Subject: Signals and Systems

Topic: Fourier Analysis

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Fourier Analysis

- The process of decomposing a musical instrument sound or any other periodic function into its constituent sine or cosine waves is called Fourier analysis.
- You can characterize the sound wave in terms of the amplitudes of the constituent sine waves which make it up. This set of numbers tells you the harmonic content of the sound and is sometimes referred to as the harmonic spectrum of the sound. The harmonic content is the most important determiner of the quality or timbre of a sustained musical note.

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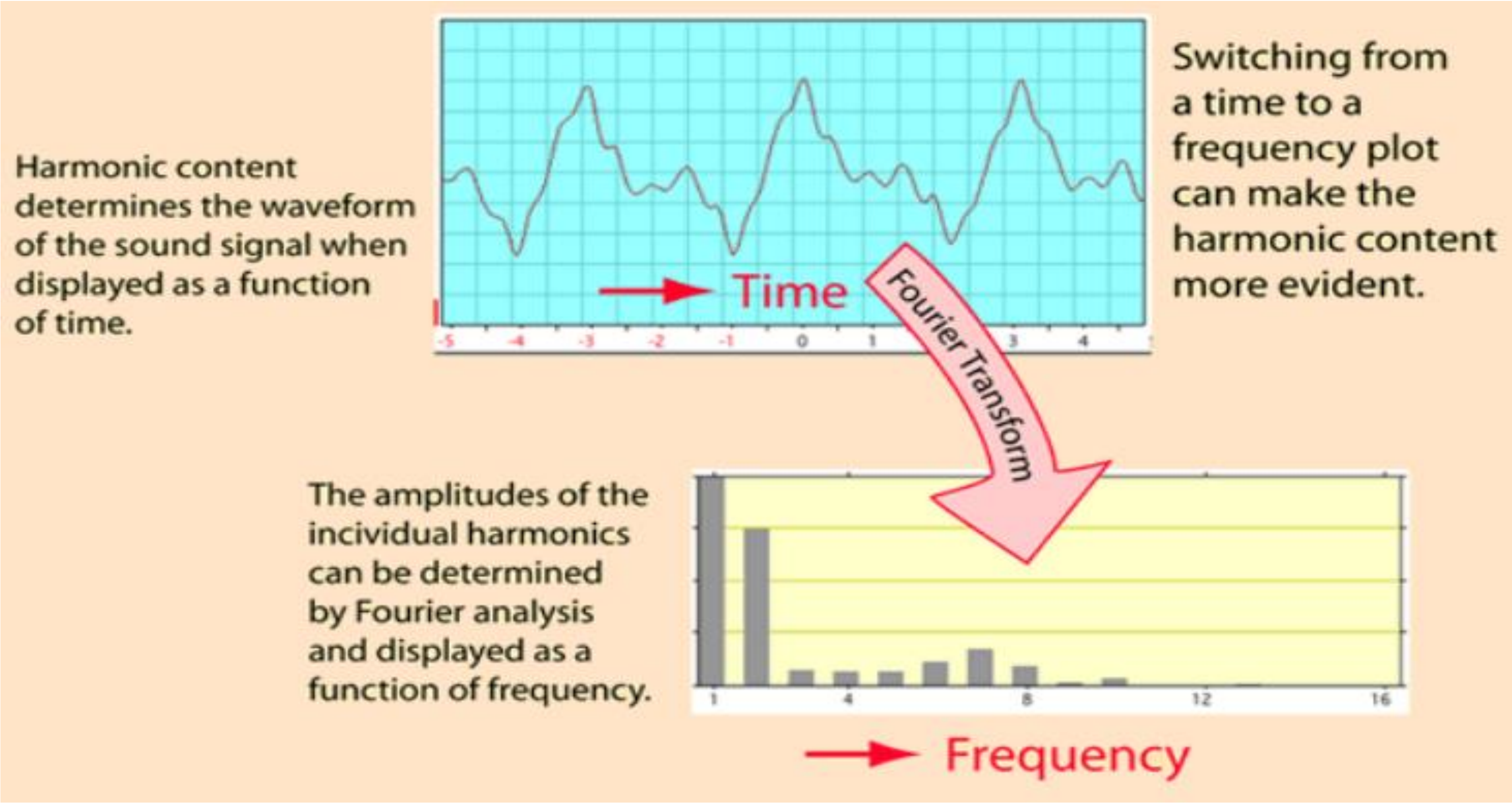

