MULTIPLE TIME SCALE

Nonlinear Plasma Physics

MS PHYSICS
LECTURE 1:

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Contents

- Multiple Scale Analysis (Multiple Time Scale Analysis)
- Example (Simple Harmonic Oscillator)
- Problem 1. Damped Oscillating Time Scale
- Problem 2. Vander Pole Oscillator (we would cover in next lecture)

Multiple Scale Analysis

In mathematics and physics, **multiple-scale analysis** contain techniques that are used to construct uniformly valid approximations to the solutions of perturbation problems, both for small as well as large values of the independent variables.

- Also called the method of **multiple scales**

- Waves can travel in two ways, faster or slower, In multiple time scale we deal with both or may be more cases at same time.

- During these motions different perturbations are produced. “In simple language these perturbations are called the disturbances”.

Multiple Scale Analysis

- We may deal with linear case or nonlinear case.

For Example in Linear cases the equation of continuity is presented by

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0$$

Where n can be replaced by $n = n_0 + n_1$ where n_0 and n_1 are presenting the equilibrium and 1st perturbation for linear cases.

However for nonlinear cases these perturbations are not restricted to one only, we may encounter other perturbations as well, i.e.

$$n = n_0 + \epsilon^1 n_1 + \epsilon^2 n_2 + \epsilon^3 n_3 + \ldots.$$

These n_1 , n_2 and n_3 factors are presenting the higher order perturbations.

Example "Simple Harmonic Oscillator"

Simple Harmonic Oscillator can b presented by using the relation of derivative i.e. $\ddot{x}_1 + x_2 = 0.$ The solution for this equation is given by following relations.

$$x_1 = a_1 \sin(\omega_1 t + \phi_1)$$
$$x_2 = a_2 \sin(\omega_2 t + \phi_2)$$

As it was 2nd order equation so it has its two solutions, the factors a_1=$a_2 = a$ represents the amplitude and it is always same for both relations and similarly

$$\ddot{x}_1 + x_2 = 0$$

Represents the superposition relation.

We can have

$$x=a_1 \sin(\omega_1 t + \phi_1)+a_2 \sin(\omega_2 t + \phi_2)$$

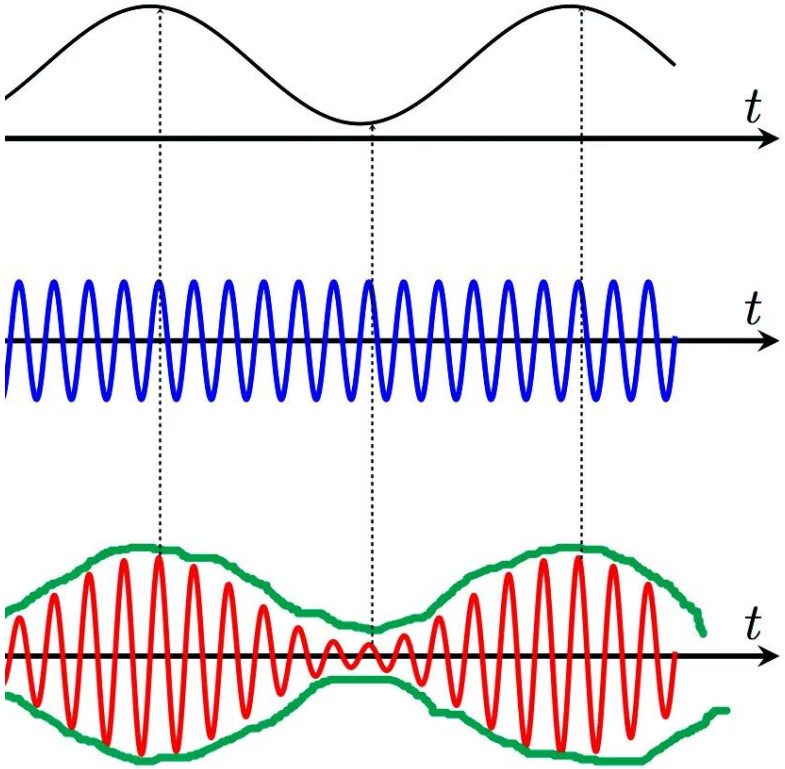
By applying the superposition principle we have

$$x = 2a \sin\left[\left(\frac{\omega_1 - \omega_2}{2}\right)t + \left(\frac{\phi_1 - \phi_2}{2}\right)\right] \sin\left[\left(\frac{\omega_1 + \omega_2}{2}\right)t + \left(\frac{\phi_1 + \phi_2}{2}\right)\right]$$

The first part of above relation shows the slow time scale and the second part shows the fast time scale.

