The Wave Equation

By maxwell's equation we know that

$$\text{curl } \vec{H} = J + \frac{\partial D}{\partial t} \quad \text{—①}$$

Taking curl of eq ①

$$\text{curl curl } H = \text{curl}\left(J + \frac{\partial D}{\partial t}\right)$$

$$= \text{curl } J + \text{curl} \frac{\partial D}{\partial t} \quad \text{—②}$$

As we know that

$D = \epsilon_0 E$ and $J = \sigma E$

put in eq ②

$$\text{curl curl } H = \sigma \,\text{curl } E + \epsilon_0 \,\text{curl} \frac{\partial E}{\partial t}$$

$$= \sigma \,\text{curl } E + \epsilon_0 \frac{\partial}{\partial t} \text{curl } E \quad \text{—③}$$

By another Maxwell's equation

$$\text{curl } E = -\frac{\partial B}{\partial t} \quad \text{—④}$$

$\because B = \mu_0 H$

$$\text{curl } E = -\mu_0 \frac{\partial H}{\partial t}$$

put this value in eq ③

$$\text{curl curl } H = -\sigma\mu_0\frac{\partial H}{\partial t} - \varepsilon_0\mu_0\frac{\partial^2 H}{\partial t^2}$$

Applying identity

$$\text{curl curl } H = \text{grad div } H - \nabla^2 H$$

$$\text{grad div} H - \nabla^2 H = -\sigma\mu_0\frac{\partial H}{\partial t} - \varepsilon_0\mu_0\frac{\partial^2 H}{\partial t^2}$$

as, $B = \mu_0 H$

$$\frac{B}{\mu_0} = H$$

$$\text{div } H = \frac{1}{\mu_0}\text{div } B$$

$$= 0$$

So,

$$\nabla^2 H - \sigma\mu_0\frac{\partial H}{\partial t} - \varepsilon_0\mu_0\frac{\partial^2 H}{\partial t^2} = 0 \quad \text{— (A)}$$

This is the wave equation for magnetic field.

Now, taking curl of the eq. ④

$$\text{curl curl } E = -\text{curl}\frac{\partial B}{\partial t}$$

$$\text{grad div } E - \nabla^2 E = -\frac{\partial}{\partial t}\text{curl } \vec{B}$$

As $\text{curl}\frac{B}{\mu} = J + \frac{\partial D}{\partial t}$

$$\text{curl } B = \mu J + \mu_0\frac{\partial D}{\partial t}$$

$$\text{grad div } E - \nabla^2 E = -\mu_0 \frac{\partial}{\partial t}\left(J + \frac{\partial D}{\partial t}\right)$$

$$\text{grad div } E - \nabla^2 E = -\mu_0 \frac{\partial}{\partial t}\left[\sigma E + \varepsilon_0 \frac{\partial E}{\partial t}\right]$$

As we know,

$$\text{div } D = \rho$$

$$\varepsilon_0 \text{ div } E = \rho$$

$$\text{div } E = 0$$

Assuming volume charge density $= \rho = 0$

$$-\nabla^2 E = -\mu_0 \sigma \frac{\partial E}{\partial t} - \varepsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2}$$

$$\nabla^2 E - \mu_0 \sigma \frac{\partial E}{\partial t} - \varepsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2} = 0 \quad \text{— (5)}$$

This is the wave equation for electric field.

let, our wave is monochromatic, and it is also depend on time and position so,

$$E(r,t) = E_0(r) e^{-i\omega t}$$

$$\frac{\partial E}{\partial t} = -i\omega E_0 e^{-i\omega t}$$

$$\frac{\partial^2 E}{\partial t^2} = -\omega^2 E_0 e^{-i\omega t}$$

put this value in eq (5)

$$\nabla^2 E_s e^{-i\omega t} + i\omega\mu_0 \sigma E_s e^{-i\omega t} + \varepsilon_0 \mu_0 \omega^2 E_s e^{-i\omega t} = 0$$

$$e^{-i\omega t}(\nabla^2 E_s + i\omega\mu_0 \sigma E_s + \varepsilon_0 \mu_0 \omega^2 E_s) = 0$$

$$\nabla^2 E_s + i\omega\mu_0 \sigma E_s + \varepsilon_0 \mu_0 \omega^2 E_s = 0 .$$

This equation shows the spatial variation of electric field.

The Monochromatic Wave In Non-Conducting Media :-

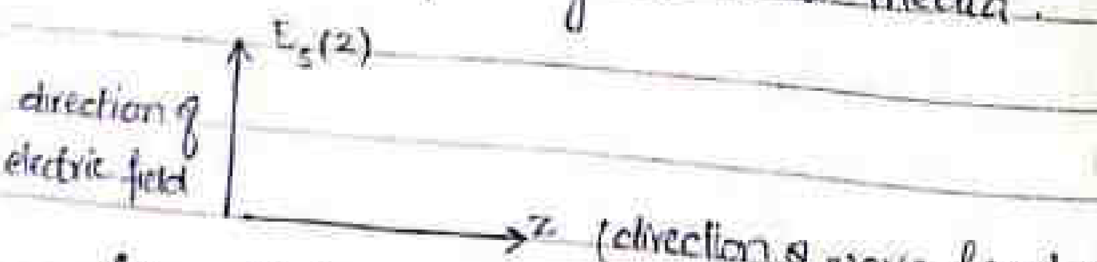
We know that we have non-conducting media

So, $\sigma = 0$

and eq becomes

$$\nabla^2 E_s + \varepsilon_0 \mu_0 \omega^2 E_s = 0$$

Plane wave is a wave, whose amplitude is same and is flowing ⊥r to its media.



So, above eq becomes

$$\frac{d^2 E_s(z)}{dz^2} + \varepsilon_0 \mu_0 \omega^2 E_s(z) = 0$$

$E_s(x,y)$ disappeared because field is not depend upon them

Take,

$$E_s(z) = E_0 e^{\pm i\omega\sqrt{\varepsilon_0\mu_0}\,z} \quad \text{— } (1)$$

E_0 ↓ constant vector

As, $\text{div } E = 0$

$\text{div } E_s = 0$

$\frac{\partial E_s(z)}{\partial z} = 0$

$$\frac{\partial E_s(z)}{\partial z} = \pm i\omega\sqrt{\varepsilon_0\mu_0}\, E_0 e^{\pm i\omega\sqrt{\varepsilon_0\mu_0}\,z}$$

$$= \pm i\omega\sqrt{\varepsilon_0\mu_0}\, E_s(z) = 0$$

It means E_s has no z-component so, our electric field vector is not $\perp r$ to wave front & is parallel to it

So, eq (1) becomes

$$E_s(z) = (\hat{i} E_0 x + \hat{j} E_0 y) e^{\pm i\omega\sqrt{\varepsilon_0\mu_0}\,z}$$

Taking curl of above eq.

$$\text{curl } E_s(z) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_0x & E_0y & 0 \end{vmatrix} e^{\pm i\omega\sqrt{\epsilon_0\mu_0}z}$$

$$\text{curl } E_s(z) = \left(\frac{-\partial}{\partial z} E_0y\,\hat{i} + \frac{\partial}{\partial z} E_0x\,\hat{j}\right) e^{\pm i\omega\sqrt{\epsilon_0\mu_0}z}$$

$$= \left(-\hat{i}\,E_0y + \hat{j}\,E_0x\right)\frac{\partial}{\partial z} e^{\pm i\omega\sqrt{\epsilon_0\mu_0}z}$$

$$\text{curl } E_s(z) = \pm i\omega\sqrt{\epsilon_0\mu_0}\left(-\hat{i}\,E_0y + \hat{j}\,E_0x\right) e^{\pm i\omega\sqrt{\epsilon_0\mu_0}z}. \quad \text{——} \; (2)$$

Now,

By Maxwell's Equation,

$$\text{curl } E = \frac{-\partial B}{\partial t}$$

$\therefore$ $\vec{E} = E_s e^{-i\omega t}$

Similarly, $\vec{B} = \vec{B}_s e^{-i\omega t}$

diff B

$$\frac{\partial B}{\partial t} = -i\omega B_s e^{-i\omega t}$$

Then

$$\text{curl } E_s e^{-i\omega t} = +i\omega B_s e^{-i\omega t} \quad \text{——} \; (3)$$

From eq (2) and (3)

$\because \hat{k}\times\hat{i} = \hat{j}$

$\hat{k}\times\hat{j} = -\hat{i}$

$$i\omega B_s = \pm i\omega\sqrt{\epsilon_0\mu_0}\left(-\hat{i}\,E_0y + \hat{j}\,E_0x\right) e^{\pm i\omega\sqrt{\epsilon_0\mu_0}z}$$

$$B_s = \pm\sqrt{\epsilon_0\mu_0}\left(\hat{k}\times\hat{j}\,E_0y + \hat{k}\times\hat{i}\,E_0x\right) e^{\pm i\omega\sqrt{\epsilon_0\mu_0}z}$$

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Taking $\hat{k}$ outside

$$\hat{k}\times(E_0 x\hat{i} + E_0 y\hat{j})$$
$$= \hat{k}\times E_0$$

So,

$$B_s = \pm\sqrt{\epsilon_0\mu_0}\,|\hat{k}\times E_0|\,e^{\pm i\omega\sqrt{\epsilon_0\mu_0}z}$$

$$\because E_s = E_0 e^{\pm i\omega\sqrt{\epsilon_0\mu_0}z}$$

$$B_s = \pm\sqrt{\epsilon_0\mu_0}\,\hat{k}\times E_s.$$

It means B is $\perp$r to E_s and z axis.

