# Subject: Signals and Systems

## Chapter: 05

## The Discrete Time Fourier Transform

**Text Book:** Signals & Systems By: Alan V. Oppenheim, Alan S. Willsky with S. Hamid Nawab, 2nd Edition

### 5.1.1 Development of the Discrete-Time Fourier Transform

Consider a general sequence $x[n]$ that is of finite duration. That is, for some integers $N_1$ and $N_2$, $x[n] = 0$ outside the range $-N_1 \leq n \leq N_2$. A signal of this type is illustrated in Figure 5.1(a). From this aperiodic signal, we can construct a periodic sequence $\tilde{x}[n]$ for which $x[n]$ is one period, as illustrated in Figure 5.1(b). As we choose the period $N$ to be larger, $\tilde{x}[n]$ is identical to $x[n]$ over a longer interval, and as $N \to \infty$, $\tilde{x}[n] = x[n]$ for any finite value of $n$.

Let us now examine the Fourier series representation of $\tilde{x}[n]$.

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}, \tag{5.1}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk(2\pi/N)n}. \tag{5.2}$$

# Cont.

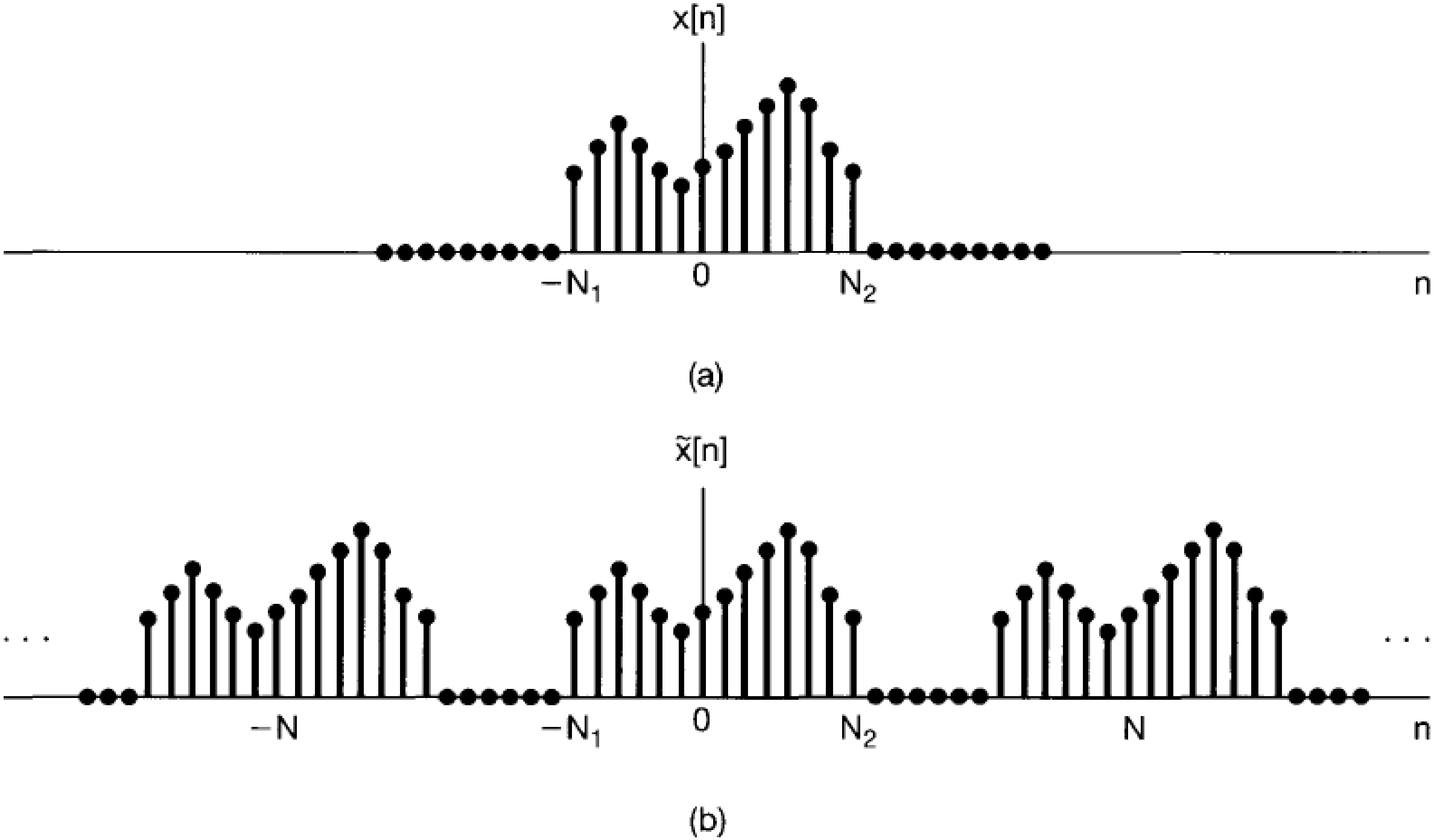



**Figure 5.1** (a) Finite-duration signal $x[n]$; (b) periodic signal $\tilde{x}[n]$ constructed to be equal to $x[n]$ over one period.