Pat Phenomenon

- Both time scales adjust each other in case of superposition as shown in figure.
- High Time.
- Slow Time.
- Modulated wave.



Explanation

- In Pat phenomenon the relation t is called regular time phenomenon and ϵt is slow time scale phenomenon. i.e.

$$\tau = \epsilon t \longrightarrow \text{Slow Time Scale}$$

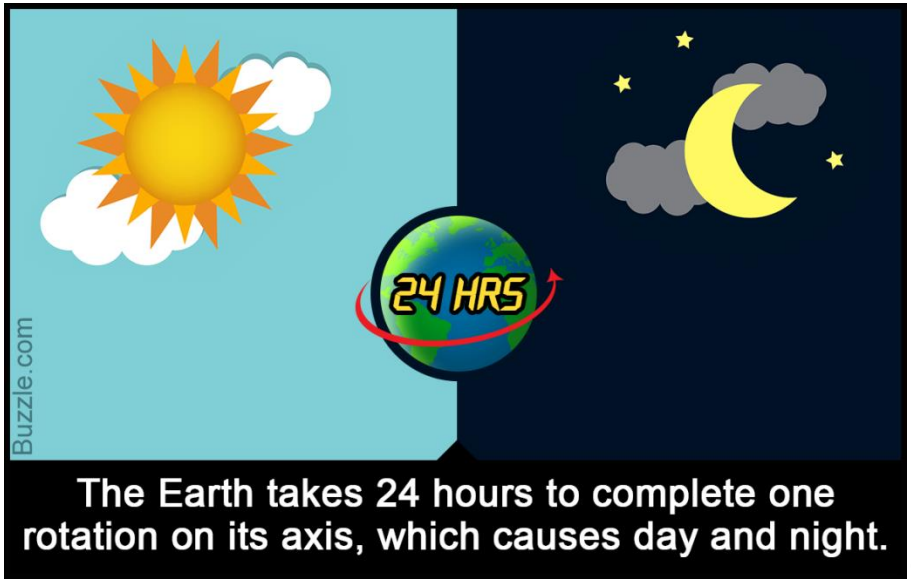
$$\text{Here } \epsilon = \frac{\omega_1 - \omega_2}{\omega_1 + \omega_2} \ll 1$$

$$x = 2a \sin\left[\left(\frac{\omega_1 - \omega_2}{2}\right)t + \left(\frac{\phi_1 - \phi_2}{2}\right)\right] \sin\left[\left(\frac{\omega_1 + \omega_2}{2}\right)t + \left(\frac{\phi_1 + \phi_2}{2}\right)\right]$$

This relation determines that we can find physically how much time scales are involved in system. Technique to find out these solutions is to find the way to convert the nonlinear problem in linear one.

Example 2

- Another example to nonlinear time scales is temperature during different times of a day. The temperature at early morning, in mid afternoon, in the evening and at night. This example represents the fast time scale in comparison to the changing seasons during whole year.



The Earth take a whole year for seasons

Problem 1

- Damped oscillating time scale.

We consider damped linear oscillator, so by using the relation.

$$\ddot{x} + x = -2\,\epsilon\dot{x}$$

$$\frac{d^2x}{dt^2} + x = -2\,\epsilon\,\frac{dx}{dt} \longrightarrow \text{Eq. 1}$$

The relation shows damping factor because if R.H.S was zero than it was simple oscillator. By using relation involving time scale we have

$$x = a\exp(-\epsilon t)\cos\left(\sqrt{1-\epsilon^2}\,t + \phi\right) \longrightarrow Eq.\ \alpha$$

The exponent factor is slowly decaying amplitude. The time scale involved are

Here $\epsilon = \frac{\omega_1 - \omega_2}{\omega_1 + \omega_2} \ll 1.$

This represents that solutions are much more smaller exponentially, so we have ϵt and $\sqrt{1-\epsilon^2}\,t$

$T_2 = \epsilon t$ shows slow time scale and $T_1 = \sqrt{1-\epsilon^2}\,t$ shows regular of fast time scale.