Taking B which is depend on position and time So,

$$B(r,t) = B_s e^{-i\omega t}$$

So, $B(r,t) = \pm\sqrt{\epsilon_0\mu_0}\,|\hat{k}\times E_0|\,e^{i\omega(\sqrt{\epsilon_0\mu_0}z - t)}$

Similarly,

$$E(r,t) = E_s e^{-i\omega t} = E_0 e^{i\omega(\sqrt{\epsilon_0\mu_0}z - t)} \quad \text{——} \quad (4)$$

Plane wave moving in z-direction can solve our problem of electric field in x and y direction.

we define another unit vector 'u' such that if we take its dot product with $r = \hat{z}$

$$\vec{u}\cdot\vec{r} = \vec{z}$$

Replace 'z' by $\vec{u}\cdot\vec{r}$ in eq (4)

$$\vec{E}(r,t) = E_0 e^{i\omega(\sqrt{\epsilon_0\mu_0}\,\vec{u}\cdot\vec{r} - t)}$$

Similarly for $\vec{B}(r,t)$, we have

$\vec{B}(r,t) = \sqrt{\epsilon_0 \mu_0}\left|\hat{u} \times E_0\right| e^{i\omega(\sqrt{\epsilon_0 \mu_0}\,\hat{u}\cdot\vec{r} - t)}$

put,

$$\omega\sqrt{\epsilon_0\mu_0}\,\hat{u} = \vec{k} \qquad \therefore \hat{u} = \frac{k}{\omega\sqrt{\epsilon_0\mu_0}}$$

$\vec{k}$ is the wave vector and $\hat{u}$ is the velocity of wave

So,

$$B(r,t) = \sqrt{\epsilon_0\mu_0}\left|\frac{k}{\omega\sqrt{\epsilon_0\mu_0}} \times E_0\right| e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$B(r,t) = \frac{k}{\omega} \times E_0 \underbrace{e^{i(\vec{k}\cdot\vec{r} - \omega t)}}_{\text{(constant value)}}$$

diff constant wrt 't'

$$(k\cdot r - \omega t)$$

$$k\cdot\frac{\partial r}{\partial t} - \omega = 0$$

$$kV_p - \omega = 0$$

V_p is the phase velocity.

$$V_p = \frac{\omega}{k}$$

put value of $\vec{k}$

$$\frac{\omega}{k} = \frac{\omega}{\omega\sqrt{\epsilon\mu}}$$

$$\frac{\omega}{k} = \frac{1}{\sqrt{\epsilon\mu}}$$

So,

$$V_p = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = c$$

as, $\frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon \mu}} = \frac{c}{c\sqrt{\varepsilon \mu}} = \frac{c}{\frac{1}{\sqrt{\varepsilon_0 \mu_0}} \sqrt{\varepsilon \mu}}$

$$= \frac{c}{\sqrt{\frac{\varepsilon \mu}{\varepsilon_0 \mu_0}}}$$

electric field constant　　magnetic field constant

$$= \frac{c}{\sqrt{K_e K_m}}$$

$$V_p = \frac{c}{\sqrt{K_e K_m}}$$

Dielectric constant of electric field is

$$K_e = \frac{\varepsilon}{\varepsilon_0}$$

Dielectric constant of magnetic field is

$$K_m = \frac{\mu}{\mu_0}$$

The product of dielectric is Refractive index.

$$\hat{Y} = \sqrt{K_e K_m}$$

The Monochromatic Wave In Conducting Media :-

$\nabla^2 E_s + \omega^2 \varepsilon \mu E_s + i\omega\sigma\mu E_s = 0$

diff w.r.t z.

$\frac{d^2 E_s}{dz^2} + \omega^2 \varepsilon \mu E_s + i\omega\sigma\mu E_s = 0$ —— ①

As we know

$E_s = E_0 e^{i\gamma z}$

$\frac{dE_s}{dz} = i\gamma E_0 e^{i\gamma z} = i\gamma E_s$

$\frac{d^2 E_s}{dz^2} = -\gamma^2 E_0 e^{i\gamma z} = -\gamma^2 E_s(z).$

put in eq ①.

$-\gamma^2 E_s + \omega^2 \varepsilon \mu E_0 e^{i\gamma z} + i\omega\sigma\mu E_0 e^{i\gamma z} = 0$

$-\gamma^2 E_0 e^{i\gamma z} + \omega^2 \varepsilon \mu E_0 e^{i\gamma z} + i\omega\sigma\mu E_0 e^{i\gamma z} = 0$

$E_0 e^{i\gamma z} (-\gamma^2 + \omega^2 \varepsilon \mu + i\omega\sigma\mu) = 0$

$-\gamma^2 + \omega^2 \varepsilon \mu + i\omega\sigma\mu = 0$

$\gamma^2 = \omega^2 \varepsilon \mu + i\omega\sigma\mu$ —— ②

where,

$\alpha = \omega^2 \varepsilon \mu$ (Real term)

$\beta = \omega\sigma\mu$ (Imaginary term)

Now,

Take,

$$\gamma^2 = A\cos 2\varphi + iA\sin 2\varphi \quad — ③$$

$A\cos 2\varphi = \omega^2 \varepsilon \mu$ — (A)

$A\sin 2\varphi = \omega \sigma \mu$ — (B) squaring & Adding A & B

$A^2\cos^2 2\varphi + A^2\sin^2 2\varphi =$

$= \omega^4 \varepsilon^2 \mu^2 + \omega^2 \sigma^2 \mu^2$

$\Rightarrow A^2(\cos^2 2\varphi + \sin^2 2\varphi) = \omega^4 \varepsilon^2 \mu^2 + \omega^2 \sigma^2 \mu^2$

$A^2 = \omega^4 \varepsilon^2 \mu^2 + \omega^2 \sigma^2 \mu^2$ — ④

$$A = \left[\omega^4 \varepsilon^2 \mu^2 + \omega^2 \sigma^2 \mu^2\right]^{1/2}$$

Take eq ③ as

$\gamma^2 = Ae^{i2\varphi}$

$\gamma = A^{1/2} \cdot e^{i\varphi}$

$\gamma = (\omega^4 \varepsilon^2 \mu^2 + \omega^2 \sigma^2 \mu^2)^{1/4} \cdot e^{i\varphi}$

$= (\omega^4 \varepsilon^2 \mu^2 + \omega^2 \sigma^2 \mu^2)^{1/4} (\cos\varphi + i\sin\varphi)$

$= (\omega^4 \varepsilon^2 \mu^2 + \omega^2 \sigma^2 \mu^2)^{1/4}\cos\varphi + i(\omega^4 \varepsilon^2 \mu^2 + \omega^2 \sigma^2 \mu^2)^{1/4}\sin\varphi$

$\gamma = \alpha + i\beta$

where,

$\alpha = (\omega^4 \varepsilon^2 \mu^2 + \omega^2 \sigma^2 \mu^2)^{1/4}\cos\varphi$

$\beta = (\omega^4 \varepsilon^2 \mu^2 + \omega^2 \sigma^2 \mu^2)^{1/4}\sin\varphi$