# Cont.

Since $x[n] = \tilde{x}[n]$ over a period that includes the interval $-N_1 \leq n \leq N_2$, it is convenient to choose the interval of summation in eq. (5.2) to include this interval, so that $\tilde{x}[n]$ can be replaced by $x[n]$ in the summation. Therefore,

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] e^{-jk(2\pi/N)n} = \frac{1}{N} \sum_{n=-\infty}^{+\infty} x[n] e^{-jk(2\pi/N)n}, \tag{5.3}$$

where in the second equality in eq. (5.3) we have used the fact that $x[n]$ is zero outside the interval $-N_1 \leq n \leq N_2$. Defining the function

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}, \tag{5.4}$$

we see that the coefficients $a_k$ are proportional to samples of $X(e^{j\omega})$, i.e.,

$$a_k = \frac{1}{N} X(e^{jk\omega_0}), \tag{5.5}$$

where $\omega_0 = 2\pi/N$ is the spacing of the samples in the frequency domain. Combining eqs. (5.1) and (5.5) yields

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n}. \tag{5.6}$$

# Cont.

Since $\omega_0 = 2\pi/N$, or equivalently, $1/N = \omega_0/2\pi$, eq. (5.6) can be rewritten as

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0})e^{jk\omega_0 n}\omega_0. \tag{5.7}$$

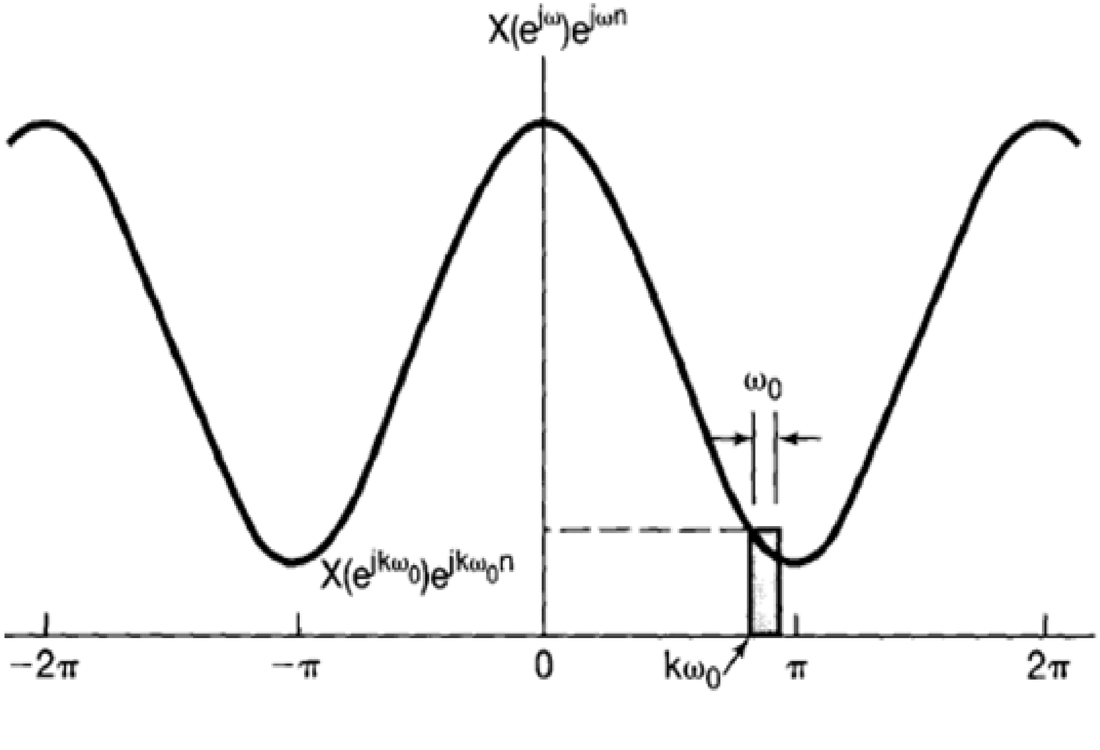



**Figure 5.2** Graphical interpretation of eq. (5.7).

# Cont.

As with eq. (4.7), as $N$ increases $\omega_0$ decreases, and as $N \to \infty$ eq. (5.7) passes to an integral. To see this more clearly, consider $X(e^{j\omega})e^{j\omega n}$ as sketched in Figure 5.2. From eq. (5.4), $X(e^{j\omega})$ is seen to be periodic in $\omega$ with period $2\pi$, and so is $e^{j\omega n}$. Thus, the product $X(e^{j\omega})e^{j\omega n}$ will also be periodic. As depicted in the figure, each term in the summation in eq. (5.7) represents the area of a rectangle of height $X(e^{jk\omega_0})e^{j\omega_0 n}$ and width $\omega_0$. As $\omega_0 \to 0$, the summation becomes an integral. Furthermore, since the summation is carried out over $N$ consecutive intervals of width $\omega_0 = 2\pi/N$, the total interval of integration will always have a width of $2\pi$. Therefore, as $N \to \infty$, $\tilde{x}[n] = x[n]$, and eq. (5.7) becomes

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega,$$

where, since $X(e^{j\omega})e^{j\omega n}$ is periodic with period $2\pi$, the interval of integration can be taken as *any* interval of length $2\pi$. Thus, we have the following pair of equations:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega, \tag{5.8}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}. \tag{5.9}$$

Equations (5.8) and (5.9) are the discrete-time counterparts of eqs. (4.8) and (4.9). The function $X(e^{j\omega})$ is referred to as the *discrete-time Fourier transform* and the pair of equations as the *discrete-time Fourier transform pair.* Equation (5.8) is the *synthesis equation,* eq. (5.9) the *analysis equation.* Our derivation of these equations indicates how an aperiodic sequence can be thought of as a linear combination of complex exponentials.

