9. Two Functions of Two Random Variables

In the spirit of the previous lecture, let us look at an immediate generalization: Suppose X and Y are two random variables with joint p.d.f $f_{XY}(x,y)$. Given two functions $g(x,y)$ and $h(x,y)$, define the new random variables

$$Z = g(X,Y) \tag{9-1}$$

$$W = h(X,Y). \tag{9-2}$$

How does one determine their joint p.d.f $f_{ZW}(z,w)$? Obviously with $f_{ZW}(z,w)$ in hand, the marginal p.d.fs $f_Z(z)$ and $f_W(w)$ can be easily determined.

The procedure is the same as that in (8-3). In fact for given *z* and *w*,

$$F_{ZW}(z,w) = P\big(Z(\xi) \le z, W(\xi) \le w\big) = P\big(g(X,Y) \le z, h(X,Y) \le w\big)$$
$$= P\big((X,Y) \in D_{z,w}\big) = \int\int_{(x,y)\in D_{z,w}} f_{XY}(x,y)\,dx\,dy, \qquad (9\text{-}3)$$

where $D_{z,w}$ is the region in the *xy* plane such that the inequalities $g(x,y) \le z$ and $h(x,y) \le w$ are simultaneously satisfied.

We illustrate this technique in the next example.

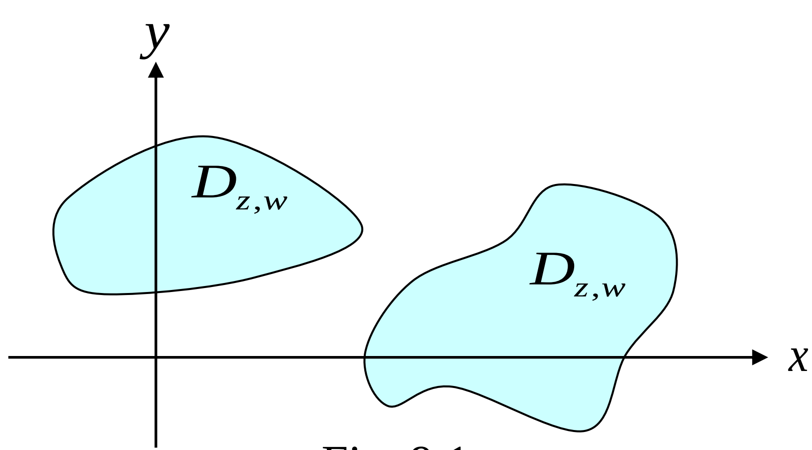


Fig. 9.1

Example 9.1: Suppose X and Y are independent uniformly distributed random variables in the interval $(0,\theta)$. Define $Z = \min(X,Y),\ \ W = \max(X,Y).$ Determine $f_{ZW}(z,w).$

Solution: Obviously both w and z vary in the interval $(0,\theta)$. Thus $F_{ZW}(z,w) = 0, \quad \text{if} \quad z < 0 \quad \text{or} \quad w < 0.$ (9-4)

$$F_{ZW}(z,w) = P(Z \le z, W \le w) = P(\min(X,Y) \le z,\ \max(X,Y) \le w). \quad (9\text{-}5)$$

We must consider two cases: $w \ge z$ and $w < z,$ since they give rise to different regions for $D_{z,w}$ (see Figs. 9.2 (a)-(b)).

