CHAPTER 30

CAPACITANCE

In many applications of electric circuits, the goal is to store electrical charge or energy in an electrostatic field. A device that stores charge is called a capacitor, *and the property that determines how much charge it can store is its* capacitance. *We shall see that the capacitance depends on the geometrical properties of the device and not on the electric field or the potential.*

In this chapter we define capacitance and show how to calculate the capacitance of a few simple devices and of combinations of capacitors. We study the energy stored in capacitors and show how it is related to the strength of the electric field. Finally, we investigate how the presence of a dielectric in a capacitor enhances its ability to store electric charge.

30-1 CAPACITORS

A *capacitor** is a device that stores energy in an electrostatic field. A flashbulb, for example, requires a short burst of electric energy that exceeds what a battery can generally provide. A capacitor can draw energy relatively slowly (over several seconds) from the battery, and it then can release the energy rapidly (within milliseconds) through the bulb. Much larger capacitors are used to produce short laser pulses in attempts to induce thermonuclear fusion in tiny pellets of hydrogen. In this case the power level during the pulse is about 10^{14} W, about 200 times the entire electrical generating capacity of the United States, but the pulses typically last only for 10^{-9} s.

Capacitors are also used to produce electric fields, such as the parallel-plate device that gives the very nearly uniform electric field that deflects beams of electrons in a TV or oscilloscope tube.

In circuits, capacitors are often used to smooth out the sudden variations in line voltage that can damage computer memories. In another application, the tuning of a radio or TV receiver is usually done by varying the capacitance of the circuit.

30-2 CAPACITANCE

Figure 30-1 shows a generalized capacitor, consisting of two conductors *a* and *b* of arbitrary shape. No matter what their geometry, these conductors are called *plates*. We assume that they are totally isolated from their surroundings. We further assume, for the time being, that the conductors exist in a vacuum.

A capacitor is said to be *charged* if its plates carry equal and opposite charges $+q$ and $-q$. Note that q is *not* the net charge on the capacitor, which is zero. In our discussion of capacitors, we let q represent the absolute value of the charge on either plate; that is, q represents a magnitude only, and the sign of the charge on a given plate must be specified.

We can "charge" a capacitor by connecting one of its plates to the positive terminal of a battery and the other plate to the negative terminal, as shown in Fig. 30-2. As we discuss in the next chapter, the flow of charge in an electrical circuit is analogous to the flow fluid, and the battery serves as a "pump" for electric charge. When we connect a battery to the capacitor (by closing the switch in the

*See "Capacitors," by Donald M. Trotter, Jr., *Scientific American*, July 1988, p. 86.

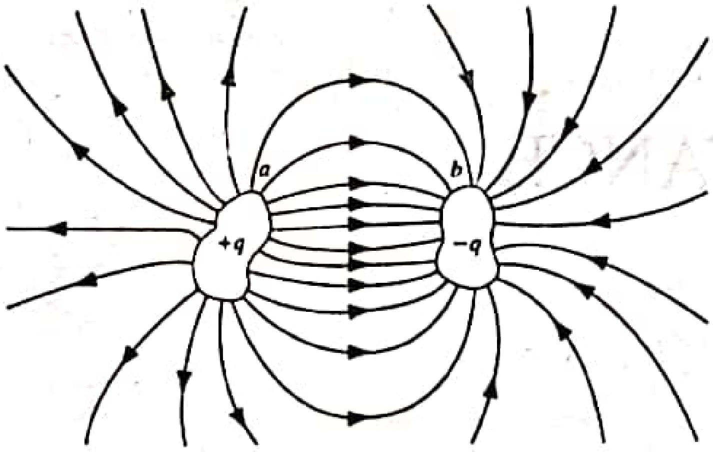


FIGURE 30-1. Two conductors, isolated from one another and from their surroundings, form a capacitor. When the capacitor is charged, the conductors carry equal but opposite charges of magnitude q. The two conductors are called *plates* no matter what their shape.

circuit), the battery "pumps" electrons from the (previously uncharged) positive plate of the capacitor to the negative plate. After the battery moves a quantity of charge of magnitude q, the charge on the positive plate is $+q$ and the charge on the negative plate is $-q$.

An ideal battery maintains a constant potential difference between its terminals. The positive plate and the wire connecting it to the positive terminal of the battery are conductors, and so (under electrostatic conditions) they must be at the same potential V_+ as the positive terminal of the battery. The negative plate and the wire connecting it to the negative terminal of the battery are also conductors, and so (when the switch is closed) they must be at the same potential V_- as the negative terminal of the battery. The potential difference $\Delta V = V_+ - V_-$ between the battery terminals is the same potential difference that appears between the capacitor plates when the switch is closed. We usually describe this as the potential difference "across" the capacitor, meaning the potential difference between its plates.