As,

$\gamma = \alpha + i\beta$ Taking sq on b/s

$\gamma^2 = \alpha^2 - \beta^2 + 2i\alpha\beta$ — ⑤

$\gamma^2 = \omega^2 \varepsilon \mu + i\omega \sigma \mu$ — ⑥

Comparing eq ⑤ and ⑥

$$\alpha^2 - \beta^2 = \omega^2 \varepsilon \mu \quad — ⑦$$

$$2\alpha\beta = \sigma\omega\mu$$

$$\beta = \frac{\sigma\omega\mu}{2\alpha} \quad — ⑧$$

put eq ⑥ in eq ⑦

$$\alpha^2 - \left(\frac{\sigma\omega\mu}{2\alpha}\right)^2 = \omega^2 \varepsilon \mu$$

$$4\alpha^4 - \sigma^2\omega^2\mu^2 = 4\alpha^2\omega^2\varepsilon\mu$$

$$4\alpha^4 - 4\alpha^2\omega^2\varepsilon\mu - \sigma^2\omega^2\mu^2 = 0$$

here, $a = 4$ $\quad b = -4\omega^2\varepsilon\mu$ $\quad c = -\omega^2\sigma^2\mu^2$

By using quadratic formula.

$$\alpha^2 = \frac{4\omega^2\varepsilon\mu \pm \sqrt{16\omega^4\varepsilon^2\mu^2 + 16\sigma^2\omega^2\mu^2}}{2(4)}$$

$$= \frac{4\omega^2\varepsilon\mu \pm \sqrt{16\omega^2\mu^2(\omega^2\varepsilon^2 + \sigma^2)}}{8}$$

$$= \frac{4\omega^2\varepsilon\mu}{8} \pm \frac{4\omega\mu}{8}\sqrt{\omega^2\varepsilon^2 + \sigma^2}$$

$$= \frac{\omega^2\varepsilon\mu}{2} \pm \frac{\omega\mu}{2}\sqrt{\omega^2\varepsilon^2 + \sigma^2}$$

Multiplying and dividing by $\omega^2\varepsilon^2$

$$\alpha^2 = \frac{\omega^2\varepsilon\mu}{2} \pm \frac{\omega^2\varepsilon\mu}{2}\sqrt{\frac{1+\sigma^2}{\omega^2\varepsilon^2}}$$

$$\alpha^2 = \frac{\omega^2\varepsilon\mu}{2}\left[1 \pm \sqrt{\frac{1+\sigma^2}{\omega^2\varepsilon^2}}\right]$$

If we consider $\phi = 45^\circ$ then we get,
If frequency is below optical range, then σ is very high (conductivity) it is possible only at 45°

$$\sigma >> \varepsilon\omega$$

then,

$$\omega^4\varepsilon^2\mu^2 << \omega^2\sigma^2\mu^2$$

then, β becomes

$$\beta = (\omega^2\sigma^2\mu^2)^{1/4}\frac{1}{\sqrt{2}}$$

$$= (\omega\sigma\mu)^{2\times 1/4}\frac{1}{\sqrt{2}}$$

$$\beta = \sqrt{\frac{\omega\sigma\mu}{2}}$$

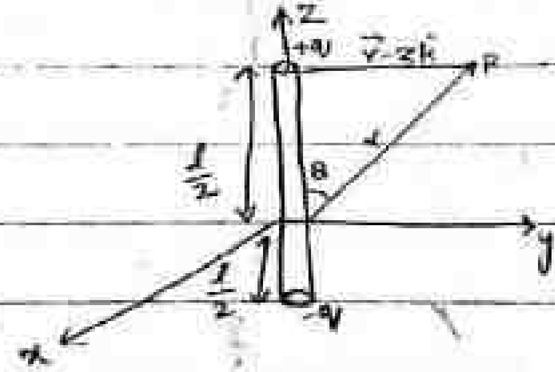
then,

$$\frac{1}{\beta} = \sqrt{\frac{2}{\omega\sigma\mu}}$$

It measures the depth, where the value of electrical field falls it is "skin depth". It is given by δ

$$\delta = \sqrt{\frac{2}{\omega \sigma \mu}}$$

Radiation from an Oscillating Dipoles :-



$Z = \pm \frac{l}{2}$

$I = \frac{dq}{dt} = \dot{q}$

$l << \lambda$

l is length of wire, λ is wavelength of radiation.

Magnetic vector potential is

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\bar{J}(r_1) \times (r_2 - r_1)}{|r_2 - r_1|^3} \, dV_1$$

$$\vec{A}(r,t) = \frac{\mu_0}{4\pi} \int_{-l/2}^{l/2} \frac{I(z, t - \frac{|r - z\hat{k}|}{c}) \, (\vec{r} - z\hat{k})}{|r - z\hat{k}|} \, dz \quad \longleftarrow (1)$$

Simplifying $|r - z\hat{k}|$

Taking square

$$= |r^2 + z^2 - 2r \cdot z\hat{k}|^{1/2}$$

$$= r\left|1 + \frac{z^2}{r^2} - \frac{2\vec{r} \cdot z\hat{k}}{r^2}\right|^{1/2}$$

$z << r$

z is the direction of orientation of dipole. and r is the distance from origin to observation point.

So,

$$= r\left|1 - \frac{2\vec{r} \cdot z\hat{k}}{r^2}\right|^{1/2}$$

$$= r\left|1 - \frac{2rz\cos\theta}{r^2}\right|^{1/2}$$

$$= r\left|1 - \frac{2z\cos\theta}{r}\right|^{1/2}$$

By binomial theorem & neglecting higher terms,

$$|r - z\hat{k}| = r\left|1 - \frac{z\cos\theta}{r}\right|$$

$$|r - z\hat{k}| = r - z\cos\theta$$

Put these values in eq (1)

$$A(r,t) = \frac{\mu_0}{4\pi}\int_{-l/2}^{l/2} \frac{I(z, t - \sqrt{\epsilon\mu}(r - z\cos\theta))\, dz}{(r - z\cos\theta)}$$

We know that, $\because v = \frac{1}{\sqrt{\epsilon\mu}}$

$$\sqrt{\epsilon\mu}\, z\cos\theta \qquad \because \sqrt{\epsilon\mu} = \frac{1}{v}$$

$$= \frac{z\cos\theta}{v}$$

So, We have to prove $\sqrt{\epsilon\mu}z\cos\theta$ is smaller than r to ignore $z\cos\theta$ from the equation.

$$\vec{A}(r,t) = \frac{\mu_0}{4\pi}\int_{-l/2}^{l/2} \frac{I(z, t - \sqrt{\epsilon\mu})(r)}{r}\, dz$$

$$\vec{A}(r,t) = \frac{\mu_0}{4\pi}\frac{I}{r}\left(t - \frac{r}{v}\right)\Big|z\Big|_{-l/2}^{l/2}$$

$z\cos \leq \frac{l}{2}$

$v - \frac{l}{2}\cos\theta$

v

$\lambda \gg \frac{l}{2}$

$$\vec{A}(r,t) = \frac{\mu_0}{4\pi} I\left(t - \frac{r}{v}\right)\frac{l}{r} \quad \text{---} \; (2)$$

Calculate the magnetic vector potential due to a dipole?

Calculate the retarding potential due to an oscillating dipole?

Calculating the scalar potential due to an oscillating dipole?

The magnetic vector potential is,

$$\vec{A}(r,t) = \frac{\mu_0}{4\pi} I\left(t - \frac{r}{v}\right)\frac{l}{r} \;\ldots\ldots \text{---} \; (2)$$

The Lorentz condition is,

$$\operatorname{div}A + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0$$

$$\vec{\nabla}\cdot\vec{A} + \varepsilon_0\mu_0 \frac{\partial \phi}{\partial t} = 0$$

$$\varepsilon_0\mu_0 \frac{\partial \phi}{\partial t} = -\vec{\nabla}\cdot\vec{A}$$

$$= -\frac{\partial A_z}{\partial z}$$

$$= -\frac{\partial}{\partial z}\left[\frac{\mu_0}{4\pi} I\left(t-\frac{r}{v}\right)\frac{l}{r}\right]$$

$$= -\frac{\mu_0}{4\pi}\frac{l}{r}\frac{\partial}{\partial z} I\left(t-\frac{r}{v}\right) - \frac{\mu_0}{4\pi} I\left(t-\frac{r}{v}\right)\frac{\partial}{\partial z}\frac{l}{r} \quad \text{— } ①$$