Fig. 1 Example of Fourier Analysis

Fourier Series and Fourier Transform

- The Fourier series is used to represent a periodic function by a discrete sum of complex exponentials, while the Fourier transform is then used to represent a general, nonperiodic function by a continuous superposition or integral of complex exponentials

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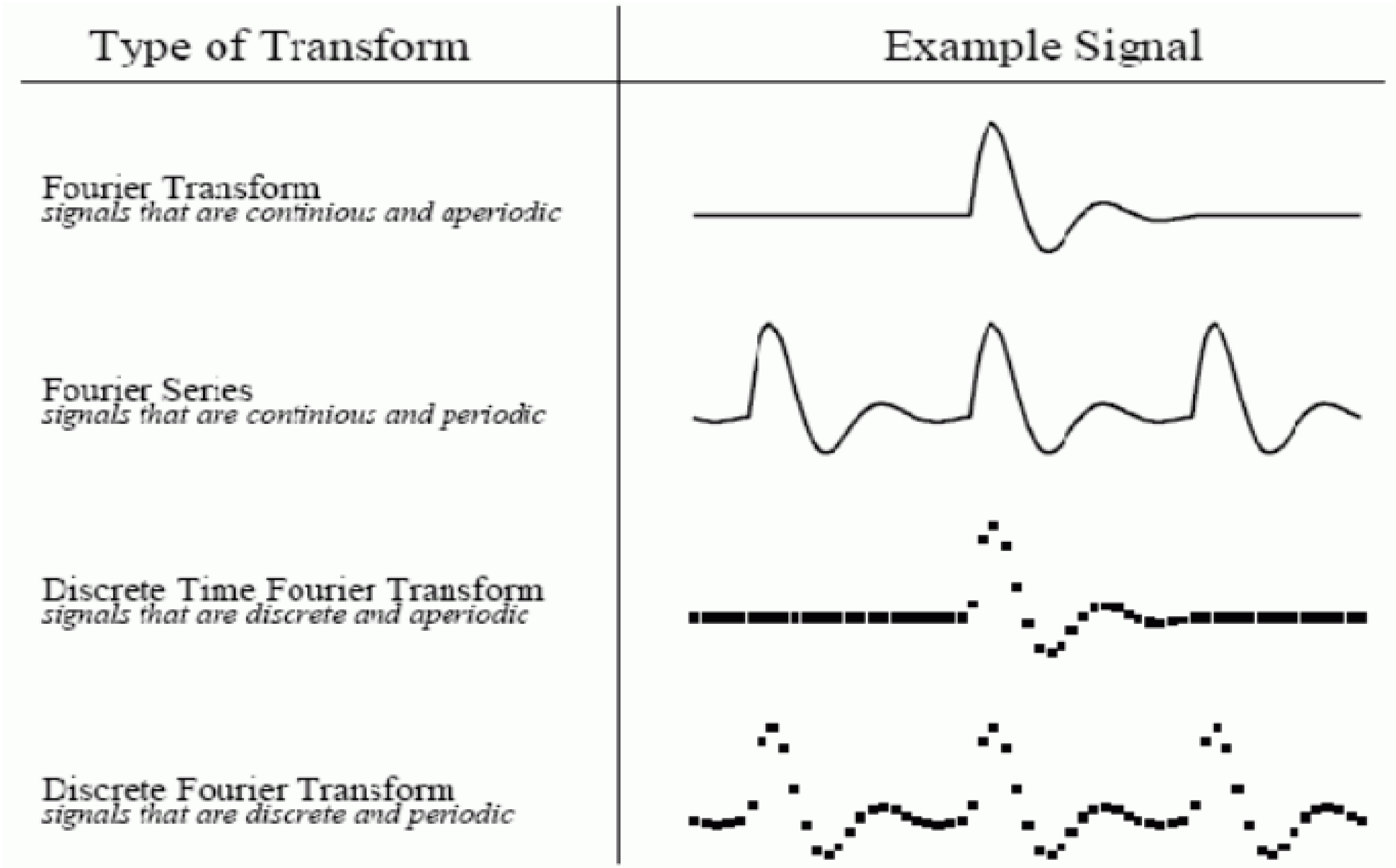

