Electron Plasma Waves:-

There is an effect that can cause plasma oscillations to propagates & that is "thermal motion".

Electrons streaming into the adjacent layers of plasma with their thermal velocities will carry information about what is happening in the oscillating regions. then plasma oscillations can properly be called plasma wave.

We can easily treat this effect by adding a term $-\nabla P_e$.

By using eq. of motion :-

$$m_e n_e \frac{\partial V_e}{\partial t} = -e n_e E - \nabla P_e \quad \text{——— (1)}$$

This eq. of motion is for $\bar{e}s$, where

$m = m_e$; $n = n_e$; $V = V_e$.

Now, linearizing eq. (1),

$$m n_0 \frac{\partial V_1}{\partial t} = -n_0 e E_1 - \nabla P_1 \quad \text{——— (2)}$$

As we know that,

$$P_e = \gamma K_B T_e n .$$

$$\gamma = \frac{N+2}{N}$$

if $N=1$; $\gamma = \frac{1+2}{1} = 3$

$$P_e = 3 K_B T_e n$$

$\nabla P_e = 3K_B T_e \nabla n$

$\nabla P_1 = 3K_B T_e \nabla n_1$ $\quad \because T_e = T$

Put this value in eq. (2),

$$mn_0 \frac{\partial V_1}{\partial t} = -n_0 e E_1 - 3K_B T_e \nabla n_1$$

$$mn_0 \frac{\partial V_1}{\partial t} = -n_0 e E_1 - 3K_B T \nabla n_1 \quad \text{——— (3)}$$

Now, continuity eq.,

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (nv) = 0$$

After linearizing, the linearized continuity eq. is :

$$\frac{\partial n_1}{\partial t} + n_0 \nabla \cdot V_1 = 0 \quad \text{——— (4)}$$

Now, Poisson's Eq.,

$$\nabla \cdot E = 4\pi e (n_i - n_e)$$

After linearizing, the linearized Poisson's eq is:

$$\nabla \cdot E_1 = -4\pi e n_1 \quad \text{——— (5)}$$

Now taking the divergence of eq. (3)

$$mn_0 \frac{\partial}{\partial t}(\nabla \cdot V_1) = -n_0 e \nabla \cdot E_1 - 3K_B T \nabla \cdot \nabla n_1$$

putting value of $\nabla \cdot E_1$ from eq. (5),

$$mn_0 \frac{\partial}{\partial t}(\nabla \cdot V_1) = -n_0 e(-4\pi e n_1) - 3K_B T \nabla^2 n_1$$

$$mn_0 \frac{\partial}{\partial t}(\nabla \cdot V_1) = n_0 e(4\pi e n_1) - 3k_B T \nabla^2 n_1 \quad \text{——— (6)}$$

Now from eq. (4) of continuity,

$$\frac{\partial n_1}{\partial t} = -n_0 \nabla \cdot V_1$$

$$\nabla \cdot V_1 = \frac{-1}{n_0}\frac{\partial n_1}{\partial t}$$

put this value in eq. (6),

$$m\cancel{n_0}\frac{\partial}{\partial t}\left(\frac{-1}{\cancel{n_0}}\frac{\partial n_1}{\partial t}\right) = +n_0 e(4\pi e n_1) - 3K_B T \nabla^2 n_1$$

$$-m\frac{\partial^2 n_1}{\partial t^2} = 4\pi e^2 n_0 n_1 - 3K_B T \nabla^2 n_1$$

$$-\frac{\partial^2 n_1}{\partial t^2} = \frac{4\pi n_0 e^2 n_1}{m} - \frac{3K_B T \nabla^2 n_1}{m}$$

Multiplying & Dividing the 2nd term on R.H.S of above eq. by '2' to make it V_{th}.

$\because V_{th}^2 = 2K_B T/m$

$$-\frac{\partial^2 n_1}{\partial t^2} = \omega_{pe}^2 n_1 - \frac{3}{2}V_{th}^2 \nabla^2 n_1$$

