Motion of Particle In Non-Uniform Magnetic Field:-

For uniform fields we were able to obtain exact answer (expression) for the guiding centre drift. If we introduce inhomogenity then to get an approximate expression it is customary to expand in the small ratio r_L/L where L is the scale length of the inhomogenity. This type of theory is called orbit theory.

Figure:-

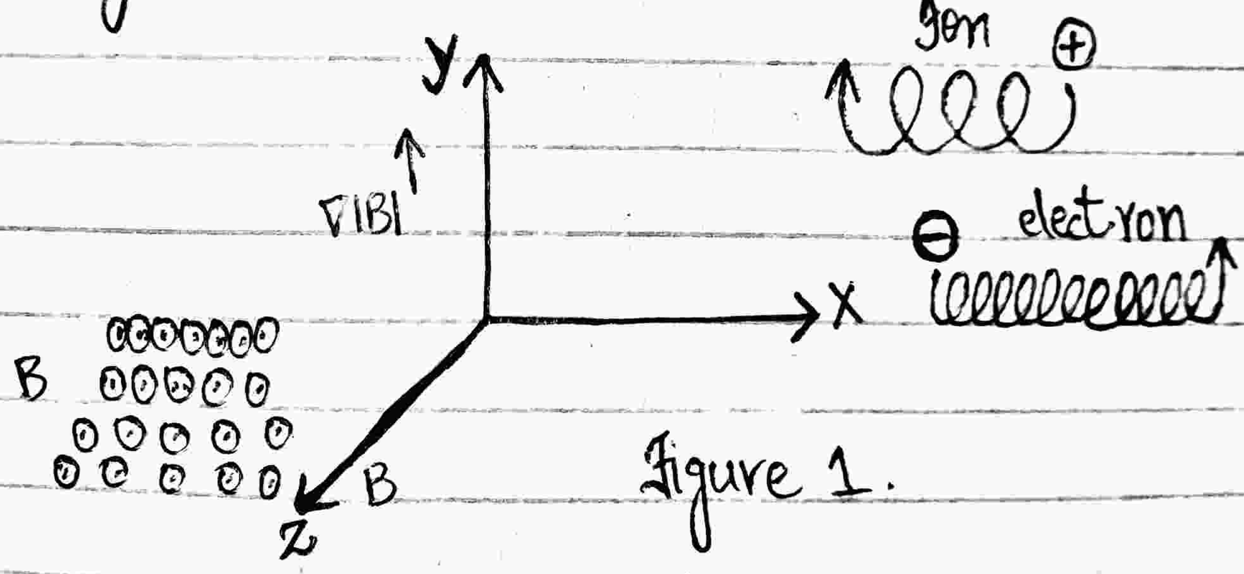

Figure 1.

Figure 1 shows the drift of a gyrating particle in an non-uniform magnetic field.

As we Know that the Larmour Radius is,

$$r_L = \frac{V_\perp m}{qB} \quad \text{—— (1)}$$

At the bottom of axis value of r_L increases hence, the magnetic field (ions) decreases.
while,
at the top of the axis value of r_L decreases hence the magnetic field ($\bar{e}s$) increases.

General Equation of Motion is,

$$F = q(V \times B) \quad \text{(Lorentz force)}$$

As we are concerned only with the magnetic field so, $E = 0$
&

The only component important for us is the y-component.

$$F_y = -qV_x B_z(y) \quad \text{—— (2)}$$

A drift in opposite direction for ions & $\bar{e}s$ & the force is along y-axis.
Now,
As we know that,

$$V_x = V_\perp e^{i\omega_c t} \quad \text{—— (3)}$$

$$V_x = V_\perp \cos\omega_c t + i\sin\omega_c t$$

but we take only its real part,

$$V_x = V_\perp \cos\omega_c t \quad \text{—— (4)}$$

Put the value of V_x from eq. (4) in eq. (2),

$$f_y = -qV_\perp \cos w_c t \, B_z(y) \quad \text{---} (5)$$

By applying Taylor's Expansion,

$$B_z(y) = \vec{B}_0 + (r \cdot \vec{\nabla}) B \quad \text{---} (6)$$

In our case,

$$r = y \quad \& \quad \nabla = \partial/\partial y$$

put in eq. (6)

$$B_z(y) = B_0 + (y \cdot \partial/\partial y) B$$

$$B_z(y) = B_0 + y \frac{\partial B}{\partial y} \quad \text{---} (7)$$

As we know that,

$$y - y_0 = r_L e^{iw_c t}$$

as we assume, x_0, y_0 is fixed.

