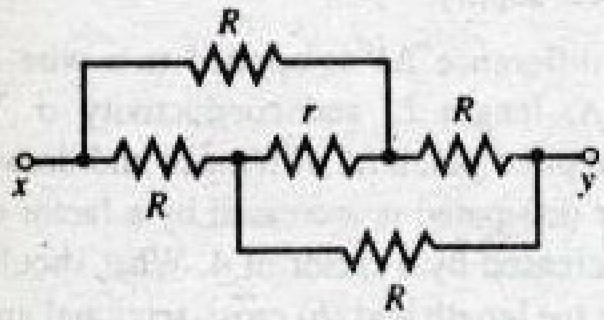


FIGURE 31-41. Problem 6.

7. Twelve resistors, each of resistance R ohms, form a cube (see Fig. 31-18a). (a) Find R_{13}, the equivalent resistance of a face diagonal. (b) Find R_{17}, the equivalent resistance of a body diagonal. See Sample Problem 31-6.

8. In Fig. 31-42, find (a) the current in each resistor and (b) the potential difference between a and b. Assume $\mathcal{E}_1 = 6.0$ V, $\mathcal{E}_2 = 5.0$ V, $\mathcal{E}_3 = 4.0$ V, $R_1 = 100\ \Omega$, and $R_2 = 50\ \Omega$.

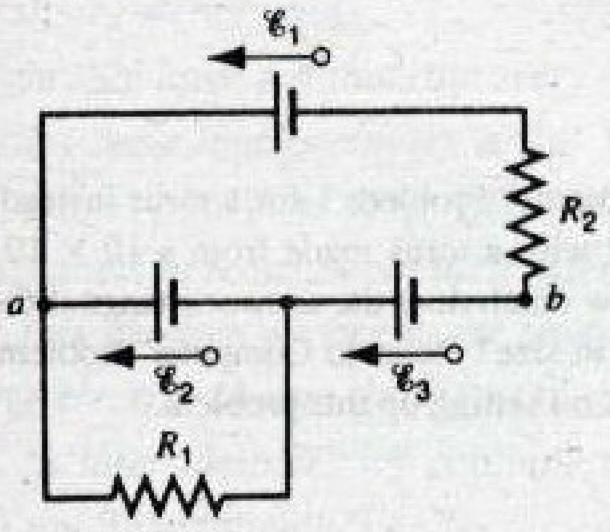


FIGURE 31-42. Problem 8.

9. A three-way, 120-V lamp bulb, rated for 100-200-300 W, burns out a filament. Afterward, the bulb operates at the same intensity on its lowest and its highest switch positions but does not operate at all on the middle position. (a) How are the two filaments wired inside the bulb? (b) Calculate the resistances of the filaments.

10. A simple ohmmeter is made by connecting a 1.50-V flashlight battery in series with a resistor R and a 1.00 mA ammeter, as shown in Fig. 31-43. R is adjusted so that when the circuit terminals are shorted together the meter deflects to its full-scale value of 1.00 mA. What external resistance across the terminals results in a deflection of (a) 10%, (b) 50%, and (c) 90% of full scale? (d) If the ammeter has a resistance of 18.5 Ω and the internal resistance of the battery is negligible, what is the value of R?

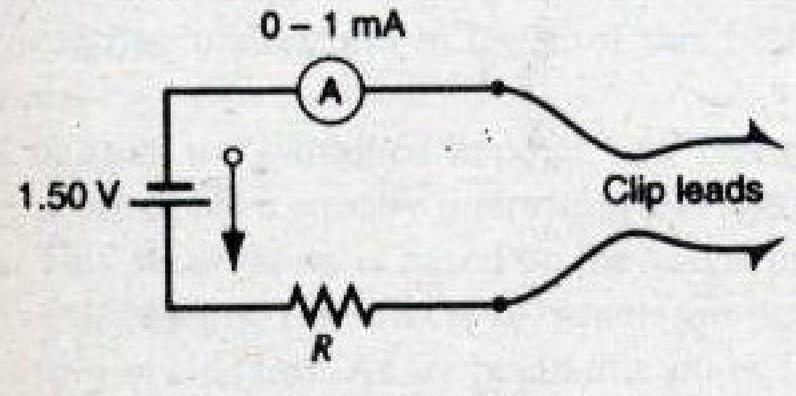


FIGURE 31-43. Problem 10.

11. In Fig. 31-44, imagine an ammeter inserted in the branch containing R_3. (a) What will it read, assuming $\mathcal{E} = 5.0$ V, $R_1 = 2.0\ \Omega$, $R_2 = 4.0\ \Omega$, and $R_3 = 6.0\ \Omega$? (b) The ammeter and the source of emf are now physically interchanged. Show that the ammeter reading remains unchanged. Assume the ammeter to have zero resistance.

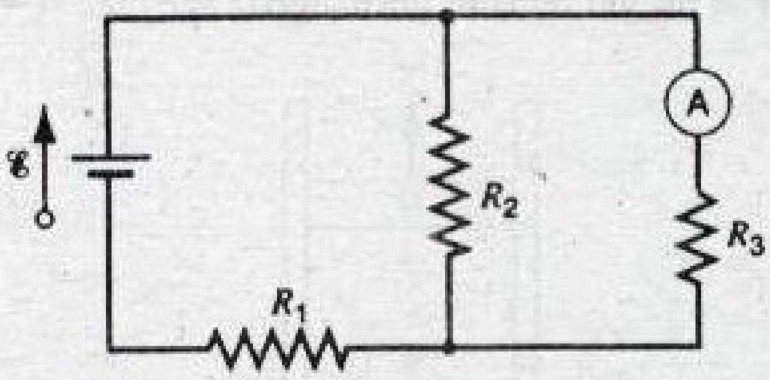


FIGURE 31-44. Problem 11.

12. In Fig. 31-45, R_s is to be adjusted in value until points a and b are brought to exactly the same potential. (One tests for this condition by momentarily connecting a sensitive ammeter between a and b; if these points are at the same potential, the ammeter will not deflect.) Show that when this adjustment is made, the following relation holds:

$$R_x = R_s(R_2/R_1).$$

An unknown resistance (R_x) can be measured in terms of a standard (R_s) using this device, which is called a *Wheatstone bridge.*

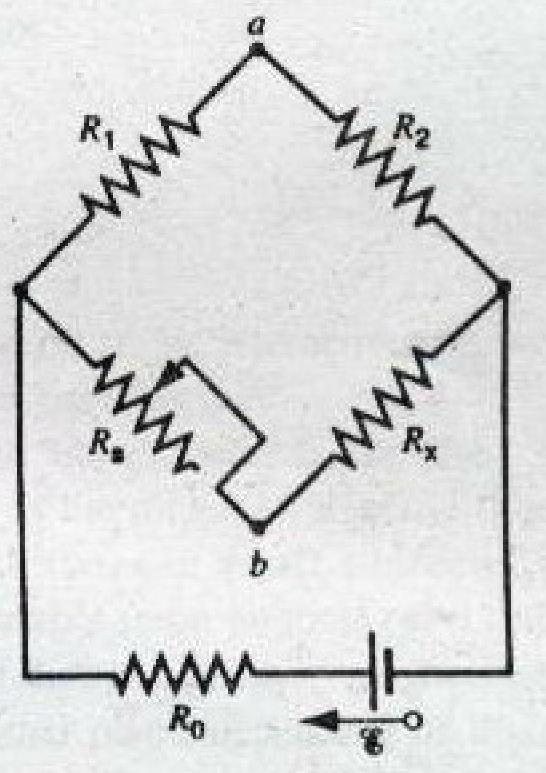


FIGURE 31-45. Problems 12 and 13.

13. If points a and b in Fig. 31-45 are connected by a wire of resistance r, show that the current in the wire is

$$i = \frac{\mathcal{E}(R_s - R_x)}{(R + 2r)(R_s + R_x) + 2R_sR_x},$$

where $\mathcal{E}$ is the emf of the battery. Assume that R_1 and R_2 are equal ($R_1 = R_2 = R$) and that R_0 equals zero. Is this formula consistent with the result of Problem 12?

14. A coil of current-carrying Nichrome wire is immersed in a liquid contained in a calorimeter. When the potential difference across the coil is 12 V and the current through the coil is 5.2 A, the liquid boils at a steady rate, evaporating at the rate of 21 mg/s. Calculate the heat of vaporization of the liquid.

15. A resistance coil, wired to an external battery, is placed inside an adiabatic cylinder fitted with a frictionless piston and containing an ideal gas. A current i = 240 mA flows through the coil, which has a resistance R = 550 Ω. At what speed v must the piston, mass m = 11.8 kg, move upward so that the temperature of the gas remains unchanged? See Fig. 31-46.

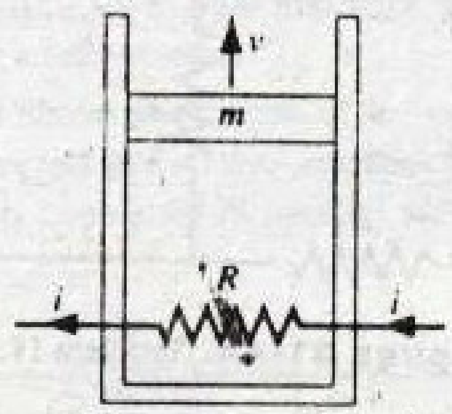


FIGURE 31-46. Problem 15.

16. A 32-μF capacitor is connected across a programmed power supply. During the interval from t = 0 to t = 3 s, the output voltage of the supply is given by $V(t)$ = (6 V) + (4 V/s)t − (2 V/s²)t^2. At t = 0.50 s, find (*a*) the charge on the capacitor, (*b*) the current into the capacitor, and (*c*) the power output from the power supply.

17. A potential difference ΔV is applied to a wire of cross-sectional area A, length L, and conductivity σ. You want to change the applied potential difference and draw out the wire so the power dissipated is increased by a factor of 30 and the current is increased by a factor of 4. What should be the new values of the (*a*) length and (*b*) cross-sectional area?

18. An initially uncharged capacitor C is fully charged by a constant emf $\mathcal{E}$ in series with a resistor R. (*a*) Show that the final energy stored in the capacitor is half the energy supplied by the emf. (*b*) By direct integration of i^2R over the charging time, show that the internal energy dissipated by the resistor is also half the energy supplied by the emf.

19. At what time after charging begins in Problem 18 is the rate of energy dissipation in the resistor equal to the rate of energy storage in the capacitor?

COMPUTER PROBLEMS

1. Use the results of Exercise 28 to find the equivalent resistance between any two points separated by a single resistor (such as A and B) for the infinite resistor array in Fig. 31-37. This problem is easiest done by iteration, and can be programmed and solved on a spreadsheet in less than a minute!

2. Repeat Computer Problem 1 for a torus instead of an infinite sheet. Start with a torus made from a 10 × 10 grid of resistors. By how much does the answer change if the original grid is doubled in size? Refer to Computer Problem 2 of Chapter 30 for hints on setting up this problem.

CHAPTER 32

THE MAGNETIC FIELD

The science of magnetism *had its origin in ancient times. It grew from the observation that certain naturally occurring stones would attract one another and would also attract small bits of one metal, iron, but not other metals, such as gold or silver. The word "magnetism" comes from the name of the district (Magnesia) in Asia Minor, one of the locations where these stones were found.*

Today we have put that discovery to great practical use, from small "refrigerator" magnets to magnetic recording tape and computer disks. The magnetism of individual atomic nuclei is used by physicians to make images of organs deep within the body. Spacecraft have measured the magnetism of the Earth and the other planets to learn about their internal structure.

In this chapter we begin our study of magnetism by considering the magnetic field and its effects on a moving electric charge. In the next chapter we consider the production of magnetic fields by electric currents. In later chapters we continue to explore the close relationship between electricity and magnetism, which are linked together under the common designation electromagnetism.

32-1 MAGNETIC INTERACTIONS AND MAGNETIC POLES

When we began our study of electrostatics in Chapter 25, we described an observation—the attraction between a plastic rod rubbed with fur and a glass rod rubbed with silk—that we could not explain based on the forces and interactions we had considered up to that point. We accounted for that attraction as a force exerted by the electric charges of one rod on the electric charges of the other. In subsequent chapters we learned that many interesting and useful phenomena can be understood in terms of this basic electrostatic force.

In this chapter we introduce another new observation, which will prove to have equally interesting and useful consequences. This observation is based on the *magnetic* interaction between objects, the effects of which are discussed in the next several chapters. As we begin this study, it is important to keep in mind many important similarities between electric and magnetic interactions as well as the important differences between them.

As early as the 8th century B.C., the Greeks had discovered that a piece of the mineral magnetite (known as lodestone, an oxide of iron) can attract a piece of iron but does not exert a measurable force on most other materials. Later it was discovered that one piece of magnetite can either attract or repel another piece, depending on their relative orientation. In these experiments and the ones we describe below, it is necessary to determine that neither of the objects carries a net electric charge, so that we can be sure these new forces cannot be of electrical origin.

By the 12th century, the following experiment was known. A small needle-shaped piece of magnetite is suspended so that it can pivot about a vertical axis. Even with no other magnetite or iron nearby, the piece will spontaneously rotate about its pivot and eventually come to rest with one end pointing roughly toward the Earth's north geographic pole. Let us identify that end by painting it red. No

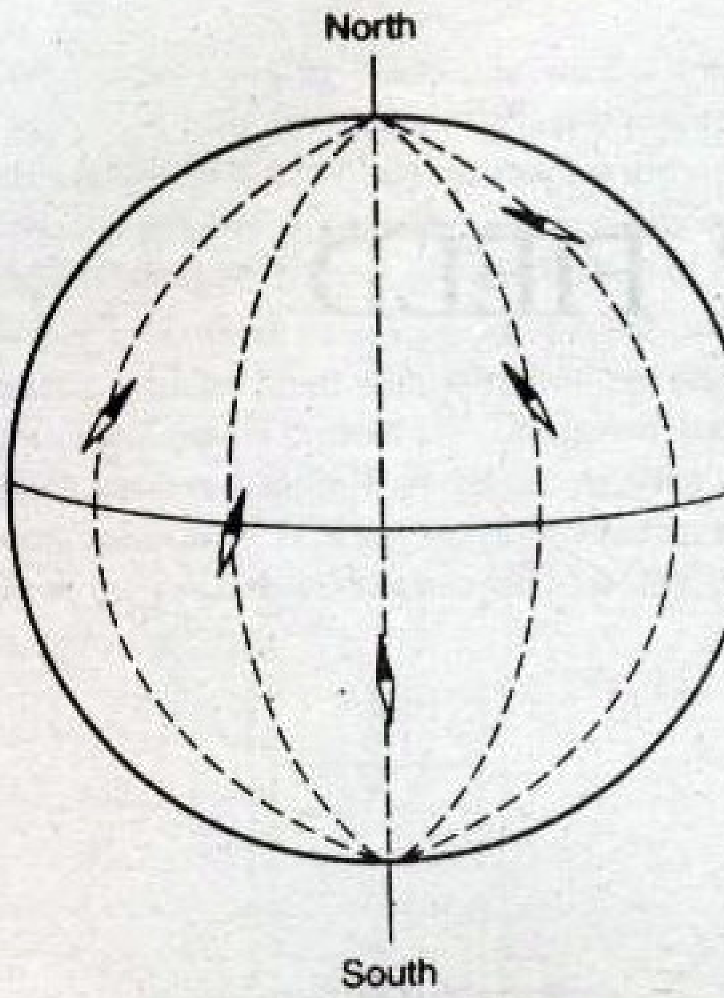


FIGURE 32-1. At any location on the Earth, a magnetic needle will point approximately in the direction of the north pole.

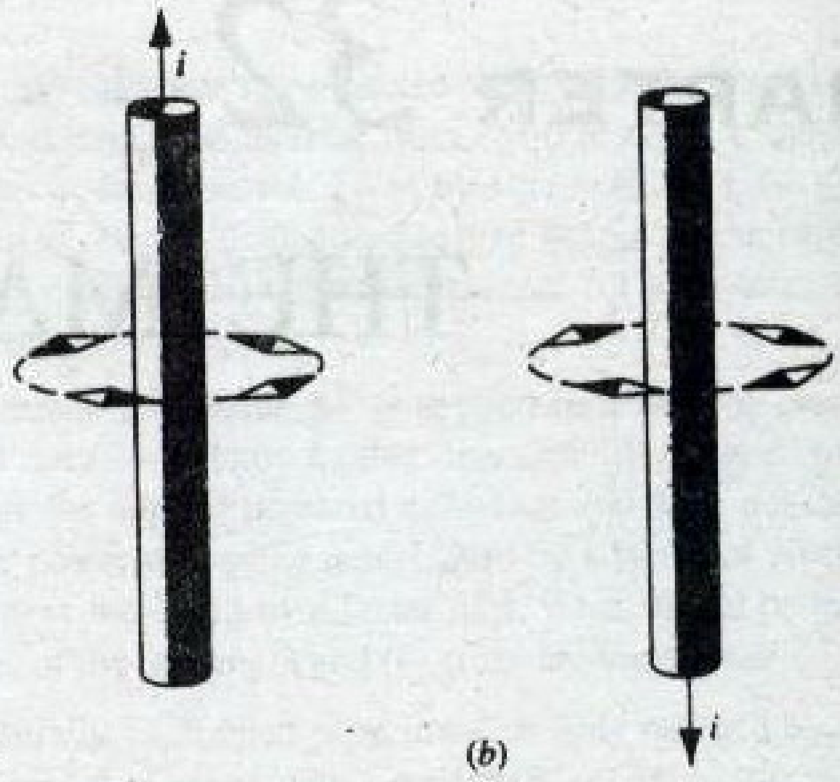


FIGURE 32-3. (*a*) A compass shows that a magnetic field surrounds a current-carrying wire. (*b*) When the current is reversed, the direction of the magnetic field reverses.

matter where we try this experiment on Earth, the red end always points north (Fig. 32-1).

The device we have constructed is of course a magnetic compass, which is responding to the magnetic field of the Earth, just as two pieces of magnetite exert forces on each other. We describe this outcome using the same language that we used for electrical interactions: one piece of magnetite (or the Earth) sets up a magnetic field, and the other piece responds to that magnetic field. The direction of a magnetic compass gives the direction of the horizontal component of the magnetic field of the Earth.

We can also use our magnetic compass to trace out the magnetic field of a bar magnet. In Fig. 32-2, one end of a bar magnet has been painted red, because that end points north if we suspend the bar magnet like a compass. By convention, we call this the "north-seeking" or simply *north* pole of the magnet, and the opposite end is the *south* pole. When we bring our compass close to the bar magnet, the compass rotates until its direction indicates the direction of the magnetic field at that point, as shown in Fig. 32-2. As we discuss in Chapter 35, the magnetic field of the Earth is in many ways similar to the magnetic field of a bar magnet.

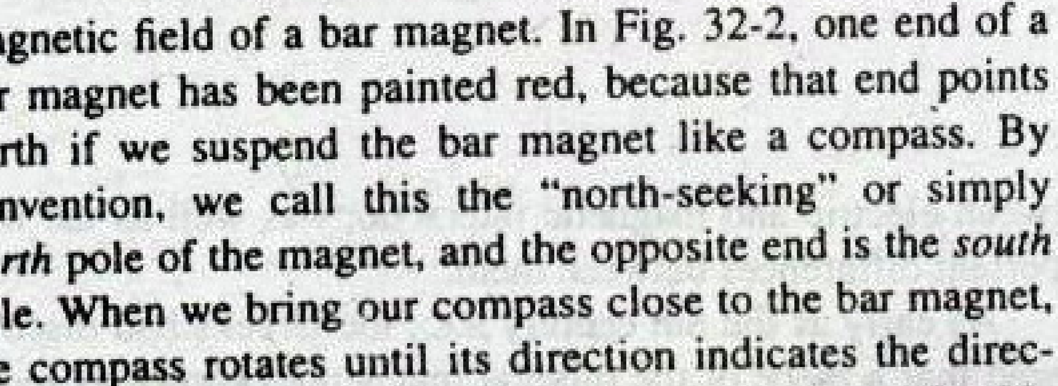


FIGURE 32-2. The compass needle shows the magnetic field surrounding a bar magnet.

Even more surprising results occur when we place our compass near a current-carrying wire, as in Fig. 32-3. When a steady current flows in the wire, the compass shows quite clearly that a magnetic field is present, and the direction of the compass needle indicates the direction of the magnetic field near the wire. If we turn off the current, there is no magnetic field. If we reverse the direction of the current, the compass needle points in the opposite direction (Fig. 32-3*b*).

These observations give a hint of the complex and fascinating relationship between electrical phenomena (such as a current in a wire) and magnetic phenomena (the deflection of a compass needle). Later in the text we discuss other examples of this relationship, which is responsible for such diverse effects as the operation of electric motors and the propagation of light.

Magnetism and Moving Charges

It is tempting to try to understand magnetic fields by following the same procedure we used for electric fields—that is, by using a test charge to probe the field. This leads immediately to questions about the relationship between electric and magnetic phenomena.

1. *Does there exist in nature a "magnetic test charge" that could be used to determine the strength and direction of the magnetic field, just as we used the force on an electric test charge to determine the electric field?* (As in $\vec{\mathbf{E}} = \vec{\mathbf{F}}/q_0$, Eq. 26-3). Theory permits the existence of isolated magnetic charges, but no one has ever found one despite intense experimental searches. We conclude that isolated magnetic charges, known as *magnetic monopoles*, are either very rare or nonexistent, and so our equations for electromagnetism are written as if there are no magnetic charges.

2. *Can we use an electric test charge to probe a magnetic field?* Yes, but only if the charge is moving relative to the source of the magnetic field. A magnetic field exerts no force on an electric charge at rest.

3. *If electric charges in motion can be used to probe magnetic fields, can electric charges in motion also produce magnetic fields?* Yes, as we illustrated in Fig. 32-3. In fact, electric charges in motion are also responsible for the magnetic fields of the Earth and the bar magnet (in the latter case, the charges are electrons moving in atoms).

4. *In electrostatics we associated an electric potential energy with a test charge in an electric field (Section 28-2). Is there a "magnetic potential energy" associated with a moving electric test charge in a magnetic field?* In general the answer is no, because forces that depend on velocity are not conservative forces. (Recall from Chapter 12 that potential energy can be defined only for conservative forces.)

Just as the space around a collection of charges is described as the location of an electric field represented by a vector $\vec{\mathbf{E}}$, the space around the Earth, a bar magnet, or a current-carrying wire is described as the location of a magnetic field* represented by a vector $\vec{\mathbf{B}}$. In analogy with the electric field, we display the magnetic field by field lines, which are close together where the field is large and far apart where it is small. Often the lines of $\vec{\mathbf{B}}$ will look similar to patterns we have previously drawn for lines of $\vec{\mathbf{E}}$—for example, in the case of a uniform field or the field of a dipole. However, as we go along you should take note of several important differences between the lines of $\vec{\mathbf{E}}$ and the lines of $\vec{\mathbf{B}}$, which we shall discuss in the next several chapters.

The basic relationship between electric charge and electric field in electrostatics can be represented as

electric fields are caused by electric charge

electric charge → electric field → electric charge.

forces are exerted on electric charges by electric fields

That is, electric charges set up an electric field, which can then exert a force of electric origin on other electric charges. We would like to be able to write a similar relationship for magnetic fields:

magnetic charge → magnetic field → magnetic charge.

However, because no isolated magnetic charge has yet been found, we must use a different relationship:

moving electric charge → magnetic field → moving electric charge.

That is, a magnetic field is established by moving electric charges, and in turn the field can exert a force (which we call a magnetic force) on other moving electric charges. In this chapter we explore the second part of this relationship (the magnetic force on a moving charge). The next chapter discusses how magnetic fields are caused by moving electric charges, including currents in wires.

*There is not general agreement on the naming of field vectors in magnetism. $\vec{\mathbf{B}}$ may be called the *magnetic induction* or *magnetic flux density*, whereas another field vector, denoted by $\vec{\mathbf{H}}$, may be called the magnetic field. We regard $\vec{\mathbf{B}}$ as the more fundamental quantity and therefore call it the magnetic field.

Magnetic Poles

By suspending a piece of magnetic material at the Earth's surface, we can identify and mark its north pole (the end that points roughly in the direction of the Earth's north geographic pole) and its south pole (the opposite end). Let us test and mark two pieces of magnetic material in this way. We can then directly study the magnetic force that one of these pieces exerts on the other in various orientations. In particular, we can study the force that one north pole exerts on another north pole or on a south pole. From experiments of this type we deduce the following rule for the interactions of magnetic poles:

Like poles repel, and unlike poles attract.

That is, a south pole attracts a north pole, but two north poles or two south poles repel one another. This rule is very similar to the rule for the interaction of electric charges (Section 25-2). Applying this rule to the behavior of a magnetic compass on the Earth's surface, as in Fig. 32-1, we conclude that to attract the north pole of the compass there must be a magnetic *south* pole near the Earth's geographic north pole.

32-2 THE MAGNETIC FORCE ON A MOVING CHARGE

To begin to understand the properties of magnetic fields, our first task is to study the force on a charged particle moving in a magnetic field. In electrostatics, the force on a charged particle in an electric field is $\vec{\mathbf{F}}_E = q\vec{\mathbf{E}}$ (Eq. 26-4). This expression gives us a way to test whether an electric field is present at various points in any region of space; if, after we account for all other nonelectric forces (gravity, etc.), we find the force on a test charge at rest to be nonzero, then we can conclude that an electric field must be present at that point. We now seek the corresponding expression for magnetic fields, which will allow us to test for the presence of magnetic fields based on the force exerted on a moving charged particle. We know in advance that we will not obtain as simple an expression as the electric force, because the magnetic force involves two vectors: the magnetic field $\vec{\mathbf{B}}$ and the velocity $\vec{\mathbf{v}}$.

Before beginning our experiments, we first test the region of interest to make sure that no electric field is present. We make this determination by placing the test charge at rest at various locations and checking that the electric force on the test charge is zero.

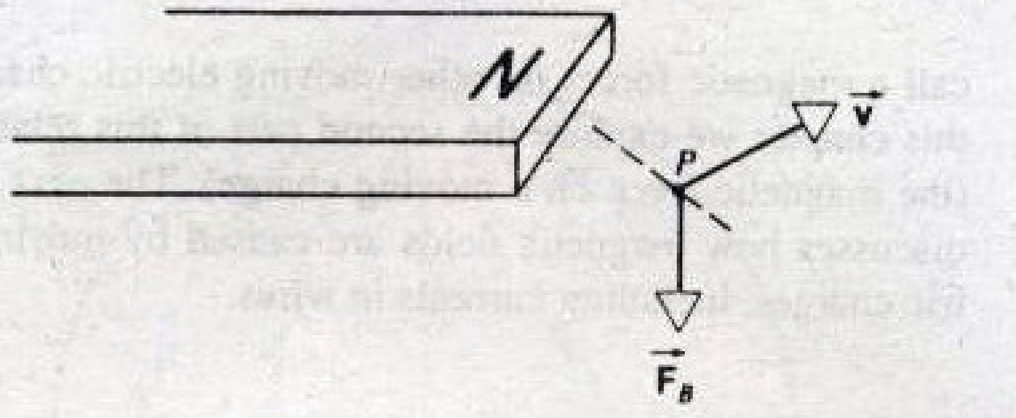


FIGURE 32-4. A charged particle moves with velocity $\vec{v}$ near a bar magnet. A magnetic force $\vec{F}_B$ is exerted on the particle. If the particle's velocity is in either direction along the dashed line, the magnetic force is zero.

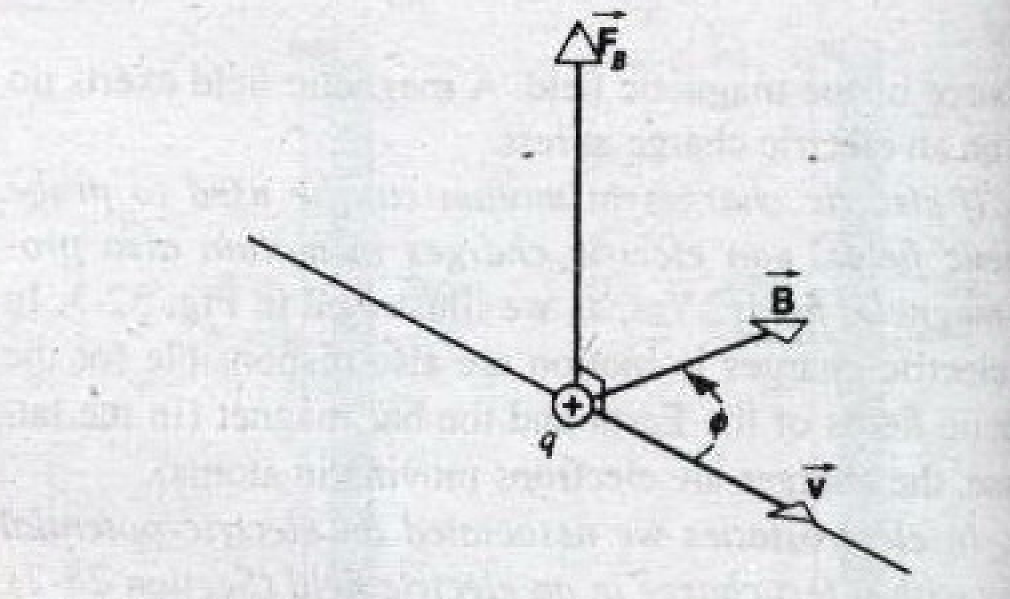


FIGURE 32-5. A particle with a positive charge q moving with velocity $\vec{v}$ through a magnetic field $\vec{B}$ experiences a magnetic deflecting force $\vec{F}_B$.

Once we have established that there is no electric field acting on the charged particle, we can use it as a probe to find the magnetic force. We pick a particular point P near a source of a magnetic field, such as a bar magnet (Fig. 32-4). We assume that we have available a device for firing charged particles through point P; the device allows us to choose the magnitude of the velocity of the particles and their direction of motion. Using this device, we can do experiments that permit a variety of observations about the magnetic force $\vec{F}_B$ that acts on the particles.

1. By firing particles through P in a variety of directions, we find that the magnetic force, if it is present, is perpendicular to the velocity $\vec{v}$, as shown in Fig. 32-4. No matter what the direction of $\vec{v}$, the magnetic force is always perpendicular to $\vec{v}$.

2. As we vary the direction of $\vec{v}$ through P (keeping the magnitude of $\vec{v}$ constant), we find that the magnetic force is zero for one particular direction of $\vec{v}$, indicated by the dashed line in Fig. 32-4. When the velocity is perpendicular to this direction, the magnetic force has its maximum value. In between these directions, the magnetic force varies as $\sin\phi$, where ϕ is the angle between the velocity and the dashed line. (Note that the dashed line actually indicates two opposite directions for which the force is zero, one corresponding to $\phi = 0$ and the other to $\phi = 180°$.)

3. As we vary the magnitude of the velocity, we find that the magnitude of $\vec{F}_B$ varies in direct proportion.

4. We also find that the magnitude of $\vec{F}_B$ is proportional to the magnitude of the test charge q, and that $\vec{F}_B$ reverses direction when q changes sign.

We now define the magnetic field $\vec{B}$ in the following way, based on these observations: the direction of $\vec{B}$ at point P is the same as one of the directions of $\vec{v}$ (to be specified shortly) for which the force is zero, and the magnitude of $\vec{B}$ is determined from the magnitude $F_{B,\max}$ of the maximum force exerted when the test charge is projected perpendicular to the direction of $\vec{B}$; that is,

$$B = \frac{F_{B,\max}}{|q|v}. \tag{32-1}$$

At arbitrary angles, our observations are summarized by the formula

$$F_B = |q|vB \sin\phi, \tag{32-2}$$

where ϕ is the smaller angle between $\vec{v}$ and $\vec{B}$. Because F_B, v, and B are vectors, Eq. 32-2 can be written as a vector product:

$$\vec{F}_B = q\vec{v} \times \vec{B}. \tag{32-3}$$

By writing $\vec{v} \times \vec{B}$ instead of $\vec{B} \times \vec{v}$ in Eq. 32-3, we have specified which of the two possible directions of $\vec{B}$ we want to use. Vector or cross products will occur frequently in our study of magnetism. You may find it helpful to review the properties of the vector product in Appendix H.

Figure 32-5 shows the geometrical relationship among the vectors $\vec{F}_B$, $\vec{v}$, and $\vec{B}$. Note that, as is always the case for a vector product, $\vec{F}_B$ is perpendicular to the plane formed by $\vec{v}$ and $\vec{B}$. Thus $\vec{F}_B$ is always perpendicular to $\vec{v}$, and the magnetic force is always a sideways deflecting force. Note also that $\vec{F}_B$ vanishes when $\vec{v}$ is either parallel or antiparallel to the direction of $\vec{B}$ (in which case $\phi = 0°$ or $180°$, and $\vec{v} \times \vec{B} = 0$), and that $\vec{F}_B$ has its maximum magnitude, equal to qvB, when $\vec{v}$ is at right angles to $\vec{B}$.

Because the magnetic force is always perpendicular to $\vec{v}$, it cannot change the magnitude of $\vec{v}$, only its direction. Equivalently, the force is always at right angles to the displacement of the particle and can do no work on it. Thus a constant magnetic field cannot change the kinetic energy of a moving charged particle. (In Chapter 34 we consider time-varying magnetic fields, which *can* change the kinetic energy of a particle. In this chapter we deal only with magnetic fields that do not vary with time.)

Equation 32-3, which serves as the definition of the magnetic field $\vec{B}$, indicates both its magnitude and its direction. We define the electric field similarly through an equation, $\vec{F}_E = q\vec{E}$, so that by measuring the electric force we can determine the magnitude *and* direction of the electric field. Magnetic fields cannot be determined quite so simply with a single measurement. As Fig. 32-5 suggests, measuring $\vec{F}_B$ for a single $\vec{v}$ is not sufficient to determine $\vec{B}$, because the direction of $\vec{F}_B$ does not indicate the direction of $\vec{B}$. We must first find the direction of $\vec{B}$ (for example, by finding the directions of $\vec{v}$ for which there is no force), and then a single additional measurement can determine its magnitude.

TABLE 32-1 Typical Values of Some Magnetic Fields[a]

Location	*Magnetic Field* (T)
At the surface of a neutron star (calculated)	10^8
Near a superconducting magnet	5
Near a large electromagnet	1
Near a small bar magnet	10^{-2}
At the surface of the Earth	10^{-4}
In interstellar space	10^{-10}
In a magnetically shielded room	10^{-14}

[a] Approximate values.

The SI unit of $\vec{\mathbf{B}}$ is the *tesla* (abbreviation T). It follows from Eq. 32-1 that

$$1 \text{ tesla} = 1 \frac{\text{newton}}{\text{coulomb} \cdot \text{meter/second}} = 1 \frac{\text{newton}}{\text{ampere} \cdot \text{meter}}.$$

An earlier (non-SI) unit for $\vec{\mathbf{B}}$, still in common use, is the *gauss*, related to the tesla by

$$1 \text{ tesla} = 10^4 \text{ gauss}.$$

Table 32-1 gives some typical values of magnetic fields.

Figure 32-6 shows the lines of $\vec{\mathbf{B}}$ of a bar magnet. Note that the lines of $\vec{\mathbf{B}}$ pass through the magnet, forming closed loops. From the clustering of field lines outside the magnet near its ends, we infer that the magnetic field in the space around the magnet has its greatest magnitude there. These ends are the *poles* of the magnet. Note that the field lines emerge from the north pole and converge toward the south pole.

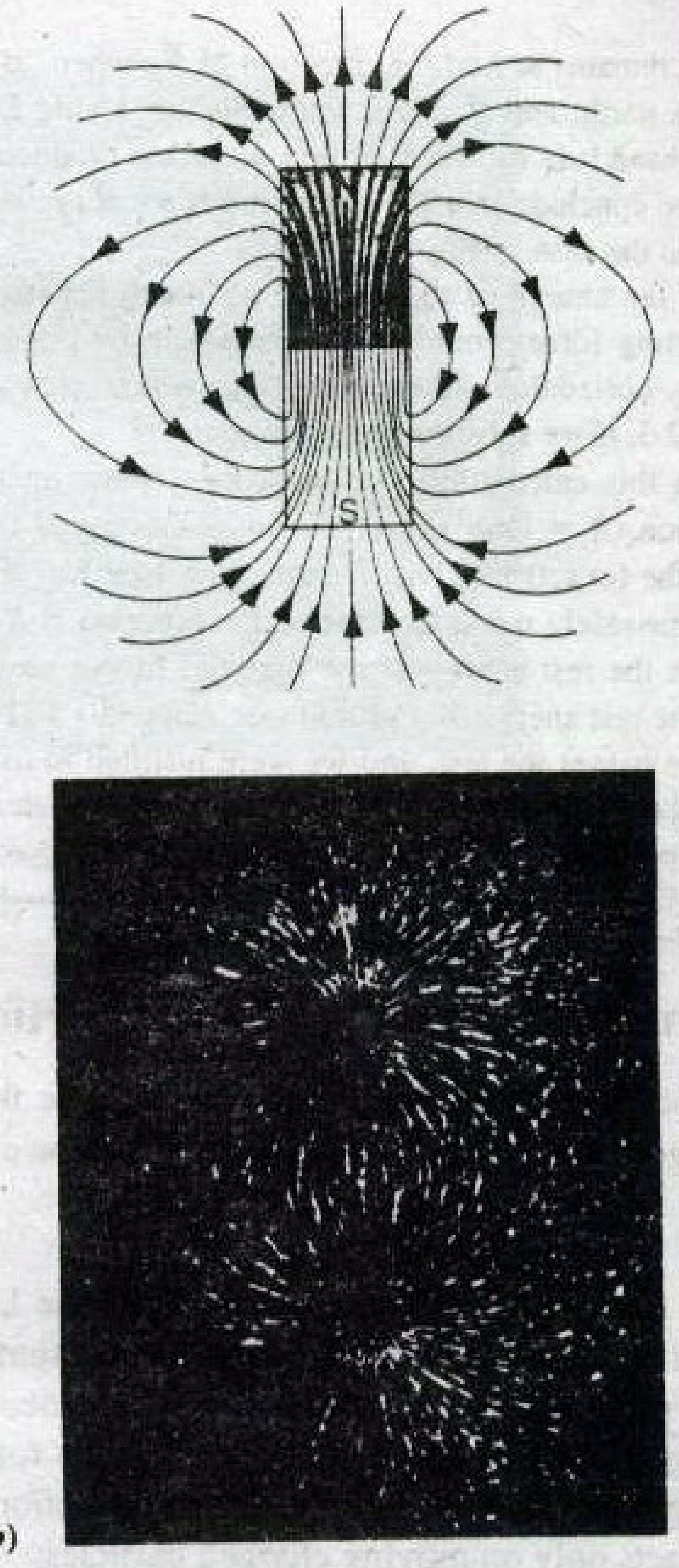


FIGURE 32-6. (*a*) The magnetic field lines for a bar magnet. The lines form closed loops, leaving the magnet at its north pole and entering at its south pole. (*b*) The field lines can be made visible by sprinkling iron filings on a sheet of paper covering a bar magnet.

SAMPLE PROBLEM 32-1. A uniform magnetic field $\vec{\mathbf{B}}$, with magnitude 1.2 mT, points vertically upward throughout the volume of the room in which you are sitting (Fig. 32-7). A proton with a kinetic energy of 5.3 MeV moves horizontally to the north through a certain point in the room. What magnetic deflecting force acts on the proton as it passes through this point? The proton mass is 1.67×10^{-27} kg.

Solution The magnetic deflecting force depends on the speed of the proton, which we can find from $K = \frac{1}{2}mv^2$. Solving for v, we find

$$v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{(2)(5.3 \text{ MeV})(1.60 \times 10^{-13} \text{ J/MeV})}{1.67 \times 10^{-27} \text{ kg}}}$$

$$= 3.2 \times 10^7 \text{ m/s}.$$

Equation 32-2 then yields

$$\begin{aligned} F_B &= |q|vB \sin \phi \\ &= (1.60 \times 10^{-19} \text{ C})(3.2 \times 10^7 \text{ m/s})(1.2 \times 10^{-3} \text{ T})(\sin 90^\circ) \\ &= 6.1 \times 10^{-15} \text{ N}. \end{aligned}$$

This may seem like a small force, but it acts on a particle of small mass, producing a large acceleration; namely,

$$a = \frac{F_B}{m} = \frac{6.1 \times 10^{-15} \text{ N}}{1.67 \times 10^{-27} \text{ kg}} = 3.7 \times 10^{12} \text{ m/s}^2.$$

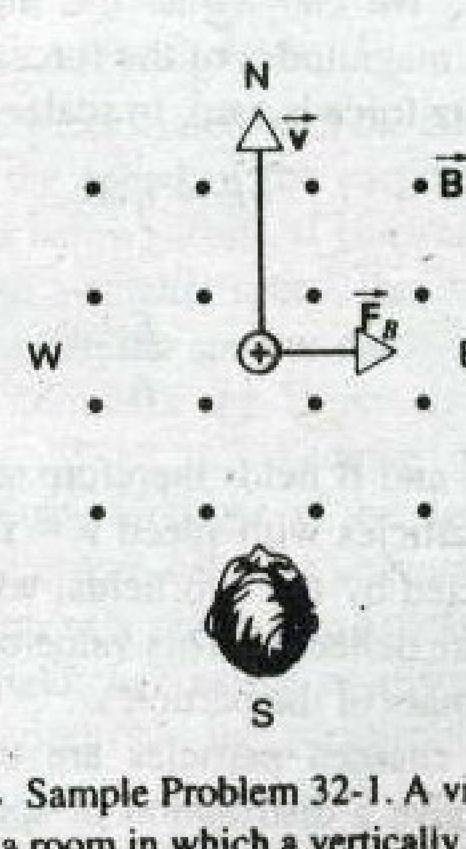


FIGURE 32-7. Sample Problem 32-1. A view from above of a student sitting in a room in which a vertically upward magnetic field deflects a moving proton toward the east. (The dots, which represent the points of arrows, symbolize magnetic field vectors pointing out of the page.)

It remains to find the direction of $\vec{F}_B$ when, as in Fig. 32-7, $\vec{v}$ points north, and $\vec{B}$ points vertically up. Using Eq. 32-3 and the right-hand rule for the direction of vector products (see Appendix H), we conclude that the deflecting force $\vec{F}_B$ must point horizontally to the east, as Fig. 32-7 shows.

If the charge of the particle had been negative, the magnetic deflecting force would have pointed in the opposite direction—that is, horizontally to the west. This is predicted automatically by Eq. 32-3, if we substitute $-q$ for q.

In this calculation, we used the (approximate) classical expression ($K = \frac{1}{2}mv^2$) for the kinetic energy of the proton rather than the (exact) relativistic expression (see Eq. 20-27). The criterion for safely using the classical expression is $K \ll mc^2$, where mc^2 is the rest energy of the particle. In this case $K = 5.3$ MeV, and the rest energy of a proton (see Appendix F) is 938 MeV. This proton passes the test, and we were justified in using the classical $K = \frac{1}{2}mv^2$ formula for the kinetic energy. In dealing with energetic particles, we must always be alert to this point.

Combined Electric and Magnetic Fields

If both an electric field $\vec{E}$ and a magnetic field $\vec{B}$ act on a charged particle, the total force on it can be expressed as

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}. \tag{32-4}$$

This force is called the *Lorentz force*. The Lorentz force is not a new kind of force; it is merely the sum of the electric and magnetic forces that may simultaneously act on a charged particle. The electric part of this force acts on any charged particle, whether at rest or in motion; the magnetic part acts only on moving charged particles.

One common application of the Lorentz force occurs when a beam of charged particles passes through a region in which the $\vec{E}$ and $\vec{B}$ fields are perpendicular to each other and to the velocity of the particles. If $\vec{E}$, $\vec{B}$, and $\vec{v}$ are oriented as shown in Fig. 32-8, then the electric force $\vec{F}_E = q\vec{E}$ is in the opposite direction to the magnetic force $\vec{F}_B = q\vec{v} \times \vec{B}$. We can adjust the magnetic and electric fields until the magnitudes of the forces are equal, in which case the Lorentz force is zero. In scalar terms,

$$qE = qvB \tag{32-5}$$

or

$$v = \frac{E}{B}. \tag{32-6}$$

The crossed $\vec{E}$ and $\vec{B}$ fields therefore serve as a *velocity selector:* only particles with speed $v = E/B$ pass through the region unaffected by the two fields, whereas particles with other speeds are deflected. This value of v is independent of the charge or mass of the particles.