Figure 30-3 shows the circuit for charging a capacitor by a battery that maintains a constant potential difference

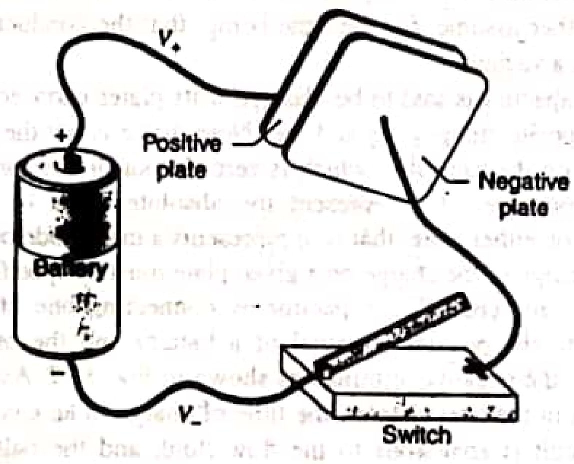


FIGURE 30-2. When the switch is closed, the capacitor becomes charged as the battery moves electrons from the positive

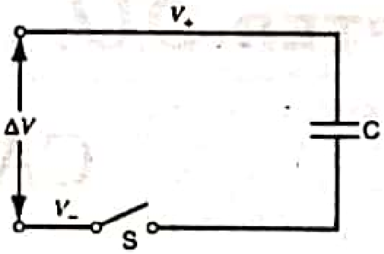


FIGURE 30-3. A schematic circuit diagram equivalent to Fig. 30-2, showing the capacitor C, switch S, and constant potential difference ΔV (supplied by a battery that is not shown in the diagram).

$\Delta V = V_+ - V_-$ between its terminals. In a circuit, a capacitor is represented by the symbol ⊣⊢, in which the two parallel lines suggest the two plates of the capacitor.

When we charge a capacitor, we find that the charge q that appears on the capacitor plates is always directly proportional to the potential difference ΔV between the plates: $q \propto \Delta V$. The *capacitance* C is the constant of proportionality necessary to make this relationship into an equation:

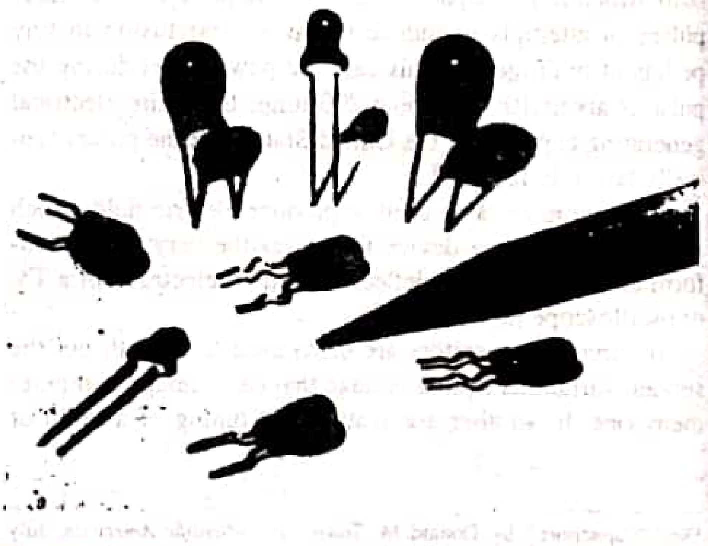
$$q = C\,\Delta V. \tag{30-1}$$

The capacitance is a geometrical factor that depends on the size, shape, and separation of the plates and on the material that occupies the space between the plates (which for now we assume is a vacuum). The capacitance of a capacitor does *not* depend on ΔV or q.

The SI unit of capacitance that follows from Eq. 30-1 is the coulomb/volt, which is given the name *farad* (abbreviation F):

$$1 \text{ farad} = 1 \text{ coulomb/volt}.$$

The unit is named in honor of Michael Faraday who, among his other contributions, developed the concept of capacitance. The submultiples of the farad, the *microfarad* ($1\ \mu F = 10^{-6}$ F) and the *picofarad* ($1\ pF = 10^{-12}$ F), are more convenient units in practice. Figure 30-4 shows some capacitors in the microfarad or picofarad range that might be found in electronic or computing equipment.



SAMPLE PROBLEM 30-1. A storage capacitor on a random access memory (RAM) chip has a capacitance of 0.055 pF. If it is charged to 5.3 V, how many excess electrons are there on its negative plate?

Solution If the negative plate has N excess electrons, it carries a net charge of magnitude $q = Ne$. Using Eq. 30-1, we obtain

$$N = \frac{q}{e} = \frac{C\,\Delta V}{e} = \frac{(0.055 \times 10^{-12}\ \text{F})(5.3\ \text{V})}{1.60 \times 10^{-19}\ \text{C}}$$

$$= 1.8 \times 10^{6}\ \text{electrons.}$$

For electrons, this is a very small number. A speck of household dust, so tiny that it essentially never settles, contains about 10^{17} electrons (and the same number of protons).