As, $\frac{\partial}{\partial z}(r) = \frac{\partial}{\partial z}(x^2+y^2+z^2)^{1/2}$

$$= \frac{1}{2}(x^2+y^2+z^2)^{1/2-1}(2z)$$

$$\frac{\partial (r)}{\partial z} = \frac{z}{(x^2+y^2+z^2)^{1/2}} = \frac{z}{r}$$

$$\frac{\partial}{\partial z}(r^{-1}) = \frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2}$$

$$= \frac{-1}{2}(x^2+y^2+z^2)^{-1/2-1}(2z)$$

$$\frac{\partial}{\partial z}(r^{-1}) = -\frac{z}{r^3}$$

put values in eq ①.

$$\mu_0\varepsilon_0\frac{\partial\phi}{\partial t} = -\frac{\mu_0}{4\pi}\frac{l}{r}I'\left(t-\frac{r}{v}\right)\left(\frac{-z}{rv}\right) - \frac{\mu_0}{4\pi}I\left(t-\frac{r}{v}\right)\left(\frac{-lz}{r^3}\right)$$

Taking μ_0 common and then eliminate.

$$\varepsilon_0\frac{\partial\phi}{\partial t} = \frac{-1}{4\pi}\frac{l}{r}I'\left(t-\frac{r}{v}\right)\left(\frac{-z}{rv}\right) - \frac{1}{4\pi}I\left(t-\frac{r}{v}\right)\left(\frac{-lz}{r^3}\right)$$

$$\frac{\partial\phi}{\partial t} = \frac{-1}{4\pi\varepsilon_0}\frac{l}{r}I'\left(t-\frac{r}{v}\right)\left(\frac{-z}{rv}\right) - \frac{1}{4\pi\varepsilon_0}I\left(t-\frac{r}{v}\right)\left(\frac{-lz}{r^3}\right)$$

$$\frac{\partial\phi}{\partial t} = \frac{-1}{4\pi\varepsilon_0}\frac{lz}{r^2}\left[I'\left(t-\frac{r}{v}\right)\left(\frac{-1}{v}\right) - I\left(t-\frac{r}{v}\right)\left(\frac{-1}{r}\right)\right]$$

Integrating

$$\phi = \frac{lz}{4\pi\varepsilon_0 r^2}\left[I\left(t-\frac{r}{v}\right)\left(\frac{1}{v}\right) + q\left(t-\frac{r}{v}\right)\left(\frac{1}{r}\right)\right]$$

As we know in sinusoidal form $I = I_0\sin\omega t$ & $q = q_0\cos\omega t$.

$$\phi = \frac{lz}{4\pi\varepsilon_0 r^2}\left[I_0\sin\omega\left(t-\frac{r}{v}\right)\left(\frac{1}{v}\right) + q_0\cos\omega\left(t-\frac{r}{v}\right)\left(\frac{1}{r}\right)\right]$$

$$q\left(t-\frac{r}{v}\right) = q_0\cos\omega\left(t-\frac{r}{v}\right)$$

$$I\left(t-\frac{r}{v}\right) = -q_0\omega\sin\omega\left(t-\frac{r}{v}\right)$$

$$\because\ -q_0\omega = I_0$$

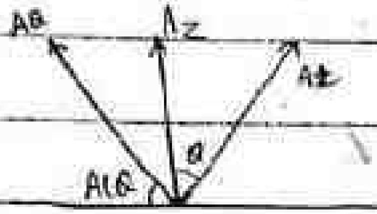
So,

$$I\left(t-\frac{r}{v}\right) = I_0\sin\omega\left(t-\frac{r}{v}\right)$$

Put this value in eq ②, we get

$$\vec{A}(r,t) = \frac{\mu_0 l}{4\pi r} I_0 \sin w\left(t - \frac{r}{v}\right)$$

This is the magnetic scalar potential of the dipole in harmonic form.



As we know current is flowing in 2 direction. In solid angle we have (r, θ, ϕ), firstly we will find r- component of scalar potential.

$$A_r = A_z \cos\theta$$

$$A_r = \frac{\mu_0 l}{4\pi r} I_0 \sin w\left(t - \frac{r}{v}\right) \cos\theta$$

Similarly,

$$A_\theta = -A_z \sin\theta$$

- sign due to 2nd quadrant

$$= -\frac{\mu_0 l}{4\pi r} I_0 \sin w\left(t - \frac{r}{v}\right) \sin\theta$$

Similarly,

$$A_\phi = 0$$

As, $\vec{B} = \text{Curl}\,\vec{A}$

In spherical coordinates,

$$B_\phi = \frac{1}{r}\frac{\partial}{\partial r}(r A_\theta) - \frac{1}{r}\frac{\partial}{\partial \theta} A_r$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(-\frac{\mu_0 l}{4\pi r}\right) I_0 \sin\omega\left(t-\frac{r}{v}\right)\sin\theta\right] - \frac{1}{r}\frac{\partial}{\partial \theta}\frac{\mu_0 l}{4\pi r} I_0 \sin\omega\left(t-\frac{r}{v}\right)\cos\theta$$

$$= \frac{1}{r}\left[-\frac{\mu_0 l}{4\pi} I_0 \cos\omega\left(t-\frac{r}{v}\right)\sin\theta\left(\frac{-\omega}{v}\right)\right] - \left[\frac{1}{r}\frac{\mu_0 l}{4\pi r} I_0 \sin\omega\left(t-\frac{r}{v}\right)(-\sin\theta)\right]$$

$$B_\phi = \frac{\mu_0 l I_0 \sin\theta}{4\pi r}\left[\cos\omega\left(t-\frac{r}{v}\right)\left(\frac{\omega}{v}\right) + \sin\omega\left(t-\frac{r}{v}\right)\frac{1}{r}\right]$$

we write electric field in terms of scalar and vector potential as

$$\vec{E} = -\text{grad}\,\phi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{E}_r = \frac{-\partial \phi}{\partial r} - \frac{\partial A_r}{\partial t}$$

putting values,

$$\vec{E}_r = -\frac{\partial}{\partial r}\frac{l z}{4\pi\varepsilon_0 r^2}\left[I_0 \sin\omega\left(t-\frac{r}{v}\right)\frac{1}{v} + q_0 \cos\omega\left(t-\frac{r}{v}\right)\frac{1}{r}\right] - \frac{\partial}{\partial t}\left[+\frac{\mu_0 l}{4\pi r} I_0 \sin\omega\left(t-\frac{r}{v}\right)\cos\theta\right]$$

$$\boxed{E_r = \frac{2 l I_0 \cos\theta}{4\pi\varepsilon_0}\left[\frac{\sin\omega(t-r/v)}{r^2 v} - \frac{\cos\omega(t-r/v)}{\omega r^3}\right]}$$

Now,

$$E_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} - \frac{\partial A_\theta}{\partial t}$$

$$E_\theta = -\frac{1}{r}\frac{\partial}{\partial \theta}\frac{lz}{4\pi\varepsilon_0 r^2}\left[I_0 \sin\omega\left(t-\frac{r}{v}\right)\frac{1}{v} + q_0 \cos\omega\left(t-\frac{r}{v}\right)\frac{1}{r}\right]$$

$$-\frac{\partial}{\partial t}\left[-\frac{\mu_0 l}{4\pi r} I_0 \sin\omega\left(t-\frac{r}{v}\right)\sin\theta\right]$$

$$E_\theta = -\frac{l I_0 \sin\theta}{4\pi\varepsilon_0}\left[\left(\frac{1}{\omega r^3} - \frac{\omega}{r v^2}\right)\cos\omega\left(t-\frac{r}{v}\right)\right]$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \phi} - \frac{\partial A_\phi}{\partial t}$$

$$E_\phi = 0$$

$$\mu_0\epsilon_0 \frac{\partial\phi}{\partial t} = -\frac{\mu_0}{4\pi}\frac{l}{r} I'\left(t-\frac{r}{v}\right)\left(\frac{-z}{rv}\right) - \frac{\mu_0}{4\pi} I\left(t-\frac{r}{v}\right)\left(-\frac{lz}{r^3}\right)$$

Taking μ_0 common and then eliminate.

$$\epsilon_0 \frac{\partial\phi}{\partial t} = \frac{-1}{4\pi}\frac{l}{r} I'\left(t-\frac{r}{v}\right)\left(\frac{-z}{rv}\right) - \frac{1}{4\pi} I\left(t-\frac{r}{v}\right)\left(-\frac{lz}{r^3}\right)$$

$$\frac{\partial\phi}{\partial t} = \frac{-1}{4\pi\epsilon_0}\frac{l}{r} I'\left(t-\frac{r}{v}\right)\left(\frac{-z}{rv}\right) - \frac{1}{4\pi\epsilon_0} I\left(t-\frac{r}{v}\right)\left(-\frac{lz}{r^3}\right)$$

$$\frac{\partial\phi}{\partial t} = \frac{-1}{4\pi\epsilon_0}\frac{lz}{r^2}\left[I'\left(t-\frac{r}{v}\right)\left(\frac{-1}{v}\right) - I\left(t-\frac{r}{v}\right)\left(\frac{-1}{r}\right)\right]$$