## Example 5.1

Consider the signal

$$x[n] = a^n u[n], \qquad |a| < 1.$$

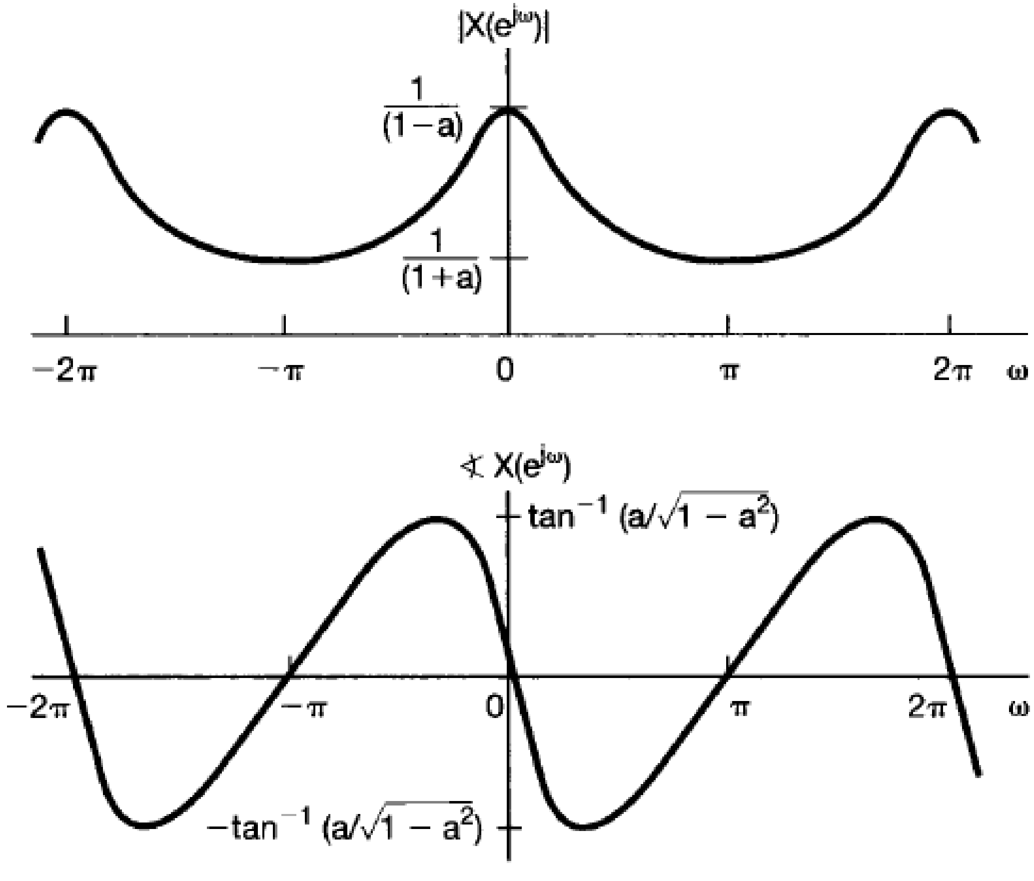
In this case,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}.$$

# Cont.

The magnitude and phase of $X(e^{j\omega})$ are shown in Figure 5.4(a) for $a > 0$ and in Figure 5.4(b) for $a < 0$. Note that all of these functions are periodic in $\omega$ with period $2\pi$.




(a)