Beams of charged particles are often prepared using methods that give a distribution of speeds (for example, a thermal distribution such as that of Fig. 22-6). Using a velocity selector we can isolate particles with a chosen speed from the beam. This principle was applied in 1897 by J. J. Thomson in his discovery of the electron and the measurement of its charge-to-mass ratio. Figure 32-9 shows a mod-

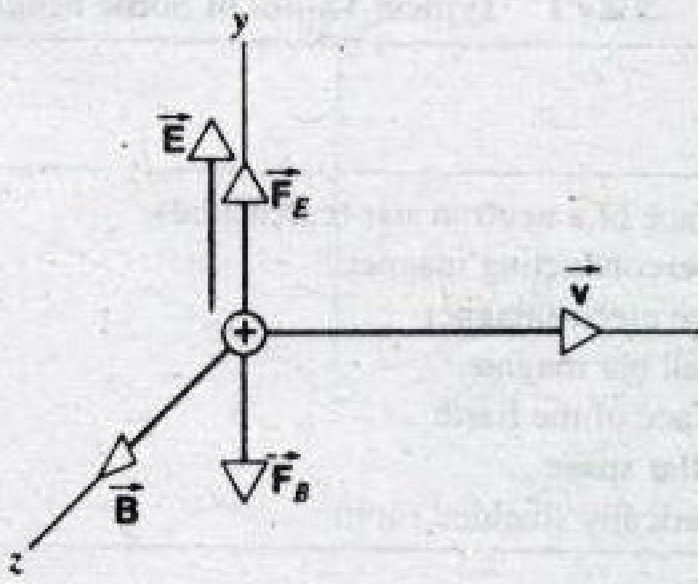


FIGURE 32-8. A positively charged particle, moving through a region in which there are electric and magnetic fields perpendicular to one another, experiences opposite electric and magnetic forces $\vec{F}_E$ and $\vec{F}_B$.

ern version of his apparatus. Thomson first measured the vertical deflection y of the beam when only the electric field was present. From Sample Problem 26-6, the deflection is

$$y = -\frac{qEL^2}{2mv^2}. \tag{32-7}$$

In this expression, as in Fig. 32-9, we take the positive y direction to be upward, and E is the *magnitude* of the electric field. The deflection y of a negatively charged particle is positive in Eq. 32-7 and Fig. 32-9.

Then the magnetic field was turned on and adjusted until the beam deflection was zero (equivalent to that measured with no fields present). In this case $v = E/B$, and solving for the charge-to-mass ratio with $q = -e$ gives

$$\frac{e}{m} = \frac{2yE}{B^2L^2}. \tag{32-8}$$

Thomson's value for e/m (expressed in modern units) was 1.7×10^{11} C/kg, in good agreement with the current value of $1.758820174 \times 10^{11}$ C/kg.

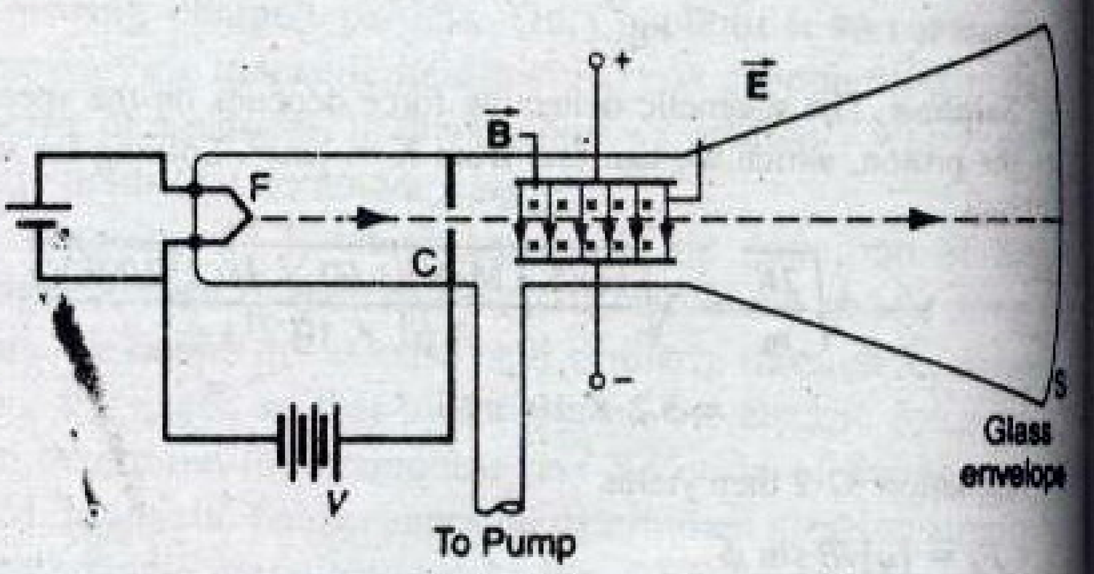


FIGURE 32-9. A modern version of J. J. Thomson's apparatus for measuring the charge-to-mass ratio of the electron. The filament F produces a beam of electrons with a distribution of speeds. The electric field $\vec{E}$ is set up by connecting a battery across the plate terminals. The magnetic field $\vec{B}$ is set up by means of current-carrying coils (not shown). The beam makes a visible spot where it strikes the screen S. (The crosses, which represent the tails of arrows, symbolize $\vec{B}$ vectors pointing into the page.)

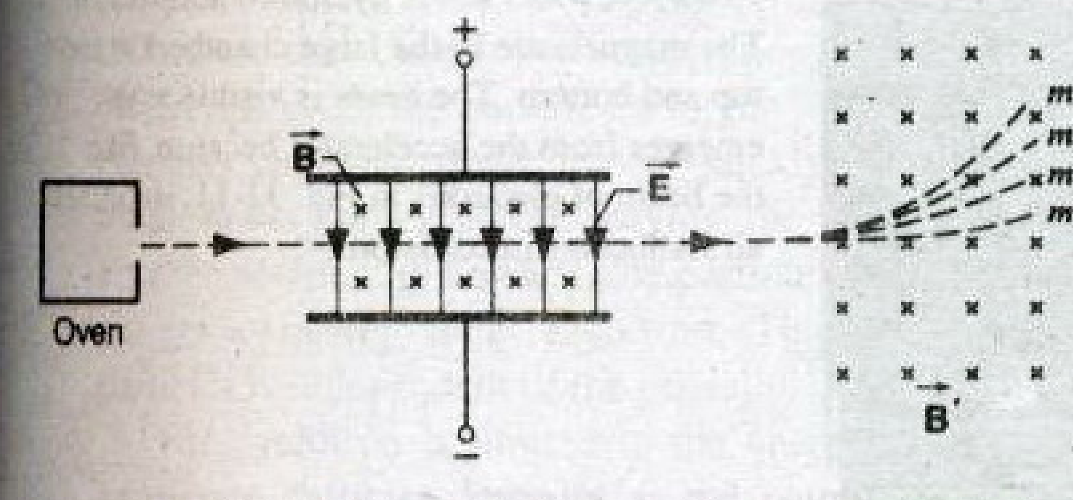


FIGURE 32-10. Schematic diagram of a mass spectrometer. A beam of ionized atoms having a mixture of different masses leaves an oven and enters a region of perpendicular $\vec{E}$ and $\vec{B}$ fields. Only those atoms with speeds $v = E/B$ pass undeflected through the region. Another magnetic field $\vec{B}'$ deflects the atoms along circular paths whose radii are determined by the masses of the atoms.

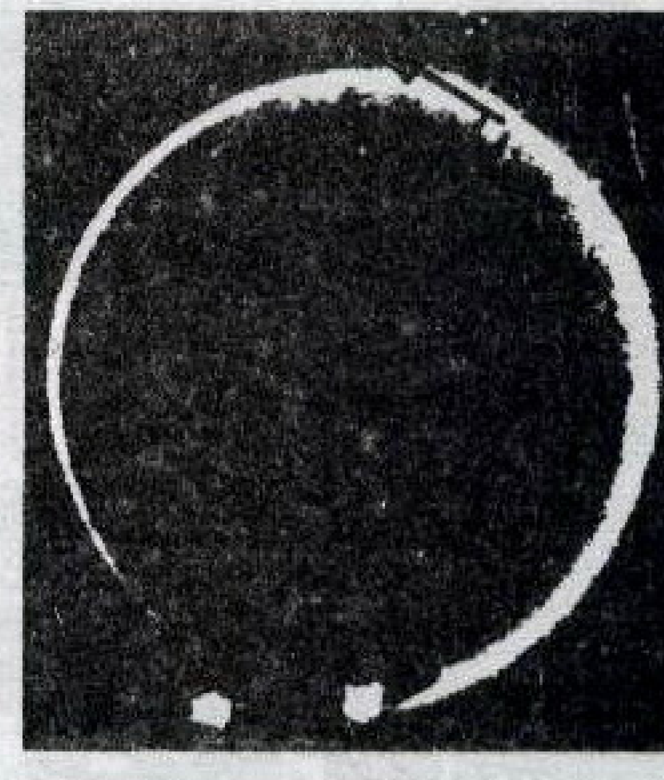


FIGURE 32-11. Electrons circulating in a chamber containing a gas at low pressure. The beam is made visible by collisions with the atoms of the gas. A uniform magnetic field $\vec{B}$, pointing out of the plane of the figure at right angles to it, fills the chamber. The magnetic force $\vec{F}_B$ is directed radially inward.

Another application of the velocity selector is in the mass spectrometer, a device for separating ions by mass. In this case a beam of ions, perhaps including species of differing masses, may be obtained from a vapor of the material heated in an oven (see Fig. 32-10). A velocity selector passes only ions of a particular speed, and when the resulting beam is then passed through another magnetic field, the paths of the particles are circular arcs (as we show in the next section) whose radii are determined by the momentum of the particles. Since all the particles have the same speed, the radius of the path is determined by the mass, and each different mass component in the beam follows a path of a different radius. These atoms can be collected and measured or else formed into a beam for subsequent experiments. See Problems 5 through 8 for other details on separating ions by their mass.

32-3 CIRCULATING CHARGES

Figure 32-11 shows a beam of electrons traveling through an evacuated chamber in which there is a uniform magnetic field $\vec{B}$ out of the plane of the figure. The magnetic deflecting force is the only important force that acts on the electrons. The beam clearly follows a circular path in the plane of the figure. Let us see how we can understand this behavior.

The magnetic deflecting force has two properties that affect the trajectories of charged particles: (1) it does not change the speed of the particles, and (2) it always acts perpendicular to the velocity of the particles. These are exactly the characteristics we require for a particle to move in a circle at constant speed, as in the case of the electrons in Fig. 32-11.

Since $\vec{B}$ is perpendicular to $\vec{v}$, the magnitude of the magnetic force can be written $|q|vB$, and Newton's second law with a centripetal acceleration of v^2/r gives

$$|q|vB = m\frac{v^2}{r} \qquad (32\text{-}9)$$

or

$$r = \frac{mv}{|q|B} = \frac{p}{|q|B}. \qquad (32\text{-}10)$$

Thus the radius of the path is determined by the momentum p of the particles, their charge, and the strength of the magnetic field. If the source of the electrons in Fig. 32-11 had projected them with a smaller speed, they would have moved in a circle of smaller radius.

The angular velocity of the circular motion is

$$\omega = \frac{v}{r} = \frac{|q|B}{m}, \qquad (32\text{-}11)$$

and the corresponding frequency is

$$f = \frac{\omega}{2\pi} = \frac{|q|B}{2\pi m}. \qquad (32\text{-}12)$$

Note that the frequency associated with the circular motion does not depend on the speed of the particle (as long as $v \ll c$, as we discuss below). Thus, if electrons in Fig. 32-11 were projected with a smaller speed, they would require the same time to complete the smaller circle that the faster electrons require to complete the larger circle. The frequency given by Eq. 32-12 is called the *cyclotron frequency*, because particles circulate at this frequency in a type of particle accelerator called a *cyclotron*. The frequency is characteristic of a particular particle moving in a particular magnetic field, just as the oscillating pendulum or the mass–spring system has its characteristic frequency.

The Cyclotron

A cyclotron (Fig. 32-12) is a device that accelerates beams of charged particles, which might be used in nuclear reaction experiments or for medical purposes. Figure 32-13 shows a schematic view of a cyclotron. It consists of two

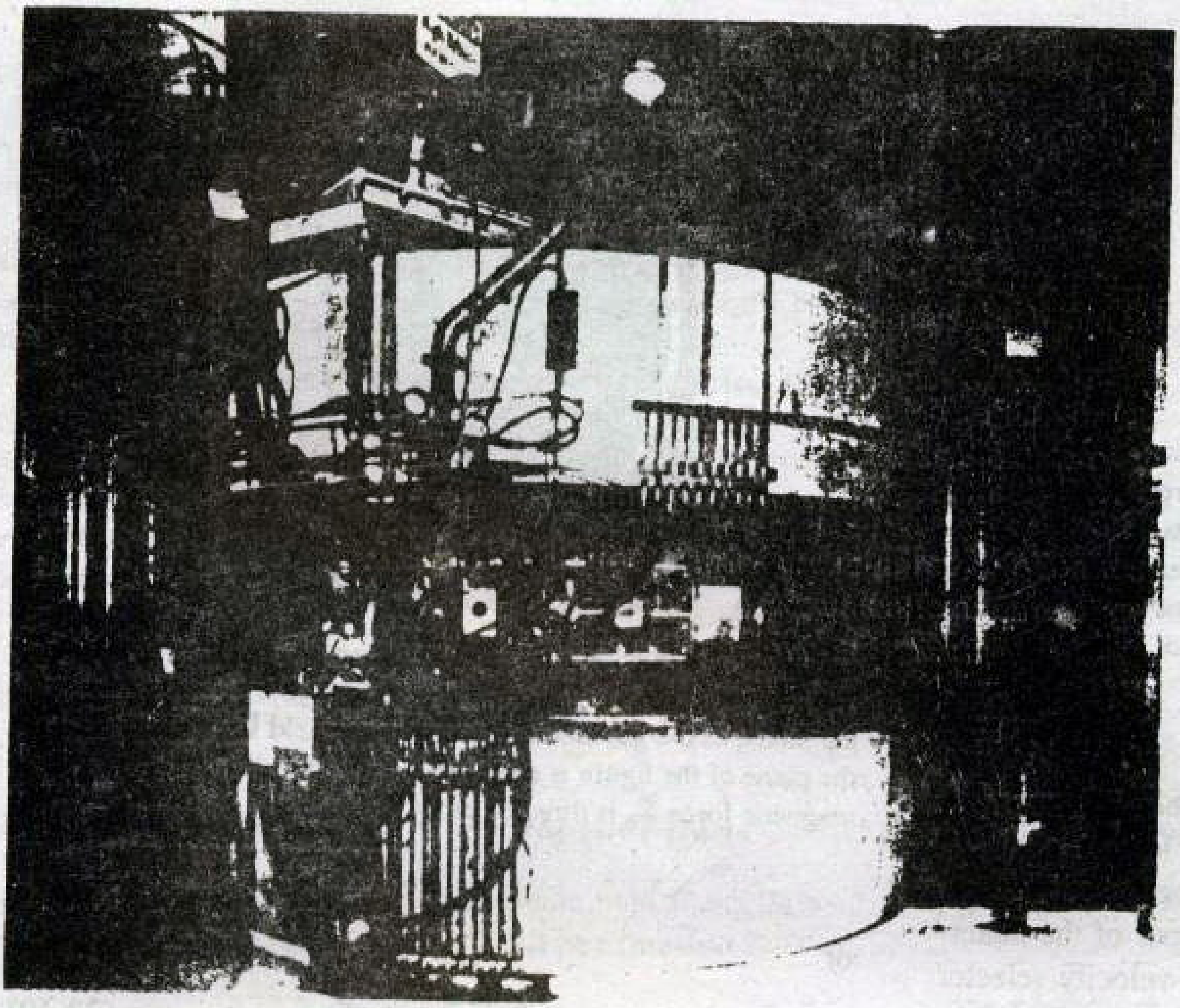


FIGURE 32-12. A cyclotron acceler... The magnets are in the large chambers at top and bottom. The beam is visible as it emerges from the accelerator because, like the beam of electrons in Fig. 32-11, it ionizes air molecules in collisions.

hollow metal D-shaped objects called *dees*. The dees are made of conducting material such as sheets of copper and are open along their straight edges. They are connected to an electric oscillator, which establishes an oscillating potential difference between the dees. A magnetic field is perpendicular to the plane of the dees. At the center of the instrument is a source that emits the ions we wish to accelerate.

When the ions are in the gap between the dees, they are accelerated by the potential difference between the dees. They then enter one of the dees, where they feel no electric field (because the electric field inside a conductor is zero), but the magnetic field (which is not shielded by the copper dees) bends their path into a semicircle. When the particles

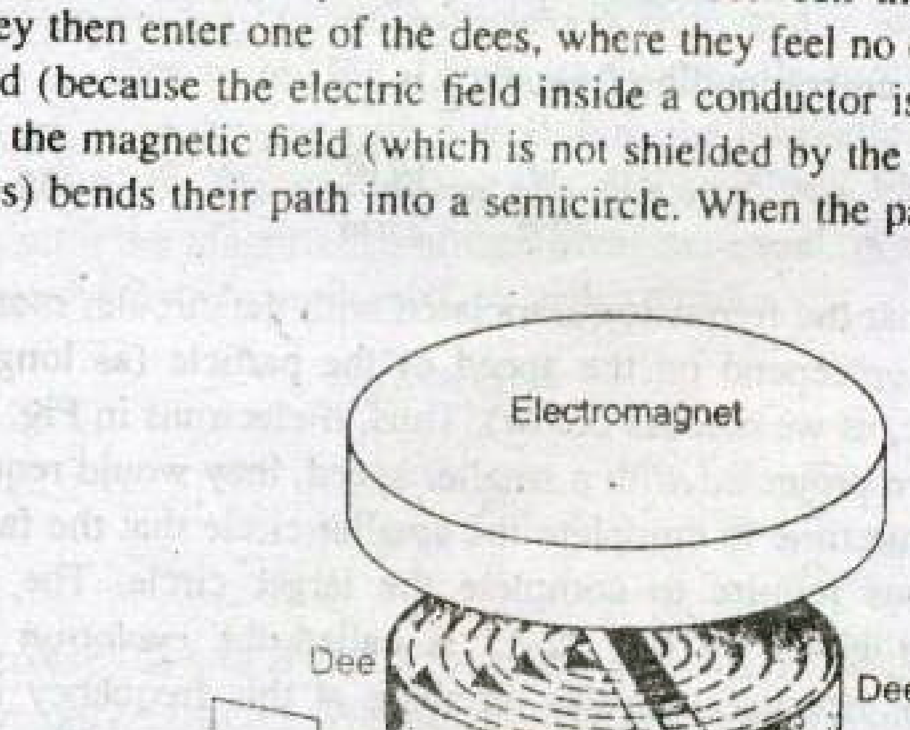


FIGURE 32-13. The elements of a cyclotron, showing the ion source S and the dees. The electromagnets provide a uniform vertical magnetic field. The particles spiral outward within the hollow dees, gaining energy every time they cross the gap between the dees.

next enter the gap, the oscillator has reversed the direction of the electric field, and the particles are again accelerated as they cross the gap. Moving with greater speed, they travel a path of greater radius, as required by Eq. 32-10. However, according to Eq. 32-12, *it takes them exactly the same amount of time to travel the larger semicircle;* this is the critical characteristic of the operation of the cyclotron. The frequency of the electric oscillator must be adjusted to be equal to the cyclotron frequency (determined by the magnetic field and the charge and mass of the particle to be accelerated according to Eq. 32-12); this equality of frequencies is called the *resonance condition.* If the resonance condition is satisfied, particles continue to accelerate in the gap and "coast" around the semicircles, gaining a small increment of energy in each circuit, until they are deflected out of the accelerator.

The final speed of the particles is determined by the radius R at which the particles leave the accelerator. From Eq. 32-10,

$$v = \frac{|q|BR}{m}, \tag{32-13}$$

and the corresponding (nonrelativistic) kinetic energy of the particles is

$$K = \tfrac{1}{2}mv^2 = \frac{q^2B^2R^2}{2m}. \tag{32-14}$$

Typical cyclotrons produce beams of protons with maximum energies in the range of 10 MeV. For a given mass, ions with higher electric charges emerge with energies that increase with the square of the charge.

It is somewhat surprising that the energy in Eq. 32-14 depends on the magnetic field, which does not accelerate the particles, but does not depend on the electric potential difference that causes the acceleration. A larger potential difference gives the particles a larger "kick" with each cycle; the radius increases more quickly, and the particles make fewer cycles before leaving the accelerator. With a smaller potential difference, the particles make more cycles but get a smaller "kick" each time. Thus the energy of the particles is independent of the potential difference.

The cyclotron is limited in the energy to which it can accelerate particles, because as the particles reach greater speeds the classical expressions for momentum ($p = mv$ in Eq. 32-10) and kinetic energy ($K = \frac{1}{2}mv^2$ in Eq. 32-14) are no longer valid, and we must use the corresponding relativistic expressions from Chapter 20. As the particles begin to move at speeds that approach the speed of light, they take a longer time to travel the circular path and so the resonance condition is lost. In practical terms, about 40 MeV is the highest proton kinetic energy that can be achieved with a conventional cyclotron, which might have a radius of the order of 1 m.

To achieve the higher energies that are needed for current research in particle physics, an accelerator of a different design, called a *synchrotron*, is used. Even though the synchrotron does not rely on the cyclotron resonance condition, it uses magnetic fields to keep particles moving in a circular path in which they can be repeatedly accelerated by an electric field. In the design of a synchrotron, both the oscillator frequency and the strength of the magnetic field are varied as bunches of particles are accelerated in a circular path of constant radius. To achieve an energy of 1 TeV ($=10^6$ MeV), as in the case of the Fermi National Accelerator Laboratory near Chicago, a path radius of 1 km is required; the protons make about 400,000 revolutions around the 6.3-km circumference in about 10 s.

The Magnetic Mirror

A nonuniform magnetic field can be used to trap a charged particle in a region of space. Figure 32-14 shows a schematic view of the operation of such a magnetic mirror.

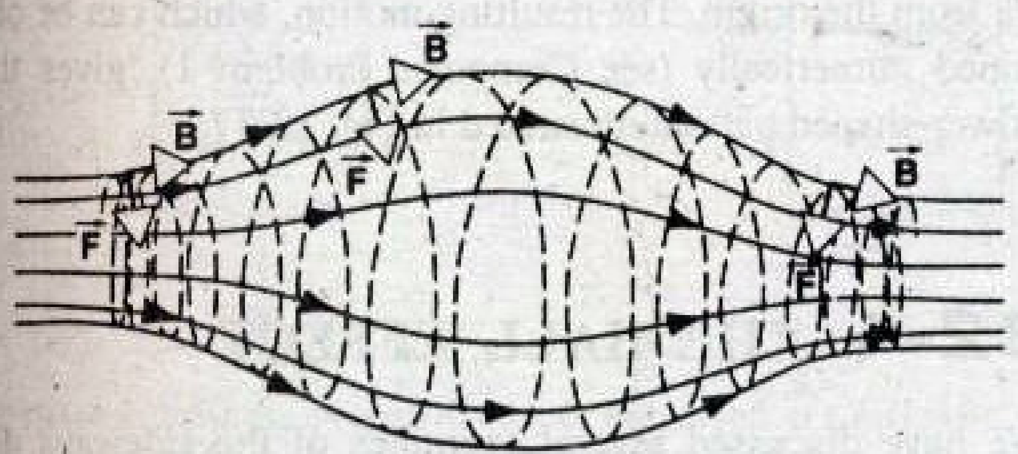


FIGURE 32-14. A charged particle spiraling in a nonuniform magnetic field. The field is greater at the left and right ends of the region than it is at the center. Particles can be trapped, spiraling back and forth between the strong-field regions at the ends. Note that the magnetic force vectors at each end of this "magnetic bottle" have components that point toward the center; it is these force components that serve to confine the particles.

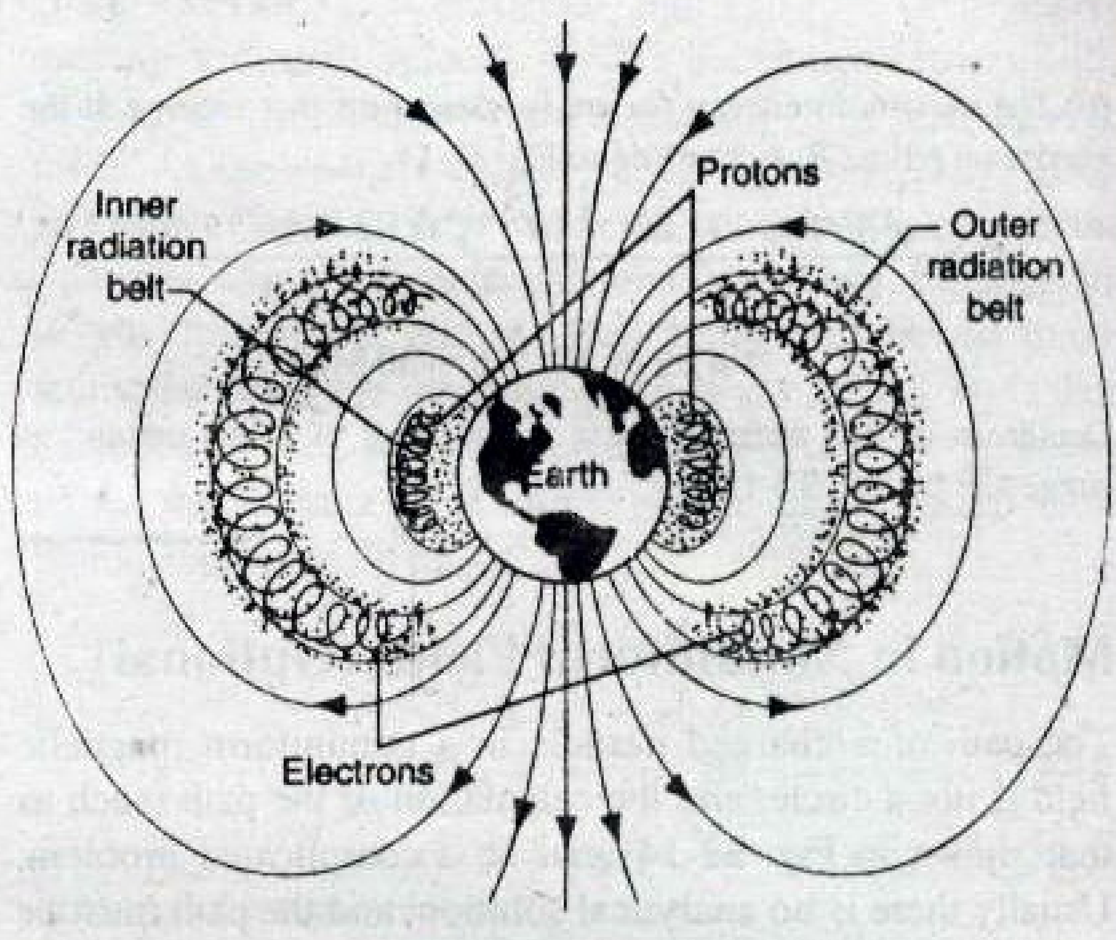


FIGURE 32-15. The Earth's magnetic field, showing protons and electrons trapped in the Van Allen radiation belts.

The charged particles tend to move in circles about the field direction. Suppose they also are drifting laterally, say, to the right in Fig. 32-14. The motion is therefore that of a helix, like a coiled spring. The field increases near the ends of the "magnetic bottle," and the force has a small component pointing toward the center of the region, which reverses the direction of the motion of the particles and causes them to spiral in the opposite direction, until they are eventually reflected from the opposite end. The particles continue to travel back and forth, confined to the space between the two high-field regions. Such an arrangement is used to confine the hot ionized gases (called *plasmas*) that are used in research into controlled thermonuclear fusion (see Section 51-8).

A similar phenomenon occurs in the Earth's magnetic field, as shown in Fig. 32-15. Electrons and protons are trapped in different regions of the Earth's field and spiral back and forth between the high-field regions near the poles in a time of a few seconds. These fast particles are responsible for the so-called Van Allen radiation belts that surround the Earth.

SAMPLE PROBLEM 32-2. A particular cyclotron is designed with dees of radius R = 75 cm and with magnets that can provide a field of 1.5 T. (*a*) To what frequency should the oscillator be set if deuterons are to be accelerated? (*b*) What is the maximum energy of deuterons that can be obtained?

Solution (*a*) A deuteron is a nucleus of heavy hydrogen, with a charge $q = +e$ and a mass of 3.34×10^{-27} kg, about twice the mass of ordinary hydrogen. Using Eq. 32-12 we can find the frequency:

$$f = \frac{|q|B}{2\pi m} = \frac{(1.60 \times 10^{-19}\ \text{C})(1.5\ \text{T})}{2\pi(3.34 \times 10^{-27}\ \text{kg})}$$
$$= 1.1 \times 10^7\ \text{Hz} = 11\ \text{MHz}.$$

(*b*) The maximum energy occurs for deuterons that emerge at the maximum radius R. According to Eq. 32-14,

$$K = \frac{q^2B^2R^2}{2m} = \frac{(1.60 \times 10^{-19}\ \mathrm{C})^2(1.5\ \mathrm{T})^2(0.75\ \mathrm{m})^2}{2(3.34 \times 10^{-27}\ \mathrm{kg})}$$
$$= 4.85 \times 10^{-12}\ \mathrm{J} = 30\ \mathrm{MeV}.$$

Deuterons of this energy have a range in air of a few meters, as suggested by Fig. 32-12.

Motion in Nonuniform Fields (Optional)

The path of a charged particle in a nonuniform magnetic field is not a circle, and the calculation of the path (such as that shown in Fig. 32-14) can be a complicated problem. Usually there is no analytical solution, and the path must be calculated numerically, in analogy with the path of a projectile including air resistance (Chapter 4). Figure 32-16 illustrates why the path in a nonuniform field is no longer a circle. The particle is initially at point P_0, where its velocity $\vec{\mathbf{v}}_0$ is along the y direction. We assume that the field is in the z direction and that its strength increases as x and y increase. Initially the field is $\vec{\mathbf{B}}_0$, and the resulting force on the particle is $\vec{\mathbf{F}}_0$ in the x direction. This force gives the particle a velocity increment in the x direction, and an instant later the particle is at point P_1 moving with velocity $\vec{\mathbf{v}}_1$. The magnetic field $\vec{\mathbf{B}}_1$ at this point has a larger magnitude, and the force $\vec{\mathbf{F}}_1$ is also larger. For circular motion, the force must have the same magnitude at all locations, so the particle's path is clearly not a circle.

We can solve this problem numerically if we know how $\vec{\mathbf{B}}$ varies in magnitude (and possibly in direction) at all points where the particle might be located. We begin by writing the complete expression for the force as a vector (cross) product (see Appendix H):

$$\begin{aligned}\vec{\mathbf{F}}_B &= q(\vec{\mathbf{v}} \times \vec{\mathbf{B}}) \\ &= q(v_yB_z - v_zB_y)\hat{\mathbf{i}} + q(v_zB_x - v_xB_z)\hat{\mathbf{j}} \\ &\quad + q(v_xB_y - v_yB_x)\hat{\mathbf{k}}.\end{aligned} \qquad (32\text{-}15)$$

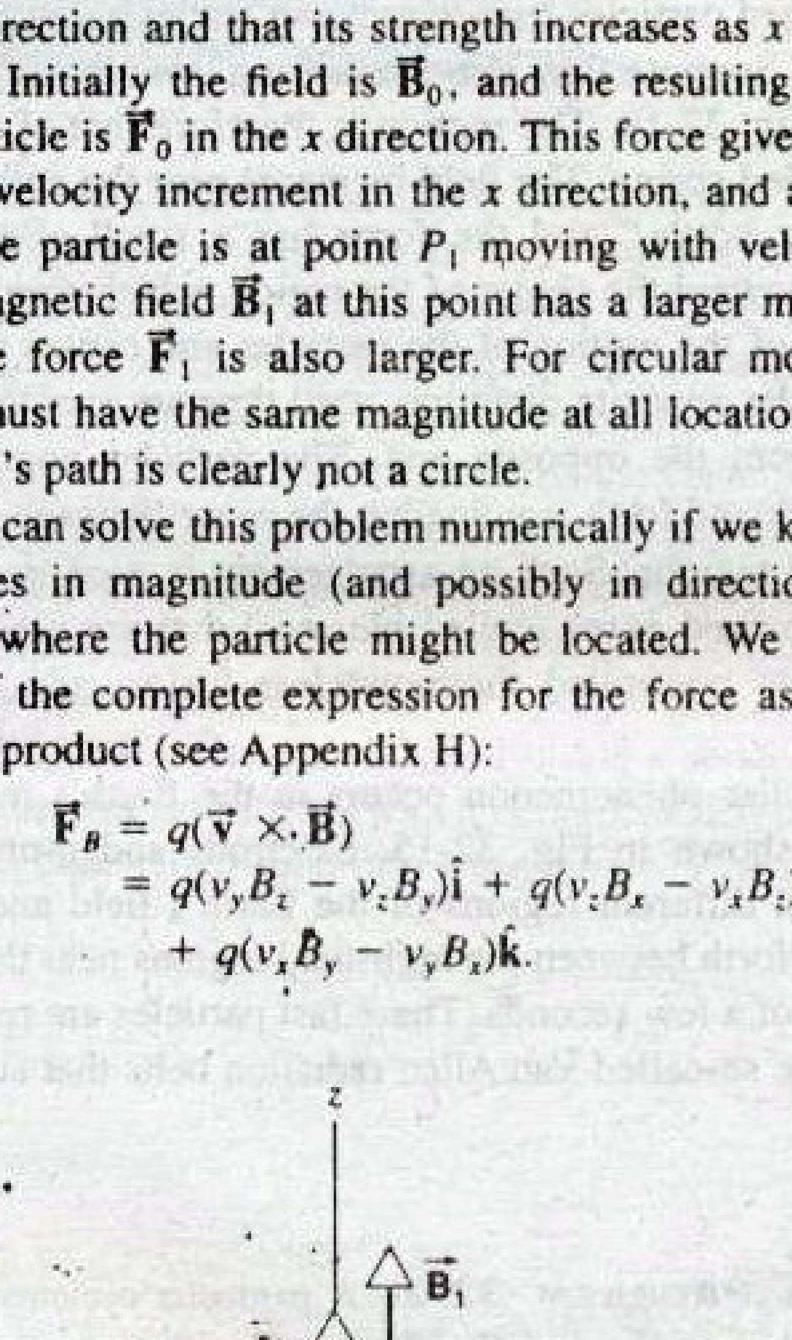


FIGURE 32-16. In a nonuniform magnetic field, the magnitude of the force varies (here as the particle moves from P_0 to P_1), and so the resulting path is not a circle.

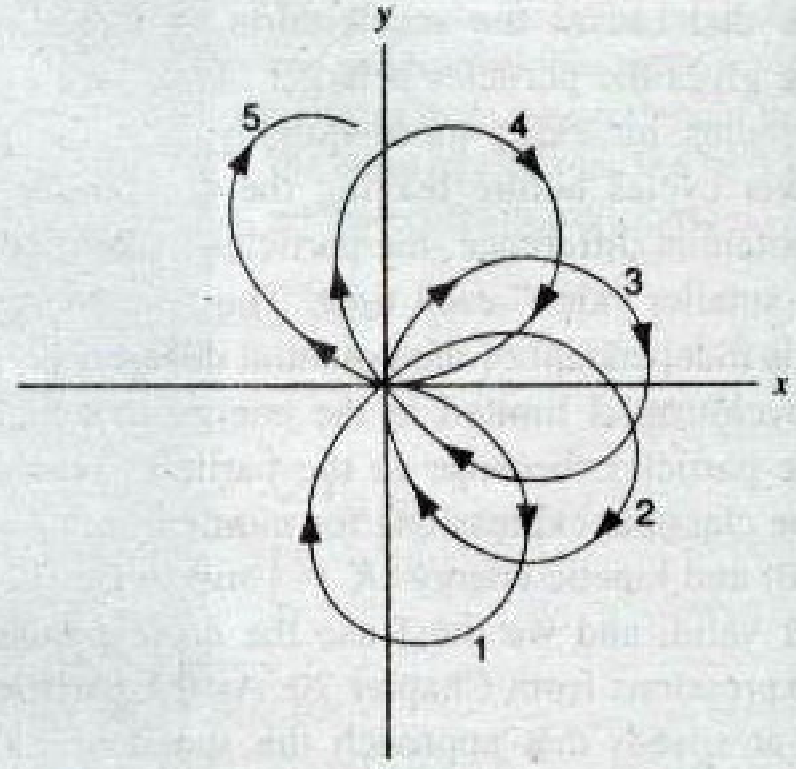


FIGURE 32-17. The path in the xy plane of a particle initially at the origin and moving in the x direction, subject to the magnetic field given by Eq. 32-17. The loops are numbered in the order they are traveled by the particle.

Using Newton's second law, $\vec{\mathbf{F}} = m\vec{\mathbf{a}} = m(a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}})$, we can equate the corresponding vector components to obtain the equations of motion, which can be solved for the path. For example, in the situation shown in Fig. 32-16, $\vec{\mathbf{B}}$ has only a z component ($B_x = B_y = 0$). In this case the equations of motion simplify to

$$\begin{aligned}F_{Bx} &= q(v_yB_z - v_zB_y) = qv_yB_z = ma_x = m\,dv_x/dt \\ F_{By} &= q(v_zB_x - v_xB_z) = -qv_xB_z = ma_y = m\,dv_y/dt \\ F_{Bz} &= q(v_xB_y - v_yB_x) = 0 = ma_z = m\,dv_z/dt.\end{aligned} \qquad (32\text{-}16)$$

(If $v_z = 0$ initially, then $v_z = 0$ at all times, because $a_z = 0$.) These expressions must be evaluated at every point of the trajectory, because the values of v_x, v_y, and B_z vary from point to point. Usually a computer is used to obtain a numerical solution for the path. For example, suppose the field varies as

$$B_z = B_0\left(1 + \frac{\sqrt{x^2 + y^2}}{R}\right), \qquad (32\text{-}17)$$

where $R = mv/qB_0$ is the radius of the path in a uniform field B_0. This field increases with the distance of the particle from the origin. The resulting motion, which can be obtained numerically (see Computer Problem 1), gives the flower-shaped pattern illustrated in Fig. 32-17. ■

32-4 THE HALL EFFECT

We have discussed several examples of the sideways deflecting force exerted on moving charged particles by a magnetic field. So far we have considered only individual particles or beams of particles moving otherwise freely. In 1879, Edwin Hall showed that the moving conduction electrons in a conductor can also be deflected by a magnetic field. The *Hall effect* provides a way to determine both the sign and the density of the charge carriers.

Consider a flat strip of material of width w carrying a current i, as shown in Fig. 32-18. The direction of the current i is the conventional one, opposite to the direction of motion of the electrons. A uniform magnetic field $\vec{\mathbf{B}}$ is established perpendicular to the plane of the strip, such as by placing the strip between the poles of an electromagnet. The charge carriers (electrons, for instance) experience a magnetic deflecting force $\vec{\mathbf{F}}_B = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$, as shown in the figure, and move to the right side of the strip. Note that positive charges moving in the direction of i experience a deflecting force in the *same* direction.

The buildup of charge along the right side of the strip (and a corresponding deficiency of charge of that sign on the opposite side of the strip), which is the Hall effect, produces an electric field $\vec{\mathbf{E}}_H$ across the strip, as shown in Fig. 32-18*b*. Equivalently, a potential difference $\Delta V_H = E_H w$, called the *Hall potential difference* (or Hall voltage), exists across the strip. We can measure ΔV_H by connecting the leads of a voltmeter to points x and y of Fig. 32-18. As we show below, the sign of ΔV_H gives the sign of the charge carriers, and the magnitude of ΔV_H gives their density (number per unit volume). If the charge carriers are electrons, for example, an excess of negative charges builds up on the right side of the strip, and point y is at a lower potential than point x. This may seem like an obvious conclusion in the case of metals; however, keep in mind that Hall's work was done nearly 20 years before Thomson's discovery of the electron, and the nature of electrical conduction in metals was not at all obvious at that time.

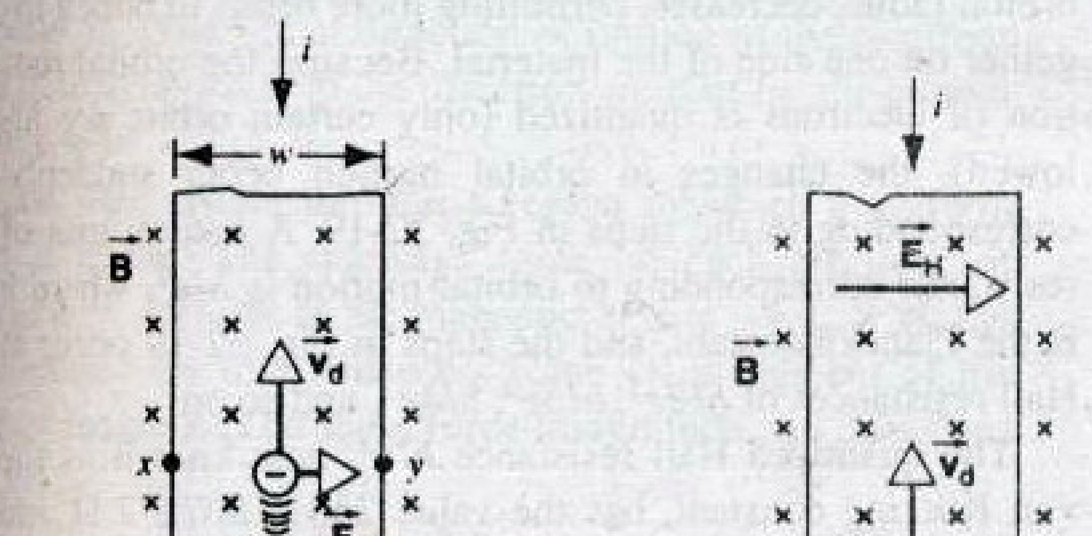


FIGURE 32-18. A strip of copper immersed in a magnetic field $\vec{\mathbf{B}}$ carries a current i. (*a*) The situation just after the magnetic field has been turned on, and (*b*) the situation at equilibrium, which quickly follows. Negative charges pile up on the right side of the strip, leaving uncompensated positive charges on the left. Point x is at a higher potential than point y.

Let us assume that conduction in the material is due to charge carriers of a particular sign (positive or negative) moving with drift velocity $\vec{\mathbf{v}}_d$. As the charge carriers drift, they are deflected to the right in Fig. 32-18 by the magnetic force. As the charges collect on the right side, they set up an electric field that acts inside the conductor to oppose the sideways motion of additional charge carriers. Equilibrium is quickly reached, and the Hall voltage reaches its maximum; the sideways magnetic force ($\vec{\mathbf{F}}_B = q\vec{\mathbf{v}}_d \times \vec{\mathbf{B}}$) is then balanced by the sideways electric force ($q\vec{\mathbf{E}}_H$). In vector terms, the Lorentz force on the charge carriers under these circumstances is zero:

$$q\vec{\mathbf{E}}_H + q\vec{\mathbf{v}}_d \times \vec{\mathbf{B}} = 0, \tag{32-18}$$

or

$$\vec{\mathbf{E}}_H = -\vec{\mathbf{v}}_d \times \vec{\mathbf{B}}. \tag{32-19}$$

Since $\vec{\mathbf{v}}_d$ and $\vec{\mathbf{B}}$ are at right angles, we can write Eq. 32-19 in terms of magnitudes as

$$E_H = v_d B. \tag{32-20}$$

From Eq. 29-6 we can write the drift speed as $v_d = j/ne$, where j is the current density in the strip and n is the density of charge carriers. The current density j is the current i per unit cross-sectional area A of the strip. If t is the thickness of the strip, then its cross-sectional area A can be written as wt. Substituting $\Delta V_H/w$ for the electric field E_H, we obtain

$$\frac{\Delta V_H}{w} = v_d B = \frac{j}{ne} B = \frac{i}{wtne} B$$

or, solving for the density of charge carriers,

$$n = \frac{iB}{et\,\Delta V_H}. \tag{32-21}$$

From a measurement of the magnitude of the Hall potential difference ΔV_H, we can find the number density of the charge carriers. Table 32-2 shows a summary of Hall effect data for several metals and semiconductors. For some

TABLE 32-2 Hall Effect Results for Selected Materials

Material	n ($10^{28}/m^3$)	Sign of ΔV_H	Number per atom[a]
Na	2.5	−	0.99
K	1.5	−	1.1
Cu	11	−	1.3
Ag	7.4	−	1.3
Al	21	−	3.5
Sb	0.31	−	0.09
Be	2.6	+	2.2
Zn	19	+	2.9
Si (pure)	1.5×10^{-12}	−	3×10^{-13}
Si (typical *n*-type)	10^{-7}	−	2×10^{-8}

[a] The number of charge carriers per atom of the material as determined from the number per unit volume and the density and molar mass of the material.

monovalent metals (Na, K, Cu, Ag) the Hall effect indicates that each atom contributes approximately one free electron to the conduction. For other metals, the number of electrons can be greater than one per atom (Al) or less than one per atom (Sb). For some metals (Be, Zn), the Hall potential difference shows that the charge carriers have a *positive* sign. In this case the conduction is dominated by *holes*, unoccupied energy levels in the valence band (see Chapter 49). The holes correspond to the absence of an electron and thus behave like positive charge carriers moving through the material. For some materials, semiconductors in particular, there may be substantial contributions from both electrons and holes, and the simple interpretation of the Hall effect in terms of free conduction by one type of charge carrier is not sufficient. In this case we must use more detailed calculations based on quantum theory.