Fig. 2 Different Types of Fourier Analysis

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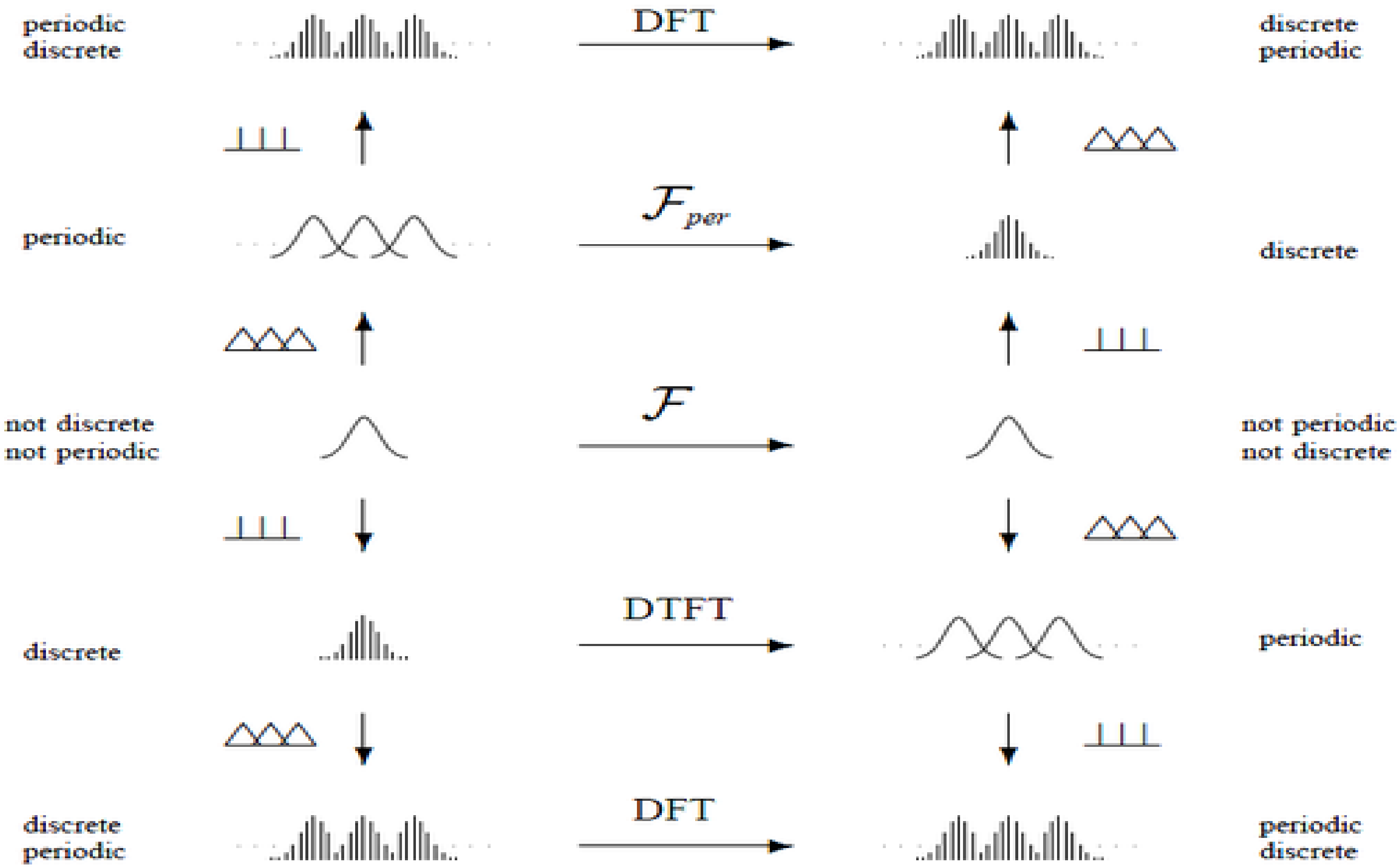

Fig. 3 Fourier Analysis linked by discretization and periodization

Cont.