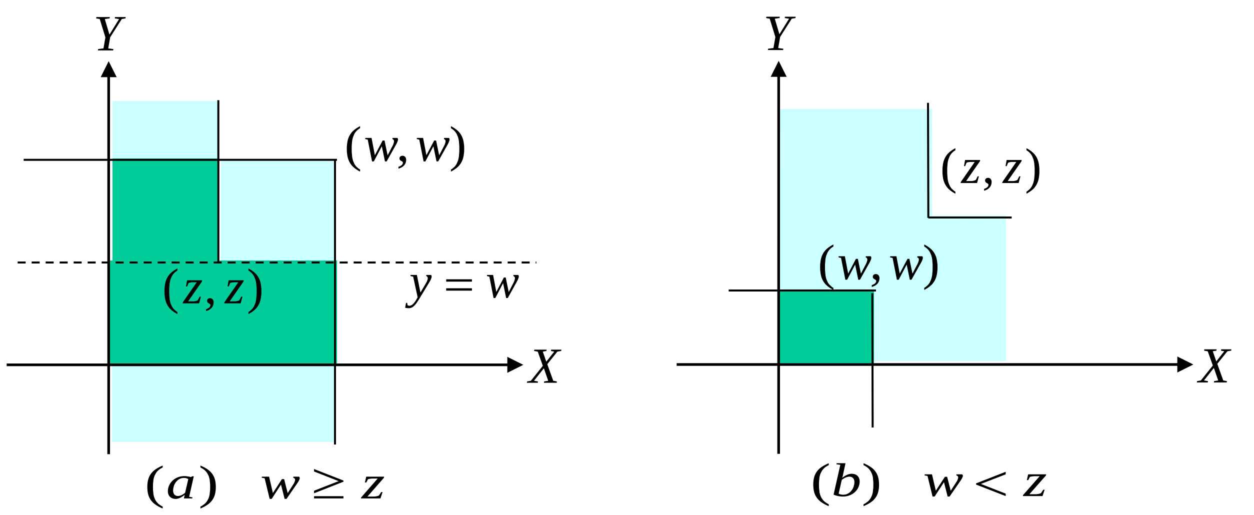


Fig. 9.2

For $w \geq z$, from Fig. 9.2 (a), the region $D_{z,w}$ is represented by the doubly shaded area. Thus

$$F_{ZW}(z,w) = F_{XY}(z,w) + F_{XY}(w,z) - F_{XY}(z,z), \quad w \geq z, \tag{9-6}$$

and for $w < z$, from Fig. 9.2 (b), we obtain

$$F_{ZW}(z,w) = F_{XY}(w,w), \quad w < z. \tag{9-7}$$

With

$$F_{XY}(x,y) = F_X(x)\, F_Y(y) = \frac{x}{\theta} \cdot \frac{y}{\theta} = \frac{xy}{\theta^2}, \tag{9-8}$$

we obtain

$$F_{ZW}(z,w) = \begin{cases} (2w - z)z / \theta^2, & 0 < z < w < \theta, \\ w^2 / \theta^2, & 0 < w < z < \theta. \end{cases} \tag{9-9}$$

Thus

$$f_{ZW}(z,w) = \begin{cases} 2/\theta^2, & 0 < z < w < \theta, \\ 0, & \text{otherwise}. \end{cases} \tag{9-10}$$

From (9-10), we also obtain

$$f_Z(z) = \int_z^\theta f_{ZW}(z,w)dw = \frac{2}{\theta}\left(1 - \frac{z}{\theta}\right), \quad 0 < z < \theta, \tag{9-11}$$

and

$$f_W(w) = \int_0^w f_{ZW}(z,w)dz = \frac{2w}{\theta^2}, \quad 0 < w < \theta. \tag{9-12}$$

If $g(x,y)$ and $h(x,y)$ are continuous and differentiable functions, then as in the case of one random variable (see (5-30)) it is possible to develop a formula to obtain the joint p.d.f $f_{ZW}(z,w)$ directly. Towards this, consider the equations

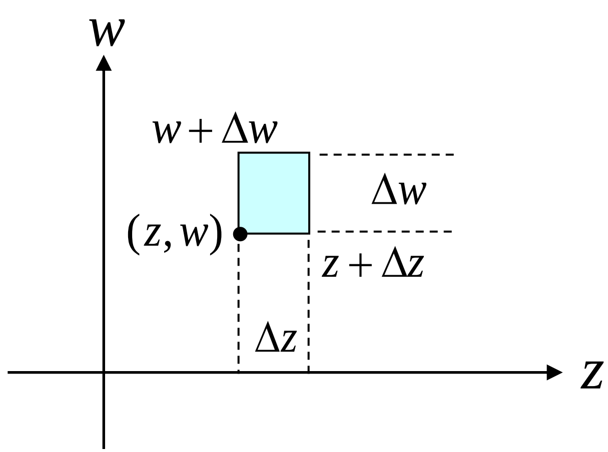
$$g(x,y) = z, \qquad h(x,y) = w. \tag{9-13}$$