Integrating

$$\phi = \frac{lz}{4\pi\epsilon_0 r^2}\left[I\left(t-\frac{r}{v}\right)\left(\frac{1}{v}\right) + q\left(t-\frac{r}{v}\right)\left(\frac{1}{r}\right)\right]$$

As we know in sinusoidal form $I = I_0 \sin\omega t$ & $q = q_0 \cos\omega t$.

$$\phi = \frac{lz}{4\pi\epsilon_0 r^2}\left[I_0 \sin\omega\left(t-\frac{r}{v}\right)\left(\frac{1}{v}\right) + q_0 \cos\omega\left(t-\frac{r}{v}\right)\left(\frac{1}{r}\right)\right]$$

$$q\left(t-\frac{r}{v}\right) = q_0 \cos\omega\left(t-\frac{r}{v}\right)$$

$$I\left(t-\frac{r}{v}\right) = -q_0 \omega \sin\omega\left(t-\frac{r}{v}\right)$$

$$\because -q_0\omega = I_0$$

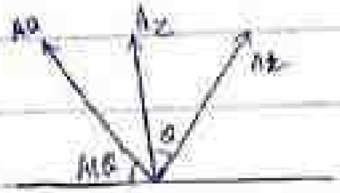
So,

$$I\left(t-\frac{r}{v}\right) = I_0 \sin\omega\left(t-\frac{r}{v}\right)$$

Put this value in eq (2), we get

$$\vec{A}(r,t) = \frac{\mu_0 l}{4\pi r} I_0 \sin w\left(t - \frac{r}{v}\right)$$

This is the magnetic scalar potential of the dipole in harmonic form.



As we know current is flowing in z direction. In solid angle we have (r, θ, ϕ), firstly we will find r- component of scalar potential,

$$A_r = A_z \cos\theta$$

$$A_r = \frac{\mu_0 l}{4\pi r} I_0 \sin w\left(t - \frac{r}{v}\right)\cos\theta$$

Similarly,

$$A_\theta = -A_z \sin\theta$$

- sign due to 2nd quadrant

$$= -\frac{\mu_0 l}{4\pi r} I_0 \sin w\left(t - \frac{r}{v}\right)\sin\theta$$

Similarly,

$$A_\phi = 0$$

As, $\vec{B} = \text{Curl}\,\vec{A}$

In spherical coordinates,

$$B_\phi = \frac{1}{r}\frac{\partial}{\partial r}(r A_\theta) - \frac{1}{r}\frac{\partial}{\partial \theta} A_r$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(-\frac{\mu_0 l}{4\pi r}\right) I_0 \sin\omega\left(t - \frac{r}{v}\right)\sin\theta\right] - \frac{1}{r}\frac{\partial}{\partial \theta}\frac{\mu_0 l}{4\pi r} I_0 \sin\omega\left(t-\frac{r}{v}\right)\cos\theta$$

$$= \frac{1}{r}\left[-\frac{\mu_0 l}{4\pi} I_0 \cos\omega\left(t-\frac{r}{v}\right)\sin\theta\left(-\frac{\omega}{v}\right)\right] - \left[\frac{1}{r}\frac{\mu_0 l}{4\pi r} I_0 \sin\omega\left(t-\frac{r}{v}\right)(-\sin\theta)\right]$$

$$B_\phi = \frac{\mu_0 l I_0 \sin\theta}{4\pi r}\left[\cos\omega\left(t-\frac{r}{v}\right)\left(\frac{\omega}{v}\right) + \sin\omega\left(t-\frac{r}{v}\right)\frac{1}{r}\right]$$

we write electric field in terms of scalar and vector potential as

$$\vec{E} = -\text{grad}\,\phi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{E}_r = \frac{-\partial \phi}{\partial r} - \frac{\partial A_r}{\partial t}$$

putting values,

$$\vec{E}_r = \frac{-\partial}{\partial r}\frac{l z}{4\pi\varepsilon_0 r^2}\left[I_0 \sin\omega\left(t-\frac{r}{v}\right)\frac{1}{v} + q_0 \cos\omega\left(t-\frac{r}{v}\right)\frac{1}{r}\right] - \frac{\partial}{\partial t}\left[+\frac{\mu_0 l}{4\pi r} I_0 \sin\omega\left(t-\frac{r}{v}\right)\cos\theta\right]$$

$$E_r = \frac{2 l I_0 \cos\theta}{4\pi\varepsilon_0}\left[\frac{\sin\omega(t - r/v)}{r^2 v} - \frac{\cos\omega(t-r/v)}{\omega r^3}\right]$$

Now,

$$E_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} - \frac{\partial A_\theta}{\partial t}$$

$$E_\theta = -\frac{1}{r}\frac{\partial}{\partial \theta}\frac{l z}{4\pi \varepsilon_0 r^2}\left[I_0 \sin\omega\left(t-\frac{r}{v}\right)\frac{1}{v} + q_0 \cos\omega\left(t-\frac{r}{v}\right)\frac{1}{r}\right]$$

$$-\frac{\partial}{\partial t}\left[\frac{-\mu_0 l}{4\pi r} I_0 \sin\omega\left(t-\frac{r}{v}\right)\sin\theta\right]$$

$$E_\theta = \frac{-l I_0 \sin\theta}{4\pi \varepsilon_0}\left[\left(\frac{1}{\omega r^3} - \frac{\omega}{r v^2}\right)\cos\omega\left(t-\frac{r}{v}\right)\right]$$

$$E_\phi = \frac{-1}{r\sin\theta}\frac{\partial \phi}{\partial \phi} - \frac{\partial A_\phi}{\partial t}$$

$$E_\phi = 0$$

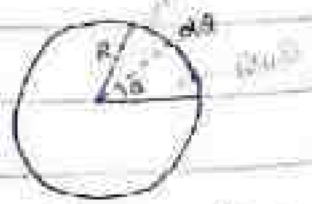
Dipole Radiation Energy :-

Poynting vector is written as
$\vec{S} = \vec{E} \times \vec{H}$

$S_z = E_\theta H_\phi$

$S_z = \frac{1}{\mu_0} E_\theta B_\phi$

As energy is radiated all around so, we take a sphere having radius 'R' making an angle θ with the origin. Consider a small portion of angle $d\theta$



$S = r\theta$

$S = r d\theta$
small distance travelled.

We will take poynting vector along the complete sphere.

$$\oint \vec{S} \cdot \hat{n} \, da = \int_0^\pi S_z \underbrace{2\pi R^2}_{\text{circum}} \sin\theta \underbrace{d\theta}_{\text{arc length}}$$

$$= 2\pi R^2 \int_0^\pi \frac{1}{\mu_0} E_\theta B_\phi \sin\theta \, d\theta$$

$$= \frac{2\pi R^2}{\mu_0} \int_0^\pi E_\theta B_\phi \sin\theta \, d\theta \quad \text{---} (1)$$

We know $l \ll r$ So, we neglect $\frac{1}{r^3}$ from the equation given below.