So, $x_0, y_0 = 0$

then,

$$y = r_L e^{iw_c t}$$

By taking its real part,

$$y = r_L \cos w_c t$$

put this value in eq. (7),

$$B_z(y) = B_0 + r_L \cos w_c t \frac{\partial B}{\partial y}$$

Put this value of $B_z(y)$ in eq. (5),

$$F_y = -qV_\perp \cos w_c t \left[B_0 + r_L \cos w_c t \frac{\partial B}{\partial y} \right]$$

$$F_y = -qV_\perp \cos\omega_c t\, B_0 - qV_\perp \cos^2\omega_c t\, r_L \frac{\partial B}{\partial y} \quad \text{—— (8)}$$

Let's take the average of eq. (8),

$$F_y = -qV_\perp \cos\omega_c t\, B_0 - r_L qV_\perp \cos^2\omega_c t \frac{\partial B}{\partial y}$$

As we know that the general formula of average is

$$\langle F \rangle = \frac{\int_0^t F\,dt}{\int_0^t dt}$$

In this case, as $\cos\omega_c t$ is the oscillating quantity so, we apply average on it,

$$\langle \cos\omega_c t \rangle = \frac{\int_0^{2\pi/\omega_c} \cos\omega_c t}{\int_0^{2\pi/\omega_c} dt}$$

$$= \frac{\left| -\sin\omega_c t \right|_0^{2\pi/\omega_c} \cdot \omega_c}{\left| t \right|_0^{2\pi/\omega_c}}$$

By applying limit,

$$\langle \cos\omega_c t \rangle = 0$$

∵ Average of $\cos\omega_c t$ or $\sin\omega_c t$ is always zero.

In eq. (8), the first term becomes zero. Now, for second term,

$$\langle \cos^2 \omega_c t \rangle = \frac{\int_0^{\frac{2\pi}{\omega_c}} \cos^2 \omega_c t \, dt}{\int_0^{\frac{2\pi}{\omega_c}} dt} \qquad \text{--- (9)}$$

As we know half angle formula is,

$$\because \cos^2 \theta = \frac{1+\cos 2\theta}{2}$$

put this in eq. (9),

$$\langle \cos^2 \omega_c t \rangle = \frac{\int_0^{\frac{2\pi}{\omega_c}} \left(\frac{1+\cos 2\omega_c t}{2}\right) dt}{\int_0^{\frac{2\pi}{\omega_c}} dt}$$

$$= \frac{\int_0^{\frac{2\pi}{\omega_c}} \frac{1}{2} \, dt}{\int_0^{\frac{2\pi}{\omega_c}} dt} + \frac{\int_0^{\frac{2\pi}{\omega_c}} \frac{1}{2} \cos 2\omega_c t \, dt}{\int_0^{\frac{2\pi}{\omega_c}} dt}$$

$$= \frac{1}{2} \frac{\cancel{\int_0^{\frac{2\pi}{\omega_c}} dt}}{\cancel{\int_0^{\frac{2\pi}{\omega_c}} dt}} + \frac{\int_0^{\frac{2\pi}{\omega_c}} \frac{1}{2} \cos 2\omega_c t \, dt}{\int_0^{\frac{2\pi}{\omega_c}} dt}$$

$$= \frac{1}{2} + 0$$

$$\langle \cos^2 \omega_c t \rangle = \frac{1}{2}$$

$\because$ Average of $\cos^2 \omega_c t$ or $\sin^2 \omega_c t$ is always $\frac{1}{2}$.

So, eq. (8) becomes,

$$F_y = 0 - q V_\perp r_L \frac{1}{2} \frac{\partial B}{\partial y}$$

$$F_y = -\frac{1}{2} q V_\perp r_L \frac{\partial B}{\partial y} \qquad \text{--- (10)}$$

As we know our general formula is,

$$V_{\nabla B} = \frac{F \times B}{qB^2}$$

$$\frac{B \times \nabla B}{B^2} = \frac{1}{B^2} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & B \\ 0 & \partial B/\partial y & 0 \end{vmatrix}$$

$$= \frac{F_y B_z}{qB^2} \hat{x}$$

$$= \frac{1}{B^2} B \frac{\partial B}{\partial y}$$

$$V_{\nabla B} = \frac{F_y}{qB} \hat{x} \quad \text{——— (11)}$$

$$= \frac{1}{B} \frac{\partial B}{\partial y}$$

Now put value of F_y from eq. (10) in eq. (11).

$$V_{\nabla B} = \frac{-\frac{1}{2} q V_\perp r_L \, \partial B/\partial y \; \hat{x}}{qB} \quad \text{——— (12)}$$

It means when magnetic field is non-uniform it would apply drift.

In vector form,

$$V_{\nabla B} = \pm \frac{1}{2} V_\perp r_L \frac{B \times \nabla B}{B^2} \quad \text{——— (13)}.$$

$\pm$ stands for sign of charges

$V_{\nabla B}$ is called gradient B drift, it is in opposite directions for ions & $\bar{e}$s & causes a current transverse to B.