$\because \omega^2 = \omega_{pe}^2 = 4\pi n_0 e^2/m$

Applying character,

$$-\frac{\partial^2}{\partial t^2}\left(n_1 e^{i(kx-\omega t)}\right) = \omega_{pe}^2 n_1 e^{i(kx-\omega t)} - \frac{3}{2}V_{th}^2\frac{\partial^2}{\partial x^2}n_1 e^{i(kx-\omega t)}$$

$$\omega^2 n_1 e^{i(kx-\omega t)} = \omega_{pe}^2 n_1 e^{i(kx-\omega t)} + \frac{3}{2}k^2 V_{th}^2 n_1 e^{i(kx-\omega t)}$$

$$\omega^2 \cancel{n_1}\cancel{e^{i(kx-\omega t)}} = \cancel{n_1}\cancel{e^{i(kx-\omega t)}}\left[\omega_{pe}^2 + \frac{3}{2}k^2 V_{th}^2\right]$$

$$\omega^2 = \omega_{pe}^2 + \frac{3}{2}k^2 V_{th}^2 \longrightarrow (7)$$

This is the dispersion relation of $\bar{e}$ plasma wave. This shows that $\bar{e}$ plasma waves are propagating due to the temperature effect.

eq.(7) can be written as:

$$\omega^2 - \frac{3}{2}K^2V_{th}^2 = \omega_{pe}^2$$

$$\frac{\omega^2}{\omega_{pe}^2} - \frac{3/2\,K^2V_{th}^2}{\omega_{pe}^2} = 1$$

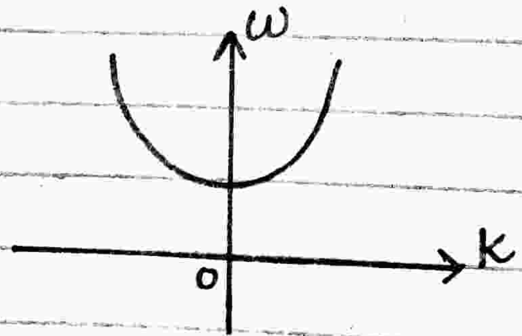
or

$$\frac{\omega^2}{\omega_{pe}^2} - \frac{K^2}{\omega_{pe}^2/\frac{3}{2}V_{th}^2} = 1 \quad \text{—— (8)}$$

Now, by using eq. of hyperbola,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

A Hyperbola →



This means that $\bar{e}$ plasma waves have the shape of a hyperbola.

Now, from eq.(7)

$$\frac{3}{2}K^2V_{th}^2 = \omega^2 - \omega_{pe}^2$$

$$K^2 = \frac{2}{3}\frac{\omega^2-\omega_{pe}^2}{V_{th}^2}$$

or,

$$K = \sqrt{\frac{2}{3}} \sqrt{\frac{\omega^2 - \omega_{pe}^2}{V_{th}^2}} \quad \text{---} \quad (9)$$

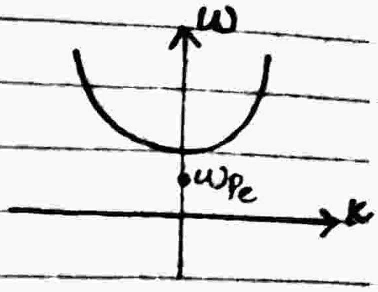
Cases:-

There are three cases of $\bar{e}$ plasma waves:

(i) If $\omega = \omega_{pe}$

Then according to eq.(9),

$$K = \sqrt{\frac{2}{3}} \sqrt{\frac{\omega^2 - \omega_{pe}^2}{V_{th}^2}}$$



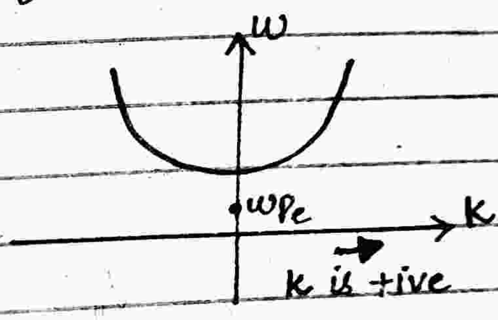
$$K = 0$$

The wave can't propagate.

(ii) If $\omega > \omega_{pe}$

Then, the value of k will be positive & k is real.

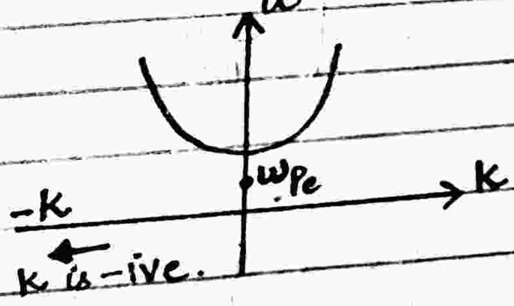
$k = +ive$



(iii) If $\omega < \omega_{pe}$

Then, k will be negative.

$k = -ive$.



Phase Velocity of Electron Plasma Wave:-

Let's calculate the phase velocity of the $\bar{e}$ plasma wave.

