Maxwell's Distribution Function

The one-dimension Maxwellian distribution is given by.

$$f(u) = A\exp\left(\frac{-mu^2}{2KT}\right) \quad \text{---} \quad (1)$$

where in $f(u) \Rightarrow f(u)$ represents no. of particles or molecules.

A is constant

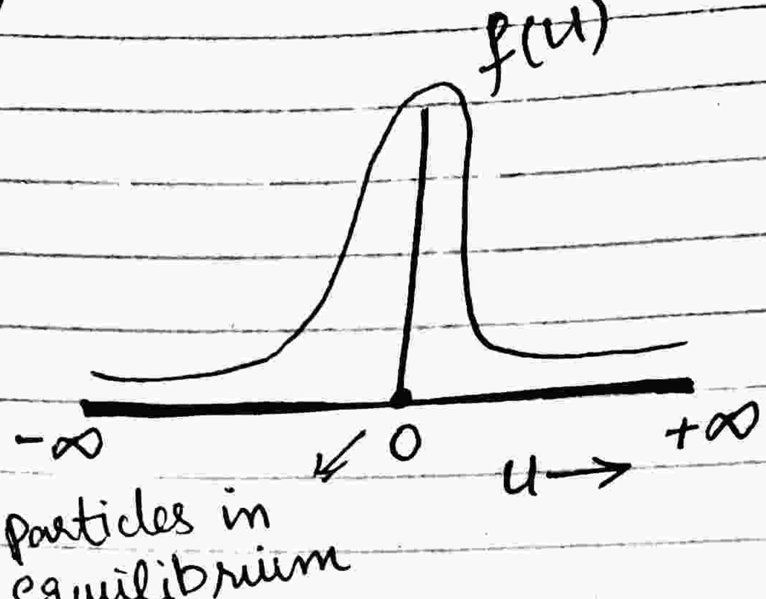
m is mass

u^2 is velocity

K is the Boltzmann constant, whose value is 1.38×10^{-23} J/K

& T is temperature.

Plotting :-



∴ As temperature increases no. of molecules or particles decreases.

No. density $= n = \dfrac{N}{V} = \dfrac{N}{cm^3} \simeq cm^{-3}$

V → volume

$$n = \int_{-\infty}^{+\infty} f(u)\, du$$

putting values from eq. ①

$$n = A \int_{-\infty}^{+\infty} \exp\left(-\frac{mu^2}{2KT}\right) du.$$

Let's integrate it to calculate no. density.

also, let $\dfrac{m}{2KT} = \alpha$

$$n = A \int_{-\infty}^{+\infty} \exp(-\alpha u^2)\, du$$

$$\because \int_{-\infty}^{+\infty} \exp(-\alpha u^2)\, du = \sqrt{\frac{\pi}{\alpha}}$$

putting values, we get

$$n = A\sqrt{\frac{\pi}{\alpha}}$$

or

$$A = n\sqrt{\frac{\alpha}{\pi}}$$

Putting value of α, we get

$$A = n\sqrt{\frac{m}{2\pi KT}} \quad \text{— (2)}$$

putting value of A from eq (2) in eq (1) we get,

$$f(u) = n\sqrt{\frac{m}{2\pi KT}} \exp\left(\frac{-mu^2}{2KT}\right)$$

As we know from eq. (1)

$$f_\alpha(u) = A \exp\left(-\frac{mu^2}{2K_BT}\right) \quad \text{— (1)}$$

$\because \alpha$ is any specie like $\bar{e}$.

$\because$ Distribution fn. is for the bulk.

& we also know,

$$n = \int_{-\infty}^{\infty} f(u)\, du$$

So, if we want to calculate the K.E of particles then,

$$E = \frac{1}{2} m u^2$$

If we want to calculate average energy in terms of distribution fn. then,

$$\langle E \rangle = \frac{\int_{-\infty}^{\infty} \frac{1}{2} m u^2 f(u)\, du}{\int_{-\infty}^{\infty} f(u)\, du}$$

∵ $f(u)$ means distribution of particles in terms of velocity ✓

putting values from eq. ① of $f(u)$, we get...

$$\frac{\int_{-\infty}^{\infty} \frac{1}{2} m u^2 A e^{-\frac{mu^2}{2KT}}\, du}{\int_{-\infty}^{\infty} A e^{-mu^2/2KT}\, du} \quad \text{——— ③}$$

Now let,

$$V_{th}^2 = \frac{2KT}{m}$$

∵ V_{th} means thermal velocity.

$$\frac{1}{V_{th}^2} = \frac{m}{2KT} \quad \text{——— ④}$$

∵ $V = \sqrt{\frac{2KT}{m}}$

∵ If temperature is multiplied by K_B then it becomes energy.

