Subject: Signals and Systems

Topic: Fourier Series Representation of Discrete Time Periodic Signals

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Linear Combinations of Harmonically Related Complex Exponentials

A discrete-time signal $x[n]$ is periodic with period N if :

$$x[n] = x[n+N]. \qquad (1)$$

The fundamental period is the smallest positive integer N for which eq. (1) holds, and $\omega_0 = 2\pi/N$ is the fundamental frequency. For example, the complex exponential $e^{j(2\pi/N)n}$ is periodic with period N. Furthermore, the set of all discrete-time complex exponential signals that are periodic with period N is given by

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk(2\pi/N)n}, \; k = 0, \pm 1, \pm 2, \ldots . \qquad (2)$$

All of these signals have fundamental frequencies that are multiples of $2\pi/N$ and thus are harmonically related. There are only N distinct signals in the set given by eq. (2).

Cont.

This is a consequence of the fact that discrete-time complex exponentials which differ in frequency by a multiple of 2π are identical. Specifically, $\phi_0[n] = \phi_N[n]$, $\phi_1[n] = \phi_{N+1}[n]$, and, in general,

$$\phi_k[n] = \phi_{k+rN}[n]. \tag{3}$$

That is, when k is changed by any integer multiple of N, the identical sequence is generated.

We now wish to consider the representation of more general periodic sequences in terms of linear combinations of the sequences $\phi_k[n]$ in eq. (2). Such a linear combination has the form

$$x[n] = \sum_k a_k \phi_k[n] = \sum_k a_k e^{jk\omega_0 n} = \sum_k a_k e^{jk(2\pi/N)n}. \tag{4}$$

Cont.

Since the sequences $\phi_k[n]$ are distinct only over a range of N successive values of k, the summation in eq. (4) need only include terms over this range. Thus, the summation is on k, as k varies over a range of N successive integers, beginning with any value of k. We indicate this by expressing the limits of the summation as $k = \langle N \rangle$. That is,

$$x[n] = \sum_{k=\langle N \rangle} a_k \phi_k[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}. \tag{5}$$

For example, k could take on the values $k = 0, 1, \ldots, N-1$, or $k = 3, 4, \ldots, N+2$. In either case, by virtue of eq. (3), exactly the same set of complex exponential sequences appears in the summation on the right-hand side of eq. (5). Equation (5) is referred to as the *discrete-time Fourier series* and the coefficients a_k as the *Fourier series coefficients*.

Determination of Fourier Series Representation of a Periodic Signal

Suppose now that we are given a sequence $x[n]$ that is periodic with fundamental period N. We would like to determine whether a representation of $x[n]$ in the form of eq. (5) exists and, if so, what the values of the coefficients a_k are. This question can be phrased in terms of finding a solution to a set of linear equations. Specifically, if we evaluate eq. (5) for N successive values of n corresponding to one period of $x[n]$, we obtain

$$\begin{aligned} x[0] &= \sum_{k=\langle N\rangle} a_k, \\ x[1] &= \sum_{k=\langle N\rangle} a_k e^{j2\pi k/N}, \\ &\vdots \\ x[N-1] &= \sum_{k=\langle N\rangle} a_k e^{j2\pi k(N-1)/N}. \end{aligned} \tag{6}$$

Cont.

Thus, eq. (6) represents a set of N linear equations for the N unknown coefficients a_k as k ranges over a set of N successive integers. It can be shown that this set of equations is linearly independent and consequently can be solved to obtain the coefficients a_k in terms of the given values of $x[n]$.

Now consider:

$$\sum_{n=\langle N\rangle} e^{jk(2\pi/N)n} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}. \tag{7}$$

Equation (7) states that the sum over one period of the values of a periodic complex exponential is zero, unless that complex exponential is a constant.

Cont.

