Ion Waves

or ion sound waves :-

In the absence of collisions, ordinary sound waves would not occur. Ions can still transmit vibrations to each other because of their charge. Since the motion of massive ions will be involved, these will be **low-frequency oscillations** & we can use the plasma approximation.

So, eq. of motion for ions is :

$$m_i n_i \frac{\partial V_i}{\partial t} = e n_i E - \nabla P_i \quad \text{—— ①}$$

linearize this eq. ①,

$$m_i n_{i_0} \frac{\partial V_{i_1}}{\partial t} = e n_{i_0} E_1 - \gamma_i K T_i \nabla n_{i_1} \quad \text{—— ②}$$

Taking divergence of eq. (2),

$$m_i n_{i_0} \frac{\partial}{\partial t} \nabla \cdot V_{i_1} = e n_{i_0} \nabla \cdot E_1 - \gamma_i K T_i \nabla \cdot \nabla n_{i_1}$$

$$m_i n_{i_0} \frac{\partial}{\partial t} \nabla \cdot V_{i_1} = e n_{i_0} \nabla \cdot E_1 - \gamma_i K T_i \nabla^2 n_{i_1} \quad \text{—— ③}$$

Now, write continuity Eq.

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i V_i) = 0$$

linearized this eq.

$$\frac{\partial n_{i_1}}{\partial t} + n_{i_0} \nabla \cdot V_{i_1} = 0$$

$$\nabla \cdot V_{i_1} = \frac{-1}{n_{i_0}} \frac{\partial n_{i_1}}{\partial t} \quad \text{—— (4)}$$

Put eq. (4) in eq. (3) we get,

$$m_i n_{i_0} \frac{\partial}{\partial t}\left(\frac{-1}{n_{i_0}} \frac{\partial n_{i_1}}{\partial t}\right) = e n_{i_0} \nabla \cdot E_1 - \gamma_i K_B T_i \nabla^2 n_{i_1}$$

As we know,

$$E = -\nabla \varphi$$

So,

$$m_i n_{i_0} \frac{\partial}{\partial t}\left(\frac{-1}{n_{i_0}} \frac{\partial n_{i_1}}{\partial t}\right) = e n_{i_0} \nabla \cdot -\nabla \varphi_1 + \gamma_i K T_i \nabla^2 n_{i_1}$$

$$-m_i \frac{\partial^2 n_{i_1}}{\partial t^2} = -e n_{i_0} \nabla^2 \varphi_1 - \gamma_i K T_i \nabla^2 n_{i_1}$$

Now, By applying character,

$$-m_i \frac{\partial^2}{\partial t^2} n_{i_1} e^{i(Kx-\omega t)} = -e n_{i_0} \nabla_x^2 \varphi_1 e^{i(Kx-\omega t)} - \gamma_i K T_i \nabla_x^2 n_{i_1} e^{i(Kx-\omega t)}$$

$$-m_i(-\omega^2) n_{i_1} e^{i(Kx-\omega t)} = e n_{i_0} K^2 \varphi_1 e^{i(Kx-\omega t)} + \gamma_i K_B T_i K^2 n_{i_1} e^{i(Kx-\omega t)}$$

$$m_i \omega^2 n_{i_1} \cancel{e^{i(Kx-\omega t)}} = \left(e n_{i_0} K^2 \varphi_1 + \gamma_i K_B T_i n_{i_1} K^2\right) \cancel{e^{i(Kx-\omega t)}}$$

$$m_i \omega^2 n_{i_1} = e n_{i_0} K^2 \varphi_1 + \gamma_i K_B T_i K^2 n_{i_1}$$

$$\omega^2 n_{i_1} = \frac{e n_{i_0} K^2 \varphi_1 + \gamma_i K_B T_i K^2 n_{i_1}}{m_i}$$

$$\omega^2 n_{i_1} - \frac{\gamma_i K_B T_i K^2 n_{i_1}}{m_i} = \frac{e n_{i_0} K^2 \varphi_1}{m_i}$$

$$\left[\omega^2 - \frac{\gamma_i K_B T_i K^2}{m_i}\right] n_{i_1} = \frac{e n_{i_0} K^2 \varphi_1}{m_i} \quad \text{———} \quad (5)$$

Assume $m_e \rightarrow 0$

$$n_e = n_0 \exp\left(\frac{e\varphi}{KT_e}\right) \qquad (6)$$

This expression is for the electron no. density.