∵ $\frac{2KT}{m} = \frac{2E}{m} = \frac{\cancel{2} \times \frac{1}{\cancel{2}} \cancel{m} u^2}{\cancel{m}} = u^2$

Putting the value of $\frac{2KT}{m}$ from eq. ④ in eq. ③
then eq ③ becomes,

$$= \frac{\int_{-\infty}^{\infty} \frac{1}{2} m u^2 A e^{-u^2/v_{th}^2} du}{\int_{-\infty}^{\infty} A e^{-u^2/v_{th}^2} du} \quad \text{——— ⑤}$$

Let,

$$y = \frac{u}{v_{th}} \Rightarrow u = v_{th} y$$

$$du = v_{th} dy$$

putting the above values in eq. ⑤ we get,

$$= \frac{\int_{-\infty}^{\infty} \frac{1}{2} m \cancel{A} v_{th}^2 y^2 e^{-y^2} \cancel{v_{th}} dy}{\int_{-\infty}^{\infty} \cancel{A} e^{-y^2} \cancel{v_{th}} dy}$$

$$= \frac{\frac{1}{2} m v_{th}^2 \int_{-\infty}^{\infty} y^2 e^{-y^2} dy}{\int_{-\infty}^{\infty} e^{-y^2} dy} \quad \text{——— ⑥}$$

Integrate numerator of eq. ⑥ by parts, we get ...

$$= y\int_{-\infty}^{\infty} e^{-y^2}\cdot y\,dy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-y^2} y\,dy\,dy \quad \text{--- (7)}$$

Let,

$$y^2 = x$$
$$2y\,dy = dx$$
$$y\,dy = \frac{1}{2}dx$$

putting values in eq. (7)

$$\int_{-\infty}^{\infty} e^{-y^2} y\,dy = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x}\,dx$$
$$= -\frac{1}{2}\left| e^{-x} \right|_{-\infty}^{\infty}$$

again putting value,

$$= -\frac{1}{2}\left| e^{-y^2} \right|_{-\infty}^{\infty}$$
$$= 0$$

$\because e^{-\infty} = 0$

$\because e^{+\infty} = \infty$

$\because e^{-(\infty)^2} - e^{-(-\infty)^2}$

$= e^{-\infty} - e^{-\infty}$

$= 0$

putting values in eq. (7)

$$= 0 - \left(-\frac{1}{2}\right)\int_{-\infty}^{\infty} e^{-y^2}\,dy$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-y^2}\,dy$$

put this value in eq. (6)

$$\langle E \rangle = \frac{\frac{1}{4} m V_{th}^2 \int_{-\infty}^{\infty} \cancel{e^{-y^2} dy}}{\int_{-\infty}^{\infty} \cancel{e^{-y^2} dy}}$$

$$\langle E \rangle = \frac{1}{4} m V_{th}^2$$

putting values in above eq.

$$\langle E \rangle = \frac{1}{\cancel{4}_2} \cancel{m} \times \frac{2KT}{\cancel{m}} \qquad \because V_{th} = \sqrt{\frac{2KT}{m}}$$

$\langle E \rangle = \frac{1}{2}$ [redacted] KT

This is the average energy of plasma particles. It means average energy depends only on temperature.

It is easy to extend this result to three dimensions. Maxwell's distribution is then,

$$f(u, v, w) = A_3 \exp\left(\frac{-m}{2KT}\left(u^2 + v^2 + w^2\right)\right)$$

Normalization constant $A = n\left(\frac{m}{2KT\pi}\right)^{1/2}$

$$A_3 = A_1 A_2 A_3 = n\left(\frac{m}{2KT\pi}\right)^{3/2}.$$

Let's calculate average energy for 3-dimensions.

$$E_{av} = \frac{A_3 \iiint_{-\infty}^{\infty} \frac{1}{2} m \left(u^2 + v^2 + w^2\right) \exp\left[\frac{-m\left(u^2 + v^2 + w^2\right)}{2KT}\right] du\,dv\,dw}{A_3 \iiint_{-\infty}^{\infty} \exp \frac{-m}{2KT}\left(u^2 + v^2 + w^2\right) du\,dv\,dw} \quad \text{—(B)}$$

