MULTIPLE TIME SCALE

Nonlinear Physics

MS PHYSICS
LECTURE 2:

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Problem 2

- **Van der Pol Oscillator**
- The simple equation for this oscillator in 2nd order differential form is given by

$$\ddot{u} + u = \epsilon(1 - u^2)\,\dot{u}$$ → Eq. 1

This equation is called Van der Pol Oscillator. It has more nonlinear effect and damping cannot take place in a specified ratio. The time scales for this relation involves three solutions and $T_0 = t, T_1 = \in t, T_2 = \in^2 t$. The roots for these time scale involve the relation

$$\mathbf{u} = u(T_0, T_1, T_2, \epsilon)$$

And its roots are

$$u = u_0(T_0, T_1, T_2) + \epsilon u_1(T_0, T_1, T_2) + \epsilon^2 u_2(T_0, T_1, T_2)$$

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2$$

Now by applying the chain rule.

$$\frac{d}{dt} = \frac{d}{dT_0}\frac{dT_0}{dt} + \frac{d}{dT_1}\frac{dT_1}{dt} + \frac{d}{dT_2}\frac{dT_2}{dt}$$

The solution for $\frac{dT_0}{dt} = 0, \frac{dT_1}{dt} = \in, \frac{dT_2}{dt} = \in^2$.

Problem 2

- By replacing the values

$$\frac{d}{dt} = \frac{d}{dT_0} + \frac{\epsilon d}{dT_1} + \frac{\epsilon^2 d}{dT_2}$$

- For 2nd order differential values and by ignoring the higher orders than 2
- .

$$\frac{d^2}{dt^2} = \left(\frac{d^2}{dT_0^2} + \frac{\epsilon d^2}{dT_0 dT_1} + \frac{\epsilon^2 d^2}{dT_0 dT_2}\right) + \left(\frac{\epsilon d^2}{dT_1 dT_0} + \frac{\epsilon^2 d^2}{dT_1^2} + \frac{\epsilon^3 d^2}{dT_1 dT_2}\right) + \left(\frac{\epsilon^2 d^2}{dT_2 dT_0} + \frac{\epsilon^3 d^2}{dT_2 dT_1} + \frac{\epsilon^4 d^2}{dT_2^2}\right)$$

- By placing the values in Eq. 1

$$\left[\left(\frac{d^2}{dT_0^2} + \frac{\epsilon d^2}{dT_0 dT_1} + \frac{\epsilon^2 d^2}{dT_0 dT_2}\right) + \left(\frac{\epsilon d^2}{dT_1 dT_0} + \frac{\epsilon^2 d^2}{dT_1^2} + \frac{\epsilon^3 d^2}{dT_1 dT_2}\right) + \left(\frac{\epsilon^2 d^2}{dT_2 dT_0} + \frac{\epsilon^3 d^2}{dT_2 dT_1} + \frac{\epsilon^4 d^2}{dT_2^2}\right)\right] (u_0 + \epsilon u_1 + \epsilon^2 u_2) + (u_0 + \epsilon u_1 + \epsilon^2 u_2) = \epsilon\left[1 - (u_0 + \epsilon u_1 + \epsilon^2 u_2)^2\right](u_0 + \epsilon u_1 + \epsilon^2 u_2)$$

Now by solving the whole relation up to 2nd order and ignoring the higher orders.