For a given point (z,w), equation (9-13) can have many solutions. Let us say

$$(x_1, y_1),\ (x_2, y_2),\ \cdots,\ (x_n, y_n),$$

represent these multiple solutions such that (see Fig. 9.3)

$$g(x_i, y_i) = z, \qquad h(x_i, y_i) = w. \tag{9-14}$$



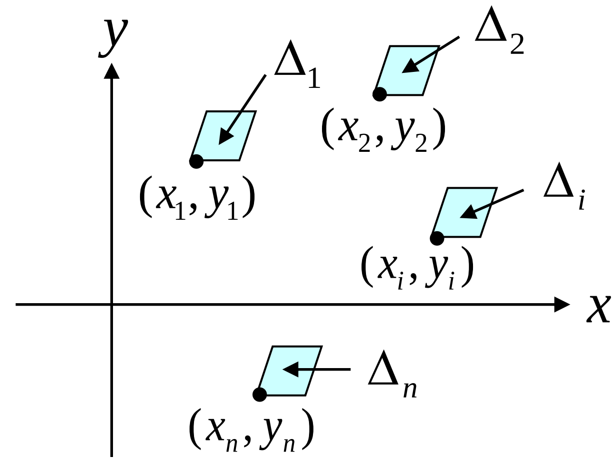


Fig. 9.3

Consider the problem of evaluating the probability

$$P\left(z < Z \le z + \Delta z, w < W \le w + \Delta w\right)$$
$$= P\left(z < g(X,Y) \le z + \Delta z, w < h(X,Y) \le w + \Delta w\right). \tag{9-15}$$

Using (7-9) we can rewrite (9-15) as

$$P\left(z < Z \le z + \Delta z, w < W \le w + \Delta w\right) = f_{ZW}(z,w)\Delta z \Delta w. \qquad (9\text{-}16)$$

But to translate this probability in terms of $f_{XY}(x,y),$ we need to evaluate the equivalent region for $\Delta z \, \Delta w$ in the *xy* plane. Towards this referring to Fig. 9.4, we observe that the point *A* with coordinates (*z*,*w*) gets mapped onto the point A' with coordinates (x_i, y_i) (as well as to other points as in Fig. 9.3(b)). As *z* changes to $z + \Delta z$ to point *B* in Fig. 9.4 (a), let B' represent its image in the *xy* plane. Similarly as *w* changes to $w + \Delta w$ to *C*, let C' represent its image in the *xy* plane.