Linearize eq. (6) we get,

$$n_{e1} = n_0 \exp\left(\frac{e\varphi_1}{KT_e}\right) \qquad (7)$$

By using Poisson's eq.,

$$\nabla \cdot E = 4\pi e(n_i - n_e)$$

linearize this eq

$$\nabla \cdot E_1 = 4\pi e\left[n_{i0} + n_{i1} - n_{e1}\right] \qquad (8)$$

put value of n_{e1} from eq. (7) in eq. (8),

$$\nabla \cdot E_1 = 4\pi e\left[n_{i0} + n_{i1} - n_0 \exp\left(\frac{e\varphi_1}{KT_e}\right)\right]$$

By applying series expansion,

$$\nabla \cdot E_1 = 4\pi e\left[n_{i0} + n_{i1} - n_0\left(1 + \frac{e\varphi_1}{KT_e} + \dots\right)\right]$$

ignoring higher terms

$$\nabla \cdot E_1 = 4\pi e\left(n_{i1} - n_{i0}\frac{e\varphi_1}{KT_e}\right) \qquad (9)$$

Now, by applying plasma approximation for low frequency:

$$\nabla \cdot E = 4\pi e(n_i - n_e)$$

(poisson's eq.)

$$-\nabla^2\phi = 4\pi e(n_i - n_e)$$

linearizing the above eq., $\because E = -\nabla\phi$

$$-\nabla^2\phi_1 = 4\pi e(n_{i_1} - n_{e_1})$$

by applying operator,

$$K^2\phi_1 = 4\pi e(n_{i_1} - n_{e_1})$$

putting value of K^2 in above eq.,

$$\frac{4\pi^2}{\lambda^2}\phi_1 = 4\pi e(n_{i_1} - n_{e_1})$$

$\because K = \frac{2\pi}{\lambda}$

$K^2 = \frac{4\pi^2}{\lambda^2}$

For low-frequency:

$$\frac{4\pi^2}{\lambda^2}\phi_1 = 0$$

$\because V = f\lambda$

$f = \frac{V}{\lambda}$

So,

$$4\pi e(n_{i_1} - n_{e_1}) = 0$$

$\because$ because f & λ are inversely proportional to each other so, when λ increases f decreases & vice versa.

When we are ignoring first term of Poisson's equation then it is called "plasma approximation".

i.e., $\nabla\cdot E_1 = 0$

So, eq. (9) becomes,

$$0 = 4\pi e\left(n_{i_1} - n_{i_0}\frac{e\phi_1}{KT_e}\right)$$

$$n_{i_1} = n_{i_0}\frac{e\phi_1}{KT_e} \qquad (10)$$

Put value of n_{i_1} from eq. (10) in eq. (5),

$$\left[\omega^2 - \frac{\gamma_i K_B T_i K^2}{m_i}\right] n_{i_0}\frac{e\phi_1}{KT_e} = \frac{e n_{i_0} K^2 \phi_1}{m_i}$$

$$\omega^2 - \frac{\gamma_i K_B T_i K^2}{m_i} = \frac{K_B T_e K^2}{m_i}$$

Taking k^2 common, from eq. (II)

$$\omega^2 = \frac{k_B T_e k^2}{m_i} + \frac{\gamma_i k_B T_i k^2}{m_i} \quad \text{———— (II)}$$

$$\omega^2 = \left[\frac{k_B T_e}{m_i} + \frac{\gamma_i k_B T_i}{m_i}\right] k^2$$

$$\frac{\omega^2}{k^2} = \left[\frac{\gamma_i k_B T_i}{m_i} + \frac{k_B T_e}{m_i}\right]$$

$$\frac{\omega}{k} = \left[\frac{\gamma_i k_B T_i}{m_i} + \frac{k_B T_e}{m_i}\right]^{1/2}$$

This is the dispersion relation for ion acoustic wave.

Ion waves are basically constant velocity waves & exist only when there are thermal motions. For ion waves the group velocity is equal to the phase velocity.