We also assume that this facor is very small so we consider

$$T_1 = t \longrightarrow \text{i}$$

$$T_2 = \epsilon t \longrightarrow \text{ii}$$

Problem 1

The roots for solution of 2nd order differential is dependent upon all variable factors so, As we have two types of time scales so by using chain rule.

$$\frac{d}{dt} = \frac{d}{dT_1}\cdot\frac{dT_1}{dt} + \frac{d}{dT_2}\cdot\frac{dT_2}{dt}$$

where $\frac{dT_1}{dt} = 1$ and $\frac{dT_2}{dt} = \epsilon$ from equation (i) and (ii), in that case our chain rule is reduced to the following

$$\frac{d}{dt} = \frac{d}{dT_1} + \frac{\epsilon d}{dT_2}$$

By taking double derivative,

$$\frac{d}{dt}\left(\frac{d}{dt}\right) = \left(\frac{d}{dT_1} + \frac{\epsilon d}{dT_2}\right)\left(\frac{d}{dT_1} + \frac{\epsilon d}{dT_2}\right)$$

$$\frac{d^2}{dt^2} = \frac{d^2}{dT_1^2} + 2\frac{\epsilon d^2}{dT_1 dT_2} + \frac{\epsilon^2 d^2}{dT_2^2}$$

By placing values in equation 1 we have.

$$\left(\frac{d^2}{dT_1^2} + 2\frac{\epsilon d^2}{dT_1 dT_2} + \frac{\epsilon^2 d^2}{dT_2^2}\right)x + x = -2\epsilon\left(\frac{d}{dT_1} + \frac{\epsilon d}{dT_2}\right)x \longrightarrow \text{Eq. 2}$$

Problem 1

- The space variable is dependent upon three factors, i.e. $x(T_1, T_2, \epsilon)$ so by using asymptotic expansion we have

$$x = x_0(T_1, T_2) + \epsilon x_1(T_1, T_2) + \epsilon^2 x_2(T_1, T_2) \ldots .$$

- We can rewrite equation in simple form as given below.

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 \ldots .$$

- By replacing values in Eq. 2 and ignoring the higher terms,

$$\left(\frac{d^2}{dT_1^2} + 2\frac{\epsilon d^2}{dT_1 dT_2} + \frac{\epsilon^2 d^2}{dT_2^2} \right)(x_0 + \epsilon x_1 + \epsilon^2 x_2) + (x_0 + \epsilon x_1 + \epsilon^2 x_2) = -2\epsilon \left(\frac{d}{dT_1} + \frac{\epsilon d}{dT_2} \right)(x_0 + \epsilon x_1 + \epsilon^2 x_2)$$

- By expanding the relation

$$\frac{d^2}{dT_1^2}(x_0 + \epsilon x_1 + \epsilon^2 x_2) + 2\frac{\epsilon d^2}{dT_1 dT_2}(x_0 + \epsilon x_1 + \epsilon^2 x_2) + \frac{\epsilon^2 d^2}{dT_2^2}(x_0 + \epsilon x_1 + \epsilon^2 x_2)$$

$$+(x_0 + \epsilon x_1 + \epsilon^2 x_2) = -2\epsilon \frac{d}{dT_1}(x_0 + \epsilon x_1 + \epsilon^2 x_2) - 2\epsilon \frac{\epsilon d}{dT_2}(x_0 + \epsilon x_1 + \epsilon^2 x_2)$$

Problem 1

$$\frac{d^2}{dT_1^2}x_0+\epsilon\frac{d^2}{dT_1^2}x_1+\epsilon^2\frac{d^2}{dT_1^2}x_2+2\epsilon\frac{d^2}{dT_1dT_2}x_0+2\epsilon^2\frac{d^2}{dT_1dT_2}x_1$$
$$+2\epsilon^3\frac{d^2}{dT_1dT_2}x_2+\left(\epsilon^2\frac{d^2}{dT_2^2}x_0+\epsilon^3\frac{d^2}{dT_2^2}x_1+\epsilon^4\frac{d^2}{dT_2^2}x_2\right)+$$
$$(x_0+\epsilon x_1+\epsilon^2x_2)=\left(-2\epsilon\frac{d}{dT_1}x_0-2\epsilon^2\frac{d}{dT_1}x_1-2\epsilon^3\frac{d}{dT_1}x_2\right)$$
$$-2\epsilon^2\frac{d}{dT_2}x_0-2\epsilon^3\frac{d}{dT_2}x_1+-2\epsilon^4\frac{d}{dT_2}x_2$$

- By arranging the orders of ϵ and ignoring the higher orders.