SAMPLE PROBLEM 32-3. A strip of copper 150 μm thick is placed in a magnetic field $B = 0.65$ T perpendicular to the plane of the strip, and a current $i = 23$ A is set up in the strip. What Hall potential difference ΔV_H would appear across the width of the strip if there were one charge carrier per atom?

Solution In Sample Problem 29-3, we calculated the number of charge carriers per unit volume for copper, assuming that each atom contributes one electron, and we found $n = 8.49 \times 10^{28}$ electrons/m^3. From Eq. 32-21 then,

$$\Delta V_H = \frac{iB}{net}$$
$$= \frac{(23\text{ A})(0.65\text{ T})}{(8.49 \times 10^{28}\text{ m}^{-3})(1.60 \times 10^{-19}\text{ C})(150 \times 10^{-6}\text{ m})}$$
$$= 7.3 \times 10^{-6}\text{ V} = 7.3\ \mu\text{V}.$$

This potential difference, though small, is readily measurable.

The Quantized Hall Effect* (Optional)

Let us rewrite Eq. 32-21 as

$$\frac{\Delta V_H}{i} = \frac{1}{etn} B. \qquad (32\text{-}22)$$

The quantity on the left has the dimension of resistance (voltage divided by current), although it is not a resistance in the conventional sense. It is commonly called the *Hall resistance*. We can determine the Hall resistance by measuring the Hall voltage ΔV_H in a material carrying a current i.

Equation 32-22 shows that the Hall resistance is expected to increase linearly with the magnetic field B for a particular sample of material (in which n and t are constants). A plot of the Hall resistance against B should be a straight line.

*See "The Quantized Hall Effect," by Bertrand I. Halperin, *Scientific American*, April 1986, p. 52.

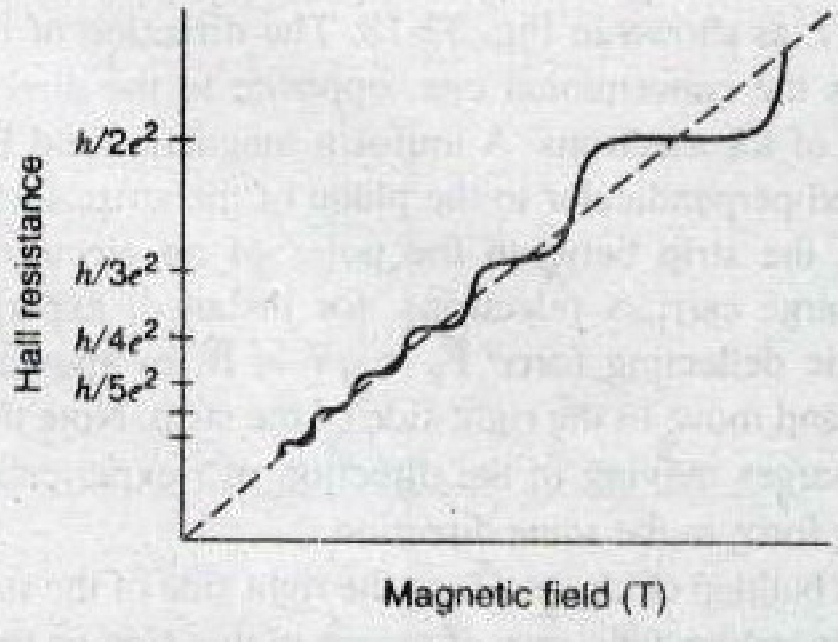


FIGURE 32-19. The quantized Hall effect. The dashed line shows the expected classical behavior. The steps show the quantum behavior.

In experiments done in 1980, German physicist Klaus von Klitzing discovered that, at high magnetic fields and low temperatures (about 1 K), the Hall resistance did not increase linearly with the field; instead, the plot showed a series of "stair steps," as shown in Fig. 32-19. This effect has become known as the *quantized Hall effect*, and von Klitzing was awarded the 1985 Nobel Prize in physics for his discovery.

The explanation for this effect involves the circular paths in which electrons are forced to move by the field. Quantum mechanics prevents the electron orbits of neighboring atoms from overlapping. As the field increases, the orbital radius decreases, permitting more orbits to bunch together on one side of the material. Because the orbital motion of electrons is quantized (only certain orbits are allowed), the changes in orbital motion occur suddenly, corresponding to the steps in Fig. 32-19. A natural unit of resistance corresponding to orbital motion is h/e^2, where h is the Planck constant, and the steps in Fig. 32-19 occur at Hall resistances of $h/2e^2$, $h/3e^2$, $h/4e^2$, and so on.

The quantized Hall resistance h/e^2, now known as the von Klitzing constant, has the value 25812.807572 Ω and has been determined to a precision of better than 1 part in 10^8, so the quantized Hall effect has provided a new standard for resistance. This standard, which can be duplicated exactly in laboratories around the world, became the new representation for the ohm in 1990. ■

32-5 THE MAGNETIC FORCE ON A CURRENT-CARRYING WIRE

A current is a collection of moving charges. Because a magnetic field exerts a sideways force on a moving charge, it should also exert a sideways force on a wire carrying a current. That is, a sideways force is exerted on the conduction electrons in the wire, but since the electrons cannot escape sideways, the force must be transmitted to the wire itself. Figure 32-20 shows a wire that passes through a region in which a magnetic field $\vec{B}$ exists. When the wire carries

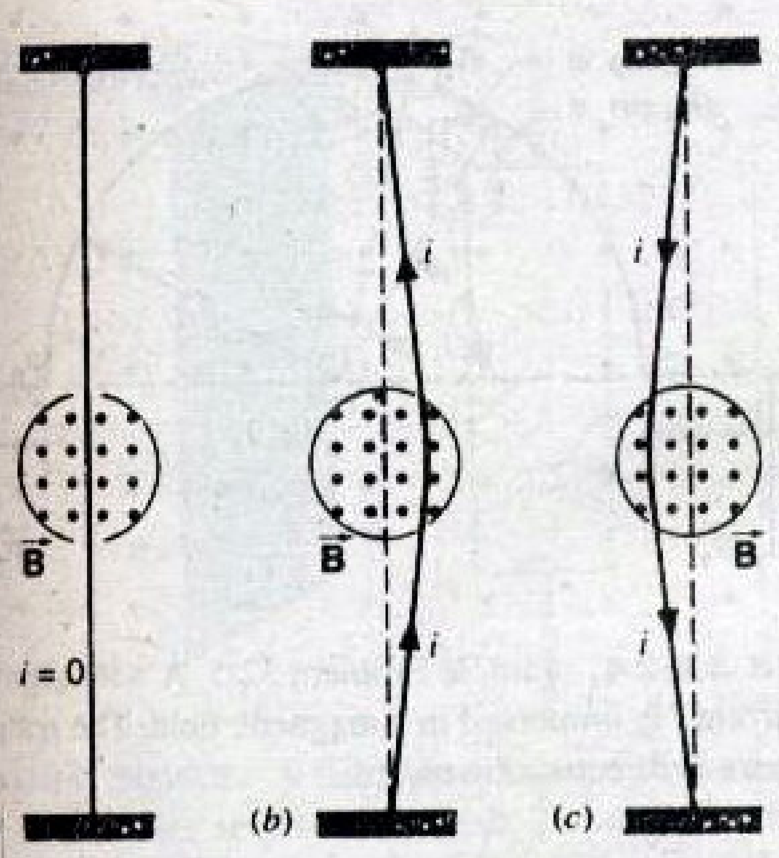


FIGURE 32-20. A flexible wire passes between the poles of a magnet. (*a*) There is no current in the wire. (*b*) A current is established in the wire. (*c*) The current is reversed.

no current (Fig. 32-20*a*), it experiences no deflection. When a current is carried by the wire, it deflects (Fig. 32-20*b*); when the current is reversed (Fig. 32-20*c*), the deflection reverses. The deflection also reverses when the field $\vec{B}$ is reversed.

To understand this effect, we consider the individual charges flowing in a wire (Fig. 32-21). We use the free-electron model (Section 29-3) for current in a wire, assuming the electrons to move with a constant velocity, the drift velocity $\vec{v}_d$. The actual direction of motion of the electrons is of course opposite to the direction we take for the current i in the wire.

The wire passes through a region in which a uniform field $\vec{B}$ exists. The sideways force on each electron (of charge $q = -e$) due to the magnetic field is $-e\vec{v}_d \times \vec{B}$. Let us consider the total sideways force on a segment of the wire of length L. The same force (magnitude and direction)

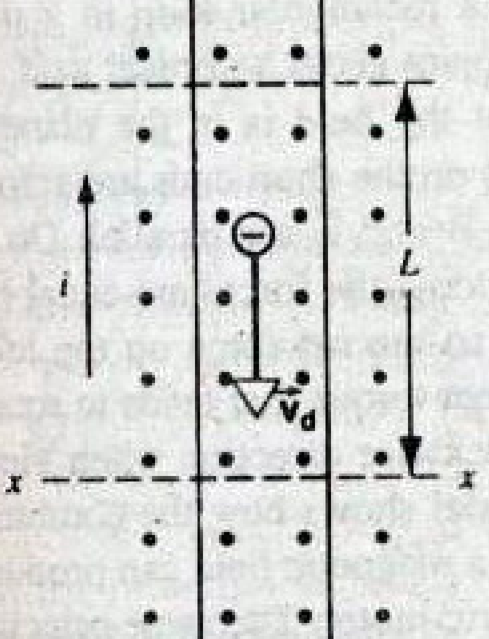


FIGURE 32-21. A close-up view of a length L of the wire of Fig. 32-20*b*. The current direction is upward, which means that electrons drift downward. A magnetic field emerges from the plane of the figure, so that the wire is deflected to the right.

acts on each electron in the segment, and the total force $\vec{F}_B$ on the segment is therefore equal to the number N of electrons times the force on each electron:

$$\vec{F}_B = -Ne\vec{v}_d \times \vec{B}. \qquad (32\text{-}23)$$

How many electrons are contained in that segment of wire? If n is the number density (number per unit volume) of electrons, then the total number N of electrons in the segment is nAL, where A is the cross-sectional area of the wire. Substituting into Eq. 32-23, we obtain

$$\vec{F}_B = -nALe\vec{v}_d \times \vec{B}. \qquad (32\text{-}24)$$

Equation 29-6 permits us to write Eq. 32-24 in terms of the current i. To preserve the vector relationship of Eq. 32-24, we define the vector $\vec{L}$ to be equal in magnitude to the length of the segment and to point in the direction of the current (opposite to the direction of electron flow). The vectors $\vec{v}_d$ and $\vec{L}$ have opposite directions, and we can write the scalar relationship $nALev_d = iL$ using vectors as

$$-nALe\vec{v}_d = i\vec{L}. \qquad (32\text{-}25)$$

Substituting Eq. 32-25 into Eq. 32-24, we obtain an expression for the force on the segment:

$$\vec{F}_B = i\vec{L} \times \vec{B}. \qquad (32\text{-}26)$$

Equation 32-26 is similar to Eq. 32-3 ($\vec{F}_B = q\vec{v} \times \vec{B}$), in that either can be taken to be the defining equation for the magnetic field. Figure 32-22 shows the vector relationship between $\vec{F}$, $\vec{L}$, and $\vec{B}$; compare with Fig. 32-5 to see the similarities between Eqs. 32-26 and 32-3.

If the field is uniform over the length of the wire segment and the direction of the current makes an angle ϕ with the field, then the magnitude of the force is (compare Eq. 32-2)

$$F_B = iLB \sin \phi. \qquad (32\text{-}27)$$

If $\vec{L}$ is parallel to $\vec{B}$, then the force is zero. If the segment is perpendicular to the direction of the field, the magnitude of the force is

$$F_B = iLB. \qquad (32\text{-}28)$$

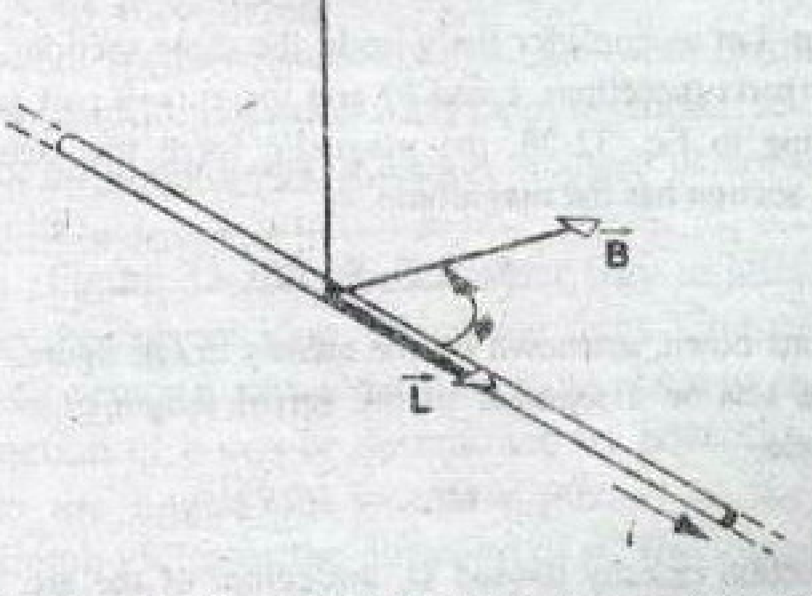


FIGURE 32-22. The magnetic force acting on a directed wire segment $\vec{L}$ that makes an angle ϕ with a magnetic field. Compare carefully with Fig. 32-5.

If the wire is not straight or the field is not uniform, we can imagine the wire to be broken into small segments of length dL; we make the segments small enough that they are approximately straight and the field is approximately uniform. The force on each segment can then be written

$$d\vec{F}_B = i\, d\vec{L} \times \vec{B}. \qquad (32\text{-}29)$$

We can find the total force on the segment of length L by doing a suitable integration over the length.

SAMPLE PROBLEM 32-4. A straight, horizontal segment of copper wire carries a current $i = 28$ A. What are the magnitude and direction of the magnetic field needed to "float" the wire—that is, to balance its weight? Its linear mass density is 46.6 g/m.

Solution Figure 32-23 shows the arrangement. For a length L of wire we have (see Eq. 32-28)

$$mg = iLB,$$

or

$$B = \frac{(m/L)g}{i} = \frac{(46.6 \times 10^{-3}\ \text{kg/m})(9.8\ \text{m/s}^2)}{28\ \text{A}}$$
$$= 1.6 \times 10^{-2}\ \text{T} = 16\ \text{mT}.$$

This is about 400 times the strength of the Earth's magnetic field.

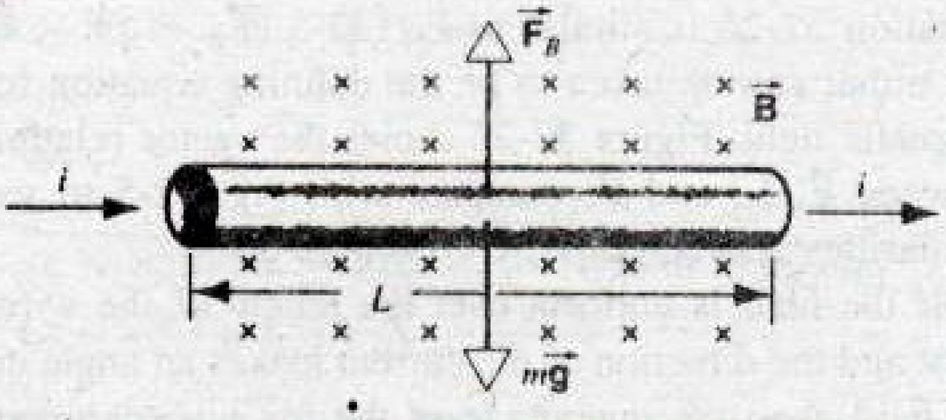


FIGURE 32-23. Sample Problem 32-4. A wire can be made to "float" in a magnetic field, with the upward magnetic force $\vec{F}_B$ balancing the downward pull of gravity. The magnetic field is into the plane of the page.

SAMPLE PROBLEM 32-5. Figure 32-24 shows a wire segment, placed in a uniform magnetic field $\vec{B}$ that points out of the plane of the figure. If the segment carries a current i, what resultant magnetic force acts on it?

Solution Let us consider the wire in the three sections—the two straight parts (secctions 1 and 3) and the curved part (section 2). According to Eq. 32-28, the magnetic force that acts on each straight section has the magnitude

$$F_1 = F_3 = iLB$$

and points down, as shown by the arrows in the figure. The force dF_B that acts on a segment of the arc of length $dL = R\,d\theta$ has magnitude

$$dF_B = iB\,ds = iB(R\,d\theta)$$

and direction radially toward O, the center of the arc. Note that only the downward component ($dF_B \sin\theta$) of this force element is effective. The horizontal component ($dF_B \cos\theta$) is cancelled by an

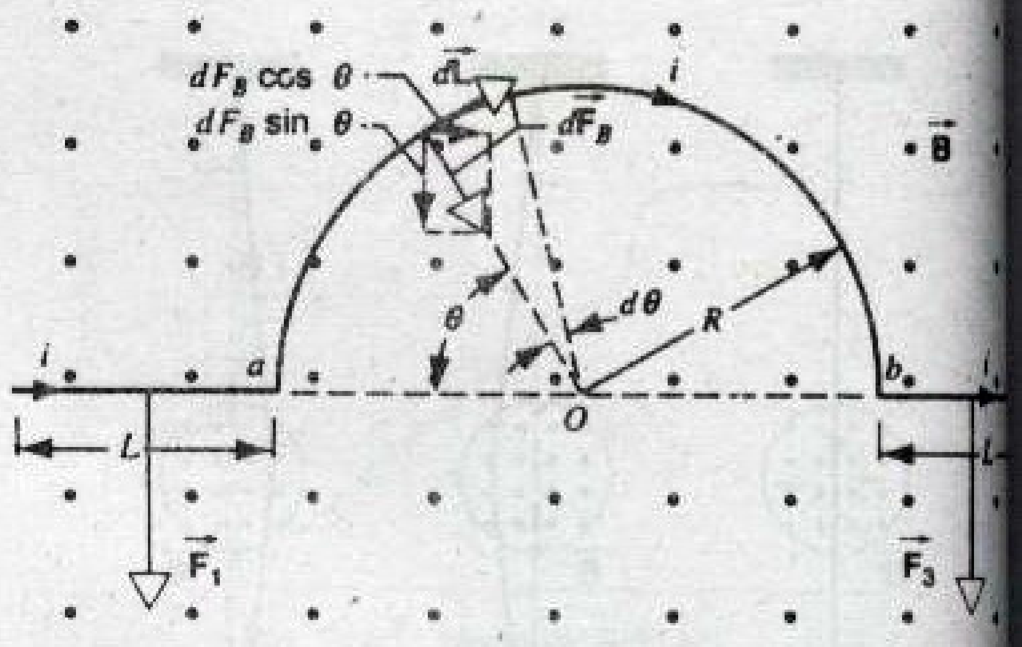


FIGURE 32-24. Sample Problem 32-5. A wire segment car ing a current i is immersed in a magnetic field. The resultant fo on the wire is directed downward.

oppositely directed horizontal component due to a symmetric located segment on the opposite side of the arc.

The total force on the central arc points down and is give

$$F_2 = \int_0^\pi dF_B \sin\theta = \int_0^\pi (iBR\,d\theta)\sin\theta$$
$$= iBR\int_0^\pi \sin\theta\,d\theta = 2iBR.$$

The resultant force on the entire wire is then

$$F = F_1 + F_2 + F_3 = iLB + 2iBR + iLB$$
$$= iB(2L + 2R).$$

The same force would also act on a wire similar to that of Fig. 32-24, with the central semicircular segment replaced by a segm of any shape (including a straight line) connecting points a and Can you convince yourself that this is so?

32-6 THE TORQUE ON A CURRENT LOOP

In an electric motor, a loop of wire carrying a current placed in a magnetic field. A simplified version is shown Fig. 32-25 for a rectangular loop in a uniform field. Th loop is free to pivot about a vertical axis. When the loop oriented so that the field is in the plane of the loop, th magnetic forces on the short ends are zero according to E 32-26, because $\vec{B}$ and $\vec{L}$ are parallel. On the long ends the rectangular loop, the forces are equal but point in opp site directions, so the net force on the loop is zero. How ever, there is a net torque that tends to rotate the loop abo its axis in a clockwise direction when viewed from abov This simple model shows how the combination of an ele tric current and a magnetic field can produce the rotary m tion of the electric motor. The same principle is responsib for the action of analog voltmeters and ammeters.

Figure 32-26 shows a rectangular loop of length a an width b carrying a current i. The plane of the loop makes angle θ with the x axis. For simplicity, only the loop itself

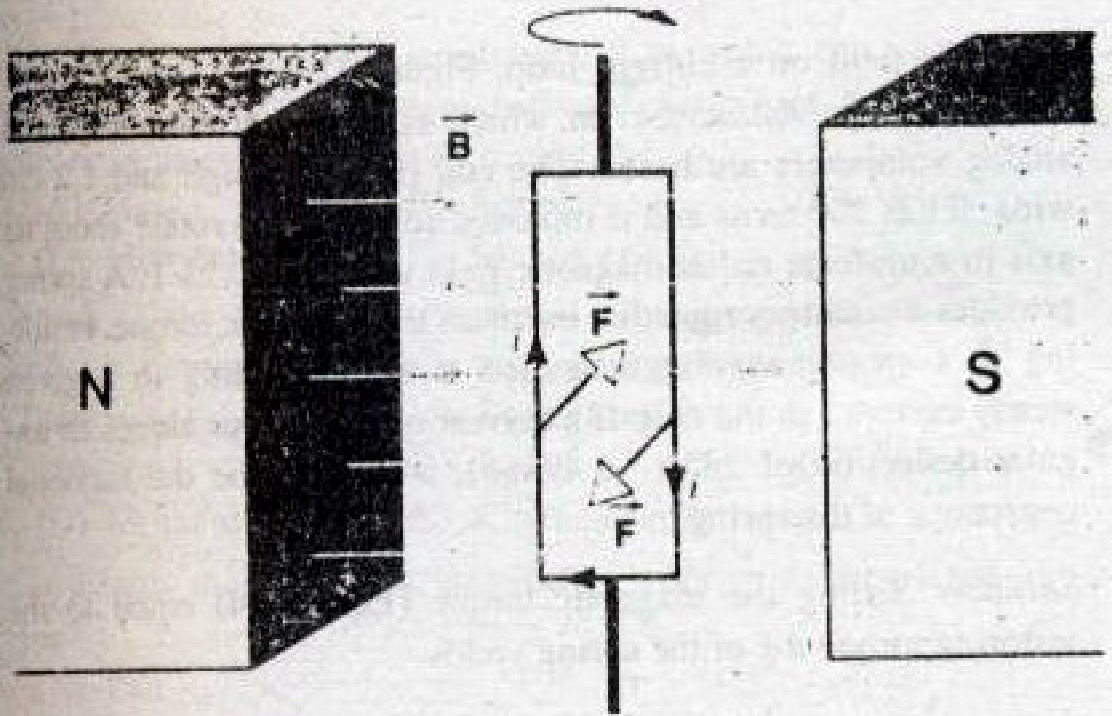


FIGURE 32-25. A simplified diagram of an electric motor. The loop carries an electric current. The magnetic forces on the long sides of the loop produce a torque that tends to rotate the loop clockwise as viewed from above.

shown; wires needed to bring current into and out of the loop are not shown. The magnetic field is taken to be uniform and in the y direction, and the z axis lies in the plane of the loop. Our goal is to find the net force and the net torque on the loop by calculating the force on each side of the loop.

In this orientation, sides 1 and 3 are perpendicular to the field. That is, if we define the vector $\vec{L}$ as being in the direction of the current, then $\vec{L}$ is perpendicular to $\vec{B}$. For those sides, we can use Eq. 32-28 for the magnitudes of the forces:

$$F_1 = F_3 = iaB, \qquad (32\text{-}30)$$

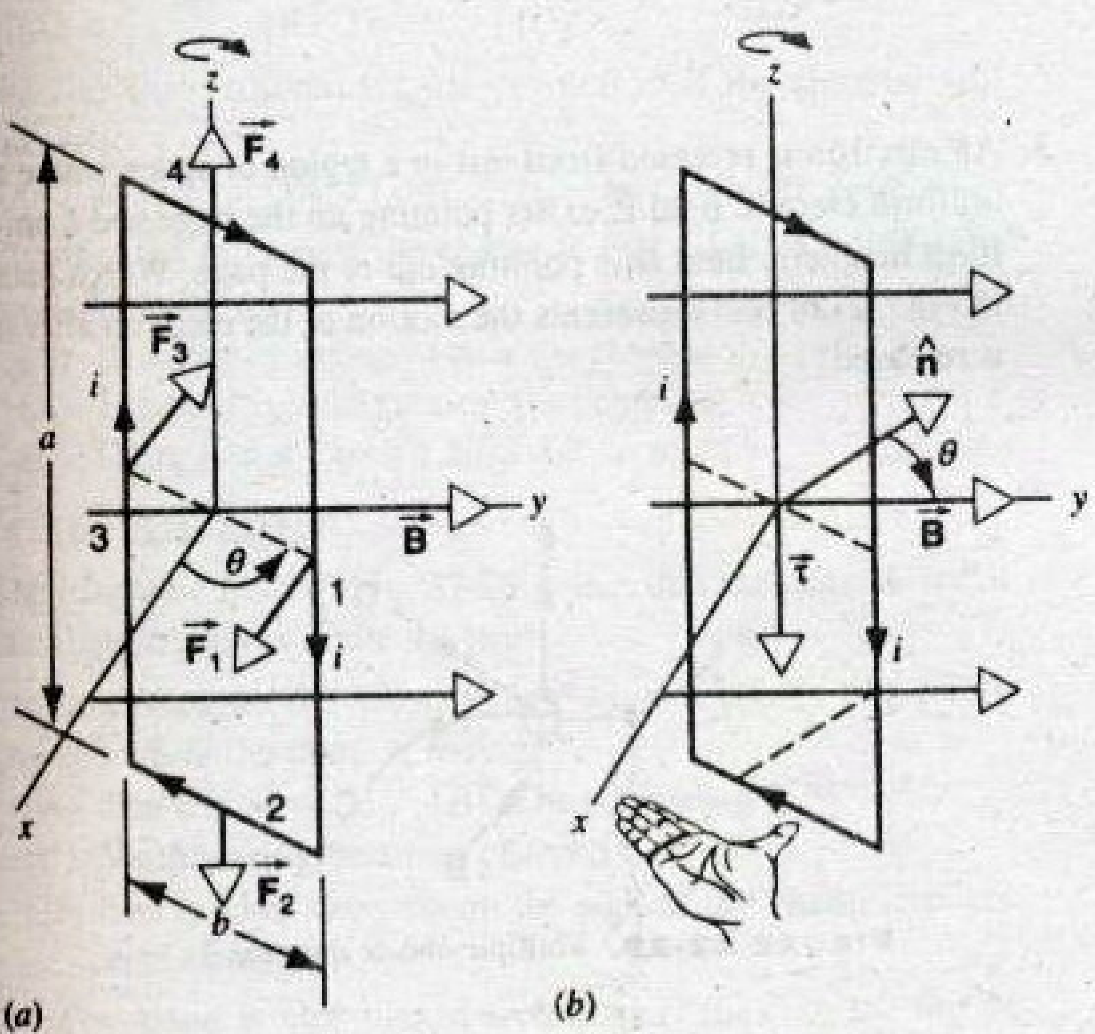


FIGURE 32-26. A rectangular loop of wire in a uniform magnetic field. (*a*) The forces on the four sides are shown. (*b*) The torque tends to rotate the loop so that the unit vector $\hat{n}$, determined from the right-hand rule and perpendicular to the plane of the loop, rotates into alignment with $\vec{B}$.

because sides 1 and 3 each have length a. The forces are parallel to the x axis of Fig. 32-26, with $\vec{F}_1$ in the positive x direction and $\vec{F}_3$ in the negative x direction.

The angle between side 2 of the wire and $\vec{B}$ is $\theta + 90°$. Using Eq. 32-27, we find the force on this segment to be

$$F_2 = ibB \sin(\theta + 90°) = ibB \cos\theta \qquad (32\text{-}31)$$

in the negative z direction. Similarly, the force on side 4 is

$$F_4 = ibB \sin(90° - \theta) = ibB \cos\theta \qquad (32\text{-}32)$$

in the positive z direction.

To find the total force on the loop, we add the forces on the four sides, being careful to take into account both their magnitudes and directions. Because F_2 and F_4 are equal in magnitude and opposite in direction, they sum to zero; the same is true for F_1 and F_3. The net force on the loop is zero, so its center of mass does not accelerate under the action of the magnetic force. This conclusion follows because the field is uniform; if the field were nonuniform, the field at opposite pairs of sides 1 and 3 or 2 and 4 might have different magnitudes, and the forces on those sides might not be equal in magnitude.

Even though the net force is zero, the net torque is nonzero. Forces F_2 and F_4 both lie along the z axis and so have the same line of action; they do not contribute to the net torque. However, F_1 and F_3 do not have the same line of action; as you can see from Fig. 32-26, they tend to rotate the loop clockwise about the z axis as viewed from above. Relative to the z axis, the forces F_1 and F_3 each have moment arms of $(b/2)\sin\theta$, and so the magnitude of the total torque is

$$\tau = 2(iaB)(b/2)\sin\theta, \qquad (32\text{-}33)$$

where the factor 2 enters because both forces contribute equally to the torque.

The torque has its maximum magnitude when the loop is oriented so that the magnetic field lies in the plane of the loop ($\theta = 90°$). The torque is zero when the magnetic field is perpendicular to the plane of the loop ($\theta = 0$).

If the loop were constructed as a coil of N turns of wire (such as might be found in a motor or an ammeter), Eq. 32-33 gives the torque on each turn, and the total torque on the coil would be

$$\tau = NiAB \sin\theta, \qquad (32\text{-}34)$$

where we have substituted A, the area of the loop, for the product ab. Equation 32-34 can be shown to hold in general for all plane loops of area A whether or not they are rectangular.

Figure 32-26*b* gives another way to interpret the torque on the current loop. Using the right-hand rule, we define a unit vector $\hat{n}$ perpendicular to the plane of the loop. The direction of $\hat{n}$ can be determined by holding your right hand so the fingers follow the direction of the current; your thumb indicates the direction of $\hat{n}$. The torque tries to rotate the loop so that $\hat{n}$ is brought into alignment with $\vec{B}$. The torque, which is in the negative z direction in Fig. 32-26*b*, is in the direction determined by the cross product $\hat{n} \times \vec{B}$. With $|\hat{n} \times \vec{B}| = B \sin\theta$, we can write Eq. 32-34 in vector form as

$$\vec{\tau} = NiA\,\hat{n} \times \vec{B}. \qquad (32\text{-}35)$$

SAMPLE PROBLEM 32-6. Analog voltmeters and ammeters, in which the reading is displayed by the deflection of a pointer over a scale, work by measuring the torque exerted by a magnetic field on a current loop. Figure 32-27 shows the rudiments of a *galvanometer*, on which both analog ammeters and analog voltmeters are based. The coil is 2.1 cm high and 1.2 cm wide; it has 250 turns and is mounted so that it can rotate about its axis in a uniform radial magnetic field with $B = 0.23$ T. A spring provides a countertorque that balances the magnetic torque, resulting in a steady angular deflection ϕ corresponding to a given steady current i in the coil. If a current of 100 μA produces an angular deflection of 28° (=0.49 rad), what must be the torsional constant κ of the spring?

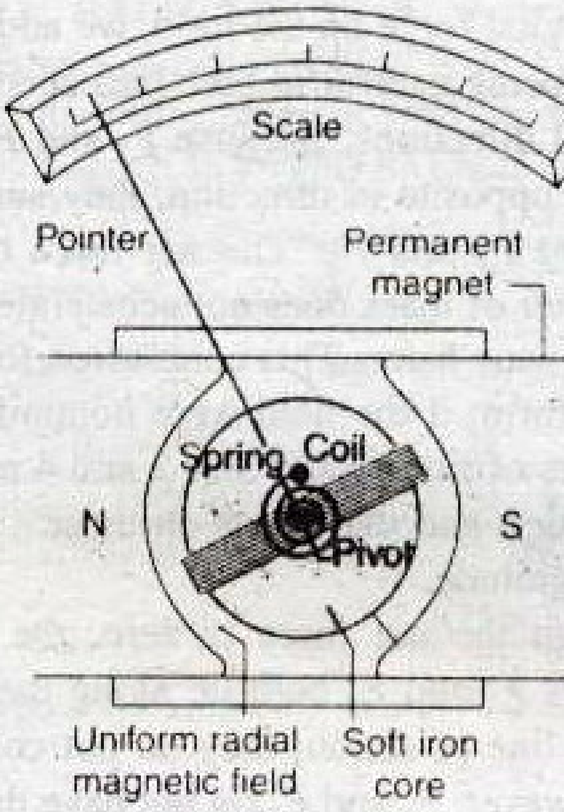


FIGURE 32-27. Sample Problem 32-6. The rudiments of a galvanometer. Depending on the external circuit, this device can act as either a voltmeter or an ammeter.

Solution Setting the magnetic torque (Eq. 32-34) equal to the restoring torque $\kappa\phi$ of the spring yields

$$\tau = NiAB \sin\theta = \kappa\phi,$$

in which ϕ is the angular deflection of the pointer and A ($=2.52 \times 10^{-4}$ m^2) is the area of the coil. Note that the normal to the plane of the coil (that is, the pointer) is always at right angles to the (radial) magnetic field so that $\theta = 90°$ for all pointer positions.

Solving for κ, we find

$$\kappa = \frac{NiAB \sin\theta}{\phi}$$

$$= \frac{(250)(100 \times 10^{-6}\text{ A})(2.52 \times 10^{-4}\text{ m}^2)(0.23\text{ T})(\sin 90°)}{0.49\text{ rad}}$$

$$= 3.0 \times 10^{-6}\text{ N}\cdot\text{m/rad}.$$

Modern ammeters and voltmeters are of the digital, direct-reading type and operate in a way that does not involve a moving coil.

MULTIPLE CHOICE

32-1 Magnetic Interactions and Magnetic Poles

32-2 The Magnetic Force on a Moving Charge

1. Of the three vectors in the equation $\vec{F}_B = q\vec{v} \times \vec{B}$, which pair(s) are always at right angles? *(There may be more than one correct answer.)*
   (A) $\vec{F}_B$ and $\vec{v}$ (B) $\vec{v}$ and $\vec{B}$ (C) $\vec{B}$ and $\vec{F}_B$
   (D) None (E) All three must be at right angles.

2. A negative charge q_1 is moving with a constant velocity $\vec{v}$ in a region where both a uniform electric field $\vec{E}$ and uniform magnetic field $\vec{B}$ exist.
   (*a*) Of the three vectors $\vec{v}$, $\vec{E}$, and $\vec{B}$, which pair(s) must be perpendicular? *(There may be more than one correct answer.)*
   (A) $\vec{E}$ and $\vec{v}$ (B) $\vec{v}$ and $\vec{B}$ (C) $\vec{B}$ and $\vec{E}$
   (D) None (E) All three must be perpendicular.
   (*b*) The negative charge is replaced with another charge q_2 moving initially with the same velocity. Under what conditions will the second charge also move with constant velocity?
   (A) q_2 must be identical to q_1.
   (B) q_2 must be negative, but can have any magnitude.
   (C) q_2 can be positive, but must have the same magnitude as q_1.
   (D) q_2 can be any charge.

3. An electron is released from rest in a region of space where a uniform electric field $\vec{E}$ exists pointing *up* the page and a uniform magnetic field $\vec{B}$ is pointing *out* of the page. Which path in Fig. 32-28 best represents the motion of the electron after it is released?

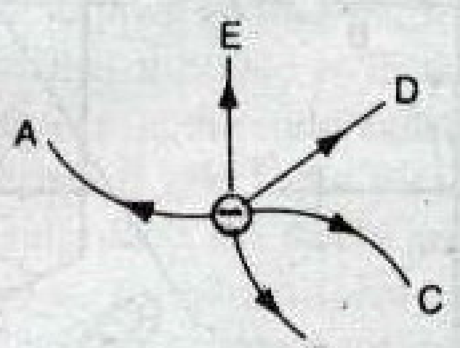


FIGURE 32-28. Multiple-choice question 3.

4. Which of the following properties of a proton can change while it moves freely in a uniform electric field $\vec{E}$? *(There may be more than one correct answer.)*
   (A) Mass (B) Speed (C) Velocity
   (D) Momentum (E) Kinetic energy

5. Which of the following properties of a proton can change while it moves freely in a uniform magnetic field $\vec{B}$? *(There may be more than one correct answer.)*
   (A) Mass (B) Speed (C) Velocity
   (D) Momentum (E) Kinetic energy

6. Which of the following properties of a proton can change while it moves freely in a *nonuniform* magnetic field $\vec{B}$? *(There may be more than one correct answer.)*
   (A) Mass (B) Speed (C) Velocity
   (D) Momentum (E) Kinetic energy

7. Can a static magnetic field do positive work on a charged particle?
   (A) Yes
   (B) Yes, but only if the particle has a positive charge
   (C) Yes, but only if the particle has an initial velocity
   (D) No

8. A region of space has a uniform electric field $\vec{E}$ directed down and a uniform magnetic field $\vec{B}$ directed east. Gravity is negligible. An electron is moving with a constant velocity $\vec{v}_1$ through the two fields.
   (*a*) In which direction could the electron be moving? *(There may be more than one correct answer.)*
   (A) North (B) South (C) Up (D) Down
   (*b*) A second electron originally follows the *direction* of the first, but is moving at a slightly slower speed $v_2 < v_1$. What is the direction of the net force on the second electron?
   (A) North (B) South (C) Up (D) Down

32-3 Circulating Charges

9. An electron with a speed $v_0 \ll c$ moves in a circle of radius r_0 in a uniform magnetic field. The time required for one revolution of the electron is T_0. The speed of the electron is now doubled to $2v_0$.
   (*a*) The radius of the circle will change to
   (A) $4r_0$. (B) $2r_0$. (C) r_0. (D) $r_0/2$.
   (*b*) The time required for one revolution of the electron will change to
   (A) $4T_0$. (B) $2T_0$. (C) T_0. (D) $T_0/2$.

10. Consider the motion of the charge in Fig. 32-17. Describe the magnetic field.
   (A) The field is strongest near the center.
   (B) The field is weakest near the center.
   (C) There is not enough information to solve the problem.

32-4 The Hall Effect

11. The magnetic field in Fig. 32-29 points into the page. A small metal plane moves down the page.
   (*a*) According to the pilot, which wing becomes negatively charged while the plane is moving?
   (A) The left wing (B) The right wing
   (C) Neither wing becomes charged.
   (D) The answer depends on the sign of the charge carriers in the plane.
   (*b*) The plane is now turned around and "flies" to the top of the page. Which wing now becomes negatively charged while the plane moves up the page?
   (A) The left wing (B) The right wing
   (C) Neither wing becomes charged.
   (D) The answer depends on the sign of the charge carriers in the plane.

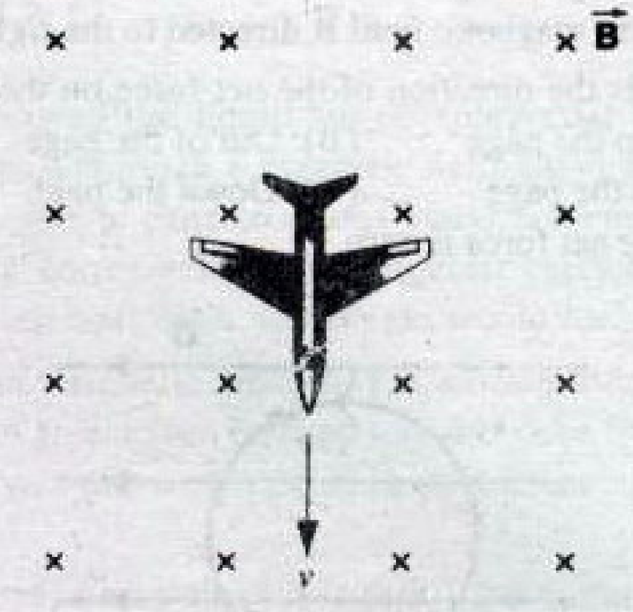


FIGURE 32-29. Multiple-choice question 11.

32-5 The Magnetic Force on a Current-Carrying Wire

12. Figure 32-30 shows several wire segments that carry equal currents from *a* to *b*. The wires are in a uniform magnetic field $\vec{B}$ directed into the page. Which wire segment experiences the largest net force?
   (A) 1 (B) 2 (C) 3
   (D) All experience the same net force.
   (E) The question cannot be answered without additional information.

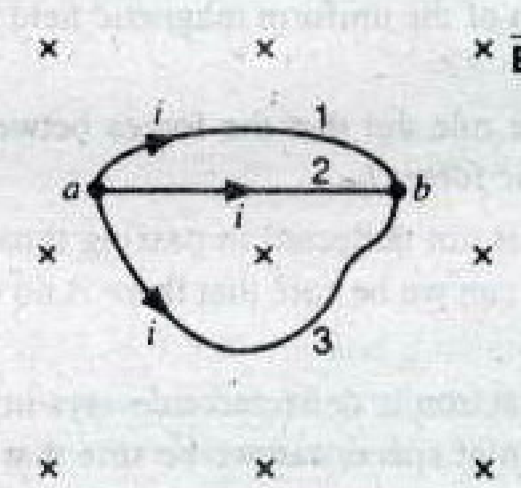


FIGURE 32-30. Multiple-choice questions 12 and 13.

13. Repeat Multiple-choice question 12, except assume that the uniform magnetic field could be pointing in *any* direction. Which wire segment experiences the largest net force?
   (A) 1 (B) 2 (C) 3
   (D) All experience the same net force.
   (E) The answer depends on the direction of the magnetic field.

32-6 The Torque on a Current Loop

14. Does Eq. 32-35 hold true for single-turn wire loops in shapes other than rectangles?
   (A) It is a reasonable approximation for shapes close to rectangular.
   (B) It is a reasonable approximation for any shape that lies in a plane.
   (C) It is true for shapes of sufficient symmetry, such as equilateral triangles or circles.
   (D) It is true for all shapes that lie in a plane.

15. The wire loop in Fig. 32-31 carries a clockwise current. There is a uniform magnetic field $\vec{B}$ directed to the right.

(*a*) What is the direction of the net force on the current loop?
(A) Into the page (B) Out of the page
(C) Up the page (D) Down the page
(E) The net force is zero.

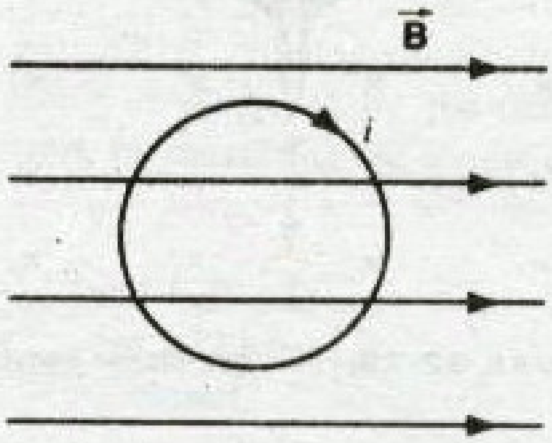


FIGURE 32-31. Multiple-choice question 15.