Problem 2

$$\frac{d^2}{dT_0^2}u_0 + \epsilon\frac{d^2}{dT_0^2}u_1 + \epsilon^2\frac{d^2}{dT_0^2}u_2 + \epsilon\frac{d^2}{dT_0dT_1}u_0$$
$$+\epsilon^2\frac{d^2}{dT_0dT_1}u_1 + \epsilon^2\frac{d^2}{dT_0dT_2}u_0 + \epsilon\frac{d^2}{dT_1dT_0}u_0 + \epsilon^2\frac{d^2}{dT_1dT_0}u_1$$
$$+\epsilon^2\frac{d^2}{dT_1^2}u_0 + \epsilon^2\frac{d^2}{dT_2dT_0}u_0 + u_0 + \epsilon u_1 + \epsilon^2 u_2$$
$$= \epsilon\frac{d}{dT_0}u_0 + \epsilon^2\frac{d}{dT_0}u_1 + \epsilon^2\frac{d}{dT_1}u_0 - \epsilon\frac{u_0^2 d}{dT_0}u_0$$
$$-\epsilon^2\frac{u_0^2 d}{dT_0}u_1 - \epsilon^2\frac{u_0^2 d}{dT_1}u_0 - \epsilon^2 2\frac{u_0 u_1 d}{dT_0}u_0$$

- By arranging the orders

$$\epsilon^0 \Rightarrow \frac{d^2}{dT_0^2}u_0 + u_0 = 0$$

$$\epsilon^1 \Rightarrow \frac{d^2}{dT_0^2}u_1 + \frac{d^2}{dT_0dT_1}u_0 + \frac{d^2}{dT_1dT_0}u_0 + u_1 = \frac{d}{dT_0}u_0 - \frac{u_0^2 d}{dT_0}u_0$$

$$\epsilon^2 \Rightarrow \frac{d^2}{dT_0^2}u_2 + \frac{d^2}{dT_0dT_1}u_1 + \frac{d^2}{dT_0dT_2}u_0 + \frac{d^2}{dT_1dT_0}u_1 + \frac{d^2}{dT_1^2}u_0 + \frac{d^2}{dT_2dT_0}u_0 + u_2 = \frac{d}{dT_0}u_1 + \frac{d}{dT_1}u_0 - \frac{u_0^2 d}{dT_0}u_1 - \frac{u_0^2 d}{dT_1}u_0 - 2\frac{u_0 u_1 d}{dT_0}u_0$$

Problem 2

- We only need to deal with order zero and one because we have 2nd order differential equation.

$\epsilon^0 \Rightarrow$	$\frac{d^2}{dT_0^2}u_0 + u_0 = 0$
$\epsilon^1 \Rightarrow$	$\frac{d^2}{dT_0^2}u_1 + \frac{2d^2}{dT_0 dT_1}u_0 + u_1 = \frac{d}{dT_0}u_0(1-u_0^2)$

So we can state that general solution for Eq. 1 includes complex and Vander Wall relations.

$$u_0 = A(T_{1,}T_2)\exp(iT_0) + \bar{A}(T_{1,}T_2)\exp(-iT_0)$$

Also

$$u_0 = A\exp(iT_0) + \bar{A}\exp(-iT_0)$$

The integral solution

$$u_0 = A\exp(iT_0) + \text{constant}$$

Similarly the general solution for 1st order relation can be found by replacing the value of u_0 in its equation.

$$\frac{d^2}{dT_0^2}u_1 + \frac{2d^2}{dT_0 dT_1}(A\exp(iT_0) + \bar{A}\exp(-iT_0)) + u_1 = \frac{d}{dT_0}$$
$$[A\exp(iT_0) + \bar{A}\exp(-iT_0)]\left[1 - [A\exp(iT_0) + \bar{A}\exp(-iT_0)]^2\right]$$

By separating the terms of u_0 and u_1 and than solving the relation w.r.t its derivative.