$$\epsilon^0\left(\frac{d^2}{dT_1^2}x_0+x_0\right)=0$$
$$\epsilon^1\left(\frac{d^2}{dT_1^2}x_1+2\frac{d^2}{dT_1dT_2}x_0+x_1-2\frac{d}{dT_1}x_0\right)=0$$
$$\epsilon^2\left(\frac{d^2}{dT_1^2}x_2+2\frac{d^2}{dT_1dT_2}x_1+\frac{d^2}{dT_2^2}x_0+x_2-2\frac{d}{dT_1}x_1-2\frac{d}{dT_2}x_0\right)=0$$

- **Results** For **order zero** it is presenting simple harmonic motion, the general solution for this is given by following $\mathrm{Eq.}\,\alpha$

$$x_0=A(T_2)\cos(T_1+\phi(T_2))$$

- Here T_2 is constant and a function of *A*, However the relation with cosine angle show the damped harmonic oscillation.

Problem 1

- The 1st order relation after rearranging it and finding the general solution from zero order. We have

$$\frac{d^2}{dT_1^2}x_1 + x_1 = -2\frac{d}{dT_1}x_0 - 2\frac{d^2}{dT_1 dT_2}x_0$$

$$\frac{d^2}{dT_1^2}x_1 + x_1 = -2\frac{d}{dT_1}[A(T_2)\cos(T_1 + \phi(T_2))] - 2\frac{d^2}{dT_1 dT_2}[A(T_2)\cos(T_1 + \phi(T_2))]$$

- By applying differential and applying the product rule of derivative.

$$\frac{d^2}{dT_1^2}x_1 + x_1 = 2A(T_2)\sin[T_1 + \phi(T_2)] - 2\frac{d}{dT_1}\left[\frac{d}{dT_2}[A(T_2)\cos(T_1 + \phi(T_2))]\right]$$

$$\frac{d^2}{dT_1^2}x_1 + x_1 = 2A(T_2)\sin[T_1 + \phi(T_2)] - 2\frac{d}{dT_1}$$
$$\left[A'(T_2)\cos(T_1 + \phi(T_2)) - A(T_2)\sin[T_1 + \phi(T_2)]\phi'(T_2)\right]$$

- By taking derivative of 2nd term w.r.t T1

$$\frac{d^2}{dT_1^2}x_1 + x_1 = 2A(T_2)\sin[T_1 + \phi(T_2)] - 2$$
$$\left[-A'(T_2)\sin(T_1 + \phi(T_2)) - A(T_2)\phi'(T_2)\cos[T_1 + \phi(T_2)]\right]$$

Problem 1

- Further by simplifying and rearranging relations.

$$\frac{d^2}{dT_1^2}x_1 + x_1 = 2A(T_2)\sin[T_1 + \phi(T_2)] + 2A'(T_2)$$

$$\sin(T_1 + \phi(T_2)) + 2A(T_2)\phi'(T_2)\cos[T_1 + \phi(T_2)]$$

- This is 2nd order differential equation and results are in inhomogeneous form. The R.H.S of this equation are presenting the harmonics of frequency of one and there are some results that grow inhomogeniously resonance also called circular terms. In order to study the phenomenon these values are need to be equalized to zero. And for that condition we assume a new relation.

$$2A(T_2)\sin[T_1 + \phi(T_2)] + 2A'(T_2)\sin(T_1 + \phi(T_2)) + 2A(T_2)\phi'(T_2)\cos[T_1 + \phi(T_2)]$$

- The 1st order differential relations in above expression are

$$A'(T_2) + A(T_2) = 0 \Rightarrow \frac{d}{dT_2}A(T_2) + A(T_2) = 0$$

$$\phi'(T_2) = 0 \Rightarrow \frac{d}{dT_2}\phi(T_2) = 0$$

- The general solution for these both 1st order differential equations can be determined by taking integral.

Problem 1

- The integral relation is

$$A(T_2) = a\exp(-T_2)$$
$$\phi(T_2) = \text{constant}$$

- Since the x_0

$$x_0 = A(T_2)\cos(T_1 + \phi(T_2))$$

- By replacing values

$$x_0 = a\exp(-T_2)\cos(T_1 + \phi(T_2))$$

- Where

$$T_1 = \sqrt{1-\epsilon^2}\,t$$
$$T_2 = \epsilon t$$

- Hence

$$x_0 = a\exp(-T_2)\cos\left(\sqrt{1-\epsilon^2}\,t + \phi(\epsilon t)\right)$$

- This is the result for Damped Harmonic relations and shows exact values for system.