# Cont.

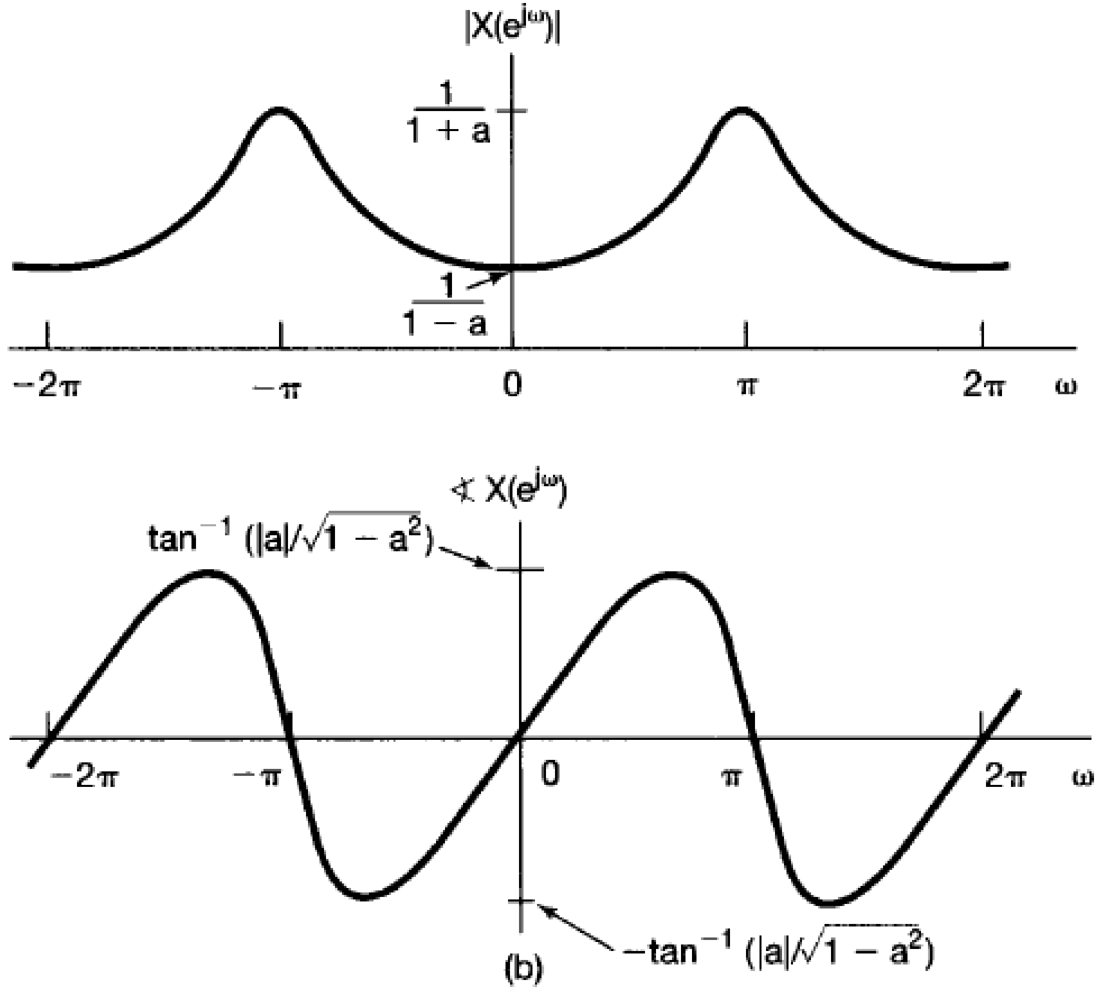



**Figure 5.4** Magnitude and phase of the Fourier transform of Example 5.1 for (a) $a > 0$ and (b) $a < 0$.

# Cont.

**Fourier Transform Formula**

$$\begin{aligned} S\left(e^{j2\pi f}\right) &= \sum_{n=-\infty}^{\infty}\left(a^n u\left(n\right)e^{-(j2\pi f n)}\right) \\ &= \sum_{n=0}^{\infty}\left(\left(ae^{-(j2\pi f)}\right)^n\right) \end{aligned}$$

This sum is a special case of the **geometric series** .

**Geometric Series**

$$\forall\alpha, |\alpha| < 1 : \sum_{n=0}^{\infty}\left(\alpha^n\right) = \frac{1}{1-\alpha}$$

Thus, as long as $|a| < 1$ , we have our Fourier transform.

$$S\left(e^{j2\pi f}\right) = \frac{1}{1 - ae^{-(j2\pi f)}}$$

Using Euler's relation, we can express the magnitude and phase of this spectrum.

$$|S\left(e^{j2\pi f}\right)| = \frac{1}{\sqrt{\left(1 - a\cos\left(2\pi f\right)\right)^2 + a^2\sin^2\left(2\pi f\right)}}$$

$$\angle\left(S\left(e^{j2\pi f}\right)\right) = -\left(\arctan\left(\frac{a\sin\left(2\pi f\right)}{1 - a\cos\left(2\pi f\right)}\right)\right)$$

## Example 5.2

Let

$$x[n] = a^{|n|}, \qquad |a| < 1.$$

This signal is sketched for $0 < a < 1$ in Figure 5.5(a). Its Fourier transform is obtained from eq. (5.9):

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}.$$

# Cont.

Making the substitution of variables $m = -n$ in the second summation, we obtain

$$X(e^{j\omega}) = \sum_{n=0}^{\infty}(ae^{-j\omega})^n + \sum_{m=1}^{\infty}(ae^{j\omega})^m.$$

Both of these summations are infinite geometric series that we can evaluate in closed form, yielding

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}} \\ &= \frac{1 - a^2}{1 - 2a\cos\omega + a^2}. \end{aligned}$$

In this case, $X(e^{j\omega})$ is real and is illustrated in Figure 5.5(b), again for $0 < a < 1$.