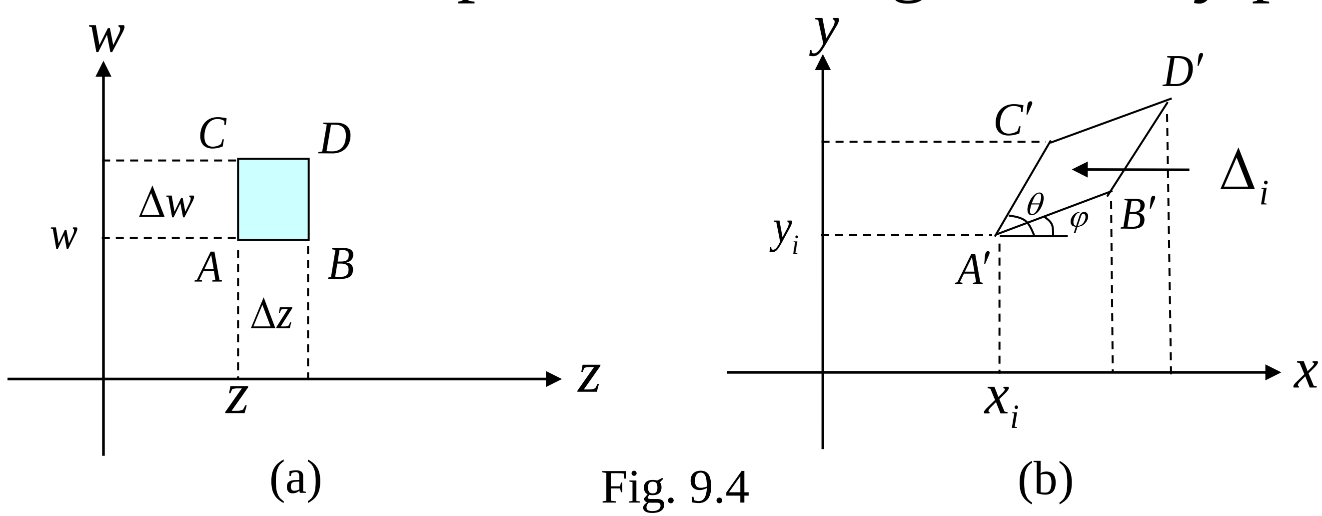


Fig. 9.4

Finally D goes to D', and $A'B'C'D'$ represents the equivalent parallelogram in the XY plane with area Δ_i. Referring back to Fig. 9.3, the probability in (9-16) can be alternatively expressed as

$$\sum_i P\big((X,Y)\in\Delta_i\big)=\sum_i f_{XY}(x_i,y_i)\Delta_i. \tag{9-17}$$

Equating (9-16) and (9-17) we obtain

$$f_{ZW}(z,w)=\sum_i f_{XY}(x_i,y_i)\frac{\Delta_i}{\Delta z\Delta w}. \tag{9-18}$$

To simplify (9-18), we need to evaluate the area Δ_i of the parallelograms in Fig. 9.3 (b) in terms of $\Delta z\Delta w.$ Towards this, let g_1 and h_1 denote the inverse transformation in (9-14), so that

$$x_i = g_1(z,w), \qquad y_i = h_1(z,w). \tag{9-19}$$

As the point (z,w) goes to $(x_i, y_i) \equiv A',$ the point $(z+\Delta z, w) \to B',$ the point $(z, w+\Delta w) \to C',$ and the point $(z+\Delta z, w+\Delta w) \to D'.$ Hence the respective x and y coordinates of B' are given by

$$g_1(z+\Delta z, w) = g_1(z,w) + \frac{\partial g_1}{\partial z}\Delta z = x_i + \frac{\partial g_1}{\partial z}\Delta z, \tag{9-20}$$

and

$$h_1(z+\Delta z, w) = h_1(z,w) + \frac{\partial h_1}{\partial z}\Delta z = y_i + \frac{\partial h_1}{\partial z}\Delta z. \tag{9-21}$$

Similarly those of C' are given by

$$x_i + \frac{\partial g_1}{\partial w}\Delta w, \quad y_i + \frac{\partial h_1}{\partial w}\Delta w. \tag{9-22}$$

The area of the parallelogram $A'B'C'D'$ in Fig. 9.4 (b) is given by

$$\begin{aligned}\Delta_i &= \left(A'B'\right)\left(A'C'\right)\sin(\theta - \varphi) \\ &= \left(A'B'\cos\varphi\right)\left(A'C'\sin\theta\right) - \left(A'B'\sin\varphi\right)\left(A'C'\cos\theta\right).\end{aligned} \tag{9-23}$$

But from Fig. 9.4 (b), and (9-20) - (9-22)

$$A'B'\cos\varphi = \frac{\partial g_1}{\partial z}\Delta z, \quad A'C'\sin\theta = \frac{\partial h_1}{\partial w}\Delta w, \tag{9-24}$$

$$A'B'\sin\varphi = \frac{\partial h_1}{\partial z}\Delta z, \quad A'C'\cos\theta = \frac{\partial g_1}{\partial w}\Delta w. \tag{9-25}$$

so that

$$\Delta_i = \left(\frac{\partial g_1}{\partial z}\frac{\partial h_1}{\partial w} - \frac{\partial g_1}{\partial w}\frac{\partial h_1}{\partial z} \right)\Delta z \Delta w \tag{9-26}$$

and

$$\frac{\Delta_i}{\Delta z \Delta w} = \left(\frac{\partial g_1}{\partial z}\frac{\partial h_1}{\partial w} - \frac{\partial g_1}{\partial w}\frac{\partial h_1}{\partial z} \right) = \det \begin{pmatrix} \frac{\partial g_1}{\partial z} & \frac{\partial g_1}{\partial w} \\ \frac{\partial h_1}{\partial z} & \frac{\partial h_1}{\partial w} \end{pmatrix} \tag{9-27}$$