Problem 2

- We find the following relations.

$$\frac{d^2}{dT_0^2}u_1 + u_1 = -\frac{2d^2}{dT_0 dT_1}(A\exp(iT_0) + \bar{A}\exp(-iT_0))$$
$$+\left[1 - [A\exp(iT_0) + \bar{A}\exp(-iT_0)]^2\right]\frac{d}{dT_0}[A\exp(iT_0) + \bar{A}\exp(-iT_0)]$$

$$\frac{d^2}{dT_0^2}u_1 + u_1 = -\frac{2d}{dT_0}\left[A\frac{d}{dT_1}\exp(iT_0) + \bar{A}\frac{d}{dT_1}\exp(-iT_0)\right]$$
$$+\left[1 - [A\exp(iT_0) + \bar{A}\exp(-iT_0)]^2\right]\frac{d}{dT_0}[A\exp(iT_0) + \bar{A}\exp(-iT_0)]$$

$$\frac{d^2}{dT_0^2}u_1 + u_1 = -2\left[A\frac{d}{dT_1}\frac{d}{dT_0}\exp(iT_0) + \bar{A}\frac{d}{dT_1}\frac{d}{dT_0}\exp(-iT_0)\right]$$
$$+\left[1 - [A\exp(iT_0) + \bar{A}\exp(-iT_0)]^2\right]\left[A\frac{d}{dT_0}\exp(iT_0) + \bar{A}\frac{d}{dT_0}\exp(-iT_0)\right]$$

$$\frac{d^2}{dT_0^2}u_1 + u_1 = -2\left[A\frac{d}{dT_1}\exp(iT_0).(i) + \bar{A}\frac{d}{dT_1}\exp(-iT_0).(-i)\right]$$
$$+\left[1 - \left[(A\exp(iT_0))^2 + (\bar{A}\exp(-iT_0))^2 + 2A\exp(iT_0)\bar{A}\exp(-iT_0)\right]\right][A\exp(iT_0)(i) + \bar{A}\exp(-iT_0)(-i)]$$

$$\frac{d^2}{dT_0^2}u_1 + u_1 = -2iA\frac{d}{dT_1}\exp(iT_0) + 2i\bar{A}\frac{d}{dT_1}\exp(-iT_0)$$
$$+\left[1 - (A\exp(iT_0))^2 - (\bar{A}\exp(-iT_0))^2 - 2A\bar{A}\right][iA\exp(iT_0) - i\bar{A}\exp(-iT_0)]$$

Problem 2

$$\frac{d^2}{dT_0^2}u_1 + u_1 = -2iA\frac{d}{dT_1}\exp(iT_0) + 2i\bar{A}\frac{d}{dT_1}\exp(-iT_0)$$
$$+\Big[1[iA\exp(iT_0) - i\bar{A}\exp(-iT_0)] - (A\exp(iT_0))^2[iA\exp(iT_0) - i\bar{A}\exp(-iT_0)]$$
$$-(\bar{A}\exp(-iT_0))^2[iA\exp(iT_0) - i\bar{A}\exp(-iT_0)] - 2A\bar{A}[iA\exp(iT_0) - i\bar{A}\exp(-iT_0)]\Big]$$

$$\frac{d^2}{dT_0^2}u_1 + u_1 = -2iA\frac{d}{dT_1}\exp(iT_0) + 2i\bar{A}\frac{d}{dT_1}\exp(-iT_0)$$
$$+iA\exp(iT_0) - i\bar{A}\exp(-iT_0) - iA^3\exp(3iT_0) + iA^2\bar{A}\exp(iT_0)$$
$$-iA\bar{A}^2\exp(-iT_0) + i\bar{A}^3\exp(-3iT_0) - 2iA^2\bar{A}\exp(iT_0) + 2iA\bar{A}^2\exp(-iT_0)$$

$$\frac{d^2}{dT_0^2}u_1 + u_1 = -2iA\frac{d}{dT_1}\exp(iT_0) + iA\exp(iT_0) - iA^3\exp(3iT_0) + iA^2\bar{A}\exp(iT_0) - 2iA^2\bar{A}\exp(iT_0)$$
$$+2i\bar{A}\frac{d}{dT_1}\exp(-iT_0) - i\bar{A}\exp(-iT_0) - iA\bar{A}^2\exp(-iT_0) + i\bar{A}^3\exp(-3iT_0) + 2iA\bar{A}^2\exp(-iT_0)$$