# Cont.

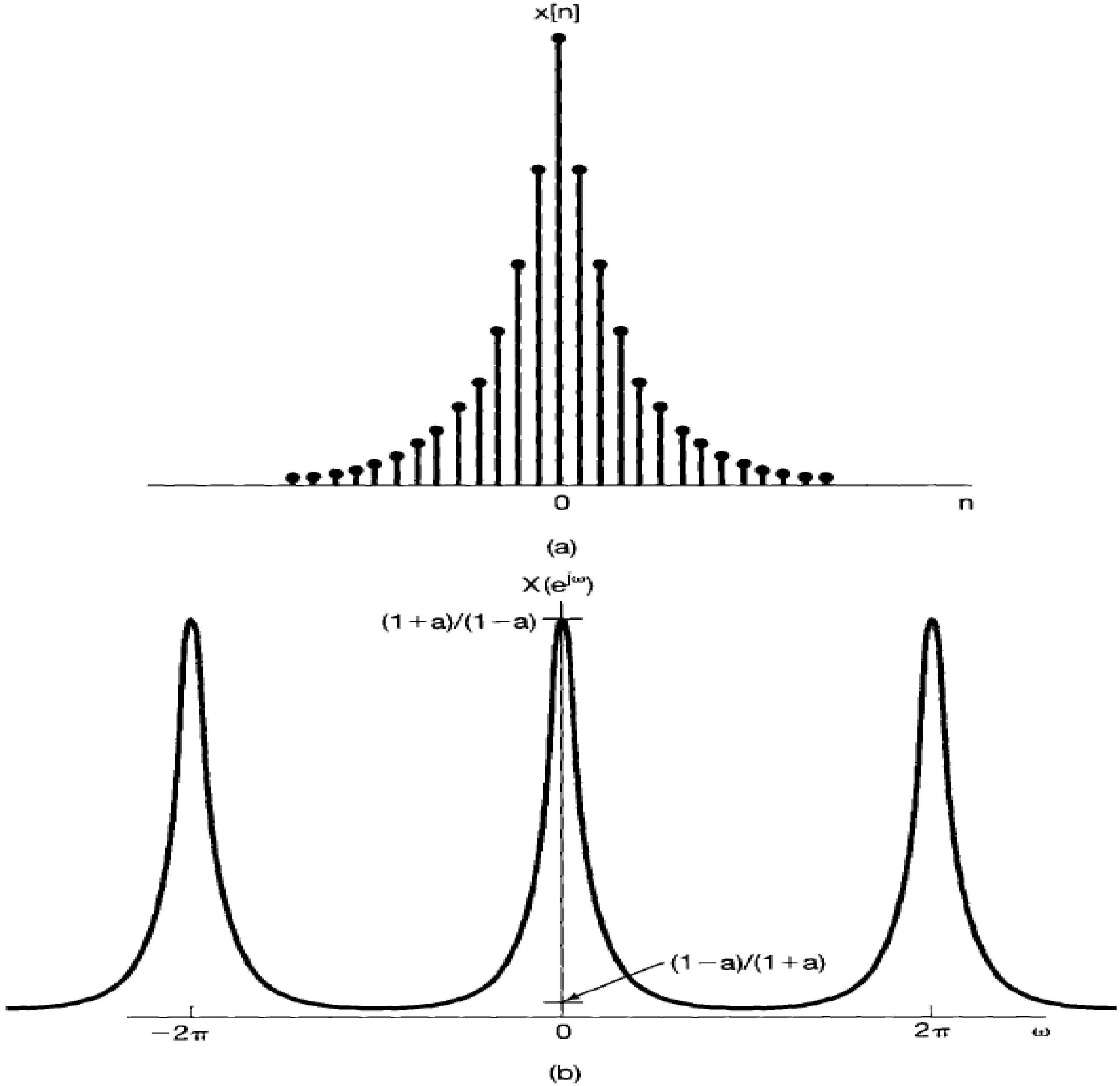



**Figure 5.5** (a) Signal $x[n] = a^{|n|}$ of Example 5.2 and (b) its Fourier transform $(0 < a < 1)$.

## 5.2 THE FOURIER TRANSFORM FOR PERIODIC SIGNALS

As in the continuous-time case, discrete-time periodic signals can be incorporated within the framework of the discrete-time Fourier transform by interpreting the transform of a periodic signal as an impulse train in the frequency domain. To derive the form of this representation, consider the signal

$$x[n] = e^{j\omega_0 n}. \tag{5.17}$$

In continuous time, we saw that the Fourier transform of $e^{j\omega_0 t}$ can be interpreted as an impulse at $\omega = \omega_0$. Therefore, we might expect the same type of transform to result for the discrete-time signal of eq. (5.17). However, the discrete-time Fourier transform must be periodic in $\omega$ with period $2\pi$. This then suggests that the Fourier transform of $x[n]$ in eq. (5.17) should have impulses at $\omega_0$, $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$, and so on. In fact, the Fourier transform of $x[n]$ is the impulse train

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\,\delta(\omega - \omega_0 - 2\pi l), \tag{5.18}$$

# Cont.

which is illustrated in Figure 5.8. In order to check the validity of this expression, we must evaluate its inverse transform. Substituting eq. (5.18) into the synthesis equation (5.8), we find that

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{2\pi}\sum_{l=-\infty}^{+\infty} 2\pi\,\delta(\omega-\omega_0-2\pi l)e^{j\omega n}\,d\omega.$$