The right side of (9-27) represents the Jacobian $J(z, w)$ of the transformation in (9-19). Thus

$$J(z,w) = \det \begin{pmatrix} \dfrac{\partial g_1}{\partial z} & \dfrac{\partial g_1}{\partial w} \\ \\ \dfrac{\partial h_1}{\partial z} & \dfrac{\partial h_1}{\partial w} \end{pmatrix}. \tag{9-28}$$

Substituting (9-27) - (9-28) into (9-18), we get

$$f_{ZW}(z,w) = \sum_i |J(z,w)|\, f_{XY}(x_i, y_i) = \sum_i \frac{1}{|J(x_i, y_i)|} f_{XY}(x_i, y_i), \tag{9-29}$$

since

$$|J(z,w)| = \frac{1}{|J(x_i, y_i)|} \tag{9-30}$$

where $J(x_i, y_i)$ represents the Jacobian of the original transformation in (9-13) given by

$$J(x_i, y_i) = \det \begin{pmatrix} \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \\ \\ \dfrac{\partial h}{\partial x} & \dfrac{\partial h}{\partial y} \end{pmatrix}_{x = x_i,\, y = y_i}. \tag{9-31}$$

Next we shall illustrate the usefulness of the formula in (9-29) through various examples:

Example 9.2: Suppose *X* and *Y* are zero mean independent Gaussian r.vs with common variance σ^2.
Define $Z = \sqrt{X^2 + Y^2}, \ W = \tan^{-1}(Y/X),$ where $|w| \le \pi/2.$
Obtain $f_{ZW}(z, w).$
Solution: Here

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}. \tag{9-32}$$

Since

$$z = g(x, y) = \sqrt{x^2 + y^2}; w = h(x, y) = \tan^{-1}(y/x), \quad |w| \le \pi/2, \tag{9-33}$$

if (x_1, y_1) is a solution pair so is $(-x_1, -y_1).$ From (9-33)

$$\frac{y}{x} = \tan w, \quad \text{or} \quad y = x \tan w. \tag{9-34}$$

Substituting this into z, we get

$$z = \sqrt{x^2 + y^2} = x\sqrt{1 + \tan^2 w} = x \ \sec w, \quad \text{or} \quad x = z \ \cos w. \qquad \text{(9-35)}$$

and

$$y = x \tan w = z \sin w. \qquad \text{(9-36)}$$

Thus there are two solution sets

$$x_1 = z\cos w, \ \ y_1 = z\sin w, \ \ x_2 = -z\cos w, \ \ y_2 = -z\sin w. \qquad \text{(9-37)}$$

We can use (9-35) - (9-37) to obtain $J(z,w)$. From (9-28)

$$J(z,w) = \begin{vmatrix} \dfrac{\partial x}{\partial z} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial z} & \dfrac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} \cos w & -z\sin w \\ \sin w & z\cos w \end{vmatrix} = z, \qquad \text{(9-38)}$$

so that

$$|J(z,w)| = z. \qquad \text{(9-39)}$$

We can also compute $J(x,y)$ using (9-31). From (9-33),

$$J(x,y) = \begin{vmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{vmatrix} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{z}. \qquad (9\text{-}40)$$

Notice that $|J(z,w)| = 1/|J(x_i,y_i)|$, agreeing with (9-30). Substituting (9-37) and (9-39) or (9-40) into (9-29), we get

$$f_{ZW}(z,w) = z\big(f_{XY}(x_1,y_1) + f_{XY}(x_2,y_2)\big)$$

$$= \frac{z}{\pi\sigma^2} e^{-z^2/2\sigma^2}, \qquad 0 < z < \infty, \quad |w| < \frac{\pi}{2}. \qquad (9\text{-}41)$$