$$\frac{d^2}{dT_0^2}u_1 + u_1 = -2iA\frac{d}{dT_1}\exp(iT_0) + iA\exp(iT_0) - iA^3\exp(3iT_0) + iA^2\bar{A}\exp(iT_0) - 2iA^2\bar{A}\exp(iT_0) + \text{constant}$$

$$\frac{d^2}{dT_0^2}u_1 + u_1 = -i\exp(iT_0)\left[2A\frac{d}{dT_1} - A - A^2\bar{A} + 2A^2\bar{A}\right] - iA^3\exp(3iT_0) + \text{constant}$$

$$\frac{d^2}{dT_0^2}u_1 + u_1 = -i\exp(iT_0)\left[2A\frac{d}{dT_1} - A + A^2\bar{A}\right] - iA^3\exp(3iT_0) + \text{constant}$$

Problem 2

- In order to solve the differential equation from above solution, we apply the approximations

$$2\frac{dA}{dT_1} - A + A^2\bar{A} = 0 \longrightarrow \text{Eq. 3}$$

- Here $\bar{A}$ is the complex conjugate of A. The complex quantity A can be replaced by

$$A = \frac{1}{2}a\exp(i\phi) \quad \text{or} \quad \bar{A} = \frac{1}{2}a\exp(-i\phi)$$

Where $a = a(T_1, T_2)$ and $\phi = \phi(T_1, T_2)$

The relations by replacing values in Eq. 3

$$2\frac{d}{dT_1}\left(\frac{1}{2}a\exp(i\phi)\right) - \left(\frac{1}{2}a\exp(i\phi)\right) + \left(\frac{1}{2}a\exp(i\phi)\right)^2\left(\frac{1}{2}a\exp(-i\phi)\right) = 0$$

$$\frac{d}{dT_1}(a\exp(i\phi)) - \left(\frac{1}{2}a\exp(i\phi)\right) + \left(\frac{1}{4}a^2\exp(i2\phi)\right)\left(\frac{1}{2}a\exp(-i\phi)\right) = 0$$

By product rule

$$\exp(i\phi)\frac{da}{dT_1} + ai\frac{d\phi}{dT_1}\exp(i\phi) - \left(\frac{1}{2}a\exp(i\phi)\right) + \left(\frac{1}{8}a^3\exp(i\phi)\right) = 0$$

Taking exponent common

$$\left[\frac{da}{dT_1} + ai\frac{d\phi}{dT_1} - \frac{1}{2}a + \frac{1}{8}a^3\right]\exp(i\phi) = 0$$

Problem 2

$$\frac{da}{dT_1} - \frac{1}{2}a + \frac{1}{8}a^3 + ai\frac{d\phi}{dT_1} = 0$$

Real Part

$$\frac{da}{dT_1} - \frac{1}{2}a + \frac{1}{8}a^3 = 0$$ → Eq. 4

Imaginary part

$$a\frac{d\phi}{dT_1} = 0$$

By solving the real part only.

$$\frac{da}{dT_1} = \frac{1}{2}a - \frac{1}{8}a^3$$

$$da = \frac{1}{2}a\left(1 - \frac{1}{4}a^2\right)dT_1$$

By taking integral at both sides

$$\int \frac{1}{\frac{1}{2}a\left(1 - \frac{1}{4}a^2\right)}da = \int dT_1$$

Solution for integral

$$\int \frac{1}{\frac{1}{2}a\left(1 + \frac{1}{2}a\right)\left(1 - \frac{1}{2}a\right)}da = T_1 + \text{constant}$$

In order to solve L.H.S we use the partial fraction method.