Analogy with Fluid Flow (Optional)

In situations involving electric circuits, it is often useful to draw analogies between the movement of electric charge and the movement of material particles such as occurs in fluid flow. In the case of a capacitor, an analogy can be made between a capacitor carrying a charge q and a rigid container of volume v (we use v rather than V for volume so as not to confuse it with potential difference) containing n moles of an ideal gas. The gas pressure p is directly proportional to n for a fixed temperature, according to the ideal gas law (Eq. 21-13)

$$n = \left(\frac{v}{RT}\right)p.$$

For the capacitor (Eq. 30-1)

$$q = C\,\Delta V.$$

Comparison shows that the capacitance C of the capacitor is analogous to the volume v of the container, assuming a fixed temperature for the gas. In fact, the word "capacitor" brings to mind the word "capacity," in the same sense that the volume of a container for gas has a certain "capacity."

We can force more gas into the container by imposing a higher pressure, just as we can force more charge into the capacitor by imposing a higher voltage. Note that any amount of charge can be put on the capacitor, and any mass of gas can be put in the container, up to certain limits. These correspond to electrical breakdown ("arcing over") for the capacitor and to rupture of the walls for the container. ■

30-3 CALCULATING THE CAPACITANCE

Our goal in this section is to calculate the capacitance of a capacitor from its geometry. We do this using the following procedure. (1) We first find the electric field in the region between the plates, using methods such as those described in Section 26-4. (2) We then use Eq. 28-15 to find the potential difference between the positive and negative plates by integrating the electric field along any convenient path connecting the plates:

$$\Delta V = V_+ - V_- = -\int_-^+ \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = \int_+^- \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}. \quad (30\text{-}2)$$

(3) The outcome of Eq. 30-2 will involve the magnitude of the charge q on the right-hand side. Using Eq. 30-1, we can then find $C = q/\Delta V$.

As we have defined it, ΔV is a positive number. Since q is an absolute magnitude, the capacitance C will always be positive.

We now illustrate this method with several examples.

A Parallel-Plate Capacitor

Figure 30-5 shows a capacitor in which the two flat plates are very large and very close together; that is, the separation d is much smaller than the length or width of the plates. We can neglect the "fringing" of the electric field that occurs near the edges of the plates and assume that the electric field has the same magnitude and direction everywhere in the volume between the plates.

We obtained the electric field for a single large uniformly charged disk at points near its center in Section 26-4: $E = \sigma/2\epsilon_0$. If the capacitor plates are very large, their shape is not important, and we can assume that the electric field due to each plate has this magnitude. The net electric field is the sum of the fields due to the two plates: $\vec{\mathbf{E}} = \vec{\mathbf{E}}_+ + \vec{\mathbf{E}}_-$. As Fig. 30-5 shows, the fields due to the positive and negative plates have the same direction, so we can write

$$E = E_+ + E_- = \sigma/2\epsilon_0 + \sigma/2\epsilon_0 = \sigma/\epsilon_0. \quad (30\text{-}3)$$

Using $\sigma = q/A$, where A is the surface area of each plate, and substituting Eq. 30-3 into Eq. 30-2, we obtain

$$\Delta V = \int_+^- E\,ds = \frac{q}{\epsilon_0 A}\int_+^- ds = \frac{qd}{\epsilon_0 A}, \quad (30\text{-}4)$$

where we have chosen an integration path along one of the lines of the electric field, so that $\vec{\mathbf{E}}$ and $d\vec{\mathbf{s}}$ are parallel (see Fig. 30-5).

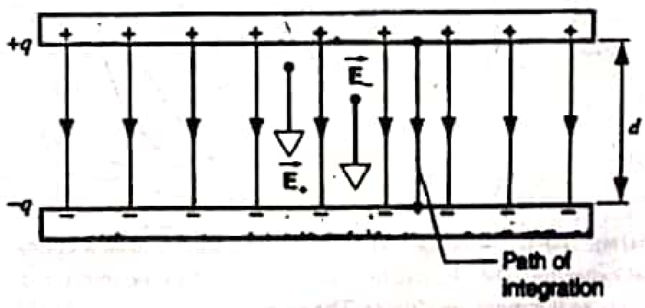


FIGURE 30-5. A parallel-plate capacitor. The path of integration for evaluating Eq. 30-4 is shown.

The capacitance is then obtained from Eq. 30-1: $C = q/\Delta V$, or

$$C = \frac{\epsilon_0 A}{d} \quad \text{(parallel-plate capacitor).} \quad (30\text{-}5)$$

You can see from this equation why we say that the capacitance depends on geometrical factors, in this case the plate separation d and area A. The capacitance does not depend on the voltage difference between the plates or the charge carried by the plates.