(*b*) What is the direction of the torque on the current loop?
(A) Into the page (B) Out of the page
(C) Up the page (D) Down the page
(E) The net force is zero.

16. Repeat Multiple-choice question 15, but now assume tha field is not uniform; it is stronger at the top of the page th the bottom.

(*a*) What is the direction of the net force on the current lo
(A) Into the page (B) Out of the page
(C) Up the page (D) Down the page
(E) The net force is zero.

(*b*) What is the direction of the torque on the current loop?
(A) Into the page (B) Out of the page
(C) Up the page (D) Down the page
(E) The net force is zero.

QUESTIONS

1. Why do we not simply define the direction of the magnetic field $\vec{B}$ to be the direction of the magnetic force that acts on a moving charge?

2. Imagine that you are sitting in a room with your back to one wall and that an electron beam, traveling horizontally from the back wall to the front wall, is deflected to your right. What is the direction of the uniform magnetic field that exists in the room?

3. How could we rule out that the forces between two magnets are electrostatic forces?

4. If an electron is not deflected in passing through a certain region of space, can we be sure that there is no magnetic field in that region?

5. If a moving electron is deflected sideways in passing through a certain region of space, can we be sure that a magnetic field exists in that region?

6. A beam of electrons can be deflected either by an electric field or by a magnetic field. Is one method better than the other? In any sense easier?

7. Electric fields can be represented by maps of equipotential surfaces. Can the same be done for magnetic fields? Explain.

8. Is a magnetic force conservative or nonconservative? Justify your answer. Could we define a magnetic potential energy as we defined electric or gravitational potential energy?

9. A charged particle passes through a magnetic field and is deflected. This means that a force acted on it and changed its momentum. Where there is a force there must be a reaction force. On what object does it act?

10. In the Thomson experiment we neglected the deflections produced by the gravitational field and magnetic field of the Earth. What errors are thereby introduced?

11. Imagine the room in which you are seated to be filled with a uniform magnetic field pointing vertically downward. At the center of the room two electrons are suddenly projected horizontally with the same initial speed but in opposite directions. (*a*) Describe their motions. (*b*) Describe their motions if one particle is an electron and one a positron—that is, a positively charged electron. (The electrons will gradually sl down as they collide with molecules of the air in the roo

12. Figure 32-32 shows the tracks of two electrons (e^-) an positron (e^+) in a bubble chamber. A magnetic field fills chamber, perpendicular to the plane of the figure. Why are tracks spirals and not circles? What can you tell about the p ticles from their tracks? What is the direction of the magne field?

FIGURE 32-32. Question 12.

13. What are the primary functions of (*a*) the electric field a (*b*) the magnetic field in the cyclotron?

14. In a given magnetic field, would a proton or an electron, tra eling at the same speed, have the greater frequency of revol tion? Consider relativistic effects.

15. What central fact makes the operation of a conventional c clotron possible? Ignore relativistic considerations.

16. A bare copper wire emerges from one wall of a room, cross the room, and disappears into the opposite wall. You are to that there is a steady current in the wire. How can you find direction? Describe as many ways as you can think of. You may use any reasonable piece of equipment, but you may not cut the wire.

17. Discuss the possibility of using the Hall effect to measure the strength B of a magnetic field.

18. (*a*) In measuring Hall potential differences, why must we be careful that points x and y in Fig. 32-18 are exactly opposite each other? (*b*) If one of the contacts is movable, what procedure might we follow in adjusting it to make sure that the two points are properly located?

19. In Section 32-5, we state that a magnetic field $\vec{B}$ exerts a sideways force on the conduction electrons in, say, a copper wire carrying a current i. We have tacitly assumed that this same force acts on the conductor itself. Are there some missing steps in this argument? If so, supply them.

20. A straight copper wire carrying a current i is immersed in a magnetic field $\vec{B}$, at right angles to it. We know that $\vec{B}$ exerts a sideways force on the free (or conduction) electrons. Does it do so on the bound electrons? After all, they are not at rest. Discuss.

21. Does Eq. 32-26 ($\vec{F}_B = i\vec{L} \times \vec{B}$) hold for a straight wire whose cross section varies irregularly along its length (i.e., a "lumpy" wire)?

22. A current in a magnetic field experiences a force. Therefore, it should be possible to pump conducting liquids by sending a current through the liquid (in an appropriate direction) and letting it pass through a magnetic field. Design such a pump. This principle is used to pump liquid sodium (a conductor, but highly corrosive) in some nuclear reactors, where it is used as a coolant. What advantages would such a pump have?

23. A uniform magnetic field fills a certain cubical region of space. Can an electron be fired into this cube from the outside in such a way that it will travel in a closed circular path inside the cube?

24. A conductor, even though it is carrying a current, has zero net charge. Why then does a magnetic field exert a force on it?

25. A rectangular current loop is in an arbitrary orientation in an external magnetic field. How much work is required to rotate the loop about an axis perpendicular to its plane?

26. Equation 32-35 shows that there is no torque on a current loop in an external magnetic field if the angle between the axis of the loop and the field is (*a*) 0° or (*b*) 180°. Discuss the nature of the equilibrium (that is, whether it is stable, neutral, or unstable) for these two positions.

27. Imagine that the room in which you are seated is filled with a uniform magnetic field pointing vertically upward. A circular loop of wire has its plane horizontal. For what direction of current in the loop, as viewed from above, will the loop be in stable equilibrium with respect to forces and torques of magnetic origin?

EXERCISES

32-1 Magnetic Interactions and Magnetic Poles

32-2 The Magnetic Force on a Moving Charge

1. Four particles follow the paths shown in Fig. 32-33 as they pass through the magnetic field there. What can one conclude about the charge of each particle?

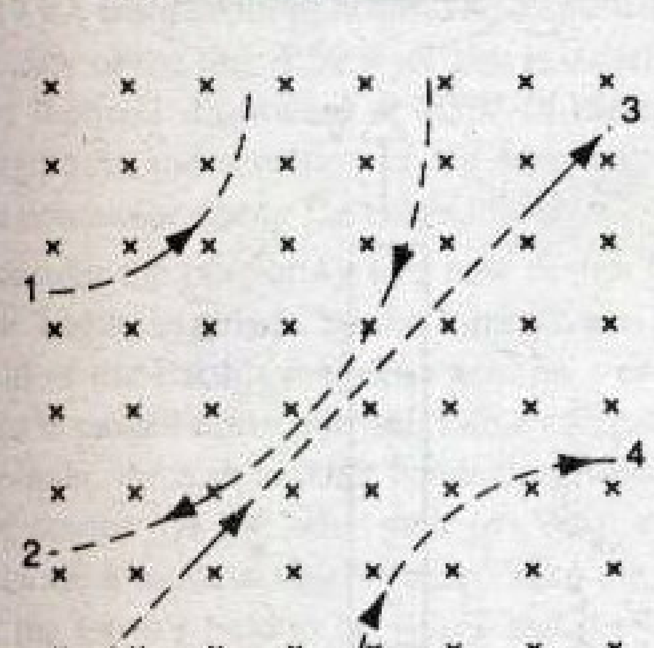


FIGURE 32-33. Exercise 1.

2. An electron in a TV camera tube is moving at 7.2×10^6 m/s in a magnetic field of strength 83 mT. (*a*) Without knowing the direction of the field, what could be the greatest and least magnitudes of the force the electron could feel due to the field? (*b*) At one point the acceleration of the electron is 4.9×10^{16} m/s^2. What is the angle between the electron's velocity and the magnetic field?

3. An electric field of 1.5 kV/m and a magnetic field of 0.44 T act on a moving electron to produce no force. (*a*) Calculate the minimum electron speed v. (*b*) Draw the vectors $\vec{E}$, $\vec{B}$, and $\vec{v}$.

4. A proton traveling at 23.0° with respect to a magnetic field of strength 2.63 mT experiences a magnetic force of 6.48×10^{-17} N. Calculate (*a*) the speed and (*b*) the kinetic energy in eV of the proton.

5. A cosmic ray proton strikes the Earth near the equator with a vertical velocity of 2.8×10^7 m/s. Assume that the horizontal component of the Earth's magnetic field at the equator is 30 μT. Calculate the ratio of the magnetic force on the proton to the gravitational force on it.

6. An electron is accelerated through a potential difference of 1.0 kV and directed into a region between two parallel plates separated by 20 mm with a potential difference of 100 V between them. If the electron enters moving perpendicular to the electric field between the plates, what magnetic field is necessary perpendicular to both the electron path and the electric field so that the electron travels in a straight line?

7. A uniform electric field $\vec{E}$ is perpendicular to a uniform magnetic field $\vec{B}$. A proton moving with velocity $\vec{v}_p$ perpendicular to both fields experiences no net force. An electron moving with velocity $\vec{v}_e$ also experiences no net force. Show that the ratio of the proton's kinetic energy to that of the electron is m_p/m_e.

8. An ion source is producing ions of ^{6}Li (mass = 6.01 u) each carrying a net charge of $+e$. The ions are accelerated by a

potential difference of 10.8 kV and pass horizontally into a region in which there is a vertical magnetic field $B = 1.22$ T. Calculate the strength of the horizontal electric field to be set up over the same region that will allow the ^{6}Li ions to pass through undeflected.

32-3 Circulating Charges

9. (*a*) In a magnetic field with $B = 0.50$ T, for what path radius will an electron circulate at a speed of $0.10c$? (*b*) What will be its kinetic energy in eV? Ignore the small relativistic effects.

10. A 1.22-keV electron is circulating in a plane at right angles to a uniform magnetic field. The orbit radius is 24.7 cm. Calculate (*a*) the speed of the electron, (*b*) the magnetic field, (*c*) the frequency of revolution, and (*d*) the period of the motion.

11. An electron is accelerated from rest by a potential difference of 350 V. It then enters a uniform magnetic field of magnitude 200 mT, its velocity at right angles to this field. Calculate (*a*) the speed of the electron and (*b*) the radius of its path in the magnetic field.

12. S. A. Goudsmit devised a method for measuring accurately the masses of heavy ions by timing their period of revolution in a known magnetic field. A singly charged ion of iodine makes 7.00 rev in a field of 45.0 mT in 1.29 ms. Calculate its mass, in atomic mass units. Actually, the mass measurements are carried out to much greater accuracy than these approximate data suggest.

13. An alpha particle ($q = +2e$, $m = 4.0$ u) travels in a circular path of radius 4.5 cm in a magnetic field with $B = 1.2$ T. Calculate (*a*) its speed, (*b*) its period of revolution, (*c*) its kinetic energy in eV, and (*d*) the potential difference through which it would have to be accelerated to achieve this energy.

14. A physicist is designing a cyclotron to accelerate protons to $0.100c$. The magnet used will produce a field of 1.40 T. Calculate (*a*) the radius of the cyclotron and (*b*) the corresponding oscillator frequency. Relativity considerations are not significant.

15. In a nuclear experiment, a proton with kinetic energy K_p moves in a uniform magnetic field in a circular path. What energy must (*a*) an alpha particle and (*b*) a deuteron have if they are to circulate in the same orbit? (For a deuteron, $q = +e$, $m = 2.0$ u; for an alpha particle, $q = +2e$, $m = 4.0$ u.)

16. A proton, a deuteron, and an alpha particle, accelerated through the same potential difference ΔV, enter a region of uniform magnetic field, moving at right angles to $\vec{\mathbf{B}}$. (*a*) Find their kinetic energies. If the radius of the proton's circular path is r_p, what are the radii of (*b*) the deuteron and (*c*) the alpha particle paths, in terms of r_p?

17. A proton, a deuteron, and an alpha particle with the same kinetic energy enter a region of uniform magnetic field, moving at right angles to $\vec{\mathbf{B}}$. The proton moves in a circle of radius r_p. In terms of r_p, what are the radii of (*a*) the deuteron path and (*b*) the alpha particle path?

18. A deuteron in a cyclotron is moving in a magnetic field with an orbit radius of 50 cm. Because of a grazing collision with a target, the deuteron breaks up, with a negligible loss of kinetic energy, into a proton and a neutron. Discuss the subsequent motions of each. Assume that the deuteron energy is shared equally by the proton and neutron at breakup.

19. (*a*) What speed would a proton need to circle the Earth [illegible] equator, if the Earth's magnetic field is everywhere horiz[illegible] there and directed along longitudinal lines? Relativistic [illegible] fects must be taken into account. Take the magnitude of [illegible] Earth's magnetic field to be 41 μT at the equator. (*b*) D[illegible] the velocity and magnetic field vectors corresponding to [illegible] situation.

20. Compute the radius of the path of a 10.0-MeV electron m[illegible] ing perpendicular to a uniform 2.20-T magnetic field. [illegible] both the (*a*) classical and (*b*) relativistic formulas. (*c*) Cal[illegible] late the true period of the circular motion. Is the result in[illegible] pendent of the speed of the electron?

21. Ionization measurements show that a particular nuclear pa[illegible] cle carries a double charge ($= 2e$) and is moving with a spe[illegible] of $0.710c$. It follows a circular path of radius 4.72 m i[illegible] magnetic field of 1.33 T. Find the mass of the particle a[illegible] identify it.

22. The proton synchrotron at Fermilab accelerates protons to [illegible] kinetic energy of 950 GeV. At this energy, calculate (*a*) t[illegible] speed, expressed as a fraction of the speed of light; a[illegible] (*b*) the magnetic field at the proton orbit that has a rad[illegible] of curvature of 750 m. (The proton has a rest energy [illegible] 938 MeV; it is necessary to use relativistic formulas here.)

23. Estimate the total path length traveled by a deuteron in a cy[illegible] clotron during the acceleration process. Assume an acceler[illegible] ing potential between the dees of 80 kV, a dee radius o[illegible] 53 cm, and an oscillator frequency of 12 MHz.

24. Consider a particle of mass m and charge q moving in the x[illegible] plane under the influence of a uniform magnetic field $\vec{\mathbf{B}}$ point[illegible] ing in the $+z$ direction. Write expressions for the coordinate[illegible] $x(t)$ and $y(t)$ of the particle as functions of time t, assuming [illegible] that the particle moves in a circle of radius R centered at the [illegible] origin of coordinates.

32-4 The Hall Effect

25. A metal strip 6.5 cm long, 0.88 cm wide, and 0.76 mm thick moves with constant speed v through a magnetic field $B =$ 1.2 mT perpendicular to the strip, as shown in Fig. 32-34. A potential difference of 3.9 μV is measured between points x and y across the strip. Calculate the speed v.

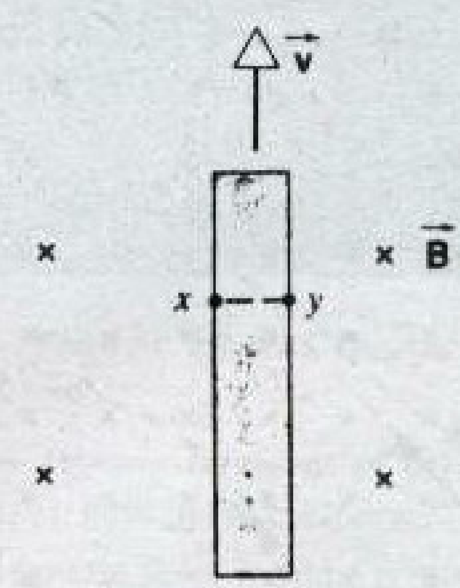


FIGURE 32-34. Exercise 25.

26. In a Hall effect experiment, a current of 3.2 A lengthwise in [illegible] conductor 1.2 cm wide, 4.0 cm long, and 9.5 μm thick pro[illegible] duces a transverse Hall voltage (across the width) of 40 μV when a magnetic field of 1.4 T acts perpendicular to the thi[illegible] conductor. From these data, find (*a*) the drift velocity of the charge carriers and (*b*) the number density of charge carriers. From Table 32-2, identify the conductor. (*c*) Show on a diagram the polarity of the Hall voltage with a given current and magnetic field direction, assuming that the charge carriers are (negative) electrons.

27. Show that, in terms of the Hall electric field E_H and the current density j, the number of charge carriers per unit volume is given by

$$n = \frac{jB}{eE_H}.$$

28. (*a*) Show that the ratio of the Hall electric field E_H to the electric field E_c responsible for the current is

$$\frac{E_H}{E_c} = \frac{B}{ne\rho}.$$

where ρ is the resistivity of the material. (*b*) Compute the ratio numerically for Sample Problem 32-3. See Table 29-1.

32-5 The Magnetic Force on a Current-Carrying Wire

29. A wire of length 62.0 cm and mass 13.0 g is suspended by a pair of flexible leads in a magnetic field of 440 mT. Find the magnitude and direction of the current in the wire required to remove the tension in the supporting leads. See Fig. 32-35.

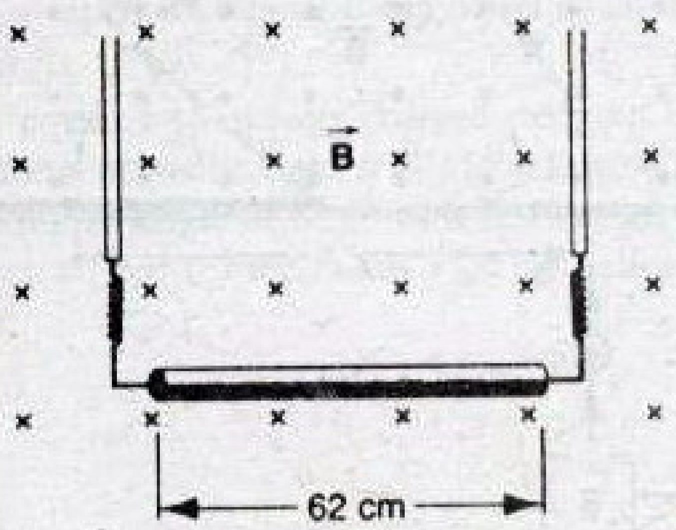


FIGURE 32-35. Exercise 29.

30. A horizontal conductor in a power line carries a current of 5.12 kA from south to north. The Earth's magnetic field in the vicinity of the line is 58.0 μT and is directed toward the north and inclined downward at 70.0° to the horizontal. Find the magnitude and direction of the magnetic force on 100 m of the conductor due to the Earth's field.

31. Consider the possibility of a new design for an electric train. The engine is driven by the force due to the vertical component of the Earth's magnetic field on a conducting axle. Current is passed down one rail, into a conducting wheel, through the axle, through another conducting wheel, and then back to the source via the other rail. (*a*) What current is needed to provide a modest 10-kN force? Take the vertical component of the Earth's field to be 10 μT and the length of the axle to be 3.0 m. (*b*) How much power would be lost for each ohm of resistance in the rails? (*c*) Is such a train totally unrealistic or just marginally unrealistic?

32. A metal wire of mass m slides without friction on two horizontal rails spaced a distance d apart, as in Fig. 32-36. The track lies in a vertical uniform magnetic field $\vec{B}$. A constant current i flows from generator G along one rail, across the wire, and back down the other rail. Find the velocity (speed and direction) of the wire as a function of time, assuming it to be at rest at $t = 0$.

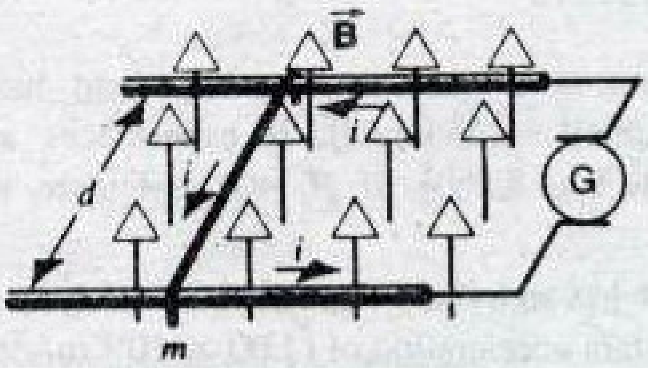


FIGURE 32-36. Exercise 32.

33. A long, rigid conductor, lying along the x axis, carries a current of 5.0 A in the $-x$ direction. A magnetic field $\vec{B}$ is present, given by $\vec{B} = (3\text{ mT})\hat{i} + (8\text{ mT/m}^2)x^2\hat{j}$. Calculate the force on the 2.0-m segment of the conductor that lies between $x = 1.2$ m and $x = 3.2$ m.

34. A 1.15-kg copper rod rests on two horizontal rails 95.0 cm apart and carries a current of 53.2 A from one rail to the other. The coefficient of static friction is 0.58. Find the smallest magnetic field (not necessarily vertical) that would cause the bar to slide.

32-6 The Torque on a Current Loop

35. A single-turn current loop, carrying a current of 4.00 A, is in the shape of a right triangle with sides 50 cm, 120 cm, and 130 cm. The loop is in a uniform magnetic field of magnitude 75.0 mT whose direction is parallel to the current in the 130-cm side of the loop. (*a*) Find the magnetic force on each of the three sides of the loop. (*b*) Show that the total magnetic force on the loop is zero.

36. Figure 32-37 shows a rectangular, 20-turn loop of wire that is 12 cm by 5.0 cm. It carries a current of 0.10 A and is hinged at one side. It is mounted with its plane at an angle of 33° to the direction of a uniform magnetic field of 0.50 T. Calculate the torque about the hinge line acting on the loop.

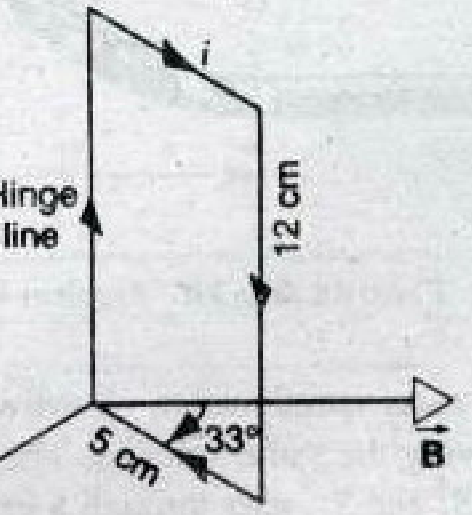


FIGURE 32-37. Exercise 36.

37. A stationary, circular wall clock has a face with a radius of 15 cm. Six turns of wire are wound around its perimeter; the wire carries a current 2.0 A in the clockwise direction. The clock is located where there is a constant, uniform external magnetic field of 70 mT (but the clock still keeps perfect time). At exactly 1:00 P.M., the hour hand of the clock points in the direction of the external magnetic field. (*a*) After how many minutes will the minute hand point in the direction of the torque on the winding due to the magnetic field? (*b*) What is the magnitude of this torque?

PROBLEMS

1. An electron in a uniform magnetic field has a velocity $\vec{v} = (40 \text{ km/s})\hat{i} + (35 \text{ km/s})\hat{j}$. It experiences a force $\vec{F} = (-4.2 \text{ fN})\hat{i} + (4.8 \text{ fN})\hat{j}$. If $B_x = 0$, calculate the magnetic field.

2. An electron has an initial velocity $(12.0 \text{ km/s})\hat{j} + (15.0 \text{ km/s})\hat{k}$ and a constant acceleration of $(2.00 \times 10^{12} \text{ m/s}^2)\hat{i}$ in a region in which uniform electric and magnetic fields are present. If $\vec{B} = (400 \ \mu\text{T})\hat{i}$, find the electric field $\vec{E}$.

3. The electrons in the beam of a television tube have a kinetic energy of 12.0 keV. The tube is oriented so that the electrons move horizontally from magnetic south to magnetic north. The vertical component of the Earth's magnetic field points down and has a magnitude of 55.0 μT. (*a*) In what direction will the beam deflect? (*b*) What is the acceleration of a given electron due to the magnetic field? (*c*) How far will the beam deflect in moving 20.0 cm through the television tube?

4. A beam of electrons whose kinetic energy is K emerges from a thin-foil "window" at the end of an accelerator tube. There is a metal plate a distance d from this window and at right angles to the direction of the emerging beam (see Fig. 32-38). (*a*) Show that we can prevent the beam from hitting the plate if we apply a magnetic field B such that

$$B \geq \sqrt{\frac{2mK}{e^2d^2}},$$

in which m and e are the electron mass and charge. (*b*) How should B be oriented?

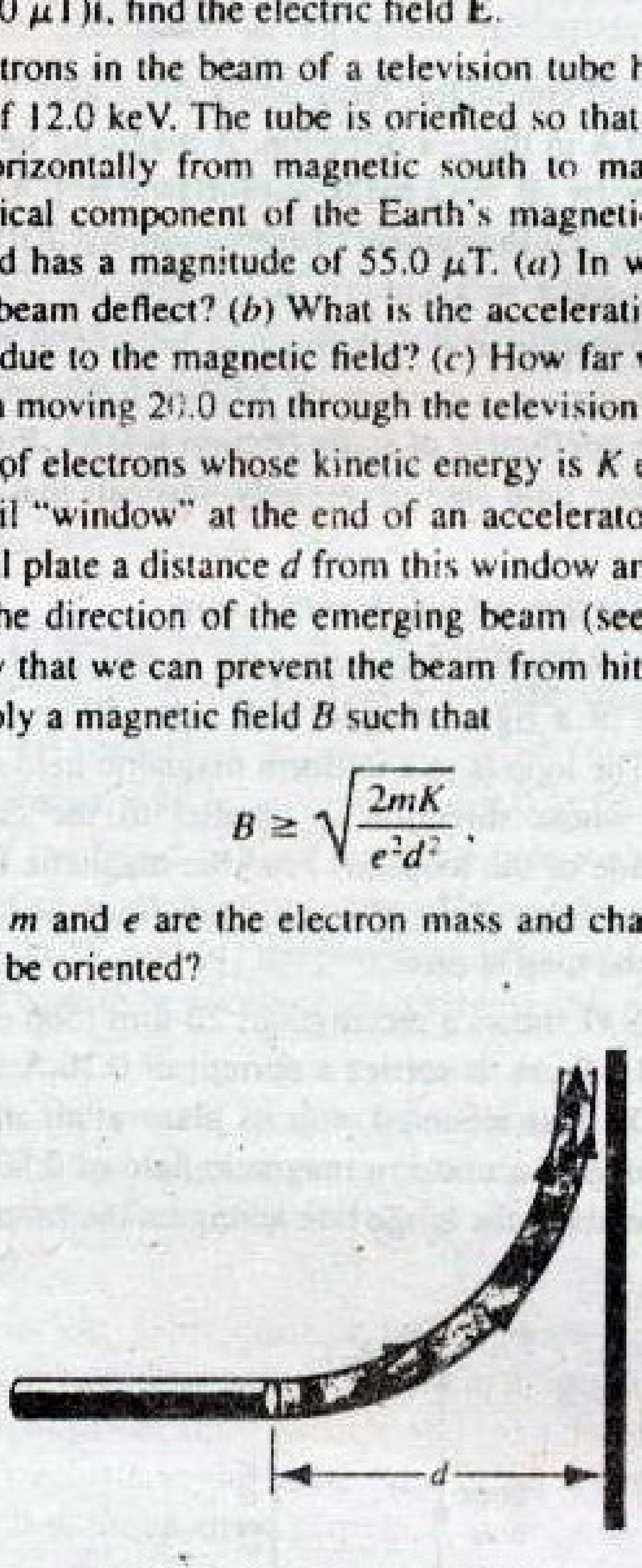


FIGURE 32-38. Problem 4.

5. Bainbridge's mass spectrometer, shown in Fig. 32-39, separates ions having the same velocity. The ions, after entering through slits S_1 and S_2, pass through a velocity selector composed of an electric field produced by the charged plates P and P′, and a magnetic field $\vec{B}$ perpendicular to the electric field and the ion path. Those ions that pass undeviated through the crossed $\vec{E}$ and $\vec{B}$ fields enter into a region where a second magnetic field $\vec{B}'$ exists, and are bent into circular paths. A photographic plate registers their arrival. Show that $q/m = E/rBB'$, where r is the radius of the circular orbit.

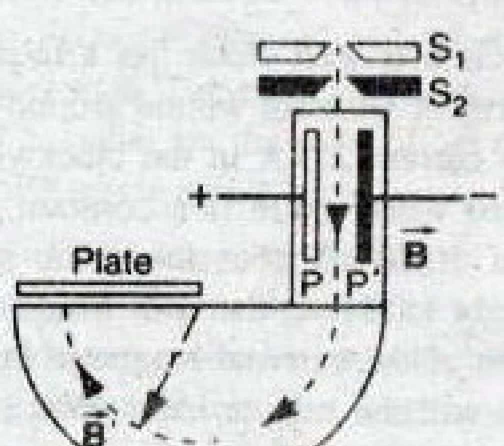


FIGURE 32-39. Problem 5.

6. Figure 32-40 shows an arrangement used to measure the masses of ions. An ion of mass m and charge $+q$ is produced essentially at rest in source S, a chamber in which a gas discharge is taking place. The ion is accelerated by potential difference ΔV and allowed to enter a magnetic field $\vec{B}$. In the field it moves in a semicircle, striking a photographic plate at distance x from the entry slit. Show that the ion mass m is given by

$$m = \frac{B^2q}{8\Delta V}x^2.$$

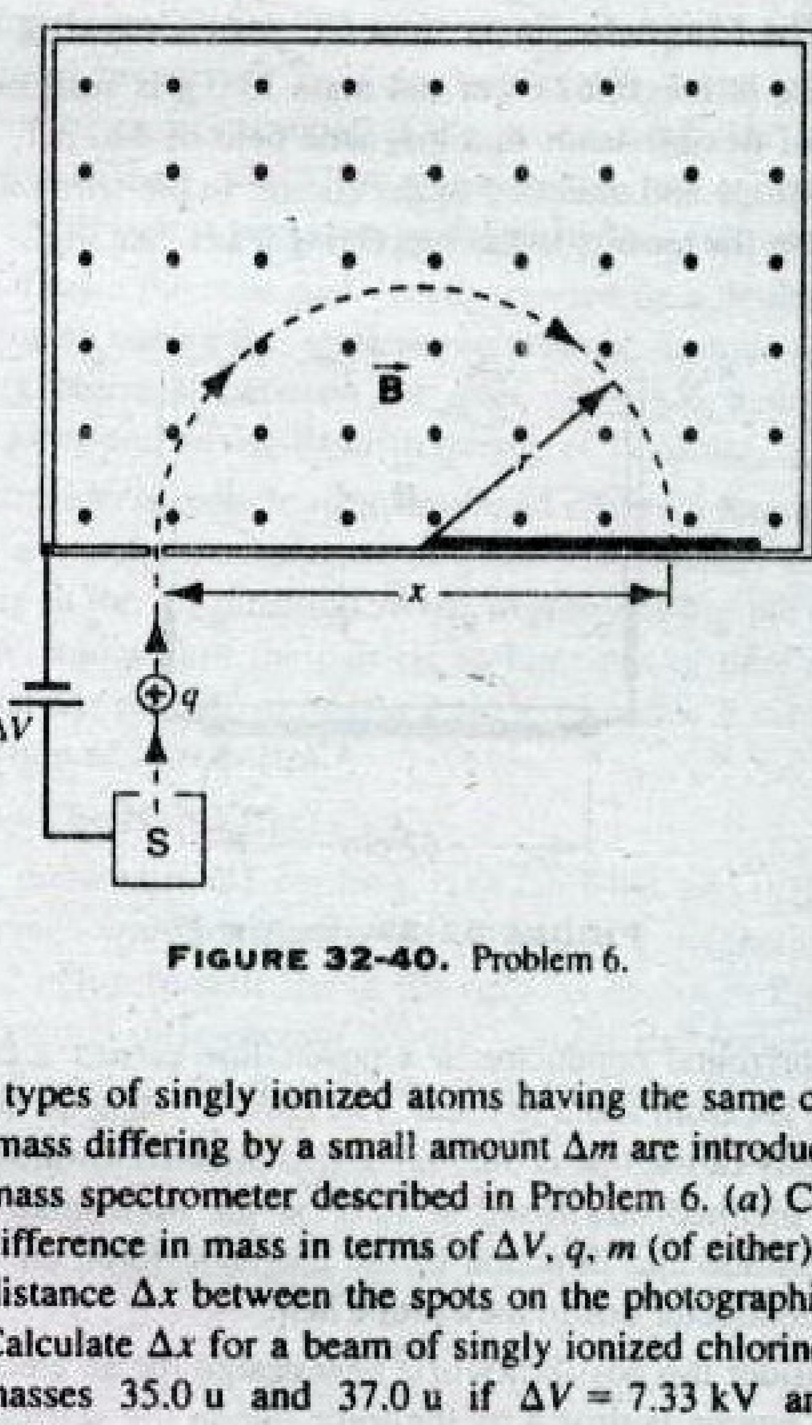


FIGURE 32-40. Problem 6.

7. Two types of singly ionized atoms having the same charge q and mass differing by a small amount Δm are introduced into the mass spectrometer described in Problem 6. (*a*) Calculate the difference in mass in terms of ΔV, q, m (of either), B, and the distance Δx between the spots on the photographic plate. (*b*) Calculate Δx for a beam of singly ionized chlorine atoms of masses 35.0 u and 37.0 u if $\Delta V = 7.33$ kV and $B = 520$ mT.

8. In a mass spectrometer (see Problem 6) used for commercial purposes, uranium ions of mass 238 u and charge $+2e$ are separated from related species. The ions are first accelerated through a potential difference of 105 kV and then pass into a magnetic field, where they travel a 180° arc of radius 97.3 cm. They are then collected in a cup after passing through a slit of width 1.20 mm and a height of 1.14 cm. (*a*) What is the magnitude of the (perpendicular) magnetic field in the separator? If the machine is designed to separate out 90.0 mg of material per hour, calculate (*b*) the current of the desired ions in the machine and (*c*) the internal energy dissipated in the cup in 1.00 h.

9. A neutral particle is at rest in a uniform magnetic field of magnitude B. At time $t = 0$ it decays into two charged parti-

cles each of mass m. (*a*) If the charge of one of the particles is $+q$, what is the charge of the other? (*b*) The two particles move off in separate paths, both of which lie in the plane perpendicular to $\vec{B}$. At a later time the particles collide. Express the time from decay until collision in terms of m, B, and q.

10. In Bohr's theory of the hydrogen atom, the electron can be thought of as moving in a circular orbit of radius r about the proton. Suppose that such an atom is placed in a magnetic field, with the plane of the orbit at right angles to $\vec{B}$. (*a*) If the electron is circulating clockwise, as viewed by an observer sighting along $\vec{B}$, will the angular frequency increase or decrease? (*b*) What if the electron is circulating counterclockwise? Assume that the orbit radius does not change. [Hint: The centripetal force is now partially electric ($\vec{F}_E$) and partially magnetic ($\vec{F}_B$) in origin.] (*c*) Show that the change in frequency of revolution caused by the magnetic field is given approximately by

$$\Delta f = \pm \frac{Be}{4\pi m}.$$

Such frequency shifts were observed by Zeeman in 1896. (Hint: Calculate the frequency of revolution without the magnetic field and also with it. Subtract, bearing in mind that because the effect of the magnetic field is very small, some—but not all—terms containing B can be set equal to zero with little error.)

11. A 22.5-eV positron (positively charged electron) is projected into a uniform magnetic field $B = 455\ \mu T$ with its velocity vector making an angle of 65.5° with $\vec{B}$. Find (*a*) the period, (*b*) the pitch p, and (*c*) the radius r of the helical path. See Fig. 32-41.

FIGURE 32-41. Problem 11.

12. Figure 32-42 shows a wire of arbitrary shape carrying a current i between points a and b. The wire lies in a plane at right angles to a uniform magnetic field $\vec{B}$. Prove that the force on the wire is the same as that on a straight wire carrying a current i directly from a to b. (Hint: Replace the wire by a series of "steps" that are parallel and perpendicular to the straight line joining a and b.)

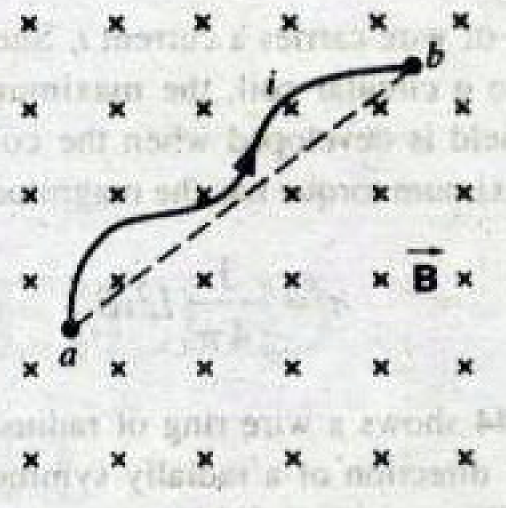


FIGURE 32-42. Problem 12.

13. Consider the particle of Exercise 24, but this time prove (rather than assume) that the particle moves in a circular path by solving Newton's second law analytically. [Hint: Solve the expression for F_y to find v_x and substitute into the expression for F_x to obtain an equation that can be solved for v_y. Do the same for v_x by substituting into the F_y equation. Finally, obtain $x(t)$ and $y(t)$ from v_x and v_y.]

14. By direct integration of

$$\vec{F}_B = \oint i\, d\vec{L} \times \vec{B},$$

show that the net force on an *arbitrary* current loop is zero in a uniform magnetic field. (*Note: an arbitrary current loop does not need to lie in a plane!*)

15. A U-shaped wire of mass m and length L is immersed with its two ends in mercury (Fig. 32-43). The wire is in a homogeneous magnetic field $\vec{B}$. If a charge—that is, a current pulse $q = \int i dt$—is sent through the wire, the wire will jump up. Calculate, from the height h that the wire reaches, the size of the charge or current pulse, assuming that the time of the current pulse is very small in comparison with the time of flight. Make use of the fact that impulse of force equals $\int F dt$, which equals mv. (Hint: Relate $\int i dt$ to $\int F dt$.) Evaluate q for $B = 0.12$ T, $m = 13$ g, $L = 20$ cm, and $h = 3.1$ m.

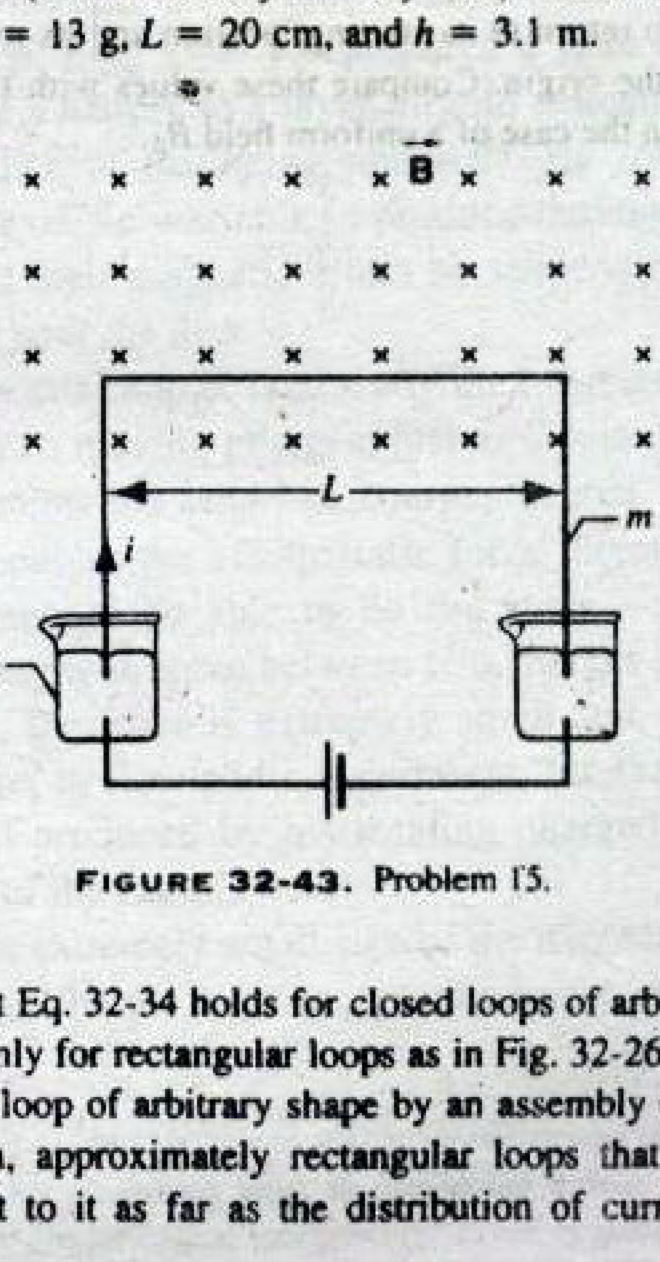


FIGURE 32-43. Problem 15.

16. Prove that Eq. 32-34 holds for closed loops of arbitrary shape and not only for rectangular loops as in Fig. 32-26. (Hint: Replace the loop of arbitrary shape by an assembly of adjacent, long, thin, approximately rectangular loops that are nearly equivalent to it as far as the distribution of current is concerned.)

17. A length L of wire carries a current i. Show that if the wire is formed into a circular coil, the maximum torque in a given magnetic field is developed when the coil has one turn only and the maximum torque has the magnitude

$$\tau = \frac{1}{4\pi} L^2 iB.$$

18. Figure 32-44 shows a wire ring of radius a at right angles to the general direction of a radially symmetric diverging magnetic field. The magnetic field at the ring is everywhere of the same magnitude B, and its direction at the ring is everywhere at an angle θ with a normal to the plane of the ring. The twisted lead wires have no effect on the problem. Find the magnitude and direction of the force the field exerts on the ring if the ring carries a current i as shown in the figure.

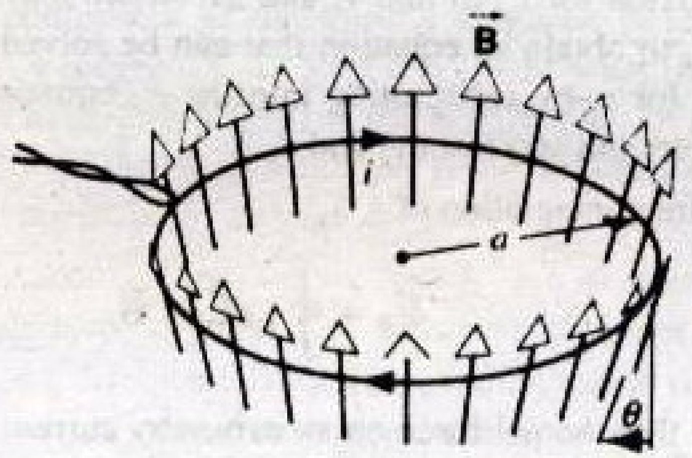


FIGURE 32-44. Problem 18.

19. Figure 32-45 shows a wooden cylinder with a mass m = 262 g and a length L = 12.7 cm, with N = 13 turns of wire wrapped around it longitudinally, so that the plane of the wire loop contains the axis of the cylinder. What is the least current through the loop that will prevent the cylinder from rolling down a plane inclined at an angle θ to the horizontal in the presence of a vertical, uniform magnetic field of 477 mT, if the plane of the windings is parallel to the inclined plane?

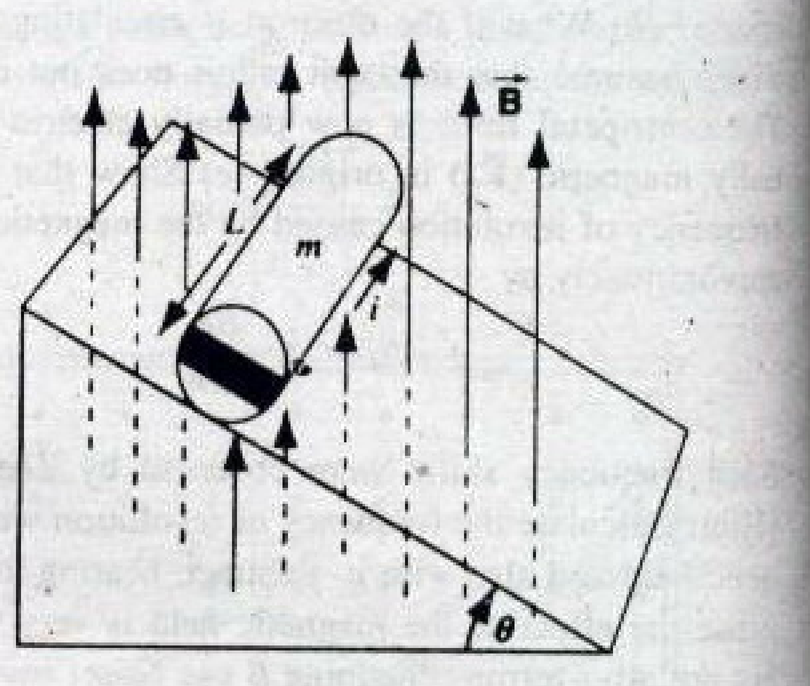


FIGURE 32-45. Problem 19.