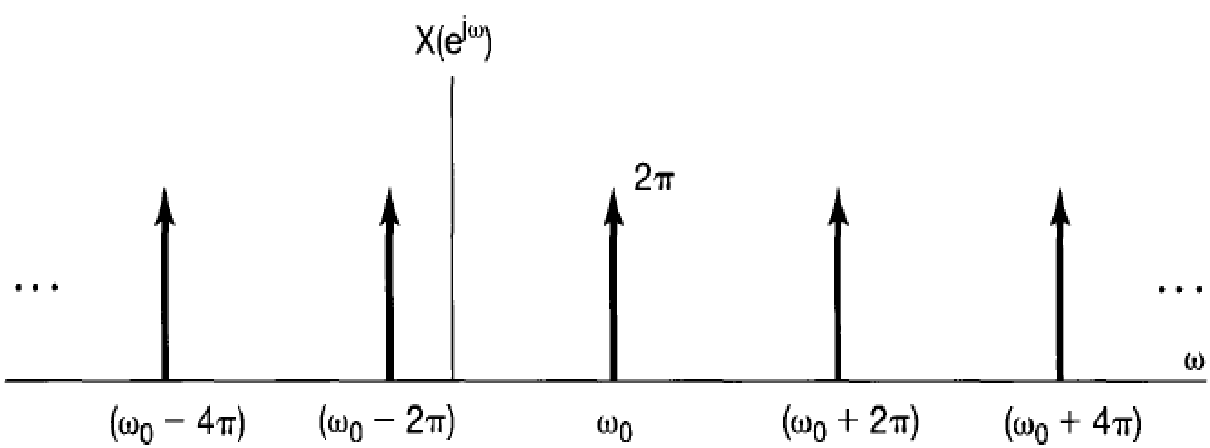



**Figure 5.8** Fourier transform of $x[n] = e^{j\omega_0 n}$.

# Cont.

Note that any interval of length $2\pi$ includes exactly one impulse in the summation given in eq. (5.18). Therefore, if the interval of integration chosen includes the impulse located at $\omega_0 + 2\pi r$, then

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega = e^{j(\omega_0+2\pi r)n} = e^{j\omega_0 n}.$$

Now consider a periodic sequence $x[n]$ with period $N$ and with the Fourier series representation

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}. \tag{5.19}$$

In this case, the Fourier transform is

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k\,\delta\left(\omega - \frac{2\pi k}{N}\right), \tag{5.20}$$

so that the Fourier transform of a periodic signal can be directly constructed from its Fourier coefficients.

## Example 5.5

Consider the periodic signal

$$x[n] = \cos \omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}, \qquad \text{with} \quad \omega_0 = \frac{2\pi}{5}. \tag{5.22}$$

From eq. (5.18), we can immediately write

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} \pi\,\delta\left(\omega - \frac{2\pi}{5} - 2\pi l\right) + \sum_{l=-\infty}^{+\infty} \pi\,\delta\left(\omega + \frac{2\pi}{5} - 2\pi l\right). \tag{5.23}$$

That is,

$$X(e^{j\omega}) = \pi\,\delta\left(\omega - \frac{2\pi}{5}\right) + \pi\,\delta\left(\omega + \frac{2\pi}{5}\right), \qquad -\pi \leq \omega < \pi, \tag{5.24}$$

and $X(e^{j\omega})$ repeats periodically with a period of $2\pi$, as illustrated in Figure 5.10.