$\dots \left(\dots - \frac{\omega}{\dots} \right) \cos\omega\left(t - \frac{r}{v}\right)$

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$$E_\theta = \frac{l I_0 \sin\theta}{4\pi\varepsilon_0} \frac{\omega}{R v^2} \cos\omega\left(t - \frac{r}{v}\right)$$

and as $l \ll r$ so we neglect $\frac{1}{r^2}$ from the eq given below

$$B_\phi = \frac{\mu_0 l I_0}{4\pi r} \sin\theta \left(\cos\omega\left(t - \frac{r}{v}\right)\frac{\omega}{v} + \frac{1}{r}\sin\omega\left(t - \frac{r}{v}\right)\right)$$

$$B_\phi = \frac{\mu_0 l I_0}{4\pi R} \sin\theta \cos\omega\left(t - \frac{r}{v}\right)\frac{\omega}{v}$$

putting these values in eq ①.

$$= \frac{2\pi}{\mu_0}\int_0^\pi R^2 \frac{l I_0 \sin\theta}{4\pi\varepsilon_0} \frac{\omega}{R v^2} \cos\omega\left(t - \frac{r}{v}\right) \cdot \frac{\mu_0 l I_0}{4\pi R} \sin\theta \cos\omega\left(t - \frac{r}{v}\right)\frac{\omega}{v} \sin\theta$$

$$\oint \vec{S}\cdot\hat{n}\,da = \frac{l^2 I_0^2}{8\pi\varepsilon_0} \cos^2\omega\left(t - \frac{r}{v}\right)\frac{\omega^2}{v^3}\int_0^\pi \sin^3\theta\, d\theta$$

$$\oint \vec{S}\cdot\hat{n}\,da = \frac{l^2 I_0^2 \cos^2\omega(t - r/v)\,\omega^2}{8\pi\varepsilon_0 v^3}\int_0^\pi \sin\theta\cdot(1 - \cos^2\theta)\, d\theta$$

$$= \frac{l^2\omega^2 I_0^2 \cos^2\omega(t - r/v)}{8\pi\varepsilon_0 v^3}\left[1 - \cos\theta\Big|_0^\pi - \left|\frac{\cos^3\theta}{3}\right|_0^\pi\right]$$

$$= \frac{l^2\omega^2 I_0^2 \cos^2\omega(t - r/v)}{8\pi\varepsilon_0 v^3}\left[-(-1-1) - \frac{1}{3}(-1-1)\right]$$

$$= \frac{l^2\omega^2 I_0^2 \cos^2\omega(t - r/v)}{8\pi\varepsilon_0 v^3}\left(2 + \frac{2}{3}\right)$$

$$\oint S\cdot\hat{n}\,da = \frac{l^2\omega^2 I_0^2 \cos^2\omega(t - r/v)}{3\pi\varepsilon_0 v^3}$$

This is the dipole radiation energy.
Now we find the power radiated

$$P = \frac{l^2 \omega^2 I_0^2}{3\pi\varepsilon_0 v^3 2} \quad (2)$$

$\because \cos 45° = \frac{1}{2}$

This is the average power radiated

As $\omega = 2\pi f$ $\Rightarrow$ $\omega = 2\pi \frac{v}{\lambda}$ or $\lambda = \frac{2\pi v}{\omega}$

$v = f\lambda$

$f = \frac{v}{\lambda}$

putting values in eq (2)

$$P = \frac{l^2}{3\pi\varepsilon_0 v^3} \left(\frac{2\pi v}{\lambda}\right)^2 \frac{I_0^2}{2}$$

$$P = \frac{l^2}{3\varepsilon_0 v} \frac{4\pi}{\lambda^2} \frac{I_0^2}{2}$$

$$P = \frac{4\pi I_0^2}{3\pi\varepsilon_0 2v} \left(\frac{l}{\lambda}\right)^2$$

$\because v = \frac{1}{\sqrt{\varepsilon \mu}}$

$$P = \frac{4\pi}{3} \left(\frac{l}{\lambda}\right)^2 \frac{I_0^2}{2} \sqrt{\frac{\varepsilon}{\mu}}$$

$\varepsilon \frac{1}{\sqrt{\varepsilon \mu}} = \frac{\varepsilon}{\sqrt{\varepsilon}} \times \frac{\sqrt{\varepsilon}}{\sqrt{\varepsilon}}$

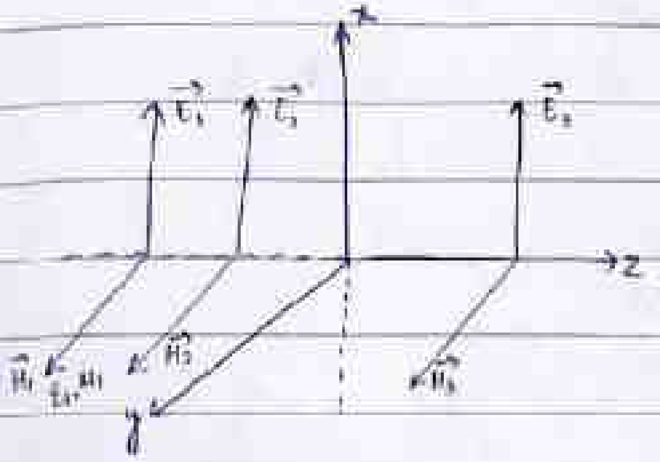
$= \frac{\varepsilon\sqrt{\varepsilon}}{\sqrt{\varepsilon^2}} = \frac{\varepsilon\sqrt{\varepsilon}}{\varepsilon\sqrt{\mu}}$

$$P = R \frac{I_0^2}{2}$$

$= \sqrt{\varepsilon/\mu}$

R is the resistance of radiation.

Reflection and Refraction at the Boundary of two Non Conducting Media :- (Normal incidence)



Equation for electric field is,

$$\vec{E_1} = \hat{i} E_{10} e^{i(\alpha_1 z - \omega t)}$$

for second wave electric field is,

$$\vec{E_2} = -\hat{i} E_{20} e^{-i(\alpha_1 z + \omega t)}$$

for third wave electric field is

$$\vec{E_3} = \hat{i} E_{30} e^{i(\alpha_2 z - \omega t)}$$

where,

$$\alpha_1 = \omega \sqrt{\epsilon_1 \mu_1}$$

and

$$\alpha_2 = \omega \sqrt{\epsilon_2 \mu_2}$$

$$\frac{\omega}{K} = \frac{1}{\sqrt{\epsilon \mu}}$$

$$\frac{K}{\omega} = \sqrt{\epsilon \mu}$$

$$K = \omega \sqrt{\epsilon \mu}$$

Equations for magnetic field vector is,

$$\text{curl} \vec{E} = -\frac{\partial B}{\partial t} \Rightarrow \text{curl} \vec{E} = -\mu \frac{\partial H}{\partial t}$$

$H = H_0 e^{-i\omega t}$

$\frac{\partial H}{\partial t} = -i\omega H$

$curl \vec{E} = i\omega\mu\vec{H}$

$$curl E = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & 0 & 0 \end{vmatrix} = i\omega\mu H$$

$$\hat{j}\frac{\partial E_x}{\partial z} - \frac{\partial E_x}{\partial y}\hat{k} = i\omega\mu\vec{H}$$

2nd term becomes zero So, we are left with

$$curl \vec{E} = \hat{j}\frac{\partial E_x}{\partial z} = i\omega\mu\vec{H}$$

$$\vec{H} = \frac{\hat{j}}{i\omega\mu}\frac{\partial E_x}{\partial z}$$

$$So, \quad \vec{H}_1 = \frac{\hat{j}}{i\omega\mu_1}\frac{\partial}{\partial z}\left[E_{10} e^{i(\alpha_1 z - \omega t)}\right]$$

$$\vec{H}_1 = \frac{\hat{j}}{i\omega\mu_1} i\alpha_1 E_{10} e^{i(\alpha_1 z - \omega t)}$$

$$\vec{H}_1 = \frac{\hat{j}}{i\omega\mu_1} i\omega\sqrt{\mu_1\varepsilon_1}\, E_{10} e^{i(\alpha_1 z - \omega t)}$$

$$\vec{H}_1 = \hat{j}\sqrt{\frac{\varepsilon_1}{\mu_1}}\, E_{10} e^{i(\alpha_1 z - \omega t)}$$

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Similarly,

$$\vec{H_2} = -\hat{j}\sqrt{\frac{\epsilon_1}{\mu_1}}\, E_{20}\, e^{-i(k_1 z + \omega t)}$$

$$\vec{H_3} = \hat{j}\sqrt{\frac{\epsilon_2}{\mu_2}}\, E_{30}\, e^{i(k_2 z - \omega t)}$$

when we are taking normal incidence then the normal component of the wave will vanish and we are left with only tangential component.

$$E_1 + E_2 = E_3$$

boundary condition at $z = 0$

$$E_{10} e^{-i\omega t} + E_{20} e^{-i\omega t} = E_{30} e^{-i\omega t}$$

$$E_{10} + E_{20} = E_{30} \quad \text{--- (1)} \quad \because e^{-i\omega t} \neq 0$$

For magnetic field vector,

$$H_1 + H_2 = H_3$$

$$\sqrt{\frac{\epsilon_1}{\mu_1}}\, E_{10} e^{-i\omega t} - \sqrt{\frac{\epsilon_1}{\mu_1}}\, E_{20} e^{-i\omega t} = \sqrt{\frac{\epsilon_2}{\mu_2}}\, E_{30} e^{-i\omega t}$$