Now consider the Fourier series representation of eq. (5). Multiplying both sides by $e^{-jr(2\pi/N)n}$ and summing over N terms, we obtain

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} = \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k e^{j(k-r)(2\pi/N)n}. \tag{8}$$

Interchanging the order of summation on the right-hand side, we have

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} = \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-r)(2\pi/N)n}. \tag{9}$$

From the identity in eq. (7), the innermost sum on n on the right-hand side of eq. (9) is zero, unless $k-r$ is zero or an integer multiple of N. Therefore, if we choose values for r over the same range as that over which k varies in the outer summation, the innermost sum on the right-hand side of eq. (9) equals N if $k = r$ and 0 if $k \neq r$. The right-hand side of eq. (9) then reduces to Na_r, and we have

$$a_r = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n}. \tag{10}$$

Cont.

This provides a closed-form expression for obtaining the Fourier series coefficients, and we have the *discrete-time Fourier series pair*:

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}, \tag{11}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}. \tag{12}$$

As in continuous time, the discrete-time Fourier series coefficients a_k are often referred to as the *spectral coefficients* of $x[n]$. These coefficients specify a decomposition of $x[n]$ into a sum of N harmonically related complex exponentials.

Referring to eq. (5), we see that if we take k in the range from 0 to $N-1$, we have

$$x[n] = a_0\phi_0[n] + a_1\phi_1[n] + \ldots + a_{N-1}\phi_{N-1}[n]. \tag{13}$$

Cont.

$$x[n] = a_1\phi_1[n] + a_2\phi_2[n] + \ldots + a_N\phi_N[n]. \quad (14)$$

From eq. (3), $\phi_0[n] = \phi_N[n]$, and therefore, upon comparing eqs. (13) and (14), we conclude that $a_0 = a_N$. Similarly, by letting k range over any set of N consecutive integers and using eq. (3), we can conclude that

$$a_k = a_{k+N}. \quad (15)$$

That is, if we consider more than N sequential values of k, the values a_k repeat periodically with period N. It is important that this fact be interpreted carefully. In particular, since there are only N distinct complex exponentials that are periodic with period N, the discrete-time Fourier series representation is a finite series with N terms. Therefore, if we fix the N consecutive values of k over which we define the Fourier series in eq. (11), we will obtain a set of exactly N Fourier coefficients from eq. (12).

Example: 01

Consider the signal

$$x[n] = \sin \omega_0 n, \tag{16}$$

$x[n]$ is periodic only if $2\pi/\omega_0$ is an integer or a ratio of integers. For the case when $2\pi/\omega_0$ is an integer N, that is, when

$$\omega_0 = \frac{2\pi}{N},$$

$x[n]$ is periodic with fundamental period N, and we obtain a result that is exactly analogous to the continuous-time case. Expanding the signal as a sum of two complex exponentials, we get

$$x[n] = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}. \tag{17}$$

Comparing eq. (17) with eq. (11), we see by inspection that

$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j}, \tag{18}$$

Cont.

and the remaining coefficients over the interval of summation are zero. As described previously, these coefficients repeat with period N; thus, a_{N+1} is also equal to $(1/2j)$ and a_{N-1} equals $(-1/2j)$. The Fourier series coefficients for this example with $N = 5$ are illustrated in Figure : 1 . The fact that they repeat periodically is indicated. However, only one period is utilized in the synthesis equation (11).