	Continuous Time	Discrete Time
Fourier Series	$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ continuous and periodic in time $a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$ discrete and aperiodic in frequency	$x[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n}$ discrete and periodic in time $a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\frac{2\pi}{N}n}$ discrete and periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$ continuous and aperiodic in time $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ continuous and aperiodic in frequency	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$ discrete and aperiodic in time $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ continuous and periodic in frequency

Fig. 4 Difference between Fourier Series and Fourier Transform

The Response of Continuous Time LTI Systems to Complex Exponentials

It is advantageous in the study of LTI systems to represent signals as linear combinations of basic signals that possess the following two properties:

1. The set of basic signals can be used to construct a broad and useful class of signals.
2. The response of an LTI system to each basic signal should be simple enough in structure to provide us with a convenient representation for the response of the system to any signal constructed as a linear combination of these basic signals.

Cont.

For continuous time LTI systems both of these advantages are provided by the set of complex exponentials of the form e^{st}, where s is a general complex number. The importance of complex exponentials in the study of LTI systems lies in the fact shown below that the response of an LTI system to a complex exponential input is the same complex exponential with only a change in the amplitude, that is:

$$e^{st} \rightarrow H(s)e^{st}$$

Where the complex exponential factor $H(s)$ will in general be a function of the complex variable s.

Eigen function

A signal for which the system output is just a (possibly complex) constant times the input is referred to as an eigenfunction of the system, and the amplitude factor is referred to as the eigenvalue.

To show that complex exponentials are indeed eigenfunction of LTI systems, let us consider an LTI system with impulse response $h(t)$. For an input $x(t)$we can determine the output through the use of the convolution integral, so that with $x(t)$ of the form $x(t) = e^{st}$, we have:

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau \qquad \text{Eq. A}$$

Cont.

Expressing $e^{s(t-\tau)}$ as $e^{st}e^{-s\tau}$ and noting that e^{st} can be moved outside the integral above Eq. A becomes:

$$y(t) = e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau \qquad \text{Eq. B}$$

The system impulse response here will be given by:

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

So we can write Eq. B as:

$$y(t) = H(s)e^{st} \qquad \text{Eq. C}$$

Cont.

The usefulness for the analysis of LTI systems of decomposing more general signals in terms of eigenfunctions can be seen from this example. Let $x(t)$ correspond to a linear combination of three complex exponentials that is:

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$$

The response to each separately is just:

$$a_1 e^{s_1 t} \rightarrow a_1 H(s_1) e^{s_1 t}$$

$$a_2 e^{s_2 t} \rightarrow a_2 H(s_2) e^{s_2 t}$$

$$a_3 e^{s_3 t} \rightarrow a_3 H(s_3) e^{s_3 t}$$

More generally,

$$\sum_k a_k e^{s_k t} \rightarrow \sum_k a_k H(s_k) e^{s_k t}$$

Linear Combination of Harmonically Related Complex Exponentials

- If a signal is periodic for some positive nonzero value of T, then we can write:

$$x(t) = x(t+T) \quad \text{for all } t \qquad \text{Eq.1}$$

- The fundamental period T_0 of $x(t)$ is the minimum positive, nonzero values of T for which Eq. 1 is satisfied and the value $2\pi/T_0$ is referred to as the fundamental frequency.

- Now consider two basic periodic signals, the sinusoid:

$$x(t) = \cos \omega_o t \qquad \text{Eq. 2}$$

And the periodic complex exponential:

$$x(t) = e^{j\omega_0 t} \qquad \text{Eq. 3}$$

Both of these signals are periodic with fundamental frequency ω_0 and fundamental period $T_o = 2\pi/\omega_0$.

Cont.

Now associated with the signal in Eq. 3 is the set of harmonically related complex exponentials given as:

$$\phi_k(t) = e^{jk\omega_0 t} \quad , k = 0. \pm 1, \pm 2, \ldots \quad \text{Eq. 4}$$

Each of these signals has a fundamental frequency that is a multiple of ω_0 , and therefore each is periodic with period T_o. Thus a linear combination of harmonically related complex exponentials of the form:

$$x\,(t) = \sum_{k=-\infty}^{+\infty} a_k\, e^{jk\omega_0 t} \qquad \text{Eq. 5}$$

Is also periodic with period T_o.