As, $e^{-i\omega t} \neq 0$

So,

$$\sqrt{\frac{\epsilon_1}{\mu_1}}\,(E_{10} - E_{20}) = \sqrt{\frac{\epsilon_2}{\mu_2}}\, E_{30}$$

$$\sqrt{\frac{\epsilon_1}{\mu_1}}\,(E_{10} - E_{20}) = \sqrt{\frac{\epsilon_2}{\mu_2}}\,(E_{10} + E_{20})$$

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$$\left(\sqrt{\frac{\varepsilon_1}{\mu_1}} - \sqrt{\frac{\varepsilon_2}{\mu_2}}\right) E_{10} = \left(\sqrt{\frac{\varepsilon_1}{\mu_1}} + \sqrt{\frac{\varepsilon_2}{\mu_2}}\right) E_{20}$$

$$\frac{\sqrt{\varepsilon_1/\mu_1} - \sqrt{\varepsilon_2/\mu_2}}{\sqrt{\varepsilon_1/\mu_1} + \sqrt{\varepsilon_2/\mu_2}} E_{10} = E_{20} \quad \text{--- (2)}$$

put value of E_{20} in eq. (1)

$$E_{30} = E_{10} + \frac{\sqrt{\varepsilon_1/\mu_1} - \sqrt{\varepsilon_2/\mu_2}}{\sqrt{\varepsilon_1/\mu_1} + \sqrt{\varepsilon_2/\mu_2}} E_{10}$$

$$E_{30} = E_{10}\left[\frac{\sqrt{\varepsilon_1/\mu_1} + \sqrt{\varepsilon_2/\mu_2} + \sqrt{\varepsilon_1/\mu_1} - \sqrt{\varepsilon_2/\mu_2}}{\sqrt{\varepsilon_1/\mu_1} + \sqrt{\varepsilon_2/\mu_2}}\right]$$

$$E_{30} = \frac{2\sqrt{\varepsilon_1/\mu_1}}{\sqrt{\varepsilon_1/\mu_1} + \sqrt{\varepsilon_2/\mu_2}} E_{10} \quad \text{--- (3)}$$

let both medium are optically transparent materials so. If both medium have same [illegible]

$$\mu_1 = \mu_2 = \mu_0$$

Put this values in eq (2)

$$E_{20} = \frac{\sqrt{\varepsilon_1/\mu_0} - \sqrt{\varepsilon_2/\mu_0}}{\sqrt{\varepsilon_1/\mu_0} + \sqrt{\varepsilon_2/\mu_0}} E_{10}$$

$$E_{20} = \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} E_{10} \quad \text{— (4)}$$

Similarly, for eq (3)

$$E_{30} = \frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} E_{10} \quad \text{— (5)}$$

Dividing eq (4) and (5) by ε_0

$$E_{20} = \frac{\sqrt{\varepsilon_1/\varepsilon_0} - \sqrt{\varepsilon_2/\varepsilon_0}}{\sqrt{\varepsilon_1/\varepsilon_0} + \sqrt{\varepsilon_2/\varepsilon_0}} E_{10} \quad \text{— (6)}$$

$$E_{30} = \frac{2\sqrt{\varepsilon_1/\varepsilon_0}}{\sqrt{\varepsilon_1/\varepsilon_0} + \sqrt{\varepsilon_2/\varepsilon_0}} E_{10} \quad \text{— (7)}$$

As,

refractive index $= \eta = \sqrt{\varepsilon/\varepsilon_0}$

So, for eq (6)

$$E_{20} = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} E_{10}$$

This is the reflected wave in terms of incident wave.

for eq (7)

$$E_{20} = \frac{2\eta_1}{\eta_1 + \eta_2} E_{10}$$

This is the refracted wave in terms of incident wave.

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Two phenomena's are going on,

$R \rightarrow$ Reflection of incident wave

$T \rightarrow$ Transmission of incident wave

and

$$R + T + \underbrace{\vec{A}}_{\text{absorption}} = 1$$

here, $A = 0$ so,

$$R + T = 1$$

We have to prove this relation, for this we calculate reflection co-efficient

$$R = \frac{\vec{E}_2 \times \vec{H}_2}{\vec{E}_1 \times \vec{H}_1} \quad \text{— (8)}$$

$$\vec{E}_2 \times \vec{H}_2 = \hat{i} E_{20} e^{-i(k_1 z + \omega t)} \times -\hat{j} \sqrt{\epsilon_1/\mu_1}\, E_{20}\, e^{-i(k_1 z + \omega t)}$$

for the sake of simplicity $\sqrt{\epsilon_1/\mu_1} = 1$

$E_2 \times H_2 = -\hat{k}\, E_{20}^2 \sqrt{\epsilon_1/\mu_1}$

$E_1 \times H_1 = \hat{k}\, E_{10}^2 \sqrt{\epsilon_1/\mu_1}$

$$R = \frac{-\sqrt{\epsilon_1/\mu_1}\, E_{20}^2}{\sqrt{\epsilon_1/\mu_1}\, E_{10}^2} = -\frac{E_{20}^2}{E_{10}^2}$$

$$R_{01} = \left(\frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}\right)^2 \frac{E_{10}^2}{E_{10}^2}$$

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$$R_n = \left(\frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}\right)^2$$

This is the co-efficient of reflection.

· Now transmission coefficient would be,

$$T_n = \frac{E_3 \times H_3}{E_1 \times H_1} \quad \text{--- (9)}$$

$$E_{30} \times H_{30} = K\, E_{30}^2 \sqrt{\epsilon_2/\mu_2}$$

$$E_{10} \times H_{10} = K\, E_{10}^2 \sqrt{\epsilon_1/\mu_1}$$

$$T_n = \frac{K\, E_{30}^2 \sqrt{\epsilon_2/\mu_2}}{K\, E_{10}^2 \sqrt{\epsilon_1/\mu_1}}$$

$$= \frac{\eta_2}{\eta_1} \frac{E_{30}^2}{E_{10}^2}$$

putting value of E_{30}^2

$$= \frac{\eta_2}{\eta_1} \left(\frac{2\eta_1}{\eta_1 + \eta_2}\right)^2 \frac{E_{10}^2}{E_{10}^2}$$

$$T_n = \frac{4\eta_1\eta_2}{(\eta_1 + \eta_2)^2}$$

Put values in $R + T = 1$

$$\left(\frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}\right)^2 + \frac{4\eta_1\eta_2}{(\eta_1 + \eta_2)^2}$$

$$= \frac{\eta_1^2 + \eta_2^2 - 2\eta_1\eta_2 + 4\eta_1\eta_2}{\eta_1^2 + \eta_2^2 + 2\eta_1\eta_2}$$

$$= \frac{\eta_1^2 + \eta_2^2 + 2\eta_1\eta_2}{\eta_1^2 + \eta_2^2 + 2\eta_1\eta_2}$$

$$= 1$$

$$R + T = 1$$

Hence proved.

for air $\eta_2 = 1.5$ ad $\eta_1 = 1$

$R = .04$ ad $T = .96$

water $\eta_2 = 1.33$

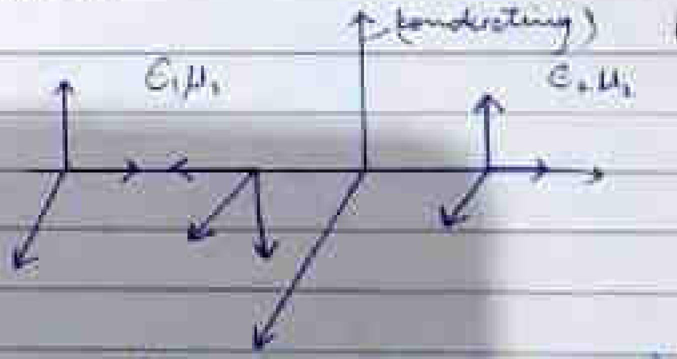
12/11/20

CONDUCTING MEDIUM.