COMPUTER PROBLEMS

1. Using the magnetic field given in Eq. 32-17 with B_0 = 0.15 T, obtain the trajectory of an alpha particle that is initially moving through the origin in the x direction with speed v_0 = 3.0×10^6 m/s (Fig. 32-17). Find the time that it takes the particle to return to its starting location and its maximum distance from the origin. Compare these values with the corresponding ones in the case of a uniform field B_0.

2. A cylindrically symmetric magnetic field in a certain region of space is given by $\vec{B} = (B_0 r/a)\hat{k}$, where r is the perpendicular distance from the z axis. Find the trajectory of an electron emitted from a point on the z axis with an initial velocity of $0.050c$ perpendicular to the axis. What is the maximum distance the electron will travel from the axis before returning?

CHAPTER 33

THE MAGNETIC FIELD OF A CURRENT

In the previous chapter we studied the effect of a magnetic field on a moving charge. We now turn to the source of the field itself, and in this chapter we study the magnetic field produced by moving charges, particularly currents in wires.

In analogy with our previous study of the electric fields of some simple charge distributions, we investigate in this chapter the magnetic fields produced by some simple current distributions: straight wires and circular loops. Finally, we show that the relationship between electric and magnetic fields is deeper than merely the similarity of equations; the relationship extends to the transformation of the fields into one another when charge or current distributions are viewed from different inertial frames.

33-1 THE MAGNETIC FIELD DUE TO A MOVING CHARGE

In the previous chapter we discussed the force experienced by a charged particle moving in a magnetic field. By analogy with the electric field, in which charged particles at rest are both the sources of the field and its probe, we might expect that moving electric charges, which we have already shown to be probes of the magnetic field, could also serve as sources of the field.

This expectation was first demonstrated in 1820 by Hans Christian Oersted*, who observed that, as illustrated in Fig. 33-1, when a compass is placed near a straight wire carrying a current, the compass needle aligns so that it is tangent to a circle drawn around the wire (neglecting the influence of the Earth's magnetic field on the compass). Oersted's discovery provided the first link between electricity and magnetism.

Direct experimental evidence for the magnetic field of a moving charge did not come until 1876 in an experiment by Henry Rowland.** Rowland's experiment is shown schematically in Fig. 33-2. He prepared a disk of charge (by connecting a battery to a layer of gold deposited on the surface of a disk of insulating material). By rotating the disk about its axis, he was able to produce moving charges, and he showed their magnetic effect by suspending a magnetized needle near the disk.

Our goal in this chapter is to study the magnetic interaction between two moving charges, just as Coulomb studied the electric interaction between charges at rest. Coulomb was able to measure the electrostatic force directly, and in principle we should be able to do the same—that is, to measure the magnetic force between two charges in motion. Unfortunately, the force is extremely small and very difficult to measure; in Rowland's experiment, for example, the magnetic field produced by his rotating charged disk was only 0.00001 of the Earth's field!

Despite the extremely small size of the magnetic field of a single moving charge, it is conceptually easier to begin our

*Hans Christian Oersted (1777–1851) was a Danish physicist and chemist. His discovery that a current-carrying wire can deflect a compass needle was made unexpectedly during a lecture at the University of Copenhagen. The unit for magnetic field intensity (H), the oersted, is named in his honor.

**Henry Rowland (1848–1901) was a U.S. physicist who is today most remembered for his pioneering work in developing diffraction gratings, which he used for precise optical spectroscopy including measurements of the wavelengths of the solar spectrum. Rowland served as the first president of the American Physical Society.

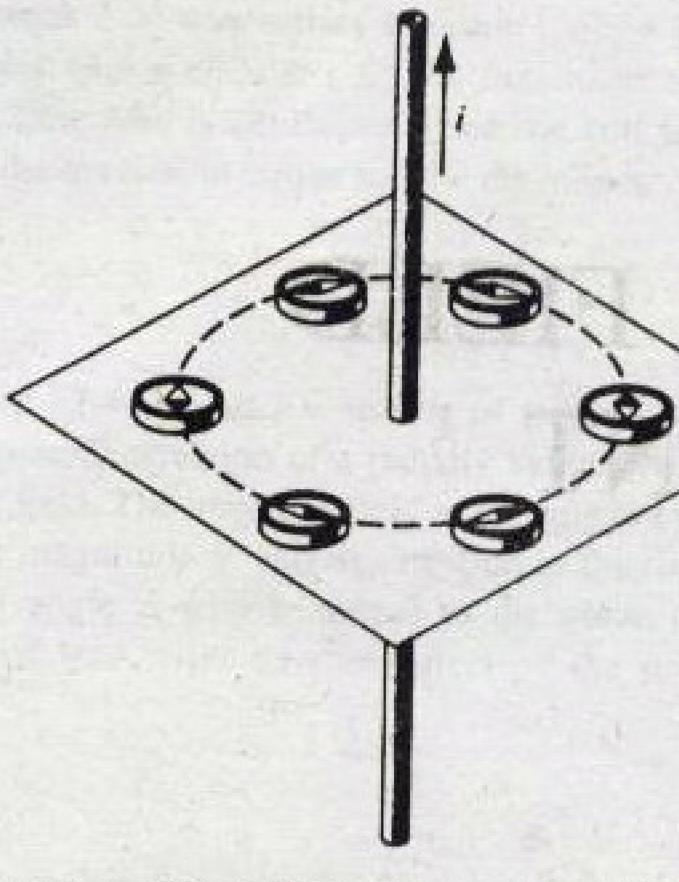


FIGURE 33-1. Oersted's experiment. The direction of the compass needle is always perpendicular to the direction of the current in the wire.

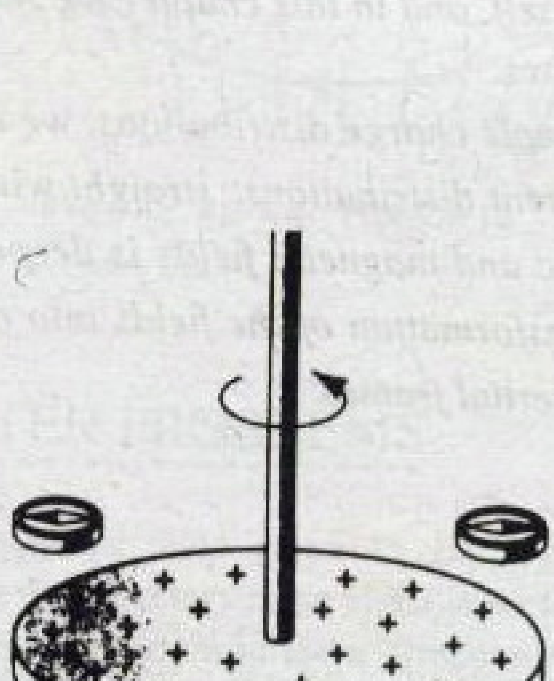


FIGURE 33-2. Schematic diagram of Rowland's experiment. The moving charges on the gold surface of the rotating disk produce a magnetic field that deflects the compass needle. In practice, the deflection is very small and requires apparatus far more sensitive than a compass to detect.

study of the sources of the magnetic field with a discussion of how a single moving charge produces a magnetic field. Later we will see why this approach is not practical and why it is easier for us to produce magnetic fields in the laboratory by using moving charges in the form of currents in wires.

Let us therefore conduct a "thought experiment" in which we project a single charge q with velocity $\vec{v}$ and detect the field with a suspended magnetic needle that is free to align in any direction. To avoid problems with relativity, we will need to keep the speed of the particle small (compared with the speed of light) in our reference frame. We set up the experiment in a region in which the Earth's magnetic field is negligible. (It is not necessary to journey to outer space to find this region; we can use current-carrying coils in our laboratories to create fields that cancel the Earth's field.) Figure 33-3*a* shows the outcome of some measurements of the magnetic field at different locations. The moving charge sets up a magnetic field $\vec{B}$, and the needle indicates the direction of the field at any location. In principle we could also determine the magnitude of the field, such as by measuring the force on a second moving charged particle, as we described in Section 32-2.

If we could actually perform these experiments, we would discover some properties of the magnetic field due to a moving charge:

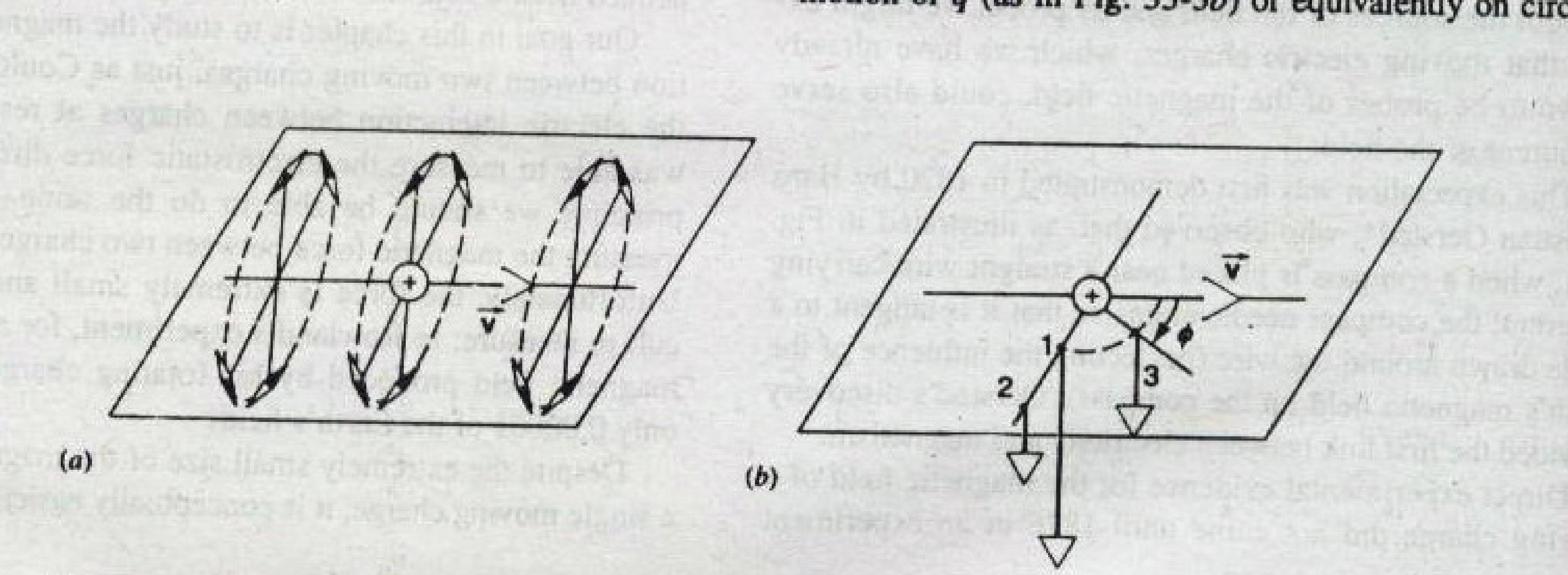
1. The field strength is directly proportional to the magnitude of the velocity v and also to the charge q.

2. If $\vec{v}$ reverses direction or q changes sign, the direction of $\vec{B}$ is reversed.

3. The field is zero at points along the direction of $\vec{v}$ (forward as well as backward). In other directions relative to $\vec{v}$, as shown in Fig. 33-3*b*, the field varies as $\sin \phi$.

4. $\vec{B}$ is tangent to circles drawn about $\vec{v}$ in planes perpendicular to $\vec{v}$, with the direction of $\vec{B}$ determined by a right-hand rule (point your thumb in the direction of $\vec{v}$, and your fingers will curl in the direction of $\vec{B}$). On any given circle, the magnitude of $\vec{B}$ is the same at all points.

5. At points on a line perpendicular to the direction of motion of q (as in Fig. 33-3*b*) or equivalently on circles of



(*a*) (*b*)

FIGURE 33-3. (*a*) A freely suspended compass needle indicates the direction of the magnetic field at various locations due to a moving electric charge. (*b*) The field at point 2 is 1/4 of the field at point 1, because point 2 is twice as far from the charge. Point 3 is the same distance from the charge as point 1, but the field at point 3 is smaller than the field at point 1 by the factor $\sin \phi$.

increasing radius drawn around the line of motion, we find that the magnitude of $\vec{B}$ decreases like $1/r^2$, where r is the distance from q to the observation point.

The simplest way of defining $\vec{B}$ that is consistent with these observations is illustrated in Fig. 33-4. At an arbitrary point P (the point at which we wish to find the magnetic field), $\vec{B}$ is perpendicular to the plane determined by $\vec{v}$ and $\vec{r}$ (the vector that locates P relative to q). We know from our observations that the magnitude of $\vec{B}$ is directly proportional to q, v, and $\sin\phi$, and inversely proportional to r^2:

$$B \propto \frac{qv \sin\phi}{r^2}. \qquad (33\text{-}1)$$

The direction of $\vec{B}$ relative to $\vec{v}$ and $\vec{r}$ reminds us of the rule for finding a vector (cross) product. In the expression $\vec{c} = \vec{a} \times \vec{b}$, the vector $\vec{c}$ is perpendicular to the plane containing $\vec{a}$ and $\vec{b}$. We can therefore write Eq. 33-1 in vector form as

$$\vec{B} = K\frac{q\vec{v} \times \hat{r}}{r^2}, \qquad (33\text{-}2)$$

where K is a constant of proportionality to be determined. Here $\hat{r}$ is the unit vector in the direction of $\vec{r}$. (You may wish to review Section 25-4, where we used a similar unit vector notation to express Coulomb's law.) Because $\hat{r} = \vec{r}/r$, we can write Eq. 33-2 as

$$\vec{B} = K\frac{q\vec{v} \times \vec{r}}{r^3}. \qquad (33\text{-}3)$$

Even though there is a factor of r^3 in the denominator, the field varies as $1/r^2$, because there is also a factor of r in the numerator.

All that remains to obtain the complete expression for the magnetic field of a moving charge is to determine the constant of proportionality in Eq. 33-3, just as we inserted the constant $1/4\pi\epsilon_0$ into Coulomb's law. The constants in the magnetic and electric field equations are not independent quantities; they are related by the speed of light, as we discuss in Chapter 38. Since the speed of light is a defined quantity, our choices are either (1) to use the electric force law (Coulomb's law) to define the electric constant and then use the value of the speed of light to obtain the magnetic constant, or (2) to use a magnetic force law (the force between current-carrying wires, which we discuss later in this chapter) to define the magnetic constant and then use the value of the speed of light to obtain the electric constant. Because the magnetic force between current-carrying wires can be measured more precisely than the electric force between charges, we choose the second method.

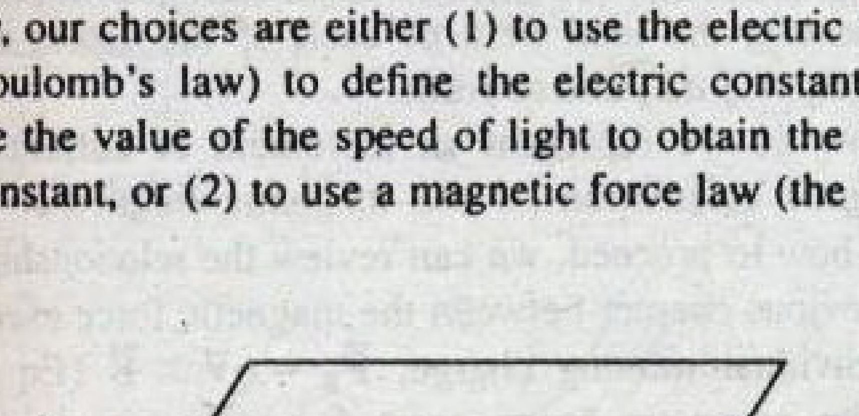


FIGURE 33-4. The magnetic field at point P due to a moving charge is perpendicular to the plane containing $\vec{v}$ and $\vec{r}$.

The proportionality constant K in Eq. 33-3 is defined in SI units to have the exact value 10^{-7} tesla · meter/ampere (T · m/A). However, as was the case in electrostatics, we find it convenient to write the constant in a different form:

$$K = \frac{\mu_0}{4\pi} = 10^{-7}\,\text{T}\cdot\text{m/A}.$$

The constant μ_0 has historically been known as the *permeability constant*, but we will refer to it simply as the *magnetic constant*. It has the exact value

$$\mu_0 = 4\pi \times 10^{-7}\,\text{T}\cdot\text{m/A}.$$

The magnetic constant μ_0 plays a role in calculating magnetic fields similar to that of the electric constant ϵ_0 in calculating electric fields. The two constants are related through the speed of light: $c = (\epsilon_0\mu_0)^{-1/2}$. Defining μ_0 and c then determines ϵ_0 exactly.

We are now able to write the complete expression for the magnetic field due to a moving charge:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{q\vec{v} \times \vec{r}}{r^3}. \qquad (33\text{-}4)$$

We can write the magnitude of $\vec{B}$ as

$$B = \frac{\mu_0}{4\pi}\frac{|q|v \sin\phi}{r^2}, \qquad (33\text{-}5)$$

where ϕ is the angle between $\vec{v}$ and $\vec{r}$.

Based on these thought experiments, we have been able to learn several properties of the magnetic field produced by a moving charge, including the important geometrical relationship between the direction of the velocity and the direction of the field. It will now be trivial to transfer this relationship to the more useful case of the magnetic field produced by currents in wires.

Why is the magnetic field specified by Eqs. 33-4 or 33-5 not especially useful? When we studied electric fields, our interest was in the steady electric field produced by charges whose locations did not change. We called this subject *electrostatics*. Now we are interested in *magnetostatics*—the production of steady magnetic fields by charges whose motion does not change. The single moving charge of Fig. 33-4 does not satisfy this criterion; an instant after the situation illustrated in the figure, the charge is at a different location with respect to point P, and there is no charge at all at its previous location. An instant later there will be a different magnetic field at P. To maintain a steady magnetic field at P due to a moving charge at the exact location shown, it would be necessary to arrange to destroy the charge as soon as it left that location and to inject a new charge into that location with the same velocity, a highly improbable scenario. On the other hand, a steady current accomplishes exactly what we want: an unchanging motion of charges that produces a steady magnetic field. In the next section we adapt Eqs. 33-4 and 33-5 to the case of steady currents.

SAMPLE PROBLEM 33-1. An alpha particle ($q = +2e$) is moving in the positive x direction with a speed of $0.0050c = 1.50 \times 10^6$ m/s. When the particle is at the origin, find the magnetic field at (*a*) P_1: $x = 0$, $y = 0$, $z = +2.0$ cm; (*b*) P_2: $x = 0$, $y = +2.0$ cm, $z = 0$; (*c*) P_3: $x = +1.0$ cm, $y = +1.0$ cm, $z = +1.0$ cm.

Solution (*a*) In Fig. 33-5, $\vec{r}_1$ is in the $+z$ direction (pointing from q to P_1). The length of $\vec{r}_1$ is the distance from the origin to P_1, or 2.0 cm. The vectors $\vec{v}$ and $\vec{r}_1$ are in the xz plane; $\vec{B}$ must be perpendicular to that plane, or in the positive or negative y direction. The direction of $\vec{v} \times \vec{r}_1$ determines that $\vec{B}$ is in the $-y$ direction. The magnitude of $\vec{B}$ is given by Eq. 33-5:

$$B = \frac{\mu_0}{4\pi}\frac{qv\sin\phi}{r_1^2}$$

$$= (10^{-7}\ \text{T}\cdot\text{m/A})\frac{(2)(1.60 \times 10^{-19}\ \text{C})(1.50 \times 10^6\ \text{m/s})(\sin 90^\circ)}{(0.020\ \text{m})^2}$$

$$= 1.2 \times 10^{-16}\ \text{T}.$$

(*b*) At P_2, $\vec{r}_2$ is in the $+y$ direction, and so $\vec{v} \times \vec{r}_2$ determines that $\vec{B}$ is in the $+z$ direction. Because the distance from q to P_2 is the same as the distance from q to P_1, the magnitude of $\vec{B}$ at P_2 is the same as we found in part (*a*) for P_1. In fact, $\vec{B}$ has this same magnitude at all points on the circle of radius $r = 2.0$ cm drawn about q in the yz plane.

(*c*) At P_3, $r_3 = \sqrt{x^2 + y^2 + z^2} = 1.73$ cm. The vector $\vec{r}_3$ from q to P_3 and the vector $\vec{v}$ form a plane that makes an angle of 45° with the y and z axes (Fig. 33-5). The angle ϕ between $\vec{v}$ and $\vec{r}_3$ is 54.7°, as you should show. The magnitude of $\vec{B}$ at P_3 is

$$B = \frac{\mu_0}{4\pi}\frac{qv\sin\phi}{r_3^2}$$

$$= (10^{-7}\ \text{T}\cdot\text{m/A})$$

$$\times \frac{(2)(1.60 \times 10^{-19}\ \text{C})(1.50 \times 10^6\ \text{m/s})(\sin 54.7^\circ)}{(0.0173\ \text{m})^2}$$

$$= 1.3 \times 10^{-16}\ \text{T}.$$

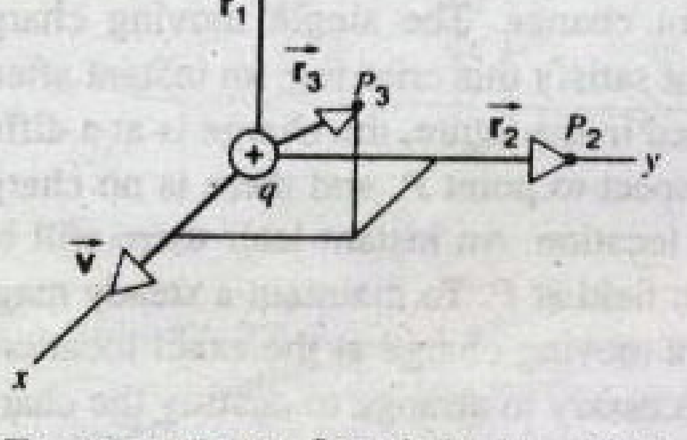


FIGURE 33-5. Sample Problem 33-1.

$\vec{B}$ is perpendicular to the plane of $\vec{r}_3$ and $\vec{v}$. Its Cartesian com nents B_y and B_z are equal in magnitude, but B_y is negative and is positive. Equivalently, $\vec{B}$ is tangent to the circle drawn ab the x axis but centered at $x = 1.0$ cm, not $x = 0$. This circle radius $\sqrt{y^2 + z^2} = 1.41$ cm. Note in this case that, in contras parts (*a*) and (*b*), r_3 is not the radius of the circle to which $\vec{B}$ tangent.

The magnetic fields we found in this problem are very sm about 12 orders of magnitude smaller than the Earth's field. can see from this calculation why it is hopeless to try to meas the force between individual charged particles in motion. Ho ever, the fields due to individual particles at distances r co sponding to atomic dimensions (10^{-10} m) are of order 1 T, wh can certainly produce measurable effects. On the scale of ato the magnetic force between moving charged particles often h observable consequences, as we discuss in Chapter 35.

33-2 THE MAGNETIC FIELD OF A CURRENT

In the laboratory we produce magnetic fields using curre carrying wires rather than the motion of individual charge In this section we extend the results of the previous sectio to permit us to find the magnetic field due to a current. O strategy is first to find the field due to the current in a shor element of the wire, and then to use integration methods t find the field due to the current in the entire wire.

This method is similar to the one we used in Sectio 26-4 to find the electric field due to a continuous charg distribution. In Chapter 26, we began with the electr field due to a point charge, which we can write as $\vec{E} = (q/4\pi\epsilon_0 r^2)\hat{r}$. To find the electric field due to a continuou charge distribution, we imagined the object to be compose of infinitesimal elements of charge dq, each of which coul be treated as a point charge in calculating its contributio $d\vec{E}$ to the electric field: $d\vec{E} = (dq/4\pi\epsilon_0 r^2)\hat{r}$. The total ele tric field is found by adding the contributions of all th charge elements: $\vec{E} = \int d\vec{E}$, which is a shorthand way of representing the total field of each of the vector compo nents: $E_x = \int dE_x$ and so forth.

How do we represent an increment of current in the analogous calculation of the magnetic field? For a clu about how to proceed, we can review the relationship from the previous chapter between the magnetic force exerted on an individual moving charge, $\vec{F}_B = q\vec{v} \times \vec{B}$ (Eq. 32-3), and the magnetic force exerted on a current elemen $d\vec{F}_B = i\,d\vec{L} \times \vec{B}$ (Eq. 32-29). In this equation, $d\vec{L}$ is a vector whose length is equal to the length of the element of the wire and whose direction is the direction of the curren in that element. That is, we can go from an expression de scribing the force on an individual moving charge to one describing the force on a current element by replacing $q\vec{v}$ by $i\,d\vec{L}$.

We can modify Eq. 33-4 in exactly the same way. We seek the contribution $d\vec{B}$ to the magnetic field due to a current element, which we can consider to be represented by a charge element dq moving with velocity $\vec{v}$:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{dq\vec{v} \times \hat{r}}{r^2}. \qquad (33\text{-}6)$$

We can write the velocity as $\vec{v} = d\vec{s}/dt$, so that the charge dq moves through the displacement $d\vec{s}$ in the time interval dt. We now have

$$dq\,\vec{v} = dq\frac{d\vec{s}}{dt} = \frac{dq}{dt}d\vec{s} = i\,d\vec{s}. \qquad (33\text{-}7)$$

Substituting Eq. 33-7 into Eq. 33-6, we obtain

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i\,d\vec{s} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{i\,d\vec{s} \times \vec{r}}{r^3}. \qquad (33\text{-}8)$$

This expression is known as the *Biot–Savart law*. The direction of $d\vec{B}$ is the same as the direction of $d\vec{s} \times \vec{r}$. The magnitude of the field element $d\vec{B}$ is

$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds\sin\phi}{r^2}, \qquad (33\text{-}9)$$

where ϕ is the angle between the vector $d\vec{s}$, which indicates the direction of the current, and the vector $\vec{r}$ from the current element to the observation point P. Figure 33-6 shows the vector relationships; compare with Fig. 33-4 and note how similar the two figures are.

To find the total field $\vec{B}$ due to the entire current distribution, we must integrate over all current elements $i\,d\vec{s}$:

$$\vec{B} = \int d\vec{B} = \frac{\mu_0}{4\pi}\int\frac{i\,d\vec{s} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi}\int\frac{i\,d\vec{s} \times \vec{r}}{r^3}. \qquad (33\text{-}10)$$

Just as we did in Chapter 26 for electric fields, we will in general have to take into account in computing this integral that not all of the elements of $d\vec{B}$ are in the same direction; see Section 26-4.

We now consider how to apply the Biot–Savart law to calculate the magnetic fields of some current-carrying wires of different shapes.

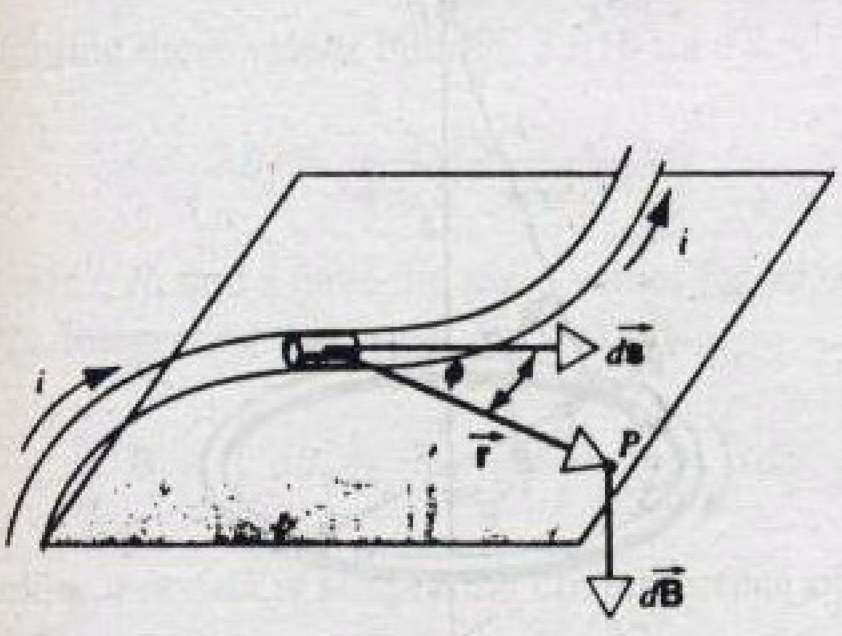


FIGURE 33-6. The magnetic field $d\vec{B}$ produced by an element of a current-carrying wire. In analogy with Fig. 33-4, the field is perpendicular to the plane containing $d\vec{s}$ and $\vec{r}$.

A Straight Wire Segment

We illustrate the use of the Biot–Savart law by applying it to the calculation of the magnetic field due to a current i in a straight wire segment of length L. Figure 33-7 shows the geometry. The wire lies along the z axis, and we want to find $\vec{B}$ at point P on the y axis, a distance d from the wire. The center of the wire is at the origin, so P is on the perpendicular bisector of the wire. The first step in the calculation is to choose an arbitrary element of the wire $i\,d\vec{s}$, which is located at coordinate z relative to the origin. The contribution $d\vec{B}$ of this element to the field at P is given by Eq. 33-8 and involves the vector cross product $d\vec{s} \times \vec{r}$. Using the right-hand rule we can show that, in the geometry of Fig. 33-7, $d\vec{s} \times \vec{r}$ is a vector that points in the negative x direction, and we see that this is true no matter where on the wire we choose the current element. Every element $i\,d\vec{s}$ of the wire gives a $d\vec{B}$ in the negative x direction, and therefore when we add up all the elements $d\vec{B}$ we will find that the total field is in the negative x direction. Since we have now obtained the direction of $\vec{B}$, we can turn to finding its magnitude using Eq. 33-9.

With $d\vec{s}$ in the z direction we have $ds = dz$, and z will be our variable of integration, ranging from $-L/2$ to $+L/2$. To integrate Eq. 33-9, we must first express ϕ and r in terms of the integration variable z:

$$r = \sqrt{z^2 + d^2}$$

and

$$\sin\phi = \sin(\pi - \phi) = \frac{d}{\sqrt{z^2 + d^2}}.$$

Making these substitutions into Eq. 33-9, we obtain

$$dB = \frac{\mu_0 i}{4\pi}\frac{dz\sin\phi}{r^2} = \frac{\mu_0 i}{4\pi}\frac{d}{(z^2 + d^2)^{3/2}}dz. \qquad (33\text{-}11)$$

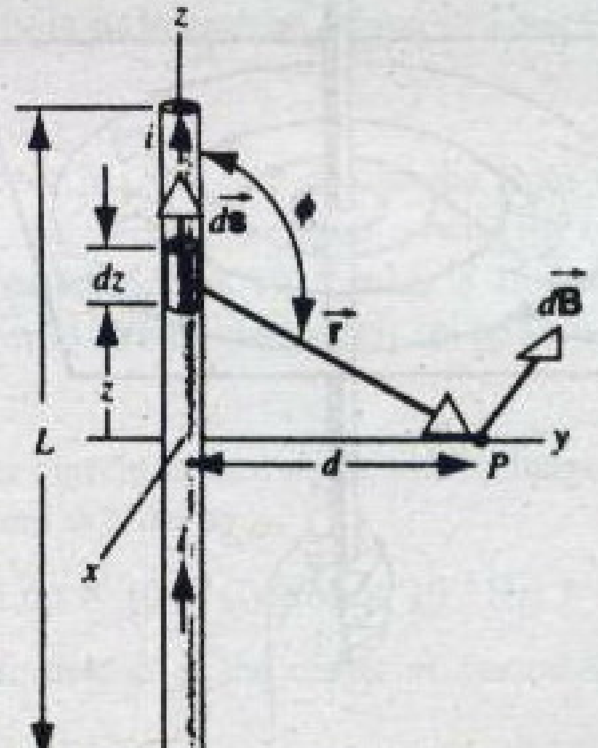


FIGURE 33-7. An element $i\,d\vec{s}$ in a straight wire segment establishes at point P a field $d\vec{B}$ in the negative x direction.

Carrying out the integral, we find the total field:

$$B = \frac{\mu_0 i d}{4\pi} \int_{-L/2}^{+L/2} \frac{dz}{(z^2 + d^2)^{3/2}} = \frac{\mu_0 i}{4\pi d} \frac{z}{(z^2 + d^2)^{1/2}} \bigg|_{z=-L/2}^{z=+L/2}$$

or

$$B = \frac{\mu_0 i}{4\pi d} \frac{L}{(L^2/4 + d^2)^{1/2}}. \qquad (33\text{-}12)$$

In the limit of a very long wire (that is, $L >> d$), Eq. 33-12 becomes

$$B = \frac{\mu_0 i}{2\pi d}. \qquad (33\text{-}13)$$

This problem reminds us of its electrostatic equivalent. We derived an expression for $\vec{E}$ due to a long charged rod by integration methods, using Coulomb's law (Section 26-4). We also solved the same problem using Gauss' law (Section 27-5). Later we consider a law of magnetic fields, Ampère's law, which is similar to Gauss' law in that it simplifies magnetic field calculations in cases (such as this one) that have a high degree of symmetry.

Just as we did for electric fields, we can represent the magnetic field of a current-carrying wire by magnetic field lines. Figure 33-8 shows a set of magnetic field lines for a long, straight wire. The field lines form concentric circles around the wire, as suggested by Oersted's experiment (Fig. 33-1) and as also indicated by the pattern of iron filings near a wire (Fig. 33-9).

At any point, the direction of $\vec{B}$ is tangent to the field line at that point. The field is large where the field lines are close together (such as near the wire) and small where the field lines are farther apart (far from the wire). Contrary to electric field lines due to charges, which begin on positive charges and end on negative charges, the magnetic field lines due to currents form continuous loops with no beginning or end. To find the direction of the field lines, we use the right-hand rule: if you were to grasp the wire in your right hand with your thumb in the direction of the current, your fingers would curl around the wire in the direction of the magnetic field.

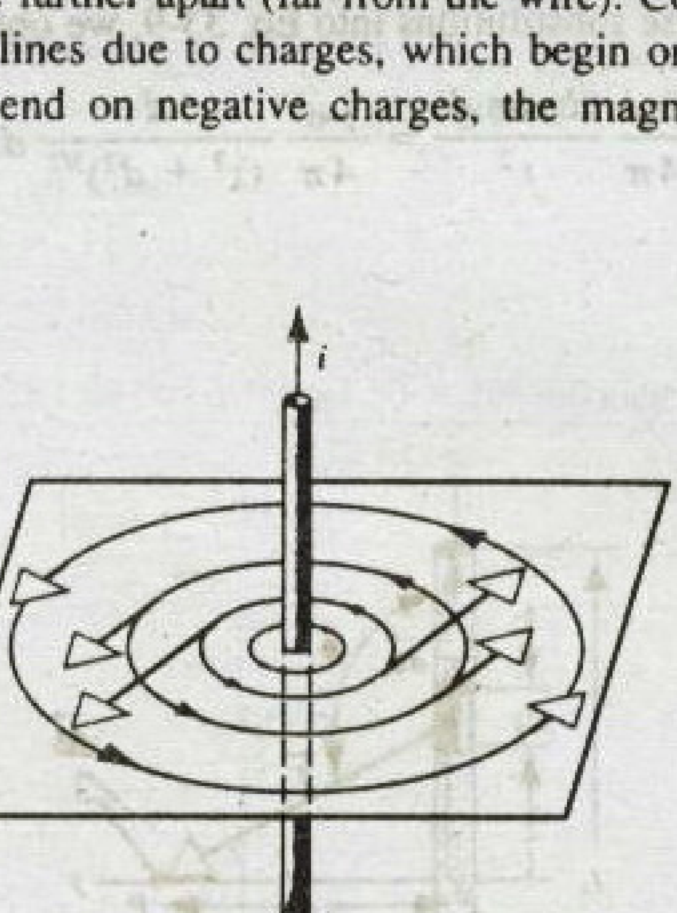


FIGURE 33-8. The lines of the magnetic field are concentric circles for a long, straight, current-carrying wire. Their direction is given by the right-hand rule.

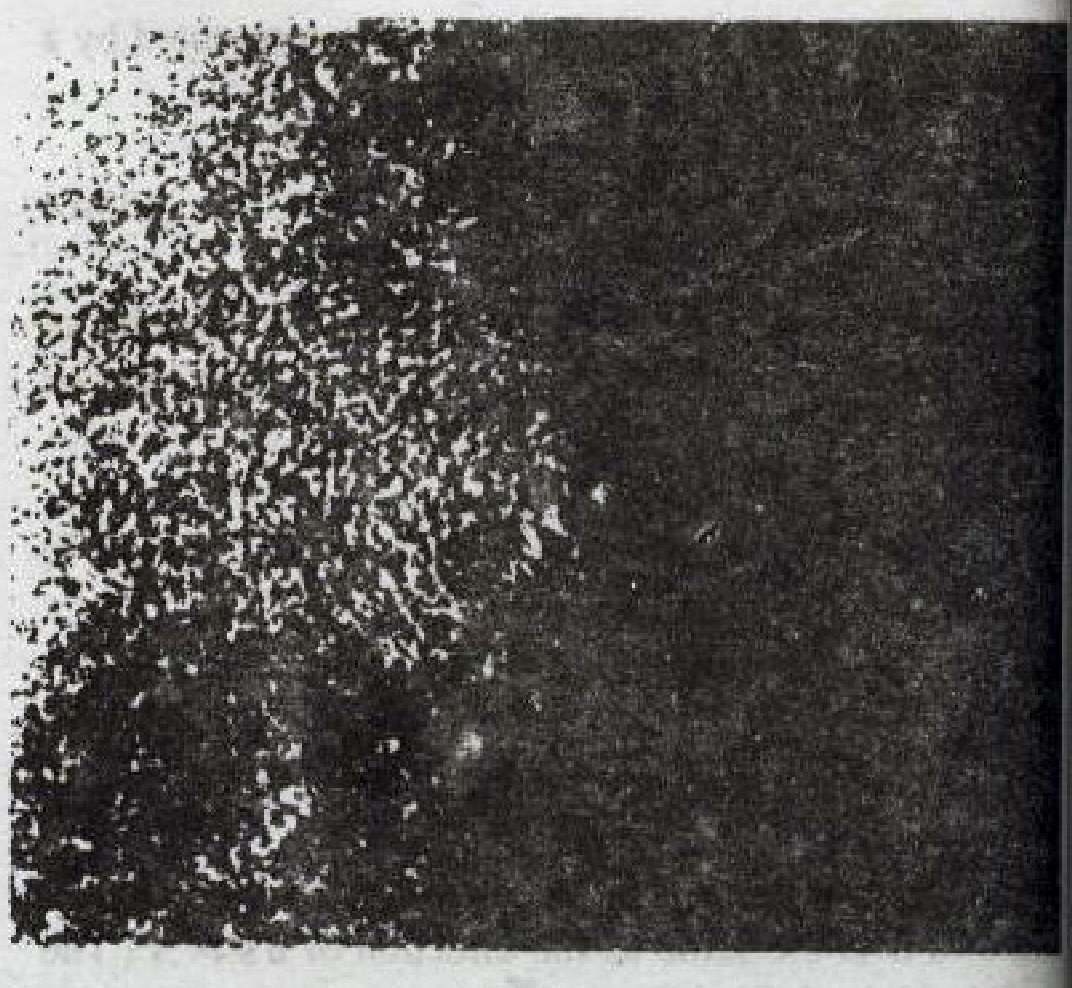


FIGURE 33-9. The vertical wire carries a current, which sets up a magnetic field. Iron filings sprinkled onto cardboard show the pattern of concentric circles that represents the field.

A Circular Current Loop

Figure 33-10 shows a circular loop of radius R carrying a current i. Let us calculate $\vec{B}$ at a point P on the axis a distance z from the center of the loop.

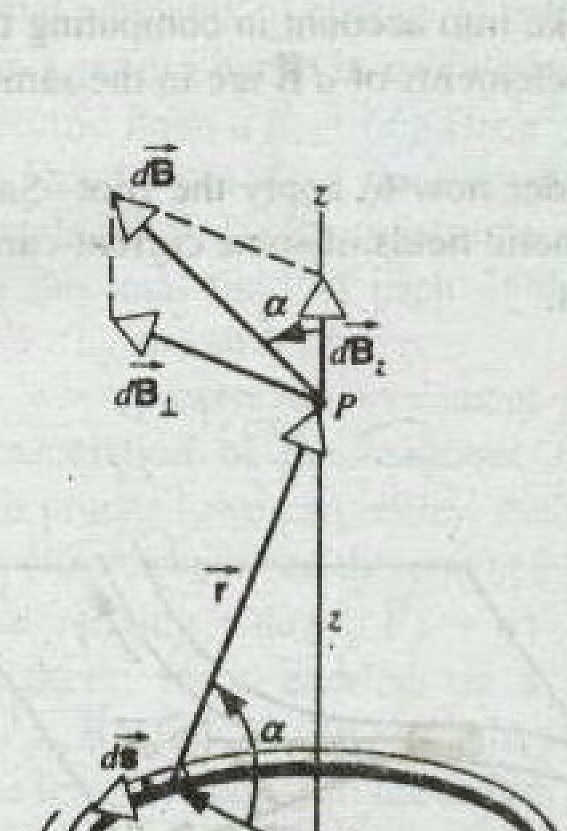


FIGURE 33-10. A circular loop of current. The element $i\,d\vec{s}$ of the loop sets up a field $d\vec{B}$ at a point P on the axis of the loop.

The angle ϕ between the current element $i\,d\vec{s}$ and $\vec{r}$ is 90°. From the Biot–Savart law, we know that the vector $d\vec{B}$ for this element is at right angles to the plane formed by $i\,d\vec{s}$ and $\vec{r}$ and thus lies at right angles to $\vec{r}$, as the figure shows.

Let us resolve $d\vec{B}$ into two components; one, $d\vec{B}_z$, along the axis of the loop and another, $d\vec{B}_\perp$, at right angles to the axis. Only $d\vec{B}_z$ contributes to the total magnetic field $\vec{B}$ at point P. This follows because the components $d\vec{B}_z$ for all current elements lie on the axis and add directly; however, the components $d\vec{B}_\perp$ point in different directions perpendicular to the axis, and the sum of all $d\vec{B}_\perp$ for the complete loop is zero, from symmetry. (A diametrically opposite current element, indicated in Fig. 33-10, produces the same $d\vec{B}_z$ but $d\vec{B}_\perp$ in the opposite direction.) We can therefore replace the vector integral over all $d\vec{B}$ with an integral over the z components only, and the magnitude of the field is given by

$$B = \int dB_z. \tag{33-14}$$

For the current element in Fig. 33-10, the Biot–Savart law (Eq. 33-9) gives

$$dB = \frac{\mu_0 i}{4\pi}\frac{ds \sin 90^\circ}{r^2}. \tag{33-15}$$

We also have

$$dB_z = dB \cos\alpha,$$

which, combined with Eq. 33-15, gives

$$dB_z = \frac{\mu_0 i \cos\alpha\, ds}{4\pi r^2}. \tag{33-16}$$

Figure 33-10 shows that r and α are not independent of each other. Let us express each in terms of z, the distance from the center of the loop to the point P. The relationships are

$$r = \sqrt{R^2 + z^2}$$

and

$$\cos\alpha = \frac{R}{r} = \frac{R}{\sqrt{R^2 + z^2}}.$$

Substituting these values into Eq. 33-16 for dB_z gives

$$dB_z = \frac{\mu_0 i R}{4\pi(R^2 + z^2)^{3/2}}\,ds. \tag{33-17}$$

Note that i, R, and z have the same values for all current elements. Integrating this equation, we obtain

$$B = \int dB_z = \frac{\mu_0 i R}{4\pi(R^2 + z^2)^{3/2}}\int ds \tag{33-18}$$

or, noting that $\int ds$ is simply the circumference of the loop ($= 2\pi R$),

$$B = \frac{\mu_0 i R^2}{2(R^2 + z^2)^{3/2}}. \tag{33-19}$$

We can repeat the above calculation to find the field at the center of the loop. In this case $r = R$ everywhere, and the Biot–Savart law gives

$$B = \int dB_z = \frac{\mu_0 i}{4\pi R^2}\int ds. \tag{33-20}$$

If we integrate around the circle, the integral is again $2\pi R$, and so

$$B = \frac{\mu_0 i}{2R}, \tag{33-21}$$

which we could have obtained by setting $z = 0$ in Eq. 33-19. However, we can use this method to obtain a more general result when the current flows not in a complete circle but in an arc of a circle. Suppose the arc subtends an angle θ at its center. Then in Eq. 33-20, the integral gives not the full circumference of the circle but only the arc length $R\theta$ (which is equal to $2\pi R$ when $\theta = 2\pi$). The field at the center of the arc is then

$$B = \frac{\mu_0 i \theta}{4\pi R}. \tag{33-22}$$

The angle θ must be expressed in radians in this equation. The right-hand rule again gives the direction of the magnetic field, which is along the z axis.