Example: 01

Consider a periodic signal $x\,(t)$, with fundamental frequency 2π, which is expressed as:

$$x\,(t) = \sum_{k=-3}^{+3} a_k\, e^{jk2\pi t} \qquad \text{Eq. 6}$$

where

$$a_0 = 1,$$

$$a_1 = a_{-1} = \frac{1}{4}$$

$$a_2 = a_{-2} = \frac{1}{2}$$

$$a_3 = a_{-3}$$

Solution:

Now we have summation from -3 to +3. So substituting value of k= -3 to k= +3 in Eq. 6 we have,

$$x(t) = \left(a_{-3}\, e^{j(-3)2\pi t}\right) + \left(a_{-2}\, e^{j(-2)2\pi t}\right) + \left(a_{-1}\, e^{j(-1)2\pi t}\right) + \left(a_{0}\, e^{j(0)2\pi t}\right) + \left(a_{1}\, e^{j(1)2\pi t}\right) + \left(a_{2}\, e^{j(2)2\pi t}\right) + \left(a_{3}\, e^{j(3)2\pi t}\right)$$

By substituting given values we have,

$$x(t) = 1 + \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right) + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right) + \frac{1}{3}\left(e^{j6\pi t} + e^{-j6\pi t}\right)$$

Equivalently by using Euler's equation we can write $x(t)$ in the form:

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t \qquad \text{Eq. 7}$$

Cont.

We can see that Eq. 7 is an example of an alternate form for the Fourier series of real periodic signals. Specifically, suppose that $x\,(t)$ is real and can be represented in the form of Eq. 6. Then since $x^*\,(t) = x(t)$, we obtain:

$$x\,(t) = \sum_{k=-\infty}^{+\infty} a_k^* \, e^{jk\omega_0 t}$$

Replacing k by –k, we have equivalently

$$x\,(t) = \sum_{k=-\infty}^{+\infty} a_{-k}^* \, e^{jk\omega_0 t}$$

which, by comparison with Eq. 5 requires that

$$a_k^* = a_{-k} \qquad \text{Eq. 8}$$

Derivation of Alternate forms of Fourier Series

To derive alternative forms of Fourier series, we first rearrange the summation in Eq. 6 as:

$$x\,(t) = a_0 + \sum_{k=1}^{\infty}[\; a_k\, e^{jk\omega_0 t} + a_{-k}\, e^{-jk\omega_0 t}\,]$$

By Eq. 8, this becomes:

$$x\,(t) = a_0 + \sum_{k=1}^{\infty}[\; a_k\, e^{jk\omega_0 t} + a_k^{*} e^{-jk\omega_0 t}\,]$$

Since the two terms inside the summation are complex conjugates of each other, this can be expressed as:

$$x\,(t) = a_0 + \sum_{k=1}^{\infty} 2\, Re\,\{\, a_k\, e^{jk\omega_0 t}\} \qquad \text{Eq. 9}$$

If a_k is expressed in polar form as:

$$a_k = \; A_k\, e^{j\theta_k}$$

Then Eq. 9 becomes:

$$x\,(t) = a_0 + \sum_{k=1}^{\infty} 2\, Re\,\{\, A_k\, e^{j(k\omega_0 t + \theta_k)}\} \qquad \text{Eq. 10}$$

Cont.

That is,

$$x\ (t) = a_0 + 2\ \sum_{k=1}^{\infty} A_k \cos\ (k\ \omega_0 t +\ \theta_k)\quad \text{Eq. 11}$$

Eq. 11 is commonly encountered form for the Fourier series of real periodic signals in continuous time. Another form is obtained by writing a_k in rectangular form as:

$$a_k = B_k + jC_k$$

where B_k and C_k are both real. With this expression for a_k, Eq. 9 takes the form:

$$x\ (t) = a_0 + 2\sum_{k=1}^{\infty}[B_k \cos\ k\ \omega_0 t - C_k \sin k\ \omega_0 t\ ]$$

In general if the input to an LTI system is periodic with period T, then the output is also periodic with the same period. We can verify this fact directly by calculating the Fourier series coefficients if the output in terms if those of the input.

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