Thus

$$f_Z(z) = \int_{-\pi/2}^{\pi/2} f_{ZW}(z,w)\,dw = \frac{z}{\sigma^2} e^{-z^2/2\sigma^2}, \qquad 0 < z < \infty, \qquad (9\text{-}42)$$

which represents a Rayleigh r.v with parameter σ^2, and

$$f_W(w) = \int_0^{\infty} f_{ZW}(z,w)\,dz = \frac{1}{\pi}, \quad |w| < \frac{\pi}{2}, \qquad (9\text{-}43)$$

which represents a uniform r.v in the interval $(-\pi/2, \pi/2)$. Moreover by direct computation

$$f_{ZW}(z,w) = f_Z(z) \cdot f_W(w) \tag{9-44}$$

implying that Z and W are independent. We summarize these results in the following statement: If X and Y are zero mean independent Gaussian random variables with common variance, then $\sqrt{X^2+Y^2}$ has a Rayleigh distribution and $\tan^{-1}(Y/X)$ has a uniform distribution. Moreover these two derived r.vs are statistically independent. Alternatively, with X and Y as independent zero mean r.vs as in (9-32), $X + jY$ represents a complex Gaussian r.v. But

$$X + jY = Ze^{jW}, \tag{9-45}$$

where Z and W are as in (9-33), except that for (9-45) to hold good on the entire complex plane we must have $-\pi < W < \pi$, and hence it follows that the magnitude and phase of a complex Gaussian r.v are independent with Rayleigh and uniform distributions $(U \sim (-\pi, \pi))$ respectively. The statistical independence of these derived r.vs is an interesting observation.

Example 9.3: Let X and Y be independent exponential random variables with common parameter λ. Define $U = X + Y$, $V = X - Y$. Find the joint and marginal p.d.f of U and V.
Solution: It is given that

$$f_{XY}(x, y) = \frac{1}{\lambda^2} e^{-(x+y)/\lambda}, \quad x > 0, \quad y > 0. \tag{9-46}$$

Now since $u = x + y$, $v = x - y$, always $|v| < u$, and there is only one solution given by

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}. \tag{9-47}$$

Moreover the Jacobian of the transformation is given by

$$J(x,y)=\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}=-2$$

and hence

$$f_{UV}(u,v)=\frac{1}{2\lambda^2}e^{-u/\lambda},\qquad 0<|v|<u<\infty, \tag{9-48}$$

represents the joint p.d.f of U and V. This gives

$$f_U(u)=\int_{-u}^{u} f_{UV}(u,v)dv=\frac{1}{2\lambda^2}\int_{-u}^{u}e^{-u/\lambda}dv=\frac{u}{\lambda^2}e^{-u/\lambda},\quad 0<u<\infty, \tag{9-49}$$

and

$$f_V(v)=\int_{|v|}^{\infty} f_{UV}(u,v)du=\frac{1}{2\lambda^2}\int_{|v|}^{\infty}e^{-u/\lambda}du=\frac{1}{2\lambda}e^{-|v|/\lambda},\ -\infty<v<\infty. \tag{9-50}$$

Notice that in this case the r.vs U and V are not independent.

As we show below, the general transformation formula in (9-29) making use of two functions can be made useful even when only one function is specified.

Auxiliary Variables:

Suppose

$$Z = g(X,Y), \tag{9-51}$$

where X and Y are two random variables. To determine $f_Z(z)$ by making use of the above formulation in (9-29), we can define an auxiliary variable

$$W = X \quad \text{or} \quad W = Y \tag{9-52}$$

and the p.d.f of Z can be obtained from $f_{ZW}(z,w)$ by proper integration.

Example 9.4: Suppose $Z = X + Y$ and let $W = Y$ so that the transformation is one-to-one and the solution is given by $y_1 = w, \quad x_1 = z - w.$

The Jacobian of the transformation is given by

$$J(x,y) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

and hence

$$f_{ZW}(x,y) = f_{XY}(x_1, y_1) = f_{XY}(z-w, w)$$

or

$$f_Z(z) = \int f_{ZW}(z,w)dw = \int_{-\infty}^{+\infty} f_{XY}(z-w,w)dw, \qquad (9\text{-}53)$$

which agrees with (8.7). Note that (9-53) reduces to the convolution of $f_X(z)$ and $f_Y(z)$ if X and Y are independent random variables. Next, we consider a less trivial example.

