3.1 Derivatives of Polynomials and Exponential Functions

Derivative of a Constant Function

$f(x) = c$.

The graph is the horizontal line $y = c$, which has slope 0, so we must have $f'(x) = 0$.

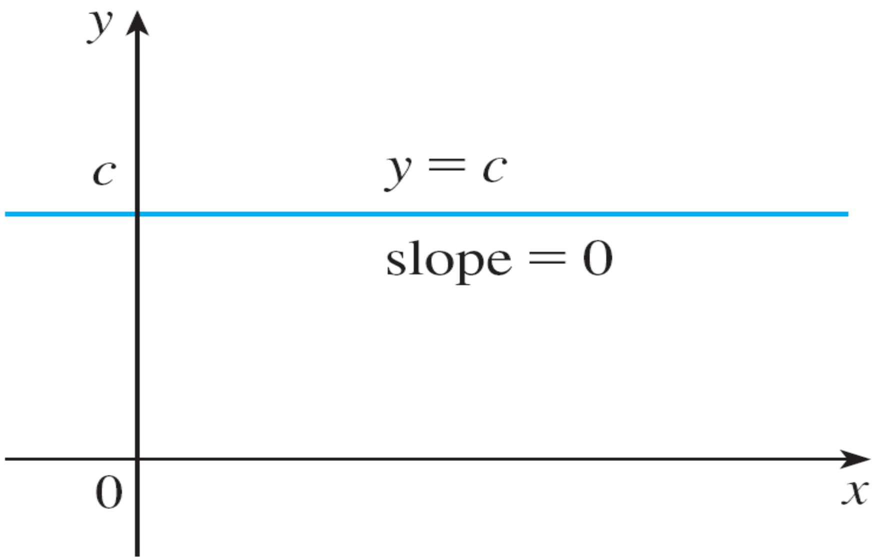

Figure 1

The graph of $f(x) = c$ is the line $y = c$, so $f'(x) = 0$.

Derivative of a Constant Function

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0$$

Derivative of a Constant Function

$$\frac{d}{dx}(c) = 0$$

Exercise – Constant Multiple Rule

Find y' if $y = \pi$

Power Rule

The Power Rule (General Version) If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Example 1 – *Using the Power Rule*

(a) If $f(x) = x^6$, then $f'(x) = 6x^5$.

(b) If $y = x^{1000}$, then $y' = 1000x^{999}$.

(c) If $y = t^4$, then $\frac{dy}{dt} = 4t^3$.

(d) $\frac{d}{dr}(r^3) = 3r^2$

Exercises – Power Rule

Find derivatives for:

$(a)\ y = x^2$

$b)\ f(x) = \frac{1}{x^2}$

$c)\ f(x) = \sqrt[3]{x}$

$d)\ y = \sqrt{x}(x^2 - 1)$

$e)\ y = \frac{3x^2+x+1}{\sqrt{x}}$

Constant Multiple Rule

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

Example 4 – *Using the Constant Multiple Rule*

(a) $$\frac{d}{dx}(3x^4) = 3\frac{d}{dx}(x^4)$$

$$= 3(4x^3)$$

$$= 12x^3$$

(b) $$\frac{d}{dx}(-x) = \frac{d}{dx}[(-1)x]$$

$$= (-1)\frac{d}{dx}(x)$$

$$= -1(1)$$

$$= -1$$

Sum Rule

The Sum Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Difference Rule

By writing $f - g$ as $f + (-1)g$ and applying the Sum Rule and the Constant Multiple Rule, we get the following formula.

The Difference Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

The Constant Multiple Rule, the Sum Rule, and the Difference Rule can be combined with the Power Rule to differentiate any polynomial, as the following examples demonstrate.

Derivatives of Polynomials

Using power, constant multiple and sum/difference rules → we can differentiate any polynomial:

Example: Find derivative of $y = x^3 - 2x + 9$

Derivatives of Polynomials-Exercise

Find derivative of $f(x) = 3x^5 - x^2 + 2x + \pi$

Exponential Functions

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

Example 8

If $f(x) = e^x - x$, find f'.

Solution:

Using the Difference Rule, we have

$$f'(x) = \frac{d}{dx}(e^x - x) = \frac{d}{dx}(e^x) - \frac{d}{dx}(x) = e^x - 1$$

Derivative of Exponential - *Exercise*

Find where the graph $y = e^x + 3x - 2$ has a horizontal tangent line.

Higher Derivatives

Higher Derivatives

Second derivative of *f* is the derivative of the derivative of *f*

$(f')' = f''$.

Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Example 7

If $f(x) = x^3 - x$, find $f''(x)$.

Solution:

Higher Derivatives - Acceleration

The instantaneous rate of change of velocity with respect to time is called the **acceleration** $a(t)$ of the object. Thus the acceleration function is the derivative of the velocity function
and is therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t)$$

or, in Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Higher Derivatives

The process can be continued. The fourth derivative f'''' is usually denoted by $f^{(4)}$.

In general, the nth derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times.

If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

Higher Derivatives - Exercises

Find first, second, third and fourth derivatives for:

(a) $f(x) = 2x^2 - 3x + 2$ (b) $g(x) = e^x + 3$

3.2 The Product and Quotient Rules

The Product Rule

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

In words, the Product Rule says that *the derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.*

Example 1 – *Using the Product Rule*

(a) If $f(x) = xe^x$, find $f'(x)$.

Solution:

(a) By the Product Rule, we have

$$f'(x) = \frac{d}{dx}(xe^x)$$

$$= x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x)$$

$$= xe^x + e^x \cdot 1 = (x + 1)e^x$$

Product Rule - Exercises

$(a)\, y = e^{x}(x^{2} + 1)$

$(b)\, y = x^{2}(2 - x)$

The Quotient Rule

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2}$$

In words, the Quotient Rule says that the *derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

Example 5 – *Using the Quotient Rule*

Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$.

Then

$$y' = \frac{(x^3 + 6)\dfrac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2)\dfrac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2}$$

$$= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2}$$

$$= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}$$

Quotient Rule - Exercises

$(a)\, y = \frac{e^x}{2-x}$

$(b)\, y = \frac{x^2}{x-1}$

$(c)\, y = \frac{x^2+\sqrt{x}}{x}$

Summary of Rules

Table of Differentiation Formulas

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1} \qquad \frac{d}{dx}(e^x) = e^x$$

$$(cf)' = cf' \qquad (f + g)' = f' + g' \qquad (f - g)' = f' - g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$