# Cont.

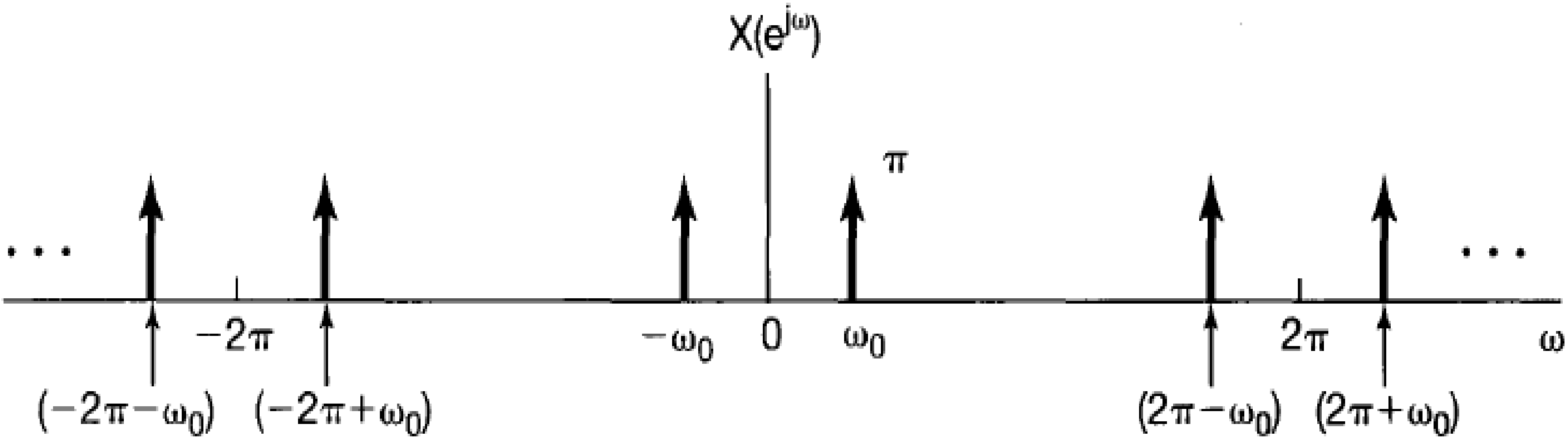



**Figure 5.10** Discrete-time Fourier transform of $x[n] = \cos \omega_0 n$.

**TABLE 5.1** PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

| Property | Aperiodic Signal | Fourier Transform |
|---|---|---|
| | $x[n]$ | $X(e^{j\omega})$ periodic with |
| | $y[n]$ | $Y(e^{j\omega})$ period $2\pi$ |
| Linearity | $ax[n] + by[n]$ | $aX(e^{j\omega}) + bY(e^{j\omega})$ |
| Time Shifting | $x[n - n_0]$ | $e^{-j\omega n_0} X(e^{j\omega})$ |
| Frequency Shifting | $e^{j\omega_0 n} x[n]$ | $X(e^{j(\omega - \omega_0)})$ |
| Conjugation | $x^*[n]$ | $X^*(e^{-j\omega})$ |
| Time Reversal | $x[-n]$ | $X(e^{-j\omega})$ |
| Time Expansion | $x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$ | $X(e^{jk\omega})$ |
| Convolution | $x[n] * y[n]$ | $X(e^{j\omega})Y(e^{j\omega})$ |
| Multiplication | $x[n]y[n]$ | $\frac{1}{2\pi} \int_{2\pi} X(e^{j\theta}) Y(e^{j(\omega - \theta)}) d\theta$ |
| Differencing in Time | $x[n] - x[n-1]$ | $(1 - e^{-j\omega}) X(e^{j\omega})$ |
| Accumulation | $\sum_{k=-\infty}^{n} x[k]$ | $\frac{1}{1 - e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$ |

# Cont.

| | | |
|---|---|---|
| Differentiation in Frequency | $nx[n]$ | $j\dfrac{dX(e^{j\omega})}{d\omega}$ |
| Conjugate Symmetry for Real Signals | $x[n]$ real | $\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$ |
| Symmetry for Real, Even Signals | $x[n]$ real an even | $X(e^{j\omega})$ real and even |
| Symmetry for Real, Odd Signals | $x[n]$ real and odd | $X(e^{j\omega})$ purely imaginary and odd |
| Even-odd Decomposition of Real Signals | $x_e[n] = \mathcal{E}v\{x[n]\}$ [$x[n]$ real]<br>$x_o[n] = \mathcal{O}d\{x[n]\}$ [$x[n]$ real] | $\mathcal{R}e\{X(e^{j\omega})\}$<br>$j\mathcal{I}m\{X(e^{j\omega})\}$ |

Parseval's Relation for Aperiodic Signals

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

**TABLE 5.2** BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

| Signal | Fourier Transform | Fourier Series Coefficients (if periodic) |
|---|---|---|
| $\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ | $2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$ | $a_k$ |
| $e^{j\omega_0 n}$ | $2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$ | (a) $\omega_0 = \frac{2\pi m}{N}$<br>$a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$<br>(b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic |
| $\cos \omega_0 n$ | $\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$ | (a) $\omega_0 = \frac{2\pi m}{N}$<br>$a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$<br>(b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic |
| $\sin \omega_0 n$ | $\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$ | (a) $\omega_0 = \frac{2\pi r}{N}$<br>$a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$<br>(b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic |
| $x[n] = 1$ | $2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$ | $a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ |

# Cont.

| | | |
|---|---|---|
| Periodic square wave $x[n] = \begin{cases} 1, & \|n\| \le N_1 \\ 0, & N_1 < \|n\| \le N/2 \end{cases}$ and $x[n+N] = x[n]$ | $2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$ | $a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \ k \neq 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1+1}{N}, \ k = 0, \pm N, \pm 2N, \ldots$ |
| $\sum_{k=-\infty}^{+\infty} \delta[n-kN]$ | $\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$ | $a_k = \frac{1}{N}$ for all $k$ |
| $a^n u[n], \quad \|a\| < 1$ | $\frac{1}{1 - ae^{-j\omega}}$ | — |
| $x[n] \begin{cases} 1, & \|n\| \le N_1 \\ 0, & \|n\| > N_1 \end{cases}$ | $\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$ | — |
| $\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$ | $X(\omega) = \begin{cases} 1, & 0 \le \|\omega\| \le W \\ 0, & W < \|\omega\| \le \pi \end{cases}$ $X(\omega)$ periodic with period $2\pi$ | — |
| $\delta[n]$ | $1$ | — |
| $u[n]$ | $\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$ | — |
| $\delta[n - n_0]$ | $e^{-j\omega n_0}$ | — |
| $(n+1)a^n u[n], \quad \|a\| < 1$ | $\frac{1}{(1 - ae^{-j\omega})^2}$ | — |
| $\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \quad \|a\| < 1$ | $\frac{1}{(1 - ae^{-j\omega})^r}$ | — |