Example 9.5: Let $X \sim U(0,1)$ and $Y \sim U(0,1)$ be independent. Define $Z = (-2\ln X)^{1/2}\cos(2\pi Y).$ (9-54)

Find the density function of Z.

Solution: We can make use of the auxiliary variable $W = Y$ in this case. This gives the only solution to be

$$x_1 = e^{-\left(z\sec(2\pi w)\right)^2/2}, \tag{9-55}$$

$$y_1 = w, \tag{9-56}$$

and using (9-28)

$$J(z,w) = \begin{vmatrix} \dfrac{\partial x_1}{\partial z} & \dfrac{\partial x_1}{\partial w} \\ \dfrac{\partial y_1}{\partial z} & \dfrac{\partial y_1}{\partial w} \end{vmatrix} = \begin{vmatrix} -z\sec^2(2\pi w)\ e^{-\left(z\sec(2\pi w)\right)^2/2} & \dfrac{\partial x_1}{\partial w} \\ 0 & 1 \end{vmatrix} \tag{9-57}$$

$$= -z\sec^2(2\pi w) e^{-\left(z\sec(2\pi w)\right)^2/2}.$$

Substituting (9-55) - (9-57) into (9-29), we obtain

$$f_{ZW}(z,w) = |z|\sec^2(2\pi w)\ e^{-\left(z\sec(2\pi w)\right)^2/2}, \qquad -\infty < z < +\infty, \quad 0 < w < 1, \tag{9-58}$$

and

$$f_Z(z) = \int_0^1 f_{ZW}(z,w)dw = e^{-z^2/2}\int_0^1 |z|\sec^2(2\pi w)\ \ e^{-\left(|z|\tan(2\pi w)\right)^2/2} dw. \quad (9\text{-}59)$$

Let $u = |z|\tan(2\pi w)$ so that $du = 2\pi\,|z|\sec^2(2\pi w)dw$. Notice that as w varies from 0 to 1, u varies from $-\infty$ to $+\infty$. Using this in (9-59), we get

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \underbrace{\int_{-\infty}^{+\infty} e^{-u^2/2} \frac{du}{\sqrt{2\pi}}}_{1} = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty, \qquad (9\text{-}60)$$

which represents a zero mean Gaussian r.v with unit variance. Thus $Z \sim N(0,1)$. Equation (9-54) can be used as a practical procedure to generate Gaussian random variables from two independent uniformly distributed random sequences.

Example 9.6 : Let X and Y be independent identically distributed Geometric random variables with

$$P(X=k)=P(Y=K)=pq^k, \qquad k=0,1,2,\cdots.$$

(a) Show that *min* (X, Y) and $X-Y$ are independent random variables.
(b) Show that *min* (X, Y) and *max* (X, Y) – *min* (X, Y) are also independent random variables.

Solution: (a) Let

$$Z=\min(X, Y), \text{ and } W=X-Y. \tag{9-61}$$

Note that Z takes only nonnegative values $\{0,1,2,\cdots\}$, while W takes both positive, zero and negative values $\{0,\pm 1,\pm 2,\cdots\}$. We have $P(Z=m, \; W=n) = P\{\min(X, Y)=m, \; X-Y=n\}$. But

$$Z=\min(X,Y)=\begin{cases} Y & X \geq Y \Rightarrow W=X-Y \text{ is nonnegative} \\ X & X<Y \Rightarrow W=X-Y \text{ is negative.} \end{cases}$$

Thus

$$P(Z=m,W=n)=P\{\min(X,Y)=m, X-Y=n,(X\geq Y \cup X<Y)\}$$
$$=P(\min(X,Y)=m, X-Y=n, X\geq Y)$$
$$+P(\min(X,Y)=m, X-Y=n, X<Y) \quad (9\text{-}62)$$

$$
\begin{aligned}
P(Z=m,W=n) &= P(Y=m, X=m+n, X \geq Y) \\
&\quad + P(X=m, Y=m-n, X<Y) \\
&= \begin{cases} P(X=m+n)P(Y=m) = pq^{m+n}\,pq^{m}, m\geq 0, n\geq 0 \\ P(X=m)P(Y=m-n) = pq^{m}\,pq^{m-n}, m\geq 0, n<0 \end{cases} \\
&= p^2 q^{2m+|n|}, \quad m=0,1,2,\cdots \quad n=0,\pm 1, \pm 2, \cdots \qquad \text{(9-63)}
\end{aligned}
$$