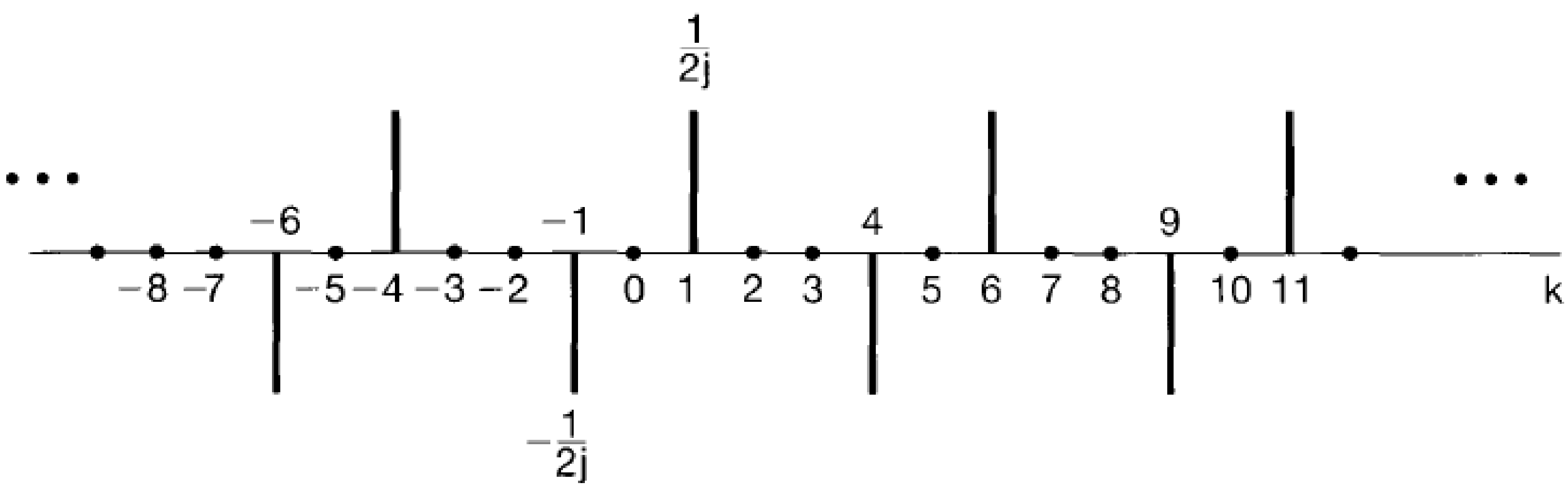

Figure : 1 Fourier coefficients for $x[n] = \sin(2\pi/5)n$.

Cont.

Consider now the case when $2\pi/\omega_0$ is a ratio of integers—that is, when

$$\omega_0 = \frac{2\pi M}{N}.$$

Assuming that M and N do not have any common factors, $x[n]$ has a fundamental period of N. Again expanding $x[n]$ as a sum of two complex exponentials, we have

$$x[n] = \frac{1}{2j}e^{jM(2\pi/N)n} - \frac{1}{2j}e^{-jM(2\pi/N)n},$$

from which we can determine by inspection that $a_M = (1/2j)$, $a_{-M} = (-1/2j)$, and the remaining coefficients over one period of length N are zero. The Fourier coefficients for this example with $M = 3$ and $N = 5$ are depicted in Figure: 2 . Again, we have indicated the periodicity of the coefficients. For example, for $N = 5$, $a_2 = a_{-3}$, which in our example equals $(-1/2j)$. Note, however, that over any period of length 5 there are only two nonzero Fourier coefficients, and therefore there are only two nonzero terms in the synthesis equation.

Cont.