If $z \gg R$, so that points close to the loop are not considered, Eq. 33-19 becomes

$$B = \frac{\mu_0 i R^2}{2z^3}. \tag{33-23}$$

This dependence of the field on the inverse cube of the distance reminds us of the electric field of an electric dipole (see Eq. 26-12 and also see Problem 1 of Chapter 26 for the field on the dipole axis). It is often convenient to consider a loop of wire to be a *magnetic dipole*. Just as the electric behavior of many molecules can be characterized in terms of their electric dipole moment, so also the magnetic behavior of atoms can be described in terms of their *magnetic dipole moment*. In the case of atoms, the current loop is due to the circulation of electrons about the nucleus. We discuss the magnetic dipole moments of atoms in Chapter 35.

SAMPLE PROBLEM 33-2. In the Bohr model of the hydrogen atom, the electron circulates around the nucleus in a circular path of radius 5.29×10^{-11} m at a frequency f of 6.60×10^{15} Hz (or rev/s). What value of B is set up at the center of the orbit?

Solution The current is the rate at which charge passes any point on the orbit and is given by

$$i = ef = (1.60 \times 10^{-19}\ \text{C})(6.60 \times 10^{15}\ \text{Hz}) = 1.06 \times 10^{-3}\ \text{A}.$$

The magnetic field B at the center of the orbit is given by Eq. 33-21,

$$B = \frac{\mu_0 i}{2R} = \frac{(4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A})(1.06 \times 10^{-3}\ \text{A})}{2(5.29 \times 10^{-11}\ \text{m})} = 12.6\ \text{T}.$$

33-3 TWO PARALLEL CURRENTS

In this section we use long wires carrying parallel (or antiparallel) currents to illustrate two properties of magnetic fields: the addition of the fields due to different wires and the force exerted by one wire on another.

First we consider the vector addition of the fields due to two different parallel wires, as shown in Fig. 33-11. Two wires are perpendicular to the plane of the figure, and they carry currents in opposite directions. We wish to find the magnetic field at point P due to the two wires. The magnetic field lines due to wire 1 form concentric circles about that wire, and the magnitude of the field at the distance r_1 is given by Eq. 33-13, $B = \mu_0 i_1/2\pi r_1$. The direction of $\vec{\mathbf{B}}_1$ is tangent to the circular arc passing through P; equivalently, $\vec{\mathbf{B}}_1$ is perpendicular to $\vec{\mathbf{r}}_1$, the radial vector from the wire to P.

Similarly, the field due to wire 2 is shown in the figure as $\vec{\mathbf{B}}_2$ and is tangent to the circular magnetic field lines and perpendicular to $\vec{\mathbf{r}}_2$. To find the net field at P, we take the vector sum of the fields due to the two individual wires: $\vec{\mathbf{B}} = \vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2$. The magnitude and direction of the total field can be found using the usual rules for vector addition.

The situation shown in Fig. 33-11 is similar to the method for calculating the total electric field due to two point charges q_1 and q_2: we find the individual fields at point P due to each charge, and then the vector sum gives the total field, $\vec{\mathbf{E}} = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2$. To observe this total electric field at P, we could measure the force on a third charged particle placed at that point. Similarly, to observe the total magnetic field at P in Fig. 33-11, we could measure the force on a charged particle moving through that point or on a third wire carrying a current through P.

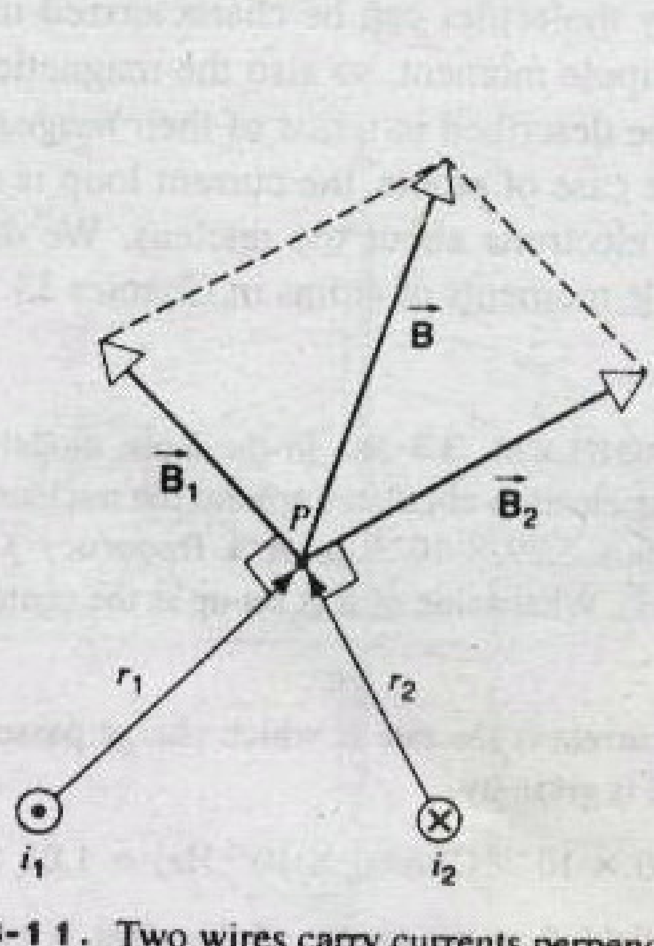


FIGURE 33-11. Two wires carry currents perpendicular to the page; i_1 is out of the page (represented by ⊙, suggesting the pointed tip of an arrow) and i_2 is into the page (represented by ⊗, suggesting the "tailfeathers" of an arrow). The total field at point P is the vector sum of $\vec{\mathbf{B}}_1$ and $\vec{\mathbf{B}}_2$.

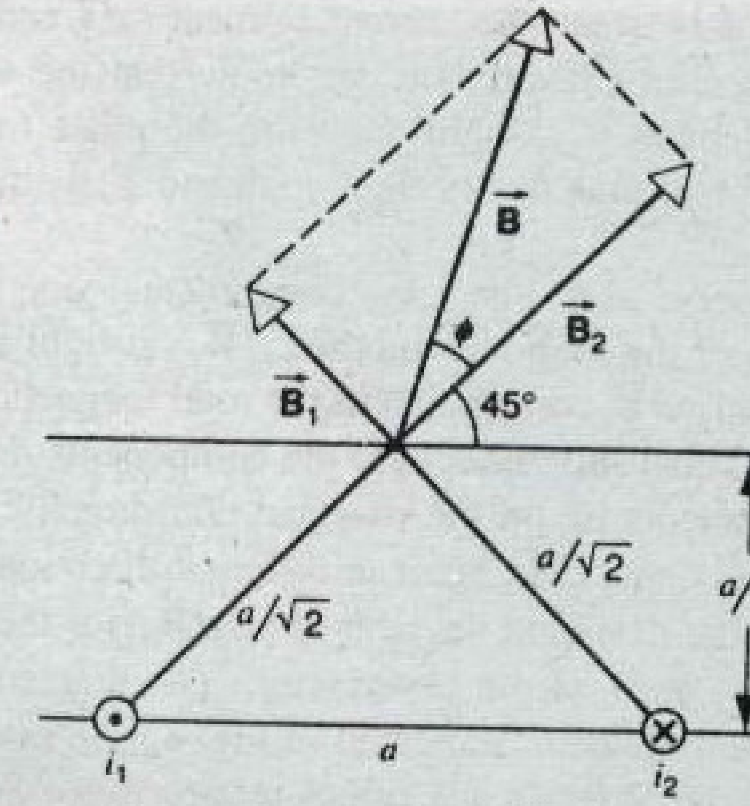


FIGURE 33-12. Sample Problem 33-3. Current i_1 is out of the page and i_2 is into the page.

SAMPLE PROBLEM 33-3. In Fig. 33-12, let i_1 = 15 A and i_2 = 32 A. The two wires are separated by a distance a = 5.3 cm. Find the total magnetic field at a point a distance $a/2$ along a line perpendicular to the line connecting the two wires.

Solution Figure 33-12 shows the geometry and the fields $\vec{\mathbf{B}}_1$ and $\vec{\mathbf{B}}_2$. With $d_1 = d_2 = a/\sqrt{2}$, the magnitudes of the fields are

$$B_1 = \frac{\mu_0 i_1}{2\pi d_1} = \frac{(4\pi \times 10^{-7}\ \mathrm{T\cdot m/A})(15\ \mathrm{A})}{2\pi(0.053\ \mathrm{m})/\sqrt{2}} = 80\ \mu\mathrm{T},$$

$$B_2 = \frac{\mu_0 i_2}{2\pi d_2} = \frac{(4\pi \times 10^{-7}\ \mathrm{T\cdot m/A})(32\ \mathrm{A})}{2\pi(0.053\ \mathrm{m})/\sqrt{2}} = 171\ \mu\mathrm{T}.$$

In the special geometry of Fig. 33-12 the two fields are perpendicular, so

$$B = \sqrt{B_1^2 + B_2^2} = 190\ \mu\mathrm{T}.$$

The angle ϕ between $\vec{\mathbf{B}}$ and $\vec{\mathbf{B}}_2$ is

$$\phi = \tan^{-1}\frac{B_1}{B_2} = 25°,$$

so the angle between $\vec{\mathbf{B}}$ and the horizontal axis is 25° + 45° = 70°.

SAMPLE PROBLEM 33-4. Two long, parallel wires a distance $2b$ apart carry equal currents i in opposite directions, as shown in Fig. 33-13a. Derive an expression for the magnetic field B at a point P on the line connecting the wires and a distance x from the point midway between them.

Solution Study of Fig. 33-13a shows that $\vec{\mathbf{B}}_1$ due to the current i_1 and $\vec{\mathbf{B}}_2$ due to the current i_2 point in the same direction at P. Each is given by Eq. 33-13 ($B = \mu_0 i/2\pi d$) so that

$$B = B_1 + B_2 = \frac{\mu_0 i}{2\pi(b+x)} + \frac{\mu_0 i}{2\pi(b-x)} = \frac{\mu_0 ib}{\pi(b^2 - x^2)}.$$

Inspection of this result shows that (1) B is symmetrical about $x = 0$, (2) B [illegible]s minimum value ($= \mu_0 i/\pi b$) at $x = 0$, and (3) $B \rightarrow \infty$ as $x \rightarrow \pm b$. This last conclusion is not correct, because Eq. 33-13 cannot be applied to points inside the wires. In re-

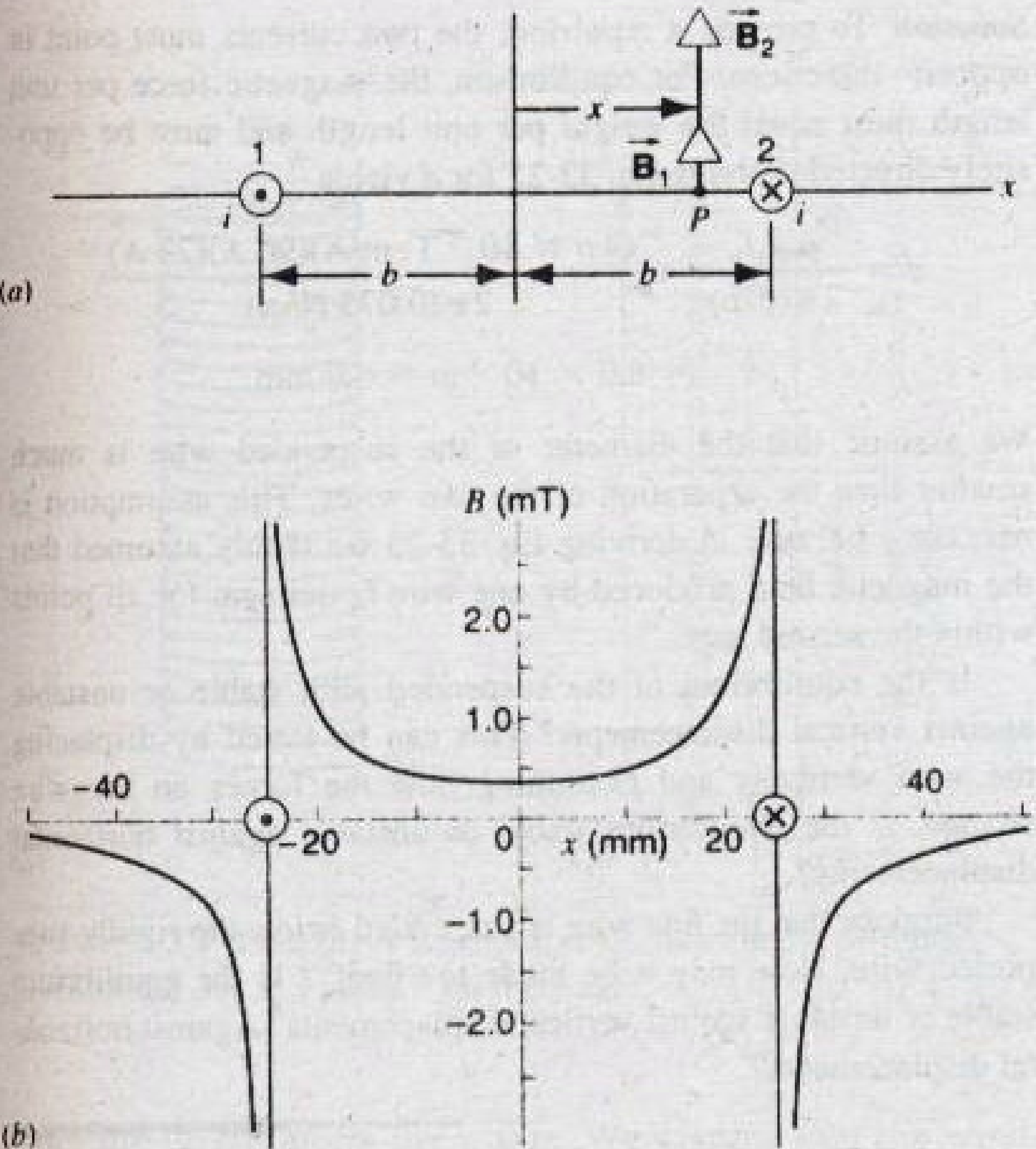


FIGURE 33-13. Sample Problem 33-4. (*a*) The magnetic fields at point P due to the currents in wires 1 and 2. (*b*) The resultant field at P, calculated for $i = 25$ A and $b = 25$ mm.

ality the field due to each wire would vanish at the center of that wire.

You should show that our result for the combined field remains valid at points where $|x| > b$. Figure 33-13*b* shows the variation of B with x for $i = 25\ A$ and $b = 25$ mm.

SAMPLE PROBLEM 33-5. Figure 33-14 shows a flat strip of copper of width a and negligible thickness carrying a current i. Find the magnetic field $\vec{\mathbf{B}}$ at point P, at a distance R from the center of the strip along its perpendicular bisector.

Solution Let us subdivide the strip into long, infinitesimal filaments of width dx, each of which may be treated as a wire carrying a current element di given by $i(dx/a)$. For the current element in the left half of the strip in Fig. 33-14, the magnitude dB of the field at P is given by the differential form of Eq. 33-13, or

$$dB = \frac{\mu_0}{2\pi}\frac{di}{d} = \frac{\mu_0}{2\pi}\frac{i(dx/a)}{R\sec\theta},$$

in which $d = R/\cos\theta = R\sec\theta$. Note that the vector $d\vec{\mathbf{B}}$ is at right angles to the line marked d.

Only the horizontal component of $d\vec{\mathbf{B}}$—namely, $dB\cos\theta$—is effective; the vertical component is canceled by the contribution of a symmetrically located current element on the other side of the strip (the second shaded element in Fig. 33-14). Thus B at point P is given by the (scalar) integral

$$B = \int dB\cos\theta = \int \frac{\mu_0 i(dx/a)}{2\pi R\sec\theta}\cos\theta$$

$$= \frac{\mu_0 i}{2\pi aR}\int\frac{dx}{\sec^2\theta}.$$

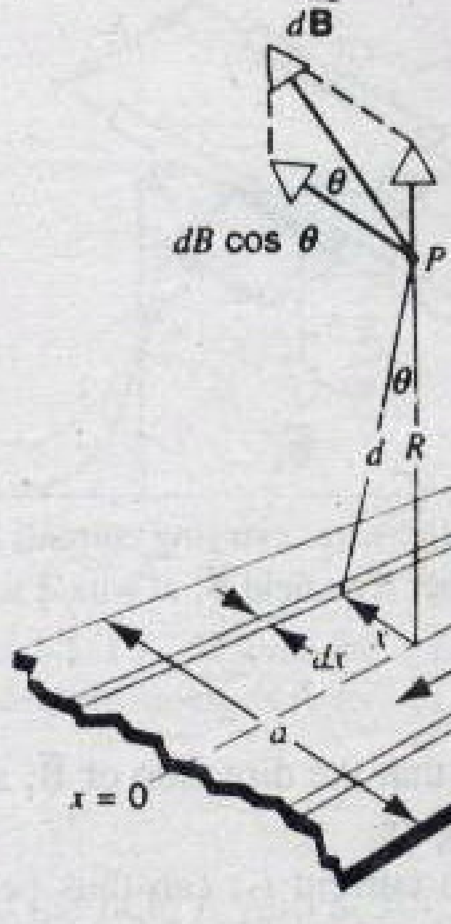


FIGURE 33-14. Sample Problem 33-5. A flat strip of width a carries a current i.

The variables x and θ are not independent, being related by

$$x = R\tan\theta$$

or

$$dx = R\sec^2\theta\, d\theta.$$

The limits on θ are $\pm\alpha$, where $\alpha = \tan^{-1}(a/2R)$. Substituting for dx in the expression for B, we find

$$B = \frac{\mu_0 i}{2\pi aR}\int\frac{R\sec^2\theta\, d\theta}{\sec^2\theta}$$

$$= \frac{\mu_0 i}{2\pi a}\int_{-\alpha}^{+\alpha} d\theta = \frac{\mu_0 i}{\pi a}\alpha = \frac{\mu_0 i}{\pi a}\tan^{-1}\frac{a}{2R}. \quad (33\text{-}24)$$

This is the general result for the magnetic field due to the strip.

At points far from the strip, α is a small angle, for which $\alpha \approx \tan\alpha = a/2R$. Thus we have, as an approximate result,

$$B \approx \frac{\mu_0 i}{\pi a}\left(\frac{a}{2R}\right) = \frac{\mu_0}{2\pi}\frac{i}{R}.$$

This result is expected because at distant points the strip cannot be distinguished from a thin wire (see Eq. 33-13).

The Interaction between Parallel Currents

We now consider a different calculation involving two long, straight wires carrying parallel (or antiparallel) currents. As a result of the magnetic field due to one wire at the location of the other wire, a magnetic force is exerted on the second wire. Similarly, the second wire sets up a magnetic field at the location of the first wire that exerts a force on that wire.

In Fig. 33-15, wire 1 carrying current i_1 produces a magnetic field $\vec{\mathbf{B}}_1$ whose magnitude at the location of the second wire is, according to Eq. 33-13,

$$B_1 = \frac{\mu_0 i_1}{2\pi d}.$$

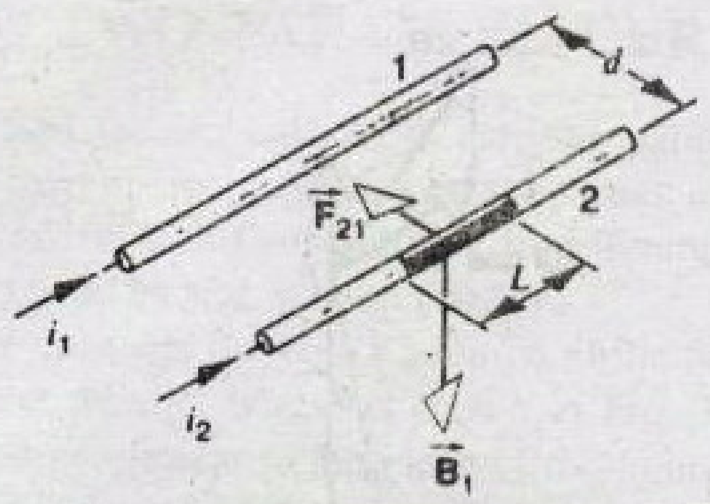


FIGURE 33-15. Two parallel wires carrying currents in the same direction attract each other. The field $\vec{B}_1$ at wire 2 is that due to the current in wire 1.

The right-hand rule shows that the direction of $\vec{B}_1$ at wire 2 is down, as shown in the figure.

Wire 2, which carries a current i_2, can thus be considered to be immersed in an *external* magnetic field $\vec{B}_1$. A length L of this wire experiences a sideways magnetic force $\vec{F}_{21} = i_2\vec{L} \times \vec{B}_1$ of magnitude

$$F_{21} = i_2 L B_1 = \frac{\mu_0 L i_1 i_2}{2\pi d}. \qquad (33\text{-}25)$$

The vector rule for the cross product shows that $\vec{F}_{21}$ lies in the plane of the wires and points toward wire 1 in Fig. 33-15.

We could equally well have started with wire 2 by first computing the magnetic field $\vec{B}_2$ produced by wire 2 at the site of wire 1 and then finding the force $\vec{F}_{12}$ exerted on a length L of wire 1 by the field of wire 2. This force on wire 1 would, for parallel currents, point toward wire 2 in Fig. 33-15. The forces that the two wires exert on each other are equal in magnitude and opposite in direction; they form an action–reaction pair according to Newton's third law.

If the currents in Fig. 33-15 were antiparallel, we would find that the forces on the wires would have the opposite directions: the wires would repel one another. The general rule is:

Parallel currents attract, and antiparallel currents repel.

This rule is in a sense opposite to the rule for electric charges, in that like (parallel) currents attract, but like (same sign) charges repel.

The force between long, parallel wires is used to define the ampere. Given two long, parallel wires of negligible circular cross section separated in vacuum by a distance of 1 meter, the ampere is defined as the current in each wire that would produce a force of 2×10^{-7} newtons per meter of length.

SAMPLE PROBLEM 33-6. A long, horizontal, rigidly supported wire carries a current i_a of 96 A. Directly above it and parallel to it is a fine wire that carries a current i_b of 23 A and weighs 0.073 N/m. How far above the lower wire should this second wire be strung if we hope to support it by magnetic repulsion?

Solution To provide a repulsion, the two currents must point in opposite directions. For equilibrium, the magnetic force per unit length must equal the weight per unit length and must be oppositely directed. Solving Eq. 33-25 for d yields

$$d = \frac{\mu_0 i_a i_b}{2\pi(F/L)} = \frac{(4\pi \times 10^{-7}\ \mathrm{T\cdot m/A})(96\ \mathrm{A})(23\ \mathrm{A})}{2\pi(0.073\ \mathrm{N/m})}$$

$$= 6.0 \times 10^{-3}\ \mathrm{m} = 6.0\ \mathrm{mm}.$$

We assume that the diameter of the suspended wire is much smaller than the separation of the two wires. This assumption is necessary because in deriving Eq. 33-25 we tacitly assumed that the magnetic field produced by one wire is uniform for all points within the second wire.

Is the equilibrium of the suspended wire stable or unstable against vertical displacements? This can be tested by displacing the wire vertically and examining how the forces on the wire change. Is the equilibrium stable or unstable against horizontal displacements?

Suppose that the fine wire is suspended *below* the rigidly supported wire. How may it be made to "float"? Is the equilibrium stable or unstable against vertical displacements? Against horizontal displacements?

33-4 THE MAGNETIC FIELD OF A SOLENOID

Sample Problem 33-5 suggests one way to obtain a uniform magnetic field (that is, a magnetic field that does not vary in either magnitude or direction). A flat strip of conductor, carrying a uniformly distributed current i, sets up a magnetic field given by Eq. 33-24. At points very close to the strip ($R \to 0$ and $\tan^{-1} a/2R \to \pi/2$), Eq. 33-24 becomes $B = \mu_0 i/2a$, which is independent of the distance R from the strip. This reminds us of the electric field near a flat plate carrying a uniform charge density, which likewise does not vary in magnitude or direction. In analogy with the parallel-plate capacitor for electric fields, we could create a device with two flat plates carrying equal currents in opposite directions, where the fields would reinforce in the region between the plates and cancel outside the plates.

A more practical way to obtain a nearly uniform magnetic field is to use a *solenoid*. As indicated in Fig. 33-16*a*, a solenoid is a helical winding on a cylindrical core. The wires carry a current i and are wound tightly together, so that there are n windings per unit length along the solenoid.

In this section we calculate the field along the central axis of the solenoid using our previous result for the magnetic field of a circular loop of wire. The calculation of the field off the axis is difficult using the Biot–Savart law, but in the next section we discuss a different and much easier way to calculate the field off axis.

Figure 33-16*b* shows the geometry for calculating the field on th... we take the symmetry axis of the solenoid to be the z axis, with the origin at the center of the solenoid. We wish to find the field at point P, which is a distance d

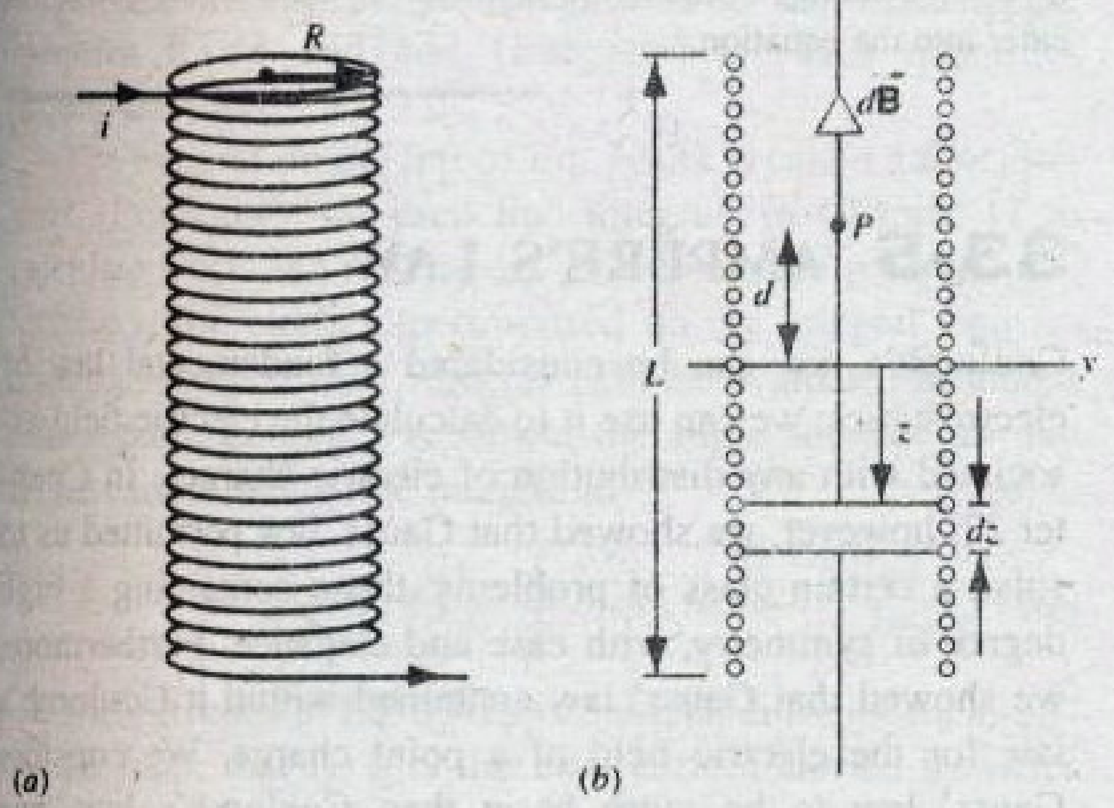


FIGURE 33-16. (*a*) A solenoid. (*b*) A thin ring of width dz gives a field $d\vec{B}$ at a point P on the z axis.

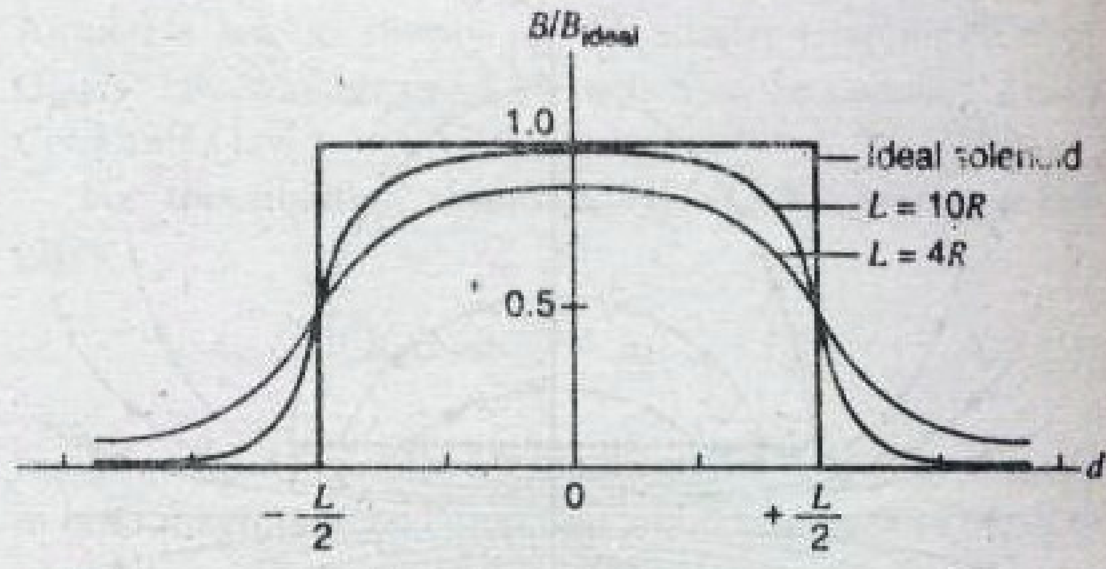


FIGURE 33-17. The magnetic fields of an ideal solenoid and two nonideal solenoids as functions of the distance d from the center.

from the origin along the z axis. We assume that the windings are so narrow that each can be considered as approximately a circular loop of wire, which we assume to be parallel to the xy plane. The solenoid has N turns of wire in a length L, so the number of turns per unit length is $n = N/L$.

Consider a thin ring of width dz. The number of turns in that ring is $n\,dz$, and so the total current carried by the ring is $ni\,dz$, since each turn carries current i. The field at P due to this ring is, using Eq. 33-19,

$$dB = \frac{\mu_0(ni\,dz)R^2}{2[R^2 + (z-d)^2]^{3/2}}, \qquad (33\text{-}26)$$

where $z - d$ is the position of the ring relative to point P. To find the total field due to all such rings, we integrate this expression from $z = -L/2$ to $z = +L/2$. Evaluating the integral (using integral 18 of Appendix I), we obtain

$$B = \frac{\mu_0 niR^2}{2}\int_{-L/2}^{+L/2}\frac{dz}{[R^2+(z-d)^2]^{3/2}}$$
$$= \frac{\mu_0 ni}{2}\left(\frac{L/2+d}{\sqrt{R^2+(L/2+d)^2}} + \frac{L/2-d}{\sqrt{R^2+(L/2-d)^2}}\right). \qquad (33\text{-}27)$$

This expression gives us the field on the axis of the solenoid at a distance d from its center. It is valid for points inside as well as outside the solenoid. The direction of the field is determined as usual using the right-hand rule, so that if the current is circulating counterclockwise as viewed from above, the field is in the positive z direction.

In an ideal solenoid, the length L is much greater than the radius R. In this case Eq. 33-27 becomes

$$B = \mu_0 ni \qquad \text{(ideal solenoid).} \qquad (33\text{-}28)$$

As we show in the next section, Eq. 33-28 gives the field of an ideal solenoid at all interior points, off axis as well as on axis, and the field is zero at all points outside the interior of the solenoid.

The field calculated from Eq. 33-27 is plotted as a function of location along the axis in Fig. 33-17 for an ideal solenoid and for two different nonideal solenoids. Note that as the solenoid becomes longer and narrower, thus approaching ideal behavior, the field along the axis becomes more nearly constant and drops more rapidly to zero beyond the ends of the solenoid.

We can begin to understand the field in the interior of a solenoid by considering the "stretched-out" solenoid illustrated in Fig. 33-18. Very close to each wire, the magnetic behavior is nearly that of a long, straight wire, with the field lines forming concentric circles around the wire. This field tends to cancel at points between adjacent wires. The figure suggests that the fields from the individual loops of wire combine to form field lines that are roughly parallel to the solenoid axis in its interior. In the limiting case of the ideal solenoid, the field becomes uniform and parallel to the axis.

At exterior points, such as point P in Fig. 33-18, the field due to the upper part of the solenoid turns (marked ⊙, because the current is out of the page) points to the left and tends to cancel the field due to the lower parts of the solenoid

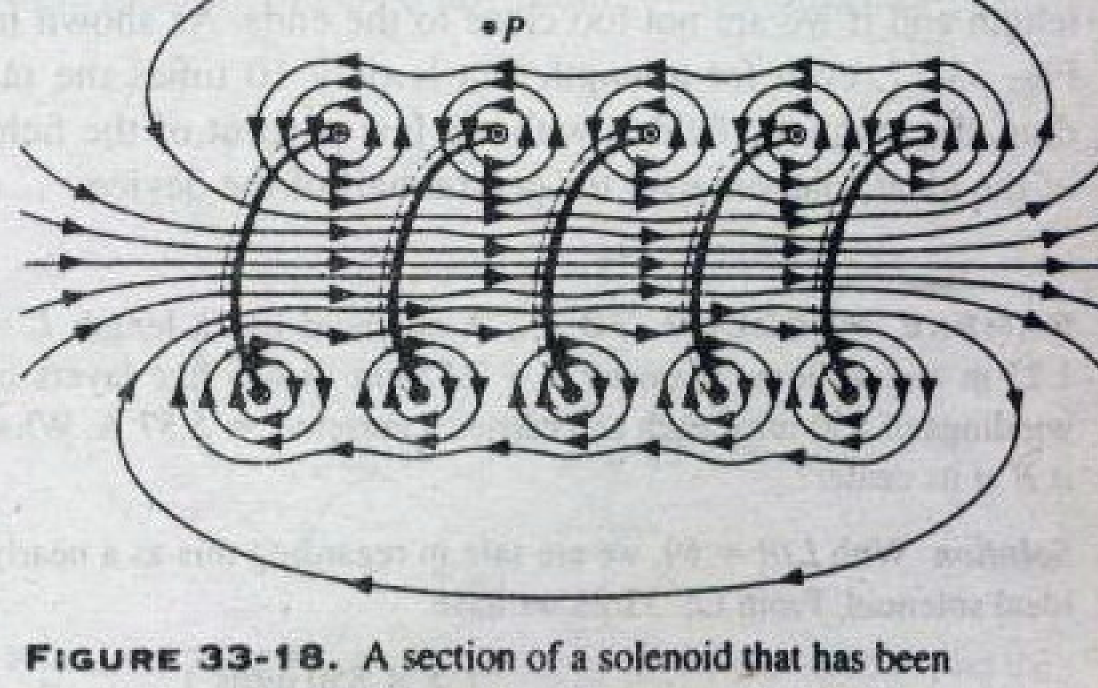


FIGURE 33-18. A section of a solenoid that has been stretched out for this illustration. The magnetic field lines are shown.

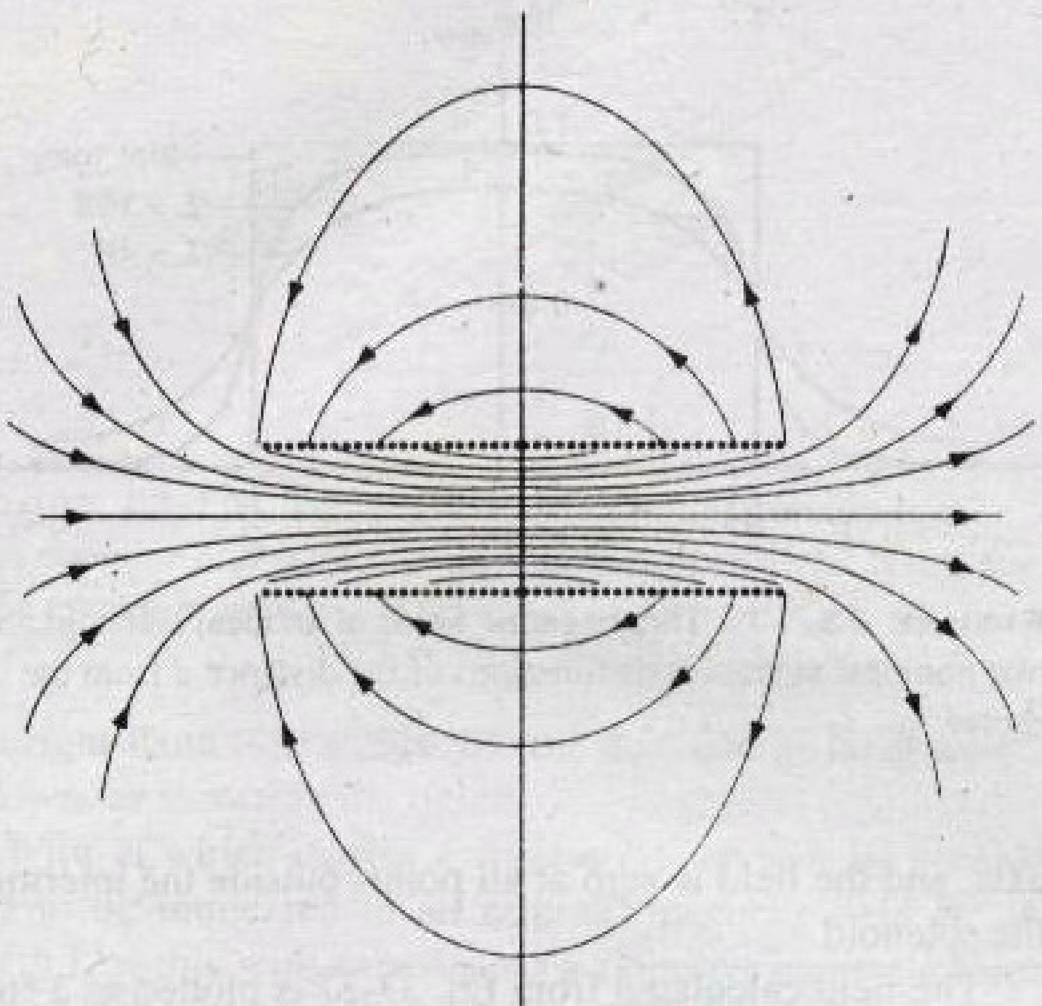


FIGURE 33-19. Magnetic field lines for a solenoid of finite length. Note that the field is stronger (indicated by the greater density of field lines) inside the solenoid than it is outside.

turns (marked ⊗, because the current is into the page), which points to the right near P. In the limiting case of the ideal solenoid, the field outside the solenoid is zero. Taking the external field to be zero is a good approximation for a real solenoid if its length is much greater than its radius and if we consider only external points such as P. Figure 33-19 shows the magnetic field lines for a nonideal solenoid. You can see from the spacing of the field lines that the field exterior to the solenoid is much weaker than the field in the interior, which is very nearly uniform over the cross section of the solenoid.

The solenoid is for magnetic fields what the parallel-plate capacitor is for electric fields: a relatively simple device capable of producing a field that is approximately uniform. In a parallel-plate capacitor, the electric field is nearly uniform if the plate separation is small compared with the dimensions of the plates, and if we are not too close to the edge of the capacitor. In the solenoid, the magnetic field is nearly uniform if the radius is small compared with the length and if we are not too close to the ends. As shown in Fig. 33-17, even for a length that is only 10 times the radius, the magnetic field is within a few percent of the field of the ideal solenoid over the central half of the device.

SAMPLE PROBLEM 33-7. A solenoid has a length L = 1.23 m and an inner diameter d = 3.55 cm. It has five layers of windings of 850 turns each and carries a current i = 5.57 A. What is B at its center?

Solution With L/R = 69, we are safe in regarding this as a nearly ideal solenoid. From Eq. 33-28 we have

$$B = \mu_0 n i = (4\pi \times 10^{-7}\ \mathrm{T \cdot m/A})\left(\frac{5 \times 850\ \text{turns}}{1.23\ \mathrm{m}}\right)(5.57\ \mathrm{A})$$

$$= 2.42 \times 10^{-2}\ \mathrm{T} = 24.2\ \mathrm{mT}.$$

Note that Eq. 33-28 applies even if the solenoid has more than one layer of windings because the diameter of the windings does not enter into the equation.

33-5 AMPÈRE'S LAW

Coulomb's law can be considered a fundamental law of electrostatics; we can use it to calculate the electric field associated with any distribution of electric charges. In Chapter 27, however, we showed that Gauss' law permitted us to solve a certain class of problems, those containing a high degree of symmetry, with ease and elegance. Furthermore, we showed that Gauss' law contained within it Coulomb's law for the electric field of a point charge. We consider Gauss' law to be more basic than Coulomb's law, and Gauss' law is one of the four fundamental (Maxwell) equations of electromagnetism.

The situation in magnetism is similar. Using the Biot–Savart law, we can calculate the magnetic field of any distribution of currents, just as we used Eq. 26-6 or Eqs. 26-13 and 26-14 (which are equivalent to Coulomb's law) to calculate the electric field of any distribution of charges. A more fundamental approach to magnetic fields uses a law that (like Gauss' law for electric fields) takes advantage of the symmetry present in certain problems to simplify the calculation of $\vec{\mathbf{B}}$. This law is considered more fundamental than the Biot–Savart law and leads to another of the four Maxwell equations.

This new result is called *Ampère's law* and is written

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 i. \qquad (33\text{-}29)$$

You will recall that, in using Gauss' law, we first constructed an imaginary closed surface (a Gaussian surface) that enclosed a certain amount of charge. In using Ampère's law we construct an imaginary closed curve (called an *Ampèrian loop*), as indicated in Fig. 33-20. The left side of Eq. 33-29 tells us to divide the curve into small segments of

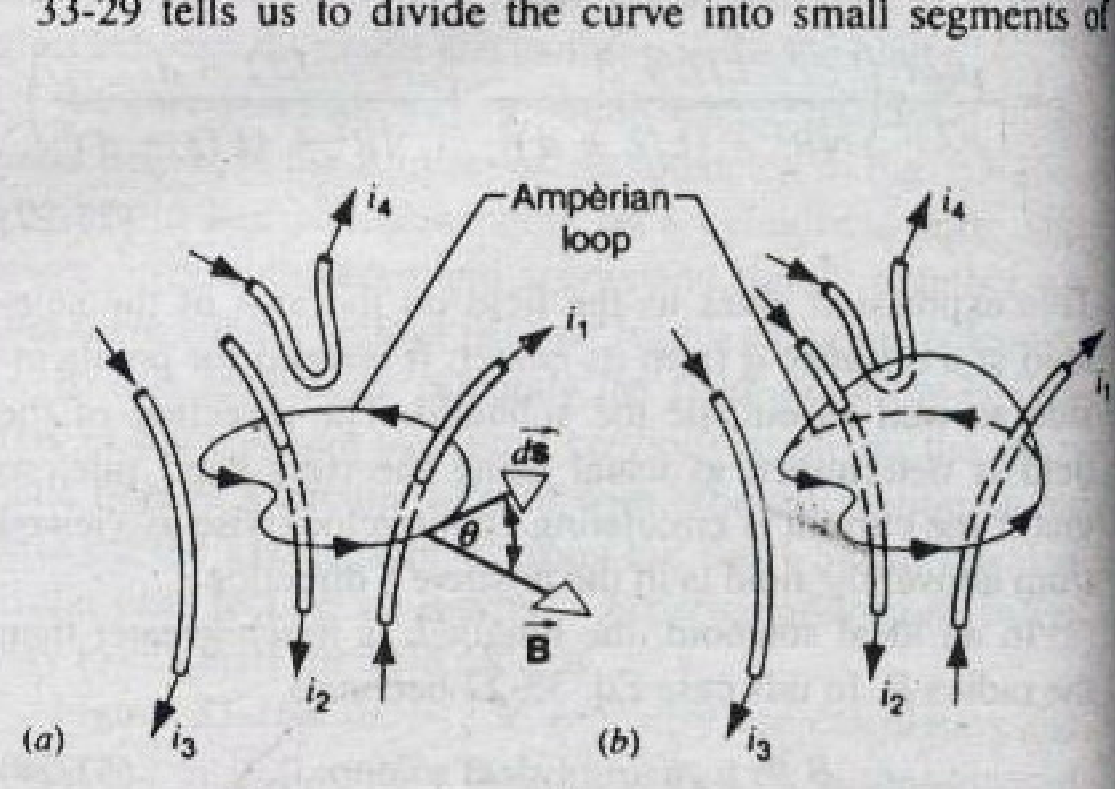


FIGURE 33-20. (*a*) In applying Ampère's law, we integrate around a closed loop. The integral is determined by the net current that passes through the surface bounded by the loop. (*b*) The surface bounded by the loop has been stretched upward.

length $d\vec{s}$. As we travel around the loop (our direction of travel determining the direction of $d\vec{s}$), we evaluate the quantity $\vec{B} \cdot d\vec{s}$ and add (integrate) all such quantities around the loop.