Eq. 5 ←

$$\frac{1}{\frac{1}{2}a\left(1 + \frac{1}{2}a\right)\left(1 - \frac{1}{2}a\right)} = \frac{8}{a(2+a)(2-a)} = \frac{A}{a} + \frac{B}{(2+a)} + \frac{C}{(2-a)}$$

Multiplying both sides by a(a+2)(a-2)

$$8 = A(2+a)(2-a) + Ba(2-a) + Ca(2+a)$$ → Eq. 6

Problem 2

By replacing a=2 in Eq. 6 $C = 1$

By replacing a=-2 in Eq. 6 $B = -1$

By replacing a=0 in Eq. 6 $A = 2$

Now by replacing the values of A, B and C in Eq. 5

$$\int \frac{2}{a} . - \frac{1}{(2+a)} . + \frac{1}{(2-a)} da = T_1 + \text{constant}_1$$

Solving integral

$$2 \ln a - \ln(2+a) - \ln(2-a) = T_1 + \text{constant}_2$$

Dividing by 2

$$\ln a - \frac{1}{2} \ln(2+a) - \frac{1}{2} \ln(2-a) = \frac{T_1}{2} + \text{constant}_2$$

Applying log rule

$$\ln a + \ln(2+a)^{-\frac{1}{2}} + \ln(2-a)^{-\frac{1}{2}} = \frac{T_1}{2} + \text{constant}_2$$

Applying log rule

$$\ln a . (2+a)^{-\frac{1}{2}} . (2-a)^{-\frac{1}{2}} = \frac{T_1}{2} + \text{constant}_2$$

Taking exponent

$$a(2+a)^{-\frac{1}{2}} (2-a)^{-\frac{1}{2}} = \exp\left(\frac{T_1}{2}\right) . \text{constant}_2$$

Taking square at B.S

$$a^2 (2+a)^{-1} (2-a)^{-1} = \exp(T_1) . \text{constant}_2$$

Problem 2

- By simplifying the whole term

$$\frac{a^2}{(2+a)(2-a)} = \exp(T_1).\text{constant}_2$$

$$\frac{a^2}{(4-a^2)} = \exp(T_1).\text{constant}_2$$

$$\frac{1}{\left(\frac{4}{a^2} - \frac{a^2}{a^2}\right)} = \exp(T_1).\text{constant}_2$$

$$\frac{1}{\left(\frac{4}{a^2} - 1\right)} = \exp(T_1).\text{constant}_2$$

$$1 = \exp(T_1).\text{constant}_2\left(\frac{4}{a^2} - 1\right)$$

$$\exp(-T_1).\text{constant}_2 = \left(\frac{4}{a^2} - 1\right)$$

$$\exp(-T_1).\text{constant}_2 + 1 = \frac{4}{a^2}$$

Problem 2

$$\frac{1}{1+\exp(-T_1).\text{constant}_2} = \frac{a^2}{4}$$

$$\frac{4}{1+\exp(-T_1).\text{constant}_2} = a^2$$

- By taking square root

$$\frac{2}{\sqrt{1+\exp(-T_1).\text{constant}_2}} = a$$

$$A = \frac{1}{2} a \exp(i\phi)$$

- Since
- W.R.T T_1

$$A = \frac{1}{2} a \exp[i\phi(T_1)]$$

- By placing the value of “a”

$$\frac{1}{2} \frac{2}{\sqrt{1+T_1\exp(-T_1).\text{constant}_2}}.\exp[i\phi(T_2)]$$

- Since

$$u_0 = A\exp(iT_0) + \text{constant} \Rightarrow \frac{1}{2}.\frac{2\exp(i\phi)\exp(iT_0)}{\sqrt{1+T_2\exp(-T_1)\text{constant}}} + \text{constant}$$

Problem 2

$$u = \frac{4}{1 + \text{constant}(T_2)\exp(-T_1)} \cos t = 0(\epsilon)$$

Which shows the initial condition $u(0) = a,$ and $\dot{u}\ (0) = 0(\epsilon),$ so for the smaller scale $\epsilon^2 t$ has not been reflected in above solution.