represents the joint probability mass function of the random variables *Z* and *W*. Also

$$
\begin{aligned}
P(Z=m) &= \sum_n P(Z=m, W=n) = \sum_n p^2 q^{2m} q^{|n|} \\
&= p^2 q^{2m}(1+2q+2q^2+\cdots) \\
&= p^2 q^{2m}(1+\tfrac{2q}{1-q}) = pq^{2m}(1+q) \\
&= p(1+q)q^{2m}, \quad m=0,1,2,\cdots. \qquad \text{(9-64)}
\end{aligned}
$$

Thus *Z* represents a Geometric random variable since $1-q^2 = p(1+q)$, and

$$P(W=n)=\sum_{m=0}^{\infty} P(Z=m, W=n)=\sum_{m=0}^{\infty} p^2 q^{2m} q^{|n|}$$
$$= p^2 q^{|n|}(1+q^2+q^4+\cdots)=p^2 q^{|n|}\frac{1}{1-q^2}$$
$$=\frac{p}{1+q} q^{|n|}, \qquad n=0, \ \pm 1, \ \pm 2, \cdots. \tag{9-65}$$

Note that

$$P(Z=m, W=n)=P(Z=m)P(W=n), \tag{9-66}$$

establishing the independence of the random variables Z and W. The independence of $X-Y$ and $min\,(X, Y)$ when X and Y are independent Geometric random variables is an interesting observation.

(b) Let

$$Z = min\,(X, Y), \quad R = max\,(X, Y) - min\,(X, Y). \tag{9-67}$$

In this case both Z and R take nonnegative integer values $0, 1, 2, \cdots$. Proceeding as in (9-62)-(9-63) we get

$$
\begin{aligned}
P\{Z=m,R=n\} &= P\{\min(X,Y)=m,\ \max(X,Y)-\min(X,Y)=n,\ X\ge Y\} \\
&\quad + P\{\min(X,Y)=m,\ \max(X,Y)-\min(X,Y)=n,\ X<Y\} \\
&= P\{Y=m,\ X=m+n,\ X\ge Y) + P(X=m, Y=m+n,\ X<Y\} \\
&= P\{X=m+n, Y=m,\ X\ge Y) + P(X=m, Y=m+n,\ X<Y\} \\
&= \begin{cases} pq^{m+n}pq^m + pq^m pq^{m+n}, & m=0,1,2,\cdots, \quad n=1,2,\cdots \\ pq^{m+n}pq^m, & m=0,1,2,\cdots, \quad n=0 \end{cases} \\
&= \begin{cases} 2p^2q^{2m+n}, & m=0,1,2,\cdots, \quad n=1,2,\cdots \\ p^2q^{2m}, & m=0,1,2,\cdots, \quad n=0. \end{cases} \qquad (9\text{-}68)
\end{aligned}
$$

Eq. (9-68) represents the joint probability mass function of Z and R in (9-67). From (9-68),

$$
\begin{aligned}
P(Z=m) &= \sum_{n=0}^{\infty} P\{Z=m,R=n\} = p^2q^{2m}\left(1+2\sum_{n=1}^{\infty}q^n\right) = p^2q^{2m}\left(1+\frac{2q}{p}\right) \\
&= p(1+q)q^{2m}, \qquad m=0,1,2,\cdots \qquad (9\text{-}69)
\end{aligned}
$$

and

$$P(R=n)=\sum_{m=0}^{\infty} P\{Z=m, R=n\}=\begin{cases} \dfrac{p}{1+q}, & n=0 \\ \dfrac{2p}{1+q} q^n, & n=1,2,\cdots. \end{cases} \tag{9-70}$$

From (9-68)-(9-70), we get

$$P(Z=m, R=n)=P(Z=m)P(R=n) \tag{9-71}$$

which proves the independence of the random variables *Z* and *R* defined in (9-67) as well.