The integral on the left of Eq. 33-29 is called a *line integral.* (Previously we used line integrals in Chapter 11 to calculate work and in Chapter 28 to calculate potential difference.) The circle superimposed on the integral sign reminds us that the line integral is to be evaluated around a *closed* path. Letting θ represent the angle between $d\vec{s}$ and $\vec{B}$, we can write the line integral as

$$\oint \vec{B} \cdot d\vec{s} = \oint B\, ds \cos\theta. \qquad (33\text{-}30)$$

The current i in Eq. 33-29 is the total current "enclosed" by the loop; that is, it is the total current carried by wires that pierce any surface bounded by the loop. In analogy with charges in the case of Gauss' law, currents outside the loop are not included. Figure 33-20*a* shows four wires carrying current. The magnetic field $\vec{B}$ at any point is the net effect of the currents in all wires. However, in the evaluation of the right side of Eq. 33-29, we include only the currents i_1 and i_2, because the wires carrying i_3 and i_4 do not pass through the surface enclosed by the loop. The two wires that pass through the loop carry currents in opposite direction. A right-hand rule is used to assign signs to currents: with the fingers of your right hand in the direction in which the loop is traveled, currents in the direction of your thumb (such as i_1) are taken to be positive, whereas currents in the opposite direction (such as i_2) are taken to be negative. The net current i in the case of Fig. 33-20*a* is thus $i = i_1 - i_2$.

The magnetic field $\vec{B}$ at points on the loop and within the loop certainly depends on the currents i_3 and i_4; however, the *integral* of $\vec{B} \cdot d\vec{s}$ around the loop does *not* depend on currents such as i_3 and i_4 that do not penetrate the surface enclosed by the loop. This is reasonable, because $\vec{B} \cdot d\vec{s}$ for the field established by i_1 or i_2 always has the same sign as we travel around the loop; however, $\vec{B} \cdot d\vec{s}$ for the fields due to i_3 or i_4 change sign as we travel around the loop, and in fact the positive and negative contributions exactly cancel one another.

Changing the shape of the surface without changing the loop does not change these conclusions. In Fig. 33-20*b* the surface has been "stretched" upward so that now the wire carrying current i_4 penetrates the surface. However, note that it does so twice, once moving downward (which would correspond to a contribution $-i_4$ to the total current through the surface, according to our right-hand rule) and once moving upward (which would contribute $+i_4$ to the total). Thus the total current through the surface does not change; this is as expected, because stretching the surface does not change $\vec{B}$ at locations along the fixed loop, and therefore the line integral on the left side of Ampère's law does not change.

Note that including the arbitrary constant of 4π in the Biot–Savart law reduces the constant that appears in Ampère's law to simply μ_0. (A similar simplification of Gauss' law was obtained by including the constant 4π in Coulomb's law.)

For the situation shown in Fig. 33-20, Ampère's law gives

$$\oint B\, ds \cos\theta = \mu_0(i_1 - i_2). \qquad (33\text{-}31)$$

Equation 33-31 is valid for the magnetic field $\vec{B}$ as it varies in both magnitude and direction around the path of the Ampèrian loop. We cannot solve that equation for B unless we can find a way to remove B from the integral. To do so, we use symmetries in the geometry to choose an Ampèrian loop for which B is constant. We used a similar trick in calculating electric fields using Gauss' law.

The following examples illustrate how Ampère's law can be used to calculate magnetic fields in cases with a high degree of symmetry.

Applications of Ampère's Law

A Long, Straight Wire (external points). We can use Ampère's law to find the magnetic field at a distance d from a long, straight wire, a problem we have already solved using the Biot–Savart law.

As illustrated in Fig. 33-21, we choose as our Ampèrian loop a circle of radius d centered on the wire with its plane perpendicular to the wire. From the symmetry of the problem, $\vec{B}$ can depend only on d (and not, for instance, on the angular coordinate around the circle). By choosing a path that is everywhere the same distance from the wire, we know that B is constant around the path.

We know from Oersted's experiments that $\vec{B}$ has only a tangential component. Thus the angle θ is zero, and the line integral becomes

$$\oint B\, ds \cos\theta = B \oint ds = B(2\pi d). \qquad (33\text{-}32)$$

Note that the integral of ds around the path is simply the length of the path, or $2\pi d$ in the case of the circle. The right side of Ampère's law is simply $\mu_0 i$ (taken as positive,

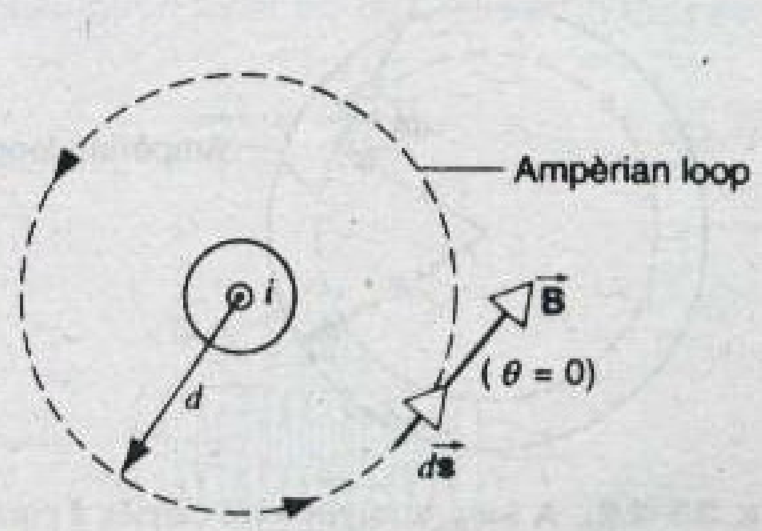


FIGURE 33-21. A circular Ampèrian loop is used to find the magnetic field set up by a current in a long, straight wire. The wire is perpendicular to the page, and the direction of the current is out of the page.

in accordance with the right-hand rule). Ampère's law gives

$$B(2\pi d) = \mu_0 i$$

or

$$B = \frac{\mu_0 i}{2\pi d}.$$

This is identical with Eq. 33-13, a result we obtained (with considerably more effort) using the Biot–Savart law.

A Long, Straight Wire (internal points). We can also use Ampère's law to find the magnetic field *inside* a wire. We assume a cylindrical wire of radius R in which a total current i is distributed uniformly over its cross section. We wish to find the magnetic field at a distance $r < R$ from the center of the wire.

Figure 33-22 shows a circular Ampèrian loop of radius r inside the wire. Symmetry suggests that **B** is constant in magnitude everywhere on the loop and tangent to the loop, so the left side of Ampère's law gives $B(2\pi r)$, exactly as in Eq. 33-32. The right side of Ampère's law involves only the current inside the radius r. If the current is distributed uniformly over the wire, the fraction of the current inside the radius r is the same as the fraction of the area inside r, or $\pi r^2/\pi R^2$. Ampère's law then gives

$$B(2\pi r) = \mu_0 i \frac{\pi r^2}{\pi R^2}, \qquad (33\text{-}33)$$

where again the right side includes only the fraction of the current that passes through the surface enclosed by the path of integration (the Ampèrian loop).

Solving for B, we obtain

$$B = \frac{\mu_0 i r}{2\pi R^2}. \qquad (33\text{-}34)$$

At the surface of the wire ($r = R$), Eq. 33-34 reduces to Eq. 33-13 (with $d = R$). That is, both expressions give the same result for the field at the surface of the wire. Figure 33-23 shows how the field depends on r at points both inside and outside the wire.

Equation 33-34 is valid only for the case in which the current is distributed uniformly over the wire. If the current density depends on r, a different result is obtained (see Problem 13). However, Eq. 33-13 for the field outside the wire remains valid whether the current density is constant or a function of r.

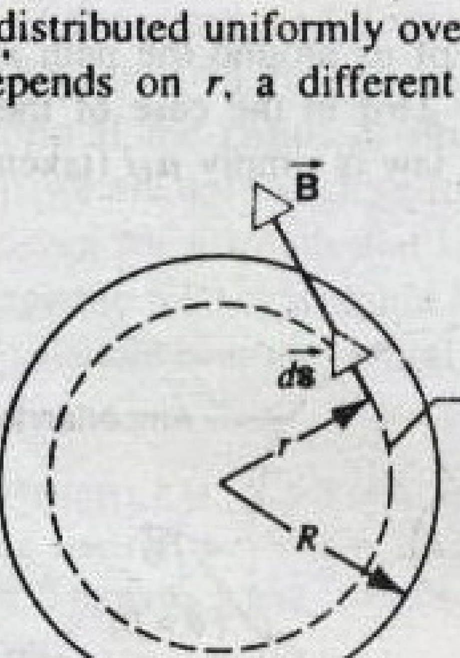


FIGURE 33-22. A long, straight wire carries a current that is emerging from the page and is uniformly distributed over the circular cross section of the wire. A circular Ampèrian loop is drawn inside the wire.

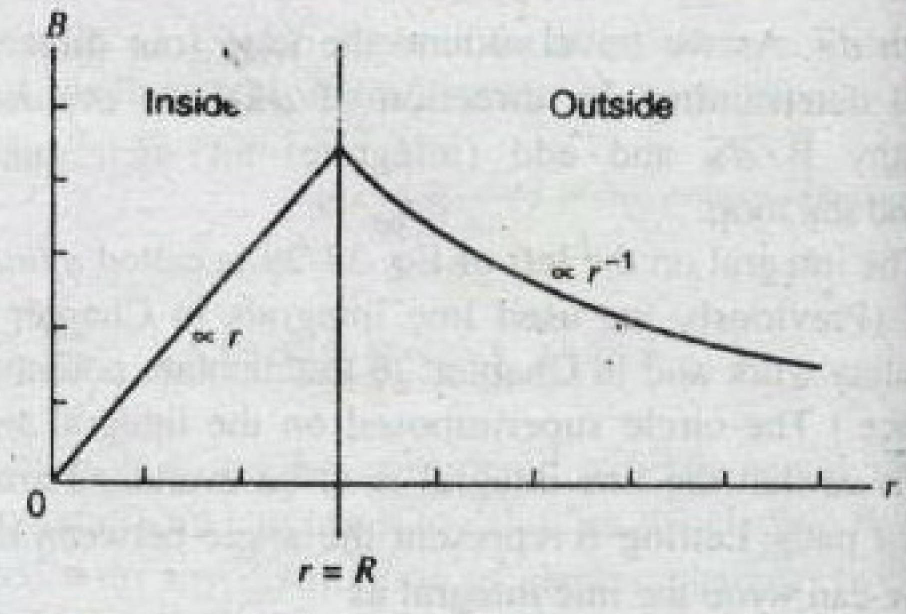


FIGURE 33-23. The magnetic field calculated for the wire shown in Fig. 33-22. Note that the largest field occurs at the surface of the wire.

A Solenoid. We consider an ideal solenoid as shown in Fig. 33-24 and choose an Ampèrian loop in the shape of the rectangle *abcda*. In this analysis we assume that the magnetic field is parallel to the axis of this ideal solenoid and constant in magnitude along line *ab*. As we shall prove, the field is also uniform in the interior (independent of the distance of *ab* from the central axis), as suggested by the equal spacing of the field lines in Fig. 33-24.

The left side of Ampère's law can be written as the sum of four integrals, one for each path segment:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \int_a^b \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} + \int_b^c \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} + \int_c^d \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} + \int_d^a \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}. \qquad (33\text{-}35)$$

The first integral on the right is Bh, where B is the magnitude of $\vec{\mathbf{B}}$ inside the solenoid and h is the arbitrary length of the path from a to b. Note that path ab, though parallel to the solenoid axis, need not coincide with it. The second and fourth integrals in Eq. 33-35 are zero because for every element of these paths $\vec{\mathbf{B}}$ is either at right angles to the path (for points inside the solenoid) or is zero (for points outside). In either case, $\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ is zero, and the integrals vanish. The third integral, which includes the part of the rectangle that lies outside the solenoid, is zero because we have taken $\vec{\mathbf{B}}$ as zero for all external points for an ideal solenoid.

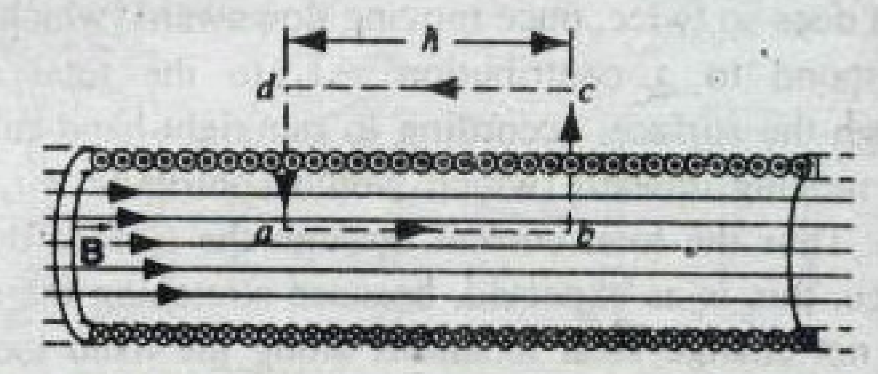


FIGURE 33-24. An Ampèrian loop (the rectangle *abcd*) is used to calculate the magnetic field of this long, idealized solenoid.

For the entire rectangular path, $\oint \vec{B} \cdot d\vec{s}$ has the value Bh. The net current i that passes through the rectangular Ampèrian loop is not the same as the current in the solenoid because the windings pass through the loop more than once. Let n be the number of turns per unit length; then nh is the number of turns inside the loop, and the total current, passing through the rectangular Ampèrian loop of Fig. 33-24 is nhi. Ampère's law then becomes

$$Bh = \mu_0 nhi$$

or

$$B = \mu_0 ni.$$

This result agrees with Eq. 33-28, which referred only to points on the central axis of the solenoid. Because line ab in Fig. 33-24 can be located at any distance from the axis, we can now conclude that the magnetic field inside an ideal solenoid is uniform over its cross section.

A Toroid. Figure 33-25 shows a toroid, which we may consider to be a solenoid bent into the shape of a doughnut. We can use Ampère's law to find the magnetic field at interior points.

From symmetry, the lines of $\vec{B}$ form concentric circles inside the toroid, as shown in the figure. Let us choose a concentric circle of radius r as an Ampèrian loop and traverse it in the clockwise direction. Ampère's law yields

$$B(2\pi r) = \mu_0 iN,$$

where i is the current in the toroid windings and N is the total number of turns. This gives

$$B = \frac{\mu_0 iN}{2\pi r}. \qquad (33\text{-}36)$$

In contrast to the solenoid, B is not constant over the cross section of a toroid. You should be able to show, from Ampère's law, that $B = 0$ for points outside an ideal toroid and in the central cavity.

Close inspection of Eq. 33-36 justifies our earlier statement that a toroid is "a solenoid bent into the shape of a doughnut." The denominator in Eq. 33-36, $2\pi r$, is the central circumference of the toroid, and $N/2\pi r$ is just n, the number of turns per unit length. With this substitution, Eq. 33-36 reduces to $B = \mu_0 in$, the equation for the magnetic field in the central region of a solenoid.

The direction of the magnetic field within a toroid (or a solenoid) follows from the right-hand rule: curl the fingers of your right hand in the direction of the current; your extended right thumb then points in the direction of the magnetic field.

Toroids form the central feature of the *tokamak*, a device showing promise as the basis for a fusion power reactor. We discuss its mode of operation in Chapter 51.

The Field Outside a Solenoid (Optional). We have so far neglected the field outside the solenoid, but even for an ideal solenoid, the field at points outside the winding is not zero. Figure 33-26 shows an Ampèrian path in the shape of a circle of radius r. Because the solenoid windings are helical, one turn of the winding pierces the surface enclosed by the circle. The product $\vec{B} \cdot d\vec{s}$ for this path depends on the tangential component of the field B_t, and thus Ampère's law gives

$$B_t(2\pi r) = \mu_0 i$$

or

$$B_t = \frac{\mu_0 i}{2\pi r}, \qquad (33\text{-}37)$$

which is the same field (in magnitude *and* direction) that would be set up by a straight wire. Note that the windings, in addition to carrying current around the surface of the solenoid, also carry current from left to right in Fig. 33-26, and in this respect the solenoid behaves like a straight wire at points outside the windings.

The tangential field is much smaller than the interior field (Eq. 33-28), as we can see by taking the ratio

$$\frac{B_t}{B} = \frac{\mu_0 i/2\pi r}{\mu_0 ni} = \frac{1}{2\pi rn}. \qquad (33\text{-}38)$$

Suppose the solenoid consists of one layer of turns in which the wires are touching one another, as in Fig. 33-24. Every interval along the solenoid of length equal to the diameter D of the wire contains one turn, and thus the number of turns per unit length n must be $1/D$. The ratio thus becomes

$$\frac{B_t}{B} = \frac{D}{2\pi r}. \qquad (33\text{-}39)$$

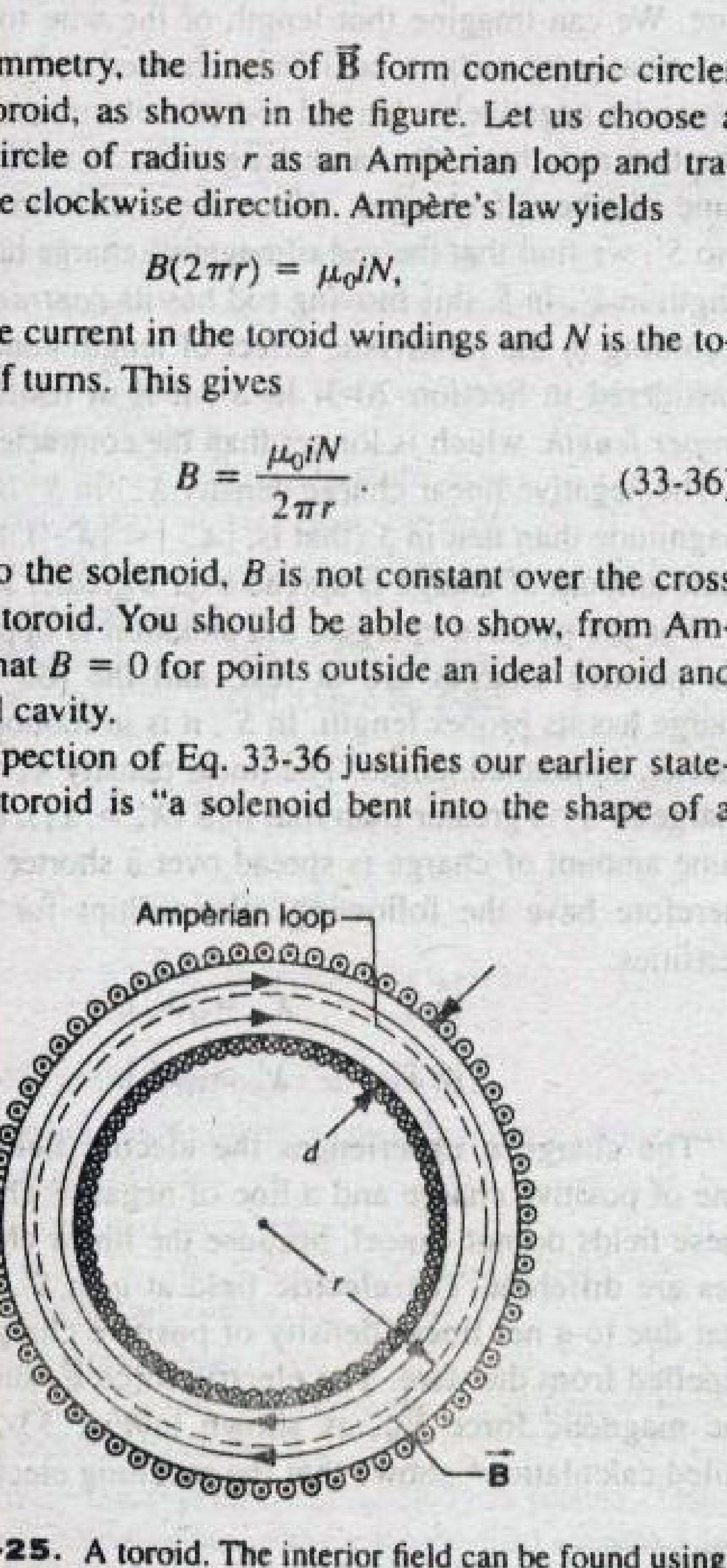


FIGURE 33-25. A toroid. The interior field can be found using the circular Ampèrian loop shown.

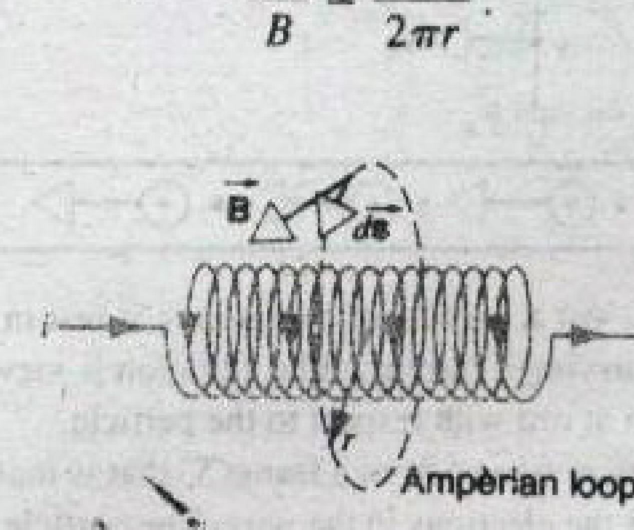


FIGURE 33-26. A circular Ampèrian loop of radius r is used to find the tangential field external to a solenoid.

For a typical wire, $D = 0.1$ mm. The distance r to exterior points must be at least as large as the radius of the solenoid, which might be a few centimeters. Thus $B_t/B \leq 0.001$, and the tangential exterior field is indeed negligible compared with the interior field along the axis. We are therefore safe in neglecting the exterior field.

By drawing an Ampèrian circle similar to that of Fig. 33-26 but with radius smaller than that of the solenoid, you should be able to show that the tangential component of the interior field is zero. ■

33-6 ELECTROMAGNETISM AND FRAMES OF REFERENCE (Optional)

Figure 33-27*a* shows a particle carrying a positive charge q at rest near a long, straight wire that carries a current i. We view the system from a frame of reference S in which the wire is at rest. Inside the wire are negative electrons moving with the drift velocity $\vec{v}_d$ and positive ion cores at rest. In any given length of wire, the number of electrons equals the number of ion cores, and the net charge is zero. The electrons can instantaneously be considered as a line of negative charge, which sets up an electric field at the location of q according to Eq. 26-17:

$$E = \frac{\lambda_-}{2\pi\epsilon_0 r},$$

where λ_- is the linear charge density of electrons (a negative number). The positive ion cores also set up an electric field given by a similar expression, depending on the linear charge density λ_+ of positive ions. Because the charge densities are of equal magnitude and opposite sign,

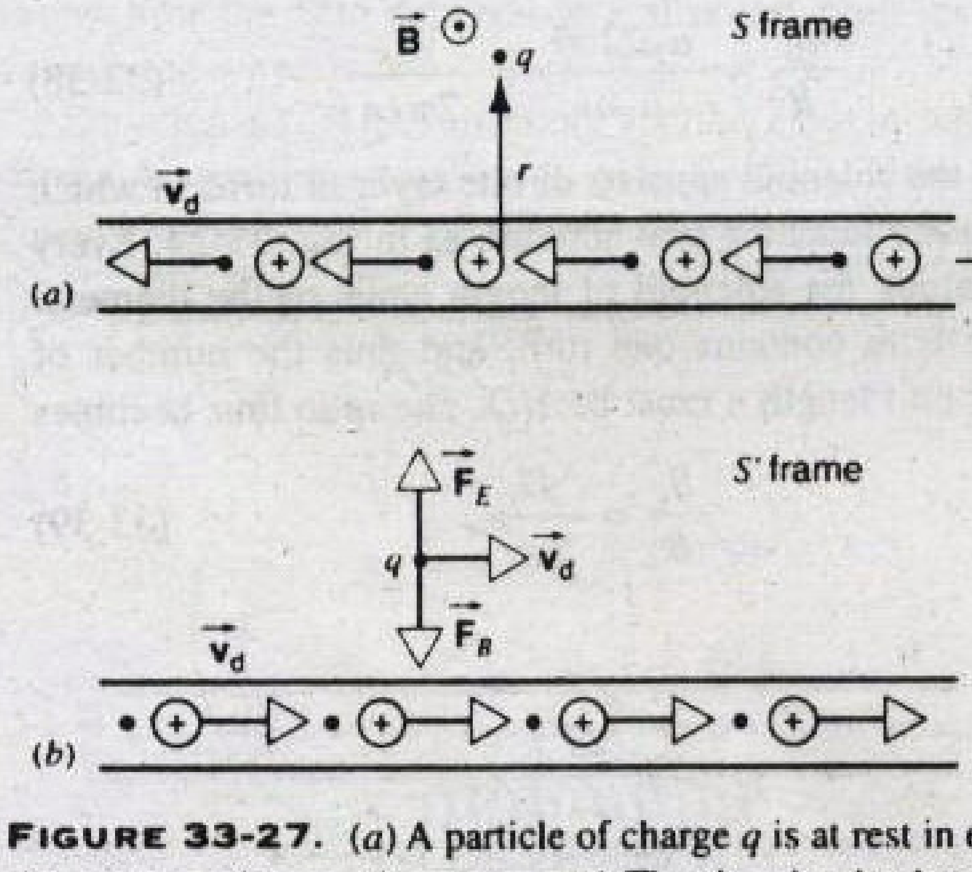


FIGURE 33-27. (*a*) A particle of charge q is at rest in equilibrium near a wire carrying a current i. The situation is viewed from a reference frame S at rest with respect to the particle. (*b*) The same situation viewed from a frame S' that is moving with the drift velocity of the electrons in the wire. The particle is also in equilibrium in this frame under the influence of the two forces $\vec{F}_E$ and $\vec{F}_B$.

$\lambda_+ + \lambda_- = 0$, and the net electric field that acts on the particle is zero.

There is a nonzero magnetic field at the location of the particle, but because the particle is at rest, there is no magnetic force. Therefore no net force of electromagnetic origin acts on the particle in this frame of reference.

Now let us consider the situation from the perspective of a frame of reference S' moving parallel to the wire with velocity $\vec{v}_d$ (the drift velocity of the electrons). Figure 33-27*b* shows the situation in this frame of reference, in which the electrons are at rest and the ion cores move to the right with velocity $\vec{v}_d$. Clearly, in this case the particle, being in motion, experiences a magnetic force $\vec{F}_B$ as shown in the figure.

Observers in different inertial frames must agree that if there is no acceleration of the charge q in the S frame, there must also be no acceleration in the S' frame. The particle must therefore experience no net force in S', and so there must be another force in addition to $\vec{F}_B$ that acts on the particle to give a net force of zero.

This additional force that acts in the S' frame must be of electric origin. Consider in Fig. 33-27*a* a length L of the wire. We can imagine that length of the wire to consist of two measuring rods, a positively charged rod (the ions) at rest and a negatively charged rod (the electrons) in motion. The two rods have the same length (in S) and contain the same number of charges. When we transform those rods into S', we find that the rod of negative charge has a greater length in S'. In S, this moving rod has its *contracted length*, according to the relativistic effect of length contraction we considered in Section 20-3. In S', it is at rest and has its *proper length*, which is longer than the contracted length in S. The negative linear charge density λ'_- in S' is smaller in magnitude than that in S (that is, $|\lambda'_-| < |\lambda_-|$), because the same amount of charge is spread over a greater length in S'.

For the positive charges, the situation is opposite. In S the positive charges are at rest, and the rod of positive charge has its proper length. In S', it is in motion and has a shorter, contracted length. The linear density λ'_+ of positive charge in S' is greater than that in S ($\lambda'_+ > \lambda_+$), because the same amount of charge is spread over a shorter length. We therefore have the following relationships for the charge densities:

$$\text{in } S: \qquad \lambda_+ = |\lambda_-|,$$

$$\text{in } S': \qquad \lambda'_+ > |\lambda'_-|.$$

The charge q experiences the electric fields due to the line of positive charge and a line of negative charge. In S' these fields do not cancel, because the linear charge densities are different. The electric field at q in S' is therefore that due to a net linear density of positive charge, and q is repelled from the wire. The electric force $\vec{F}_E$ on q opposes the magnetic force $\vec{F}_B$, as shown in Fig. 33-27*b*. A detailed calculation* shows that the resulting electric force

*See, for example, R. Resnick, *Introduction to Special Relativity* (Wiley, 1968), Chapter 4.

exactly equal to the magnetic force, and the net force in S' is zero. Thus the particle experiences no acceleration in either reference frame. We can extend this result to situations other than the special case we considered here, in which S' moves at velocity $\vec{v}_d$ with respect to S. In other frames of reference, the electric force and the magnetic force have values different from their values in S'; however, in every frame they are equal and opposite to one another and the net force on the particle is zero in *every* frame of reference.

This is a remarkable result. According to special relativity, electric and magnetic fields do not have separate existences. A field that is purely electric or purely magnetic in one frame of reference has both electric and magnetic components in another frame. Using relativistic transformation equations, we can easily pass back and forth from one frame to another, and we can often solve difficult problems by choosing a frame of reference in which the fields have a simpler character and then transforming the result back to the original frame. Special relativity can be of great practical value in solving such problems, because the techniques of special relativity may turn out to be simpler than the classical techniques.

In mathematical language, we say that the laws of electromagnetism (Maxwell's equations) are invariant with respect to the Lorentz transformation. Recall our discussion in Section 11-6 about *invariant* physical laws: we write down the law in one frame of reference, transform to another frame, and obtain a law of exactly the same mathematical form. For example, Gauss' law, one of the four Maxwell equations, has exactly the same form in every frame of reference.

Einstein's words are direct and to the point: "The force acting on a body in motion in a magnetic field is nothing else but an electric field." (In fact, Einstein's original 1905 paper, in which he first presented the ideas of special relativity, was titled "On the Electrodynamics of Moving Bodies.") In this context, we can regard magnetism as a relativistic effect, depending on the velocity of the charge relative to the observer. However, unlike other relativistic effects, it has substantial observable consequences at speeds far smaller than the speed of light. ■

MULTIPLE CHOICE

33-1 The Magnetic Field due to a Moving Charge

1. Two positive charges q_1 and q_2 are moving to the right in Fig. 33-28.
   (*a*) What is the direction of the force on charge q_1 due to the magnetic field produced by q_2?
   (A) Into the page (B) Out of the page
   (C) Up the page (D) Down the page
   (*b*) What is the direction of the force on charge q_2 due to the magnetic field produced by q_1?
   (A) Into the page (B) Out of the page
   (C) Up the page (D) Down the page

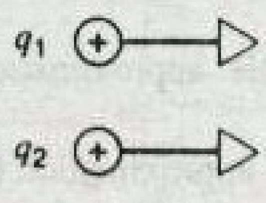


FIGURE 33-28. Multiple-choice question 1.

33-2 The Magnetic Field of a Current

2. Consider the magnitude of the magnetic field $B(z)$ on the axis of a circular current loop.
   (*a*) $B(z)$ will be a maximum where
   (A) $z = 0$. (B) $0 < |z| < \infty$. (C) $|z| = \infty$.
   (D) (A) and (C) are correct.
   (*b*) $B(z)$ can be zero where
   (A) $z = 0$. (B) $0 < |z| < \infty$. (C) $|z| = \infty$.
   (D) (A) and (C) are correct.
3. The negatively charged disk in Fig. 33-29 is rotated clockwise. What is the direction of the magnetic field at point A in the plane of the disk?
   (A) Into the page (B) Out of the page
   (C) Up the page (D) Down the page

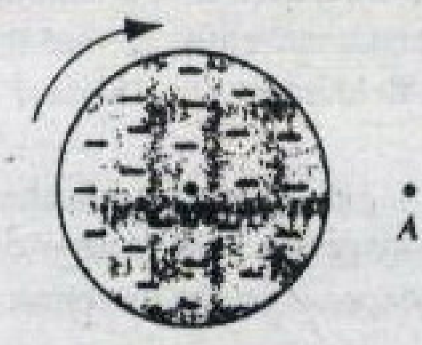


FIGURE 33-29. Multiple-choice question 3.

4. A loop of wire of length L carrying a current i can be wound once as in Fig. 33-30*a*, or twice as in Fig. 33-30*b*. The ratio of the magnetic field strength B_1 at the center of the single loop to the strength B_2 at the center of the double loop is
   (A) 2. (B) 1. (C) 1/2. (D) 1/4.

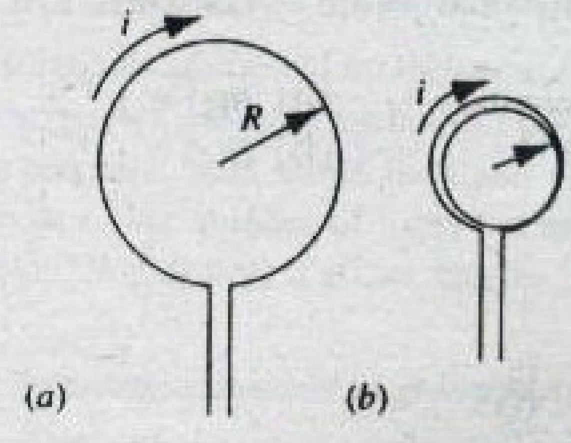


FIGURE 33-30. Multiple-choice question 4.

33-3 Two Parallel Currents

5. A long, straight wire carries a current to the north. A second long, straight wire 0.5 m vertically above the first wire carries

an identical current to the east. Both wires are long enough to be considered infinite in length.

(*a*) What is the direction of the net force on the top wire because of the current in the bottom wire?

(A) Up (B) Down (C) North (D) South
(E) The net force is zero.

(*b*) What is the direction of the torque on the top wire because of the current in the bottom wire?

(A) Up (B) Down (C) North (D) South
(E) The torque is zero.

6. Two parallel currents are directed out of the page. Compare the magnitude of the magnetic field B_2 at any arbitrary point equidistant from the wires to the magnitude of the field B_1 at that point from one wire alone.
   (A) $B_2 > B_1$ for all equidistant points.
   (B) $B_2 = B_1$ for all equidistant points.
   (C) $B_2 < B_1$ for all equidistant points.
   (D) $B_2 > B_1$ for closer equidistant points only.
   (E) $B_2 < B_1$ for closer equidistant points only.
7. Antiparallel currents are directed so that one is out of the page and the other is into the page. Compare the magnitude of the magnetic field B_2 at any arbitrary point equidistant from the wires to the magnitude of the field B_1 at that point from one wire alone.
   (A) $B_2 > B_1$ for all equidistant points.
   (B) $B_2 = B_1$ for all equidistant points.
   (C) $B_2 < B_1$ for all equidistant points.
   (D) $B_2 > B_1$ for closer equidistant points only.
   (E) $B_2 < B_1$ for closer equidistant points only.

33-4 The Magnetic Field of a Solenoid

8. A metal "Slinky" can be used as a solenoid. The "Slinky" is stretched slightly, and a current is passed through it. Will the resulting magnetic field cause the "Slinky" to collapse or to stretch out further?
   (A) Collapse (B) Stretch out further
   (C) Neither, the magnetic field is zero outside a solenoid.
   (D) The answer depends on the direction of the current.
9. Consider a solenoid with $R \ll L$. The magnetic field at the center of the solenoid is B_0. A second solenoid is constructed that has twice the radius, twice the length, and carries twice the current as the original solenoid, but has the same number of turns per meter. The magnetic field at the center of the second solenoid is
   (A) $B_0/2$. (B) B_0. (C) $2B_0$. (D) $4B_0$.
10. How does the magnetic field $B(z)$ behave for points z along the axis of a solenoid with $z \gg L$, where L is the length of the solenoid?
    (A) $B(z)$ is constant. (B) $B(z) \propto z^{-1}$
    (C) $B(z) \propto z^{-2}$ (D) $B(z) \propto z^{-3}$

33-5 Ampère's Law

11. Solve, without integrating,

$$\int_{-\infty}^{\infty} B\,dz,$$

where B is the magnetic field along the axis of a circular lo
of current as given by Eq. 33-19. What is the result?
    (A) $i/2R$ (B) $2\mu_0 i$ (C) $\mu_0 i$
    (D) The expression cannot be solved without integratin
12. What is $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ for the path shown in Fig. 33-31?
    (A) $-8\pi \times 10^{-7}$ T·m (B) $-4\pi \times 10^{-7}$ T·m
    (C) $+8\pi \times 10^{-7}$ T·m (D) $+32\pi \times 10^{-7}$ T·m

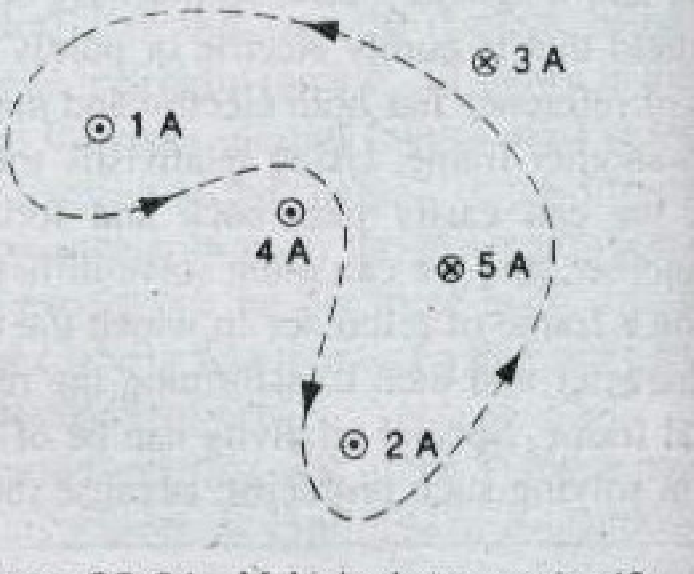


FIGURE 33-31. Multiple-choice question 12.

13. What is $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ for the path shown in Fig. 33-32?
    (A) $+56\pi \times 10^{-7}$ T·m (B) $-24\pi \times 10^{-7}$ T·m
    (C) $+328\pi \times 10^{-7}$ T·m (D) $+80\pi \times 10^{-7}$ T·m

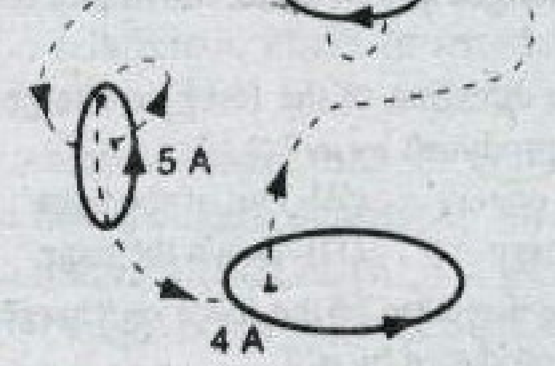


FIGURE 33-32. Multiple-choice question 13.

14. Is $B = \mu_0 n i$ true for infinite solenoids that have noncircul
cross sections?
    (A) It is a reasonable approximation for cross sectio
close to circles.
    (B) It is a reasonable approximation for any cross-section
shape.
    (C) It is true for cross-sectional shapes of sufficient sym
metry (such as equilateral triangles or squares).
    (D) It is true for any cross-sectional shape.

33-6 Electromagnetism and Frames of Reference

QUESTIONS

1. A beam of 20-MeV protons emerges from a cyclotron. Do these particles cause a magnetic field?
2. Discuss analogies and differences between Coulomb's law and the Biot–Savart law.
3. Consider a magnetic field line. Is the magnitude of $\vec{\mathbf{B}}$ consta
or variable along such a line? Can you give an example
each case?
4. In electronics, wires that carry equal but opposite currents a

often twisted together to reduce their magnetic effect at distant points. Why is this effective?

5. Consider two charges, first (*a*) of the same sign and then (*b*) of opposite signs, that are moving along separated parallel paths with the same velocity. Compare the directions of the mutual electric and magnetic forces in each case.

6. Is there any way to set up a magnetic field other than by causing charges to move?

7. Give details of three ways in which you can measure the magnetic field $\vec{B}$ at a point *P*, a perpendicular distance *r* from a long, straight wire carrying a constant current *i*. Base them on (*a*) projecting a particle of charge *q* through point *P* with velocity $\vec{v}$ parallel to the wire; (*b*) measuring the force per unit length exerted on a second wire, parallel to the first wire and carrying a current *i*′; (*c*) measuring the torque exerted on a small magnetic dipole located a perpendicular distance *r* from the wire.

8. Is $\vec{B}$ uniform for all points within a circular loop of wire carrying a current? Explain.

9. Two long, parallel conductors carry equal currents *i* in the same direction. Sketch roughly the resultant lines of $\vec{B}$ due to the action of both currents. Does your figure suggest an attraction between the wires?

10. A current is sent through a vertical spring from whose lower end a weight is hanging. What will happen?

11. Equation 33-13 ($B = \mu_0 i/2\pi d$) suggests that a strong magnetic field is set up at points near a long wire carrying a current. Since there is a current *i* and magnetic field $\vec{B}$, why is there not a force on the wire in accord with the equation $\vec{F}_B = i\vec{L} \times \vec{B}$?

12. Two long, straight wires pass near one another at right angles. If the wires are free to move, describe what happens when currents are sent through both of them.

13. Two fixed wires cross each other perpendicularly so that they do not actually touch but are close to each other, as shown in Fig. 33-33. Equal currents *i* exist in each wire in the directions indicated. In what region(s) will there be some points of zero net magnetic field?

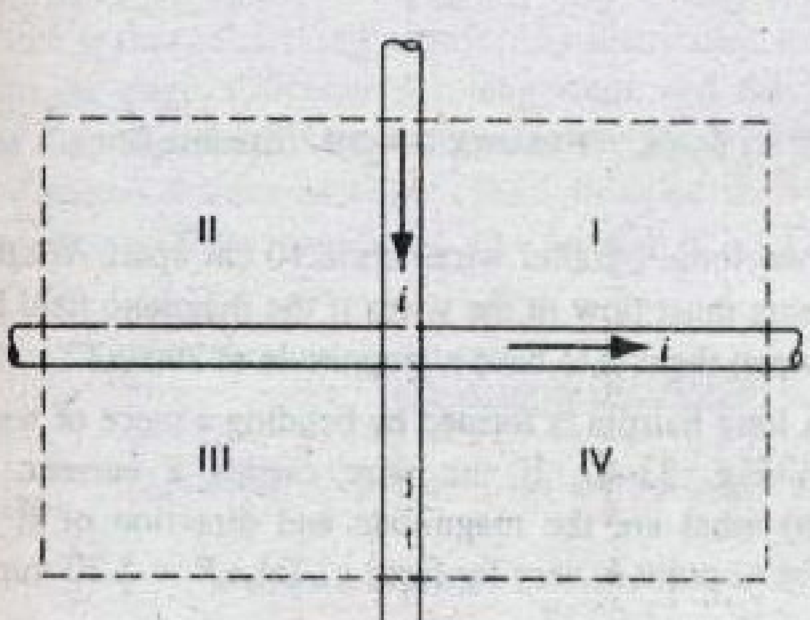


FIGURE 33-33. Question 13.

14. A messy loop of limp wire is placed on a frictionless table and anchored at points *a* and *b* as shown in Fig. 33-34. If a current *i* is now passed through the wire, will it try to form a circular loop or will it try to bunch up further?