**TABLE 5.3** SUMMARY OF FOURIER SERIES AND TRANSFORM EXPRESSIONS

| | Continuous time | | Discrete time | |
|---|---|---|---|---|
| | **Time domain** | **Frequency domain** | **Time domain** | **Frequency domain** |
| Fourier Series | $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$<br>continuous time<br>periodic in time | $a_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t}$<br>discrete frequency<br>aperiodic in frequency<br>duality | $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$<br>discrete time<br>periodic in time<br>duality | $a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n] e^{-jk(2\pi/N)n}$<br>discrete frequency<br>periodic in frequency |
| Fourier Transform | $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$<br>continuous time<br>aperiodic in time<br>duality | $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$<br>**continuous frequency**<br>**aperiodic in frequency** | $x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n}$<br>**discrete time**<br>**aperiodic in time** | $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$<br>continuous frequency<br>periodic in frequency |

# Systems Characterized By Linear Constant-Coefficient Difference Equations

A general linear constant-coefficient difference equation for an LTI system with input $x[n]$ and output $y[n]$ is of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \qquad (5.78)$$

The class of systems described by such difference equations is quite an important and useful one. In this section, we take advantage of several of the properties of the discrete-time Fourier transform to determine the frequency response $H(e^{j\omega})$ for an LTI system

# Cont.

Let $X(e^{j\omega})$, $Y(e^{j\omega})$, and $H(e^{j\omega})$ denote the Fourier transforms of the input $x[n]$, output $y[n]$, and impulse response $h[n]$, respectively. The convolution property, eq. (5.48), of the discrete-time Fourier transform then implies that

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}. \tag{5.79}$$

Applying the Fourier transform to both sides of eq. (5.78) and using the linearity and time-shifting properties, we obtain the expression

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega}),$$

or equivalently,

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}. \tag{5.80}$$

# Cont.

Comparing eq. (5.80) with eq. (4.76), we see that, as in the case of continuous time, $H(e^{j\omega})$ is a ratio of polynomials, but in discrete time the polynomials are in the variable $e^{-j\omega}$. The coefficients of the numerator polynomial are the same coefficients as appear on the right side of eq. (5.78), and the coefficients of the denominator polynomial are the same as appear on the left side of that equation. Therefore, the frequency response of the LTI system specified by eq. (5.78) can be written down by inspection.

The difference equation (5.78) is generally referred to as an $N$th-order difference equation, as it involves delays in the output $y[n]$ of up to $N$ time steps. Also, the denominator of $H(e^{j\omega})$ in eq. (5.80) is an $N$th-order polynomial in $e^{-j\omega}$.

# Example:

Consider the causal LTI system that is characterized by the difference equation

$$y[n] - ay[n-1] = x[n], \tag{5.81}$$

with $|a| < 1$. From eq. (5.80), the frequency response of this system is

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}. \tag{5.82}$$

Comparing this with Example 5.1, we recognize it as the Fourier transform of the sequence $a^n u[n]$. Thus, the impulse response of the system is

$$h[n] = a^n u[n]. \tag{5.83}$$

# Example:

Consider a causal LTI system that is characterized by the difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]. \quad (5.84)$$

From eq. (5.80), the frequency response is

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}. \quad (5.85)$$

As a first step in obtaining the impulse response, we factor the denominator of eq. (5.85):

$$H(e^{j\omega}) = \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}. \quad (5.86)$$

# Cont.

$$H(e^{j\omega}) = \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}. \tag{5.86}$$

$H(e^{j\omega})$ can be expanded by the method of partial fractions, as in Example A.3 in the appendix. The result of this expansion is

$$H(e^{j\omega}) = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}}. \tag{5.87}$$

The inverse transform of each term can be recognized by inspection, with the result that

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]. \tag{5.88}$$

The procedure followed in Example 5.19 is identical in style to that used in continuous time. Specifically, after expanding $H(e^{j\omega})$ by the method of partial fractions, we can find the inverse transform of each term by inspection. The same approach can be applied to the frequency response of any LTI system described by a linear constant-coefficient difference equation in order to determine the system impulse response. Also, as illustrated in the next example, if the Fourier transform $X(e^{j\omega})$ of the input to such a system is a ratio of polynomials in $e^{-j\omega}$, then $Y(e^{j\omega})$ is as well. In this case, we can use the same technique to find the response $y[n]$ to the input $x[n]$.

# Example

Consider the LTI system

$$x[n] = \left(\frac{1}{4}\right)^n u[n].$$

Then,

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \left[\frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}\right]\left[\frac{1}{1-\frac{1}{4}e^{-j\omega}}\right] \tag{5.89}$$

$$= \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-e\omega})^2}.$$

$$Y(e^{j\omega}) = \frac{B_{11}}{1-\frac{1}{4}e^{-j\omega}} + \frac{B_{12}}{(1-\frac{1}{4}e^{-j\omega})^2} + \frac{B_{21}}{1-\frac{1}{2}e^{-j\omega}}, \tag{5.90}$$

# Cont.

$$B_{11} = -4, \quad B_{12} = -2, \quad B_{21} = 8,$$

so that

$$Y(e^{j\omega}) = -\frac{4}{1-\frac{1}{4}e^{-j\omega}} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2} + \frac{8}{1-\frac{1}{2}e^{-j\omega}}. \tag{5.91}$$

$$y[n] = \left\{-4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n\right\} u[n]. \tag{5.92}$$

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