Let's calculate the phase velocity by writing eq. (7),

$$\omega^2 = \omega_{pe}^2 + \frac{3}{2} k^2 V_{th}^2$$

According to eq.(9),

$$k = \sqrt{\frac{2}{3}} \sqrt{\frac{\omega^2 - \omega_{pe}^2}{V_{th}^2}}$$

$$k = \sqrt{\frac{2}{3}}\, \omega \sqrt{\frac{1 - \omega_{pe}^2/\omega^2}{V_{th}^2}}$$

$$\frac{k}{\omega} = \sqrt{\frac{2}{3}} \sqrt{\frac{1 - \omega_{pe}^2/\omega^2}{V_{th}^2}}$$

So,

$$\text{phase velocity} = V_\varphi = \frac{\omega}{k}$$

$$V_\varphi = \frac{\omega}{k} = \frac{\sqrt{3/2}\, V_{th}}{\sqrt{1 - \frac{\omega_{pe}^2}{\omega^2}}} \qquad \text{———— (10)}$$

Group Velocity:-

Let's calculate the group velocity of the $\bar{e}$ plasma wave.

Differentiate eq. (7),

$$2\omega d\omega = \frac{3}{2} \cdot 2k \, dk \, V_{th}^2$$

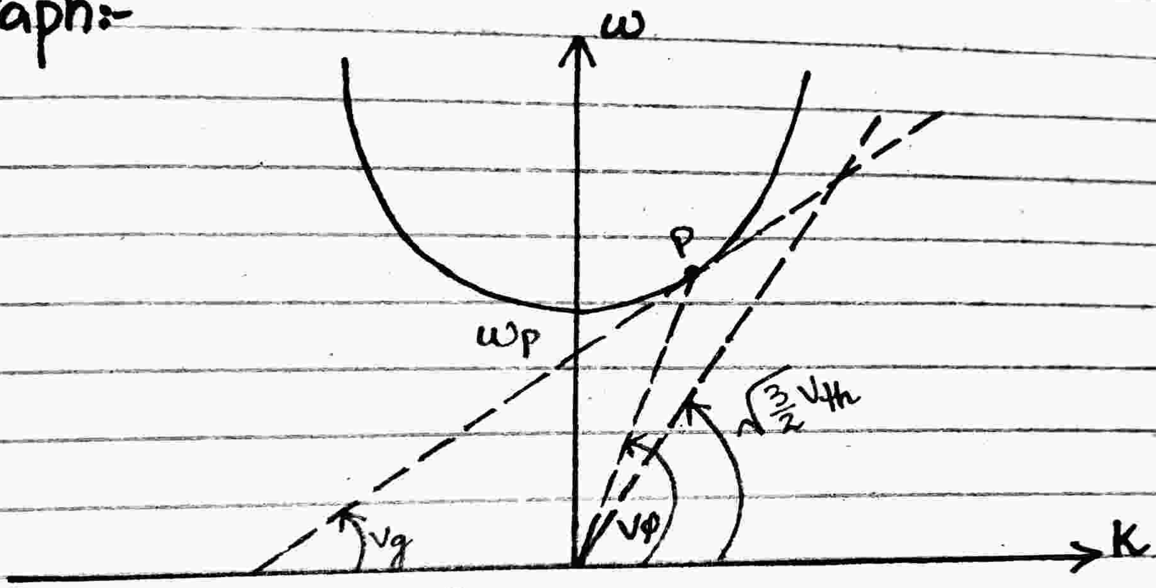
$$\frac{d\omega}{dk} = \frac{3/2 \, V_{th}^2}{\omega/k}$$

$$\frac{d\omega}{dk} = \frac{3/2 \, V_{th}^2}{V_\varphi} \quad \text{———— (11)}$$

Now,

plot a graph b/w ω & k.

Graph:-



This graph is plot for the eq. (7)

$$\omega^2 = \omega_{pe}^2 + \frac{3}{2} k^2 V_{th}^2$$

which is the dispersion relation for $\bar{e}$ plasma wave.

At any point 'P' on this curve, the slope of a line drawn from the origin gives the phase velocity ω/k.

The slope of the wave at 'P' gives the group velocity $d\omega/dk$. This is clearly always less than $(3/2)^{1/2} V_{th}$ which is much less than c.

From figure, it shows that:

$$V_g < V_\varphi < V_{th}$$

The large k (small λ) information travels essentially at the thermal velocity.
At small k (large λ) information travels more slowly than V_{th} even though V_φ is greater than V_{th}. This is because the density gradient is small at large λ, & thermal motions carry very little net momentum into adjacent layers.