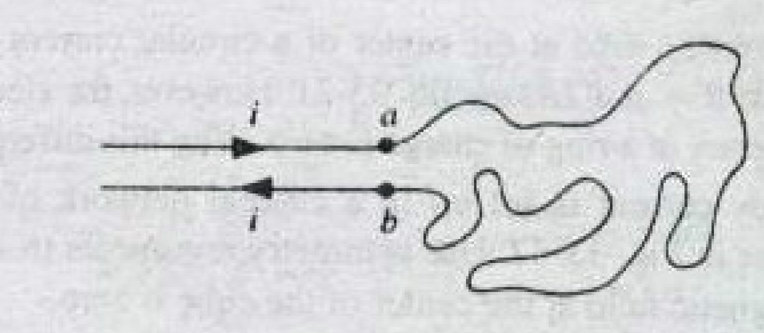


FIGURE 33-34. Question 14.

15. Can the path of integration around which we apply Ampère's law pass through a conductor?

16. Suppose we set up a path of integration around a cable that contains 12 wires with different currents (some in opposite directions) in each wire. How do we calculate *i* in Ampère's law in such a case?

17. Apply Ampère's law qualitatively to the three paths shown in Fig. 33-35.

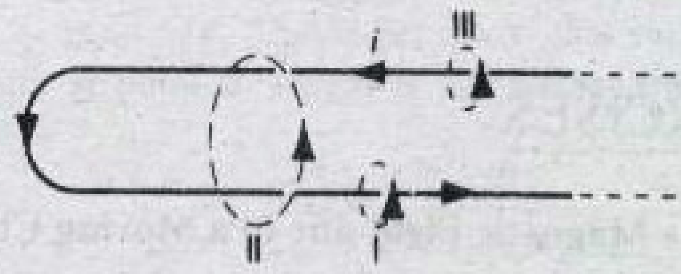


FIGURE 33-35. Question 17.

18. Discuss analogies and differences between Gauss' law and Ampère's law.

19. Does it necessarily follow from symmetry arguments alone that the lines of $\vec{B}$ around a long, straight wire carrying a current *i* must be concentric circles?

20. A steady, longitudinal, uniform current is set up in a long copper tube. Is there a magnetic field (*a*) inside and/or (*b*) outside the tube?

21. A very long conductor has a square cross section and contains a coaxial cavity also with a square cross section. Current is distributed uniformly over the material cross section of the conductor. Is the magnetic field in the cavity equal to zero? Justify your answer.

22. A long, straight wire of radius *R* carries a steady current *i*. How does the magnetic field generated by this current depend on *R*? Consider points both outside and inside the wire.

23. A long, straight wire carries a constant current *i*. What does Ampère's law require for (*a*) a loop that encloses the wire but is not circular, (*b*) a loop that does not enclose the wire, and (*c*) a loop that encloses the wire but does not all lie in one plane?

24. Two long solenoids are nested on the same axis, as in Fig. 33-36. They carry identical currents but in opposite directions. If there is no magnetic field inside the inner solenoid, what can you say about *n*, the number of turns per unit length, for the two solenoids? Which one, if either, has the larger value?

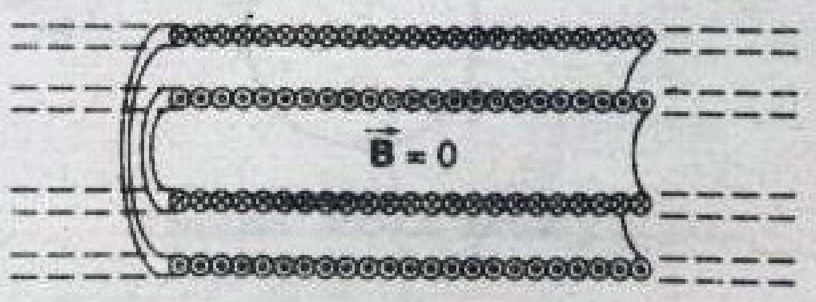


FIGURE 33-36. Question 24.

25. The magnetic field at the center of a circular current loop has the value $B = \mu_0 i/2R$; see Eq. 33-21. However, the electric field at the center of a ring of charge is zero. Why this difference?

26. A steady current is set up in a cubical network of resistive wires, as in Fig. 33-37. Use symmetry arguments to show that the magnetic field at the center of the cube is zero.

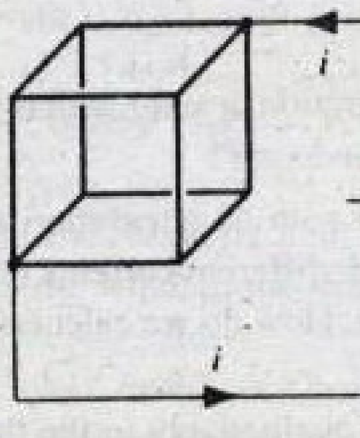


FIGURE 33-37. Question 26.

EXERCISES

33-1 The Magnetic Field due to a Moving Charge

1. (*a*) What does nonrelativistic physics predict would be the speed of two protons moving side by side a distance d apart so that the magnetic force is exactly balanced by the electric force? (*b*) Comment on the appropriateness of using nonrelativistic expressions for this problem.

33-2 The Magnetic Field of a Current

2. A surveyor is using a magnetic compass 6.3 m below a power line in which there is a steady current of 120 A. Will this interfere seriously with the compass reading? The horizontal component of Earth's magnetic field at the site is 210 μT.

3. A #10 bare copper wire (2.6 mm in diameter) can carry a current of 50 A without overheating. For this current, what is the magnetic field at the surface of the wire?

4. At a location in the Philippines, the Earth's magnetic field has a value of 39.0 μT and is horizontal and due north. The net field is zero 8.13 cm above a long, straight, horizontal wire that carries a steady current. (*a*) Calculate the current and (*b*) find its direction.

5. The 25-kV electron gun in a TV tube fires an electron beam 0.22 mm in diameter at the screen, 5.6×10^{14} electrons arriving each second. Calculate the magnetic field produced by the beam at a point 1.5 mm from the axis of the beam.

6. A straight conductor carrying a current i is split into identical semicircular turns as shown in Fig. 33-38. What is the magnetic field strength at the center C of the circular loop so formed?

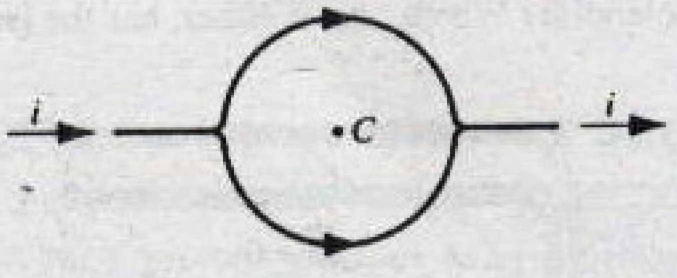


FIGURE 33-38. Exercise 6.

7. A long, straight wire carries a current of 48.8 A. An electron, traveling at 1.08×10^7 m/s, is 5.20 cm from the wire. Calcu-

27. Does Eq. 33-28 ($B = \mu_0 ni$) hold for a solenoid of
cross section?

28. A toroid is described as a solenoid bent into the shap
doughnut. The magnetic field outside an ideal solenoi
zero. What can you say about the strength of the ma
field outside an ideal toroid?

29. Drifting electrons constitute the current in a wire and
netic field is associated with this current. What curre
magnetic field would be measured by an observer m
along the wire at the electron drift velocity?

late the force that acts on the electron if the electron ve
is directed (*a*) toward the wire, (*b*) parallel to the curren
(*c*) at right angles to the directions defined by (*a*) and (*b*)

8. Two long, straight, parallel wires, separated by 0.75 cm
perpendicular to the plane of the page as shown in Fig. 3
Wire W_1 carries a current of 6.6 A into the page. What
be the current (magnitude and direction) in wire W_2 for t
sultant magnetic field at point P to be zero?

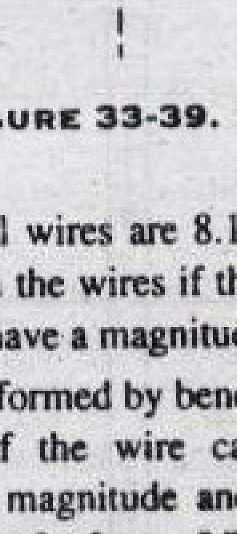


FIGURE 33-39. Exercise 8.

9. Two long, parallel wires are 8.10 cm apart. What eq
rents must flow in the wires if the magnetic field half
tween them is to have a magnitude of 296 μT?

10. A long hairpin is formed by bending a piece of wire a
in Fig. 33-40. If the wire carries a current i =
(*a*) what are the magnitude and direction of $\vec{B}$ at
(*b*) At point b, very far from a? Take R = 5.20 mm.

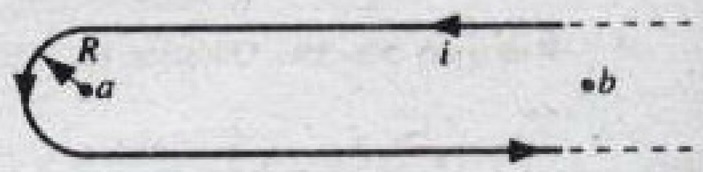


FIGURE 33-40. Exercise 10.

11. A student makes an electromagnet by winding 320 turn
around a wooden cylinder of diameter 4.80 cm. The c

nected to a battery, producing a current of 4.20 A in the wire. At what axial distance $z \gg d$ will the magnetic field of the coil be 5.0 μT (approximately one-tenth the Earth's magnetic field)?

12. A wire carrying current i has the configuration shown in Fig. 33-41. Two semi-infinite straight sections, each tangent to the same circle, are connected by a circular arc, of angle θ, along the circumference of the circle, with all sections lying in the same plane. What must θ be in order for B to be zero at the center of the circle?

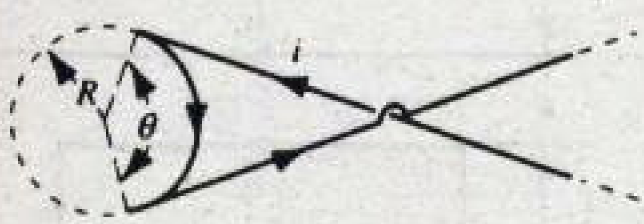


FIGURE 33-41. Exercise 12.

13. Consider the circuit of Fig. 33-42. The curved segments are arcs of circles of radii a and b. The straight segments are along the radii. Find the magnetic field $\vec{B}$ at P, assuming a current i in the circuit.

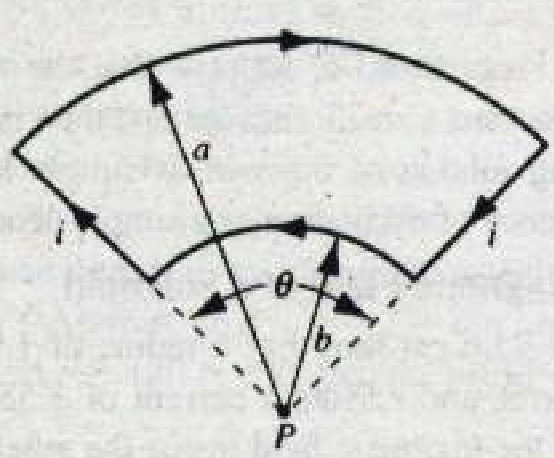


FIGURE 33-42. Exercise 13.

14. Show that B at the center of a rectangular loop of wire of length L and width W, carrying a current i, is given by

$$B = \frac{2\mu_0 i}{\pi} \frac{(L^2 + W^2)^{1/2}}{LW}.$$

Show that this reduces to a result consistent with Sample Problem 33-4 for $L \gg W$.

15. Figure 33-43 shows a cross section of a long, thin ribbon of width w that is carrying a uniformly distributed total current i into the page. Calculate the magnitude and the direction of the magnetic field $\vec{B}$ at a point P in the plane of the ribbon at a distance d from its edge. (Hint: Imagine the ribbon to be constructed from many long, thin, parallel wires.)

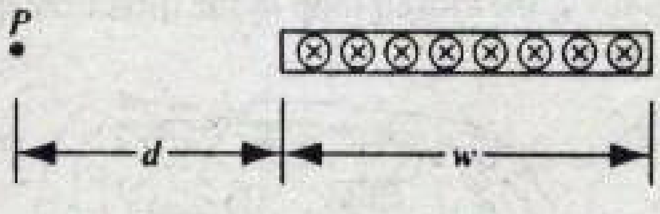


FIGURE 33-43. Exercise 15.

16. Two long, straight, parallel wires 12.2 cm apart each carry a current of 115 A. Figure 33-44 shows a cross section, with the wires running perpendicular to the page and point P lying on the perpendicular bisector of a. Find the magnitude and direction of the magnetic field at P when the current in the left-hand wire is out of the page and the current in the right-hand wire is (a) out of the page and (b) into the page.

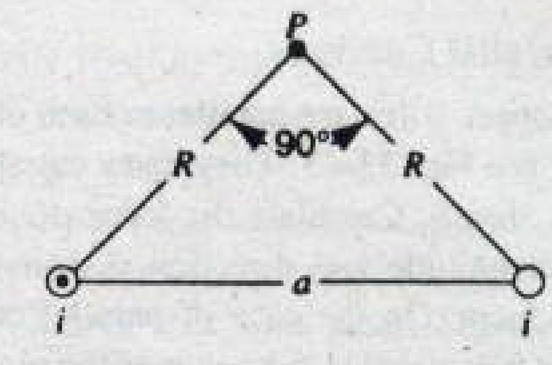


FIGURE 33-44. Exercise 16.

17. In Fig. 33-13a assume that both currents are in the same direction, out of the plane of the figure. Show that the magnetic field in the plane defined by the wires is

$$B = \frac{\mu_0 i x}{\pi(x^2 - b^2)}.$$

Assume that i = 25 A and b = 2.5 cm in Fig. 33-13a and plot B for the range $-2.5\text{ cm} < x < +2.5\text{ cm}$. Assume that the wire diameters are negligible.

18. Two long wires a distance b apart carry equal antiparallel currents i, as in Fig. 33-45. (a) Show that the magnetic field strength at point P, which is equidistant from the wires, is given by

$$B = \frac{2\mu_0 i b}{\pi(4R^2 + b^2)}.$$

(b) In what direction does $\vec{B}$ point?

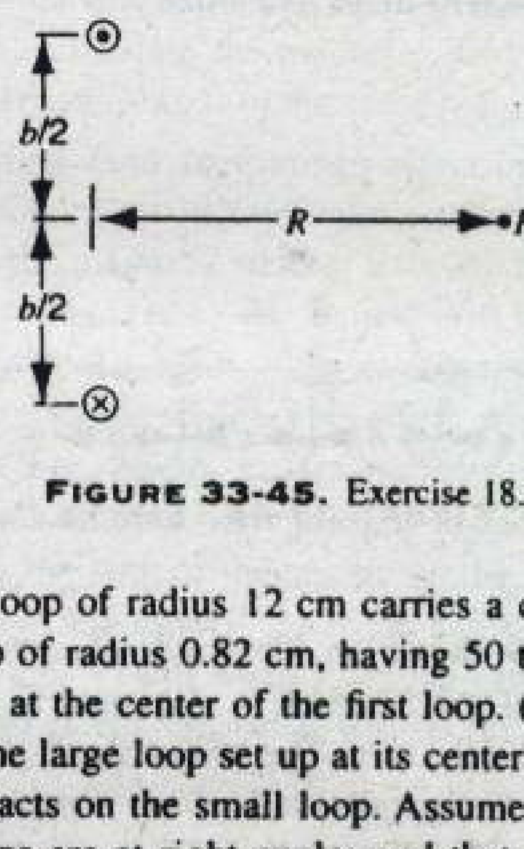


FIGURE 33-45. Exercise 18.

19. A circular loop of radius 12 cm carries a current of 13 A. A second loop of radius 0.82 cm, having 50 turns and a current of 1.3 A, is at the center of the first loop. (a) What magnetic field does the large loop set up at its center? (b) Calculate the torque that acts on the small loop. Assume that the planes of the two loops are at right angles and that the magnetic field due to the large loop is essentially uniform throughout the volume occupied by the small loop.

20. (a) A long wire is bent into the shape shown in Fig. 33-46, without cross contact at P. The radius of the circular section is R. Determine the magnitude and direction of $\vec{B}$ at the center C of the circular portion when the current i is as indicated. (b) The circular part of the wire is rotated without distortion about the dashed line one quarter turn clockwise as viewed from above, so that the plane of the circular loop is now perpendicular to the plane of the page. Determine $\vec{B}$ at C in this case.

FIGURE 33-46. Exercise 20.

33-3 Two Parallel Currents

21. Four long copper wires are parallel to each other and arranged in a square; see Fig. 33-47. They carry equal currents i out of the page, as shown. Calculate the force per meter on any one wire; give magnitude and direction. Assume that $i = 18.7$ A and $a = 24.5$ cm. (In the case of parallel motion of charged particles in a plasma, this is known as the pinch effect.)

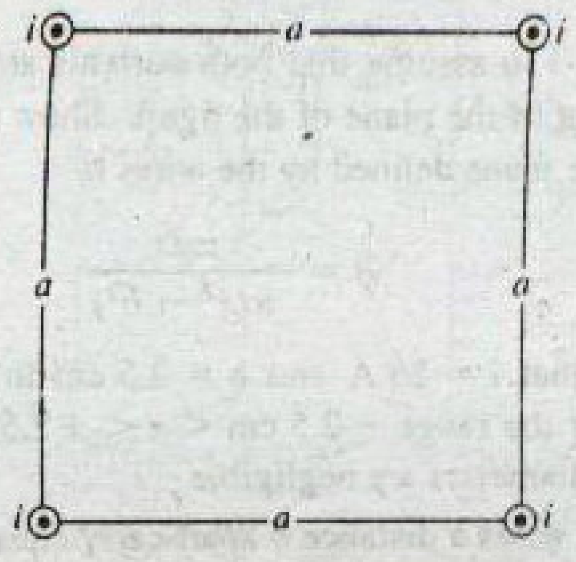


FIGURE 33-47. Exercise 21.

22. Figure 33-48 shows five long, parallel wires in the xy plane. Each wire carries a current $i = 3.22$ A in the positive x direction. The separation between adjacent wires is $a = 8.30$ cm. Find the magnetic force per meter, magnitude and direction, exerted on each of these five wires.

FIGURE 33-48. Exercise 22.

23. Figure 33-49 shows an idealized schematic of an "electromagnetic rail gun," which is designed to fire projectiles at speeds up to 10 km/s. The projectile P sits between and in contact with two parallel rails along which it can slide. A generator G provides a current that flows up one rail, across the projectile, and back down the other rail. (a) Let w be the distance between the rails, r the radius of the rails (presumed circular), and i the current. Show that the force on the projectile is to the right and given approximately by

$$F = \frac{1}{2}\left(\frac{i^2\mu_0}{\pi}\right)\ln\left(\frac{w+r}{r}\right).$$

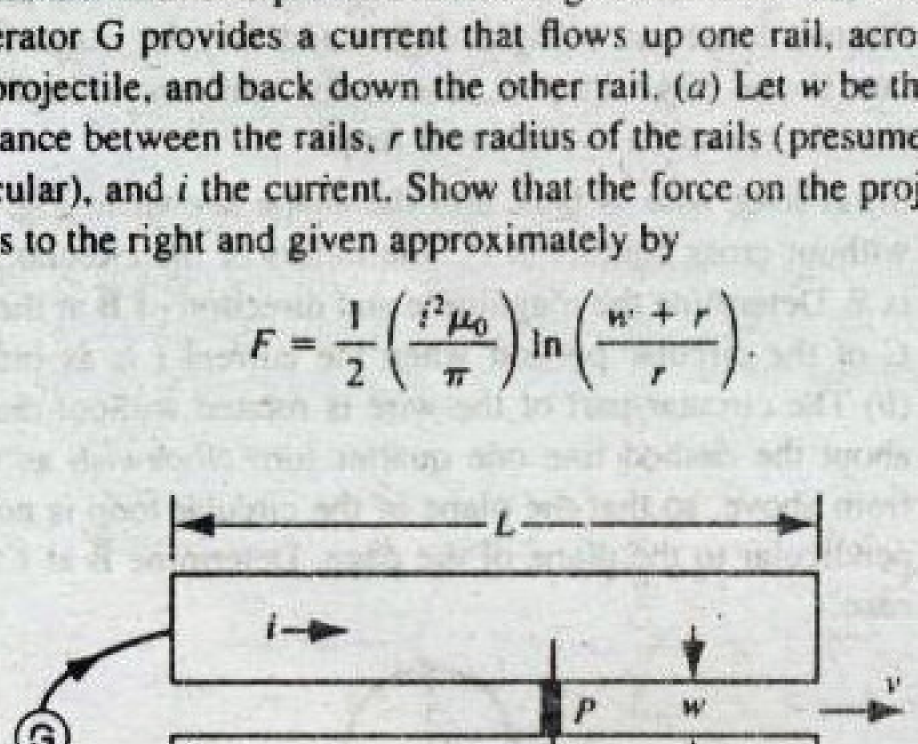


FIGURE 33-49. Exercise 23.

(b) If the projectile starts from the left end of the rail at rest, find the speed v at which it is expelled at the right. Assume that $i = 450$ kA, $w = 12$ mm, $r = 6.7$ cm, $L = 4.0$ m, and that the mass of the projectile is $m = 10$ g.

24. Figure 33-50 shows a long wire carrying a current i_1. The rectangular loop carries a current i_2. Calculate the resultant force acting on the loop. Assume that $a = 1.10$ cm, $b = 9.20$ cm, $L = 32.3$ cm, $i_1 = 28.6$ A, and $i_2 = 21.8$ A.

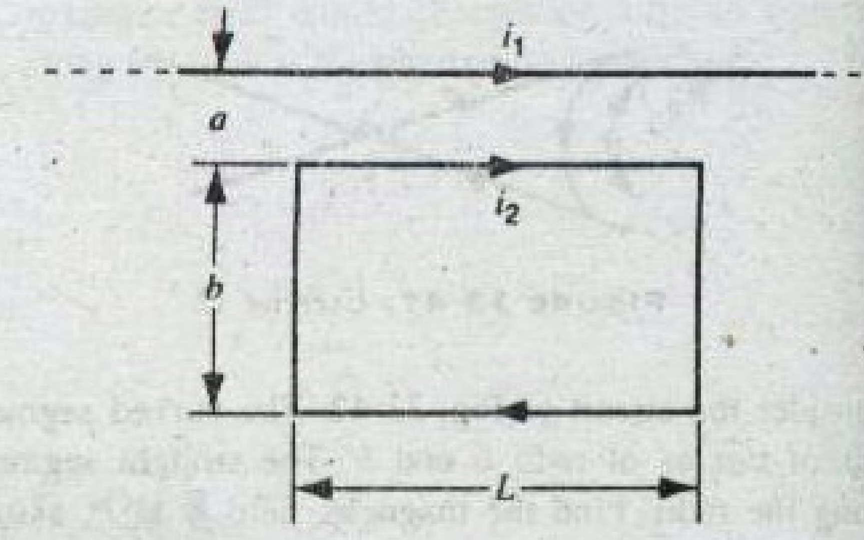


FIGURE 33-50. Exercise 24.

25. In Sample Problem 33-6, suppose that the upper wire is displaced downward a small distance and then released. Show that the resulting motion of the wire is simple harmonic with the same frequency of oscillation as a simple pendulum of length d.

33-4 The Magnetic Field of a Solenoid

26. A solenoid 95.6 cm long has a radius of 1.90 cm, a winding of 1230 turns, and carries a current of 3.58 A. Calculate the strength of the magnetic field inside the solenoid.

27. A solenoid 1.33 m long and 2.60 cm in diameter carries a current of 17.8 A. The magnetic field inside the solenoid is 22.4 mT. Find the length of the wire forming the solenoid.

28. A long solenoid with 115 turns/cm and a radius of 7.20 cm carries a current of 1.94 mA. A current of 6.30 A flows in a straight conductor along the axis of the solenoid. (a) At what radial distance from the axis will the direction of the resulting magnetic field be at 40.0° from the axial direction? (b) What is the magnitude of the magnetic field?

29. Verify the integration in Eq. 33-27, and show that as $L \to \infty$, Eq. 33-27 approaches Eq. 33-28.

33-5 Ampère's Law

30. Eight wires cut the page perpendicularly at the points shown in Fig. 33-51. A wire labeled with the integer k ($k = 1, 2, \ldots, 8$) bears the current ki_0. For those with odd k, the current is out of the page; for those with even k it is into the page. Evaluate $\oint \vec{B} \cdot d\vec{s}$ along the closed loop in the direction shown.

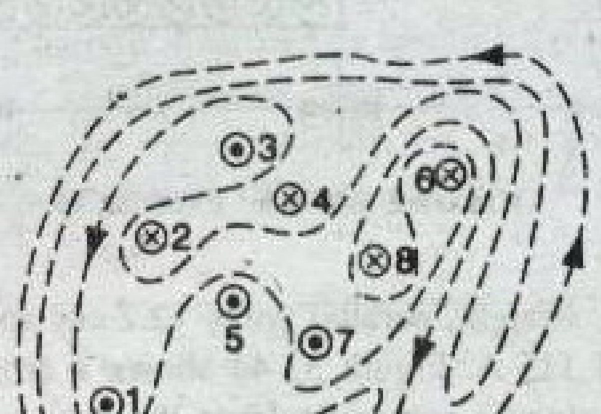


FIGURE 33-51. Exercise 30.

31. Each of the indicated eight conductors in Fig. 33-52 carries 2.0 A of current into or out of the page. Two paths are indicated for the line integral $\vec{B} \cdot d\vec{s}$. What is the value of the integral for (*a*) the dotted path and (*b*) the dashed path?

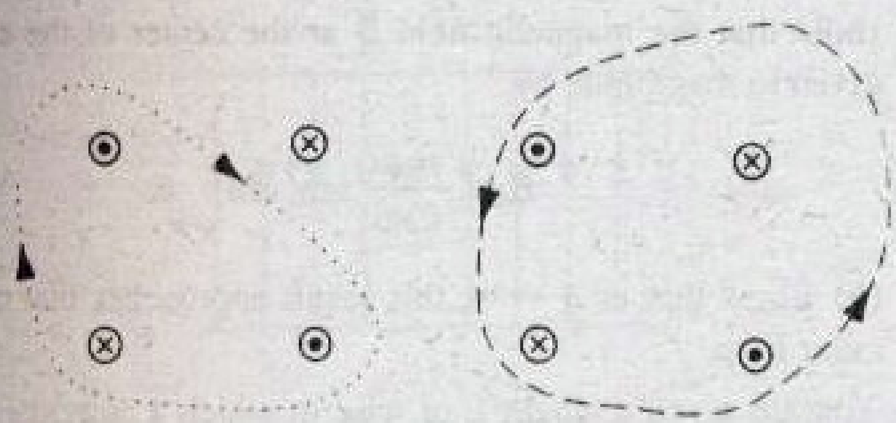


FIGURE 33-52. Exercise 31.

32. Consider a long, cylindrical wire of radius R carrying a current i distributed uniformly over the cross section. At what two distances from the axis of the wire is the magnetic field strength, due to the current, equal to one-half the value at the surface?

33. Figure 33-53 shows a cross section of a long conductor of a type called a coaxial cable of radii a, b, and c. Equal but antiparallel, uniformly distributed currents i exist in the two conductors. Derive expressions for $B(r)$ in the ranges (*a*) $r < c$, (*b*) $c < r < b$, (*c*) $b < r < a$, and (*d*) $r > a$. (*e*) Test these expressions for all the special cases that occur to you. (*f*) Assume that $a = 2.0$ cm, $b = 1.8$ cm, $c = 0.40$ cm, and $i = 120$ A and plot $B(r)$ over the range $0 < r < 3$ cm.

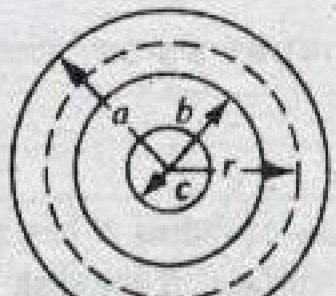


FIGURE 33-53. Exercise 33.

34. Figure 33-54 shows a cross section of a hollow, cylindrical conductor of radii a and b, carrying a uniformly distributed current i. (*a*) Using the circular Ampèrian loop shown, verify that $B(r)$ for the range $b < r < a$ is given by

$$B(r) = \frac{\mu_0 i}{2\pi(a^2 - b^2)} \frac{r^2 - b^2}{r}.$$

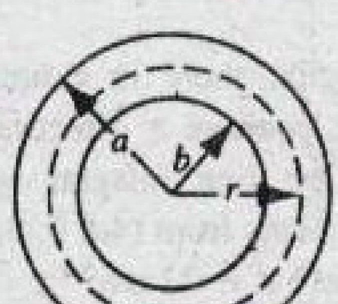


FIGURE 33-54. Exercise 34.

(*b*) Test this formula for the special cases of $r = a$, $r = b$, and $b = 0$. (*c*) Assume that $a = 2.0$ cm, $b = 1.8$ cm, and $i = 100$ A and plot $B(r)$ for the range $0 < r < 6$ cm.

35. A long, circular pipe, with an outside radius of R, carries a (uniformly distributed) current of i_0 (into the paper as shown in Fig. 33-55). A wire runs parallel to the pipe at a distance $3R$ from center to center. Calculate the magnitude and direction of the current in the wire that would cause the resultant magnetic field at the point P to have the same magnitude, but the opposite direction, as the resultant field at the center of the pipe.

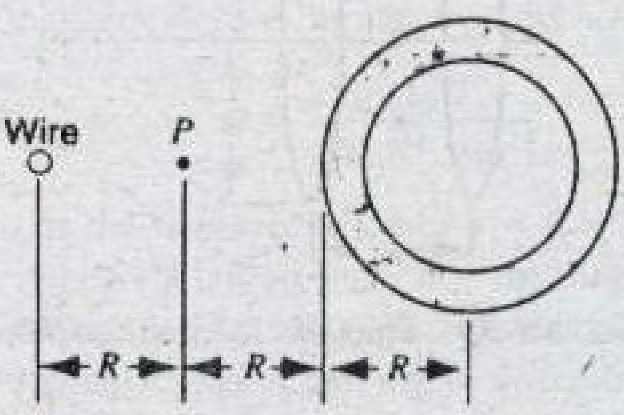


FIGURE 33-55. Exercise 35.

36. A toroid having a square cross section, 5.20 cm on edge, and an inner radius of 16.2 cm has 535 turns and carries a current of 813 mA. Calculate the magnetic field inside the toroid at (*a*) the inner radius and (*b*) the outer radius of the toroid.

37. An interesting (and frustrating) effect occurs when one attempts to confine a collection of electrons and positive ions (a plasma) in the magnetic field of a toroid. Particles whose motion is perpendicular to the $\vec{B}$ field will not execute circular paths because the field strength varies with radial distance from the axis of the toroid. This effect, which is shown (exaggerated) in Fig. 33-56, causes particles of opposite sign to drift in opposite directions parallel to the axis of the toroid. (*a*) What is the sign of the charge on the particle whose path is sketched in the figure? (*b*) If the particle path has a radius of curvature of 11 cm when its radial distance from the axis of the toroid is 125 cm, what will be the radius of curvature when the particle is 110 cm from the axis?

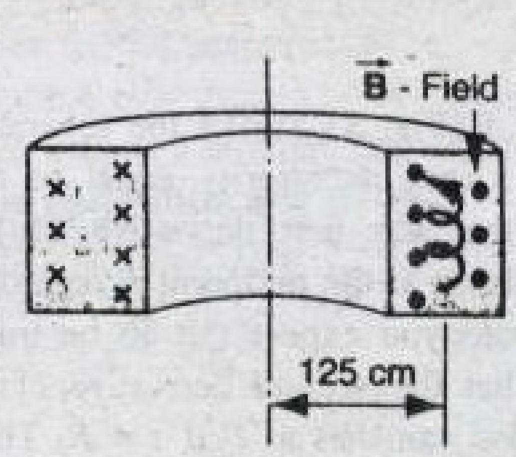


FIGURE 33-56. Exercise 37.

33-6 Electromagnetism and Frames of Reference

PROBLEMS

1. Figure 33-57 shows an arrangement known as a *Helmholtz coil.* It consists of two circular coaxial coils each of N turns and radius R, separated by a distance R. They carry equal currents i in the same direction. Find the magnetic field at P, midway between the coils.

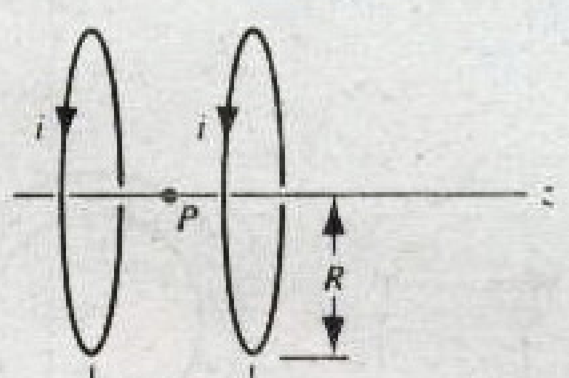


FIGURE 33-57. Problems 1, 3, and computer problem 1.

2. A straight section of wire of length L carries a current i. (*a*) Show that the magnetic field associated with this segment at P, a perpendicular distance D from one end of the wire (see Fig. 33-58), is given by

$$B = \frac{\mu_0 i}{4\pi D} \frac{L}{(L^2 + D^2)^{1/2}}.$$

(*b*) Show that the magnetic field is zero at point Q, along the line of the wire.

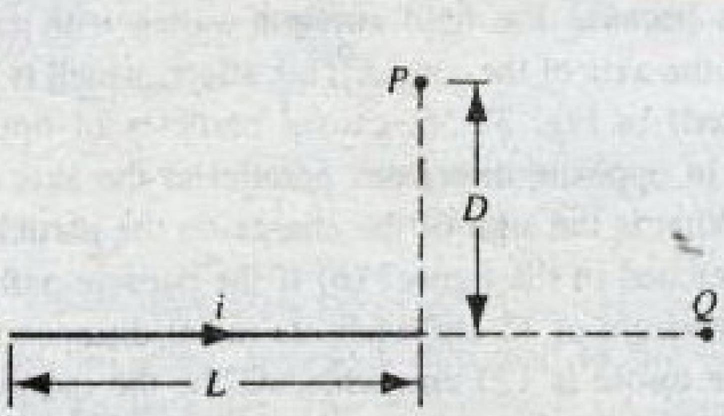


FIGURE 33-58. Problem 2.

3. In Problem 1 (Fig. 33-57) let the separation of the coils be a variable s (not necessarily equal to the coil radius R). (*a*) Show that the first derivative of the magnetic field (dB/dz) vanishes at the midpoint P regardless of the value of s. Why would you expect this to be true from symmetry? (*b*) Show that the second derivative of the magnetic field (d^2B/dz^2) also vanishes at P if $s = R$. This accounts for the uniformity of B near P for this particular coil separation.

4. A square loop of wire of edge a carries a current i. (*a*) Show that B for a point on the axis of the loop and a distance z from its center is given by

$$B(z) = \frac{4\mu_0 i a^2}{\pi(4z^2 + a^2)(4z^2 + 2a^2)^{1/2}}.$$

(*b*) To what does this reduce at the center of the loop?

5. (*a*) A wire in the form of a regular polygon of n sides is j[illegible] enclosed by a circle of radius a. If the current in this wire i[illegible] show that the magnetic field $\vec{B}$ at the center of the circl[illegible] given in magnitude by

$$B = \frac{\mu_0 n i}{2\pi a} \tan(\pi/n).$$

(*b*) Show that as $n \to \infty$, this result approaches that of a ci[illegible]cular loop.

6. You are given a length L of wire in which a current i may b[illegible] established. The wire may be formed into a circle or a squar[illegible] Show that the square yields the greater value for B at the cen[illegible]tral point.

7. (*a*) Calculate $\vec{B}$ at point P in Fig. 33-59. (*b*) Is the fi[illegible] strength at P greater or less than at the center of the squar[illegible]

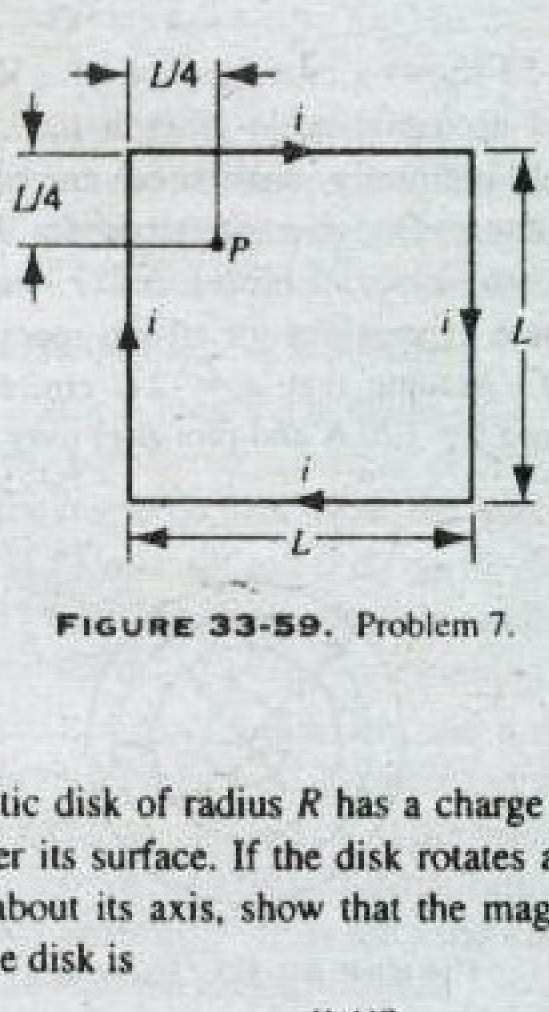


FIGURE 33-59. Problem 7.

8. A thin plastic disk of radius R has a charge q uniformly di[illegible]tributed over its surface. If the disk rotates at an angular fre[illegible]quency ω about its axis, show that the magnetic field at th[illegible] center of the disk is

$$B = \frac{\mu_0 \omega q}{2\pi R}.$$

(Hint: The rotating disk is equivalent to an array of curre[illegible] loops.)

9. A long solenoid has 100 turns per centimeter. An electro[illegible] moves within the solenoid in a circle of radius 2.30 cm pe[illegible]pendicular to the solenoid axis. The speed of the electron [illegible] $0.0460c$ (c = speed of light). Find the current in the sole[illegible]noid.

10. In a certain region there is a uniform current density [illegible] 15 A/m² in the positive z direction. What is the value [illegible] $\oint \vec{B} \cdot d\vec{s}$ when the line integral is taken along the thr[illegible] straight-line segments from $(4d, 0, 0)$ to $(4d, 3d, 0)$ to $(0, 0, 0)$ to $(4d, 0, 0)$, where d = 23 cm?

11. Show that a uniform magnetic field $\vec{B}$ cannot drop abruptly [illegible] zero as one moves at right angles to it, as suggested by th[illegible] horizontal arrow through point a in Fig. 33-60. (Hint: Ap[illegible] Ampère's [illegible] the rectangular path shown by the dashe[illegible] lines.) In actual magnets, "fringing" of the lines of $\vec{B}$ alwa[illegible] occurs, which means that $\vec{B}$ approaches zero in a gradu[illegible]

manner. Modify the $\vec{B}$ lines in the figure to indicate a more realistic situation.

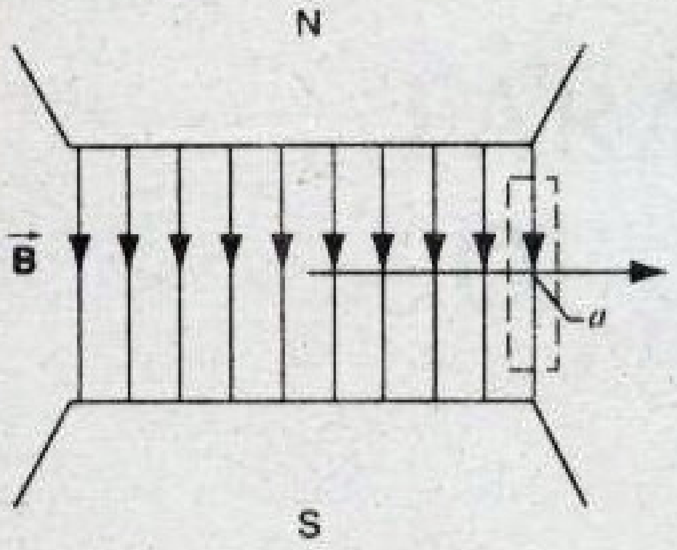


FIGURE 33-60. Problem 11.

12. A conductor consists of an infinite number of adjacent wires, each infinitely long and carrying a current i. Show that the lines of $\vec{B}$ are as represented in Fig. 33-61 and that B for all points above and below the infinite current sheet is given by

$$B = \tfrac{1}{2}\mu_0 n i,$$

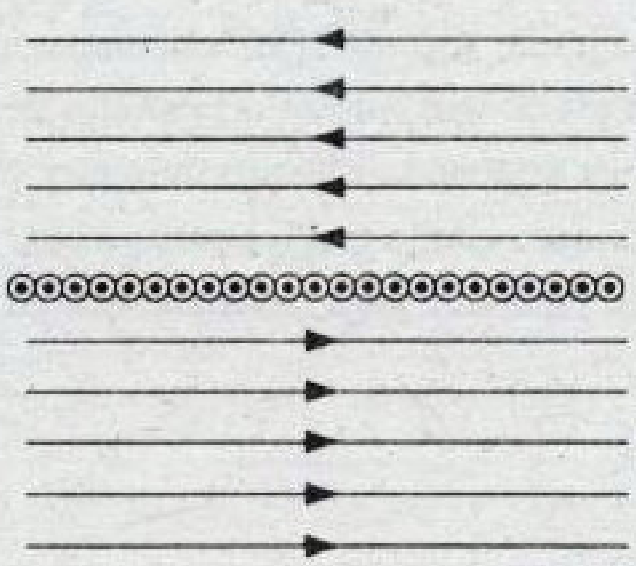


FIGURE 33-61. Problem 12.

where n is the number of wires per unit length. Derive both by direct application of Ampère's law and by considering the problem as a limiting case of Sample Problem 33-5.

13. The current density inside a long, solid, cylindrical wire of radius a is in the direction of the axis and varies linearly with radial distance r from the axis according to $j = j_0 r/a$. Find the magnetic field inside the wire. Express your answer in terms of the total current i carried by the wire.

14. Figure 33-62 shows a cross section of a long, cylindrical conductor of radius R containing a long, cylindrical hole of radius a. The axes of the two cylinders are parallel and are a distance b apart. A current i is uniformly distributed over the shaded area in the figure. (a) Use superposition ideas to show that the magnetic field at the center of the hole is

$$B = \frac{\mu_0 i b}{2\pi(R^2 - a^2)}.$$

(b) Discuss the two special cases $a = 0$ and $b = 0$. (c) Can you use Ampère's law to show that the magnetic field in the hole is uniform? (Hint: Regard the cylindrical hole as filled with two equal currents moving in opposite directions, thus canceling each other. Assume that each of these currents has the same current density as that in the actual conductor. Thus we superimpose the fields due to two complete cylinders of current, of radii R and a, each cylinder having the same current density.)

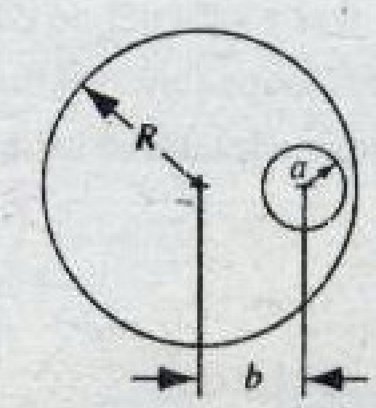


FIGURE 33-62. Problem 14.

COMPUTER PROBLEMS

1. Two 300-turn coils each carry a current i. They are arranged a distance apart equal to their radius, as in Fig. 33-57. (This is the Helmholtz coil geometry; see Problem 1.) For $R = 5.0$ cm and $i = 50$ A, plot B as a function of distance z along the common axis over the range $z = -5$ cm to $z = +5$ cm, taking $z = 0$ at the midpoint P. Such coils provide an especially uniform field B near point P.

2. Design a double Helmholtz coil so that d^4B/dz^4 also vanishes at the center. This is a problem best solved on a computer program such as Mathematica or MAPLE.