Integrate along 'u' taking other terms constants, we have

$$\frac{1}{2}m\int_{-\infty}^{\infty} u^2 \exp\left(\frac{-mu^2}{2KT}\right)du \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(v^2+w^2)\exp\left(\frac{-m}{2KT}(v^2+w^2)\right)dv\,dw$$

$$= \overline{\int_{-\infty}^{\infty} \exp\frac{-mu^2}{2KT}du \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{-m}{2KT}(v^2+w^2)\,dv\,dw}.$$

Now integrate along 'u',

$$\frac{1}{2}m\int_{-\infty}^{\infty} u^2 \exp\left(-\frac{mu^2}{2KT}\right)du \Big/ \int_{-\infty}^{\infty} \exp\frac{-mu^2}{2KT}du \quad \text{— (8)}$$

Let $V_{th}^2 = \frac{2KT}{m}$

$$\frac{1}{2}m\int_{-\infty}^{\infty} u^2 e^{-u^2/V_{th}^2}du \Big/ e^{-u^2/V_{th}^2}du \quad \text{— (9)}$$

let $y = \frac{u}{V_{th}}$ $\Rightarrow u^2 = y^2 V_{th}^2$

$du = V_{th}dy$

put values in eq. (9);

$$\frac{1}{2}m\int_{-\infty}^{\infty} V_{th}^2 y^2 e^{-y^2}\cancel{V_{th}}dy$$

$$= \overline{e^{-y^2}dy\cancel{V_{th}}}.$$

$$\frac{1}{2} m V_{th}^2 \int_{-\infty}^{\infty} y^2 e^{-y^2}\, dy$$

$$\int_{-\infty}^{\infty} e^{-y^2}\, dy \qquad \text{— (10A)}$$

again integrate by parts

$$\int_{-\infty}^{\infty} y^2 e^{-y^2}\, dy = \int_{-\infty}^{\infty} y \cdot e^{-y^2} y\, dy$$

$$= y \int_{-\infty}^{\infty} e^{-y^2} y\, dy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-y^2} y\, dy\, dy \qquad \text{— (10)}$$

let $y^2 = x \Rightarrow 2y\,dy = dx \Rightarrow y\,dy = \frac{1}{2} dx$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{-x}\, dx$$

$$= \frac{-1}{2} \left| e^{-x} \right|_{-\infty}^{\infty}$$

$$= \frac{-1}{2} \left| e^{-y^2} \right|_{-\infty}^{\infty}$$

apply limit

$$\frac{-1}{2} \left| e^{-y^2} \right|_{-\infty}^{\infty} = 0$$

put in (10).

$$y(0) - \left(\frac{-1}{2}\right) \int_{-\infty}^{\infty} e^{-y^2}\, dy$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{-y^2}\, dy \qquad \text{put in (10A)}$$

$$= \frac{\frac{1}{4} m V_{th}^2 \int_{-\infty}^{\infty} e^{-y^2} dy}{\int_{-\infty}^{\infty} e^{-y^2} dy}$$

$$E_{av} = \frac{1}{4} m V_{th}^2$$

put value of V_{th}^2, we get

$$= \frac{1}{\cancel{4}_2} \cancel{m} \times \frac{\cancel{2}KT}{\cancel{m}}$$

$$E_{av} = \frac{1}{2} KT$$

Similarly, integration along v & w also gives the same result, then by putting all the three values in eq. (B) we get,

$$E_{av} = \frac{1}{2} KT + \frac{1}{2} KT + \frac{1}{2} KT$$

$$E_{av} = \frac{3}{2} KT$$ This is total average energy in 3-dimension

Since, T & E_{av} are so closely related, it is customary in plasma physics to give temperature in units of energy. To avoid confusion on the no. of dimensions involved, it is not E_{av} but the energy corresponding to KT that is used to denote the temperature.

For $KT = 1eV = 1.6 \times 10^{-19} J$,

we have

$$T = \frac{1.6 \times 10^{-19} J}{1.38 \times 10^{-23} J/K}$$

$$T = 11,600 \, K$$

minimum temperature needed for a gas to called it plasma.