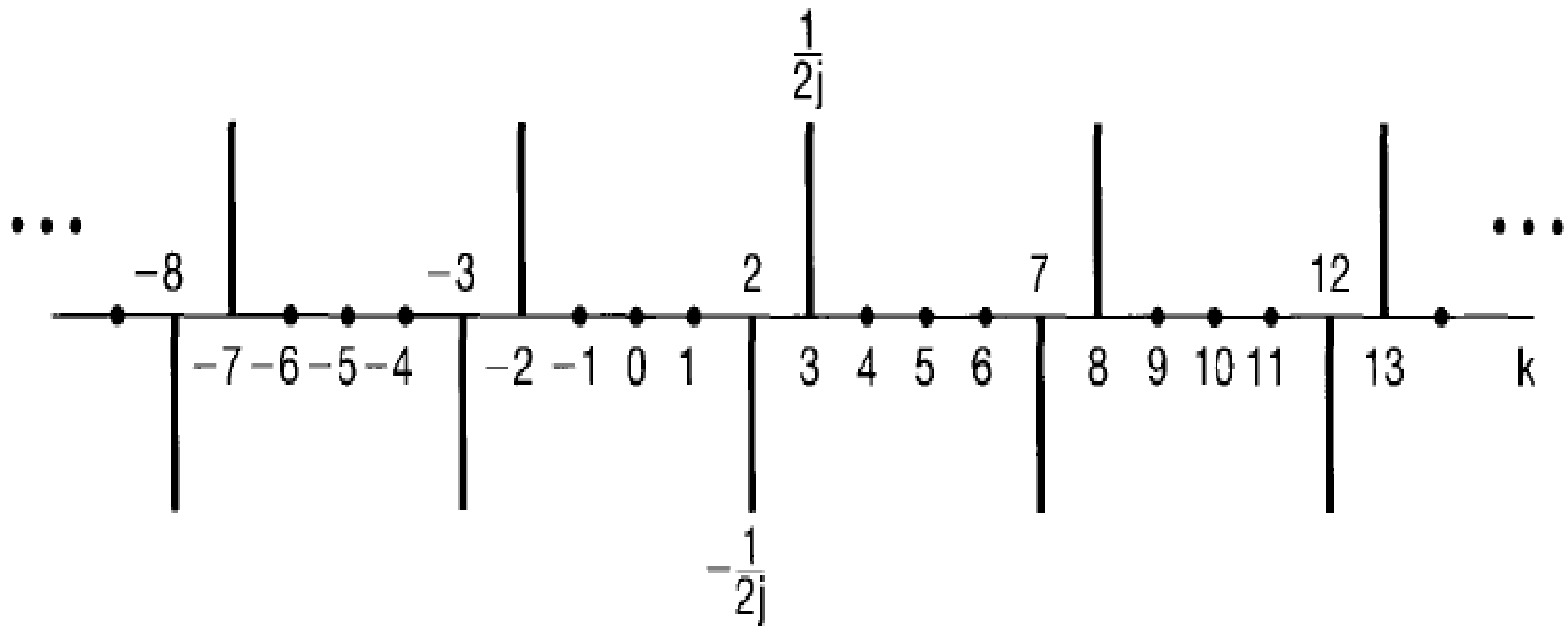

Figure: 2 Fourier coefficients for $x[n] = \sin 3(2\pi/5)n$.

Example: 02

Consider the signal

$$x[n] = 1 + \sin\left(\frac{2\pi}{N}\right)n + 3\cos\left(\frac{2\pi}{N}\right)n + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right).$$

This signal is periodic with period N, and, as in Example 3.10, we can expand $x[n]$ directly in terms of complex exponentials to obtain

$$x[n] = 1 + \frac{1}{2j}[e^{j(2\pi/N)n} - e^{-j(2\pi/N)n}] + \frac{3}{2}[e^{j(2\pi/N)n} + e^{-j(2\pi/N)n}]$$
$$+ \frac{1}{2}[e^{j(4\pi n/N + \pi/2)} + e^{-j(4\pi n/N + \pi/2)}].$$

Collecting terms, we find that

$$x[n] = 1 + \left(\frac{3}{2} + \frac{1}{2j}\right)e^{j(2\pi/N)n} + \left(\frac{3}{2} - \frac{1}{2j}\right)e^{-j(2\pi/N)n}$$
$$+ \left(\frac{1}{2}e^{j\pi/2}\right)e^{j2(2\pi/N)n} + \left(\frac{1}{2}e^{-j\pi/2}\right)e^{-j2(2\pi/N)n}.$$

Cont.

Thus the Fourier series coefficients for this example are

$$\begin{aligned} a_0 &= 1, \\ a_1 &= \frac{3}{2} + \frac{1}{2j} = \frac{3}{2} - \frac{1}{2}j, \\ a_{-1} &= \frac{3}{2} - \frac{1}{2j} = \frac{3}{2} + \frac{1}{2}j, \\ a_2 &= \frac{1}{2}j, \\ a_{-2} &= -\frac{1}{2}j, \end{aligned}$$

with $a_k = 0$ for other values of k in the interval of summation in the synthesis equation (12). Again, the Fourier coefficients are periodic with period N, so, for example, $a_N = 1$, $a_{3N-1} = \frac{3}{2} + \frac{1}{2}j$, and $a_{2-N} = \frac{1}{2}j$.

TABLE 1 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$

Cont.

Property	Periodic Signal	Fourier Series Coefficients
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k]$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \|a_k\| = \|a_{-k}\| \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} \quad [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} \quad [x[n] \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

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