Note that the right-hand side of Eq. 30-5 has the form of ϵ_0 times a quantity with the dimension of length (A/d). We will find that all expressions for capacitance have essentially this same form, which suggests that the units of ϵ_0 can be expressed as capacitance divided by length:

$$\epsilon_0 = 8.85 \times 10^{-12}\ \text{F/m} = 8.85\ \text{pF/m}.$$

These units for ϵ_0 are often more useful for calculations of capacitance than our previous (and equivalent) units of $C^2/N \cdot m^2$.

A Spherical Capacitor

Figure 30-6 shows a cross section of a spherical capacitor, in which the inner conductor is a solid sphere of radius a, and the outer conductor is a hollow spherical shell of inner radius b. We assume that the inner sphere carries a charge $+q$ and that the outer sphere has a charge $-q$. From our analysis of conductors using Gauss' law (see Section 27-6), we know that the charge on the inner conductor resides on its surface and that the charge on the outer conductor resides on its inner surface. (Draw a spherical Gaussian surface of radius slightly larger than b; the surface lies entirely within the outer conductor, so $E = 0$ everywhere on the surface and the flux through the surface is zero. Therefore the surface encloses no net charge, as Fig. 30-6 shows.)

In the region $a < r < b$, we can use Gauss' law to determine that, in the region between the conductors, the electric field depends only on the charge on the inner sphere,

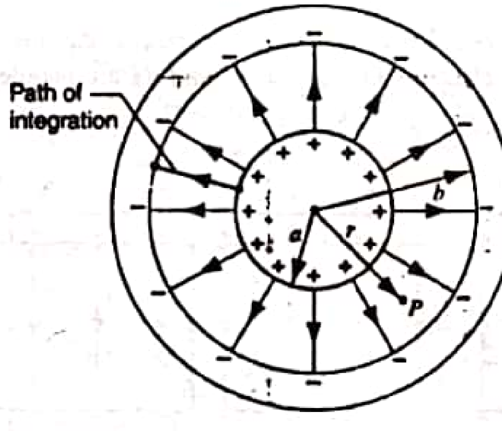


FIGURE 30-6. A cross section through a spherical or a cylindrical capacitor. The electric field at any point P in the interior is due only to the inner conductor. The path of integration for evaluating Eq. 30-7 or Eq. 30-10 is shown.

and that this field is the same as that of a point charge at its center (recall the shell theorems discussed in Section 27-5). We therefore have

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \qquad a < r < b. \qquad (30\text{-}6)$$

Substituting this expression for the electric field into Eq. 30-2 and integrating along the path shown in Fig. 30-6 from the positive plate to the negative plate, we obtain

$$\Delta V = \int_+^- E\,ds = \int_a^b \frac{q}{4\pi\epsilon_0}\frac{dr}{r^2} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\frac{b-a}{ab}. \qquad (30\text{-}7)$$

Because the path of integration is in the radial direction, we have $\vec{E} \cdot d\vec{s} = E\,ds$ and $ds = dr$.

Using $C = q/\Delta V$, we now find

$$C = 4\pi\epsilon_0 \frac{ab}{b-a} \qquad \text{(spherical capacitor).} \qquad (30\text{-}8)$$

Note that the capacitance again has the form of ϵ_0 times a quantity with the dimension of length.

A Cylindrical Capacitor

Figure 30-6 can also represent the cross section of a cylindrical capacitor, in which the inner conductor is a solid rod of radius a carrying a charge $+q$ uniformly distributed over its surface, and the outer conductor is a coaxial cylindrical shell of inner radius b carrying a charge of $-q$ uniformly distributed over its inner surface. The capacitor has length L, and we assume $L >> b$ so that, as was the case with the parallel-plate capacitor, we can neglect the "fringing" field at the ends of the capacitor.

Just as we used Gauss' law in the spherical geometry to obtain the two shell theorems, we can obtain two similar results in the cylindrical geometry. If only the uniformly charged outer cylindrical conductor were present, we could construct a Gaussian surface in the shape of a long cylinder of radius $r < b$ having the same axis as the outer cylinder. This surface encloses no net charge, so we conclude that $E = 0$ everywhere on the Gaussian surface. As in the case of the spherical shell, a uniformly charged cylindrical shell produces no electric field in its interior. Using a cylindrical Gaussian surface with $r > a$, we can deduce that the inner cylinder behaves just like a uniform line of charge, for which the field points radially outward from the axis and has a magnitude that we calculated in Section 26-4 (Eq. 26-17):

$$E = \frac{1}{2\pi\epsilon_0}\frac{q}{Lr} \qquad a < r < b, \qquad (30\text{-}9)$$

where we have replaced the linear charge density λ with q/L and the distance y with the radial coordinate r. Equation 30-2 now gives