$E_1 = i E_{10} e^{i(k_1 z - \omega t)}$

$E_2 = -i E_{20} e^{-i(k_1 z + \omega t)}$

$E_3 = i E_{30} e^{i(\gamma_2 z - \omega t)}$



using maxwell eq.

$$\text{curl } E_3 = -\frac{\partial B_3}{\partial t} = -\mu \frac{\partial H_3}{\partial t}$$

(eq for transmitted wave)

$$= -\mu_2 (-i\omega) H_3 \qquad H = H e^{i(kz - \omega t)}$$

$$i \mu_2 \omega H_3 = \vec{\nabla} \times \vec{E_3} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & 0 & 0 \end{vmatrix}$$

$$H_3 = \frac{j}{i\omega\mu_2} \frac{\partial E_{3x}}{\partial z} \quad \text{(propagation along z)}$$

$$= \frac{j}{i\omega\mu_2} i \gamma_2 E_{30} e^{i(\gamma_2 z - \omega t)}$$

$$H_3 = \frac{j}{\omega\mu_2} \gamma_2 E_{30} e^{i(\gamma_2 z - \omega t)}$$

we know $\gamma_2 = \alpha_2 + i\beta_2$

$$\alpha_2 = \pm \omega \sqrt{\epsilon_2 \mu_2} \left[\frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{\sigma^2}{\epsilon_2^2 \omega^2}} \right]^{1/2}$$

$$\beta = \omega \sigma_2 \mu_2 / 2 \alpha_2$$

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$$H_1 = j\sqrt{\epsilon_1/\mu_1}\, E_{10}\, e^{i(k_1 z - \omega t)}$$

$$H_2 = -j\sqrt{\epsilon_1/\mu_1}\, E_{20}\, e^{-i(k_1 z + \omega t)}$$

using boundary condition i.e $z = 0$

$$H_1 + H_2 = H_3$$

and $E_1 + E_2 = E_3$

Substituting values

$$iE_{10}\, e^{i(k_1 z - \omega t)} - i E_{20}\, e^{-i(k_1 z + \omega t)} = i E_{30}\, e^{i(\gamma_2 z - \omega t)}$$

$$j\sqrt{\epsilon_1/\mu_1}\, E_{10}\, e^{i(k_1 z - \omega t)} - j\sqrt{\epsilon_1/\mu_1}\, E_{20}\, e^{-i(k_1 z + \omega t)} = j/\omega\mu_2\; \gamma_2\, E_{30}\, e^{i(\gamma_2 z - \omega t)}$$

$$j\left(\sqrt{\epsilon_1/\mu_1}\, E_{10}\, e^{-i\omega t} - \sqrt{\epsilon_1/\mu_1}\, E_{20}\, e^{-i\omega t}\right) = j/\omega\mu_2\; \gamma_2\, E_{30}\, e^{-i\omega t}$$

$$\sqrt{\epsilon_1/\mu_1}\, E_{10} - \sqrt{\epsilon_1/\mu_1}\, E_{20} = \frac{\gamma_2}{\omega\mu_2} E_{30}$$

We express E_{20} in terms of E_{10} to get ~~transmission~~ reflection coefficient

$$\sqrt{\epsilon_1/\mu_1}\, E_{10} - \sqrt{\epsilon_1/\mu_1}\, E_{20} = \frac{\gamma_2}{\omega\mu_2}\left(E_{10} + E_{20}\right)$$

$$\sqrt{\epsilon_1/\mu_1}\, E_{10} - \frac{\gamma_2}{\omega\mu_2} E_{10} = \frac{\gamma_2}{\omega\mu_2} E_{20} + \sqrt{\epsilon_1/\mu_1}\, E_{20}$$

$$\frac{\left(\sqrt{\epsilon_1/\mu_1} - \gamma_2/\omega\mu_2\right) E_{10}}{\left(\sqrt{\epsilon_1/\mu_1} + \gamma_2/\omega\mu_2\right)} = E_{20}$$

dividing by $\sqrt{\epsilon_1/\mu_1}$

$$\frac{\left(1 - \frac{\gamma_2}{\omega\mu_2}\sqrt{\frac{\mu_1}{\epsilon_1}}\right) E_{10}}{1 + \frac{\gamma_2}{\omega\mu_2}\sqrt{\frac{\mu_1}{\epsilon_1}}} = E_{20}$$

For transmission co-efficient

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}\, E_{10} - \sqrt{\varepsilon_1/\mu_1}\,(E_{?} - E_{?}) = \frac{\gamma_2}{\omega\mu_2} E_{30}$$

$$\sqrt{\varepsilon_1/\mu_1}\, E_{20} + \sqrt{\varepsilon_1/\mu_1}\, E_{10} = \frac{\gamma_2}{\omega\mu_2} E_{30} + \sqrt{\frac{\varepsilon_1}{\mu_1}} E_{30}$$

$$\frac{\left(2\sqrt{\varepsilon_1/\mu_1}\right)}{\left(\gamma_2/\omega\mu_2 + \sqrt{\varepsilon_1/\mu_1}\right)} E_{10} = E_{30}$$

dividing by $\sqrt{\varepsilon_1/\mu_1}$

$$\frac{2}{1 + \frac{\gamma_2}{\omega\mu_2}\sqrt{\frac{\mu_1}{\varepsilon_1}}} E_{10} = E_{30}$$

Case 1

If $\sigma_2 = \infty$ (conductivity infinite) so $\gamma_2 = \infty$

So,

$$E_{20} = \frac{1-\infty}{1+\infty} E_{10} = \frac{-\infty}{\infty} E_{10} = -E_{10}$$

$$E_{30} = \frac{2}{\infty} E_{10} = 0$$

for super conductors transmission is zero and reflection occurs.

Case 2

$\sigma_2/\omega\varepsilon_2 \gg 1$

$$\alpha_2 = \pm\omega\sqrt{\varepsilon_2\mu_2}\left[\tfrac{1}{2} \pm \tfrac{1}{2}\,\sigma_2/\omega\varepsilon_2\right]^{1/2}$$

$$= \pm\omega\sqrt{\varepsilon_2\mu_2}\left[\pm \tfrac{1}{2}\,\sigma_2/\omega\varepsilon_2\right]^{1/2}$$

$$= \omega\sqrt{\varepsilon_2\mu_2}\left(\frac{\sigma_2}{2\omega\varepsilon_2}\right)^{1/2}$$

$$= \sqrt{\frac{\sigma_2}{2}\mu_2\omega}$$

$$\beta_2 = \frac{\omega \sigma_2 \mu_2}{2\alpha} = \frac{\omega \sigma_2 \mu_2}{2\sqrt{\frac{\sigma_2 \mu_2 \omega}{2}}} = \frac{\omega \sigma_2 \mu_2}{2 \cdot \frac{1}{\sqrt{2}} \sqrt{\sigma_2 \mu_2 \omega}}$$

$$\beta_2 = \sqrt{\frac{\omega \sigma_2 \mu_2}{2}} = \alpha_2$$

So $\gamma_2 = \alpha_2 + i\beta_2 = (1+i)\sqrt{\dfrac{\omega \sigma_2 \mu_2}{2}}$

$$E_{20} = \frac{\left(1 - \frac{\gamma_2}{\omega\mu_2}\sqrt{\mu_1/\epsilon_1}\right)}{1 + \frac{\gamma_2}{\omega\mu_2}\sqrt{\frac{\mu_1}{\epsilon_1}}} E_{10}$$

$$= \frac{1 - (1+i)\sqrt{\omega\sigma_2\mu_2/2}\,\frac{1}{\omega\mu_2}\sqrt{\mu_1/\epsilon_1}}{1 + (1+i)\sqrt{\frac{\omega\sigma_2\mu_2}{2}}\,\frac{1}{\omega\mu_2}\sqrt{\frac{\mu_1}{\epsilon_1}}} E_{10}$$

$$= \frac{\left(1 - (1+i)\sqrt{\frac{\mu_1\sigma_2}{2\mu_2\omega\epsilon_1}}\cdot\frac{\sqrt{2}}{\sqrt{2}}\right) E_{10}}{1 + (1+i)\sqrt{\frac{\mu_1\sigma_2}{2\mu_2\omega\epsilon_1}}\,\frac{\sqrt{2}}{\sqrt{2}}}$$

$$= \left[1 - \frac{(1+i)}{2}\sqrt{\frac{2\mu_1\sigma_2}{\mu_2\omega\epsilon_1}}\right] E_{10} \Big/ 1 + \frac{(1+i)}{2}\sqrt{\frac{2\mu_1\sigma_2}{\mu_2\omega\epsilon_1}}$$

$$= \left[\left(1 - \frac{(1+i)}{2}\sqrt{\frac{2\mu_1\sigma_2}{\mu_2\omega\epsilon_1}}\right)^2\right]^{1/2} \times \left[\left(1 + \frac{(1+i)}{2}\sqrt{\frac{2\mu_1\sigma_2}{\mu_2\omega\epsilon_1}}\right)^{2}\right]^{-1/2}$$

using binomial on square $(1-x)^n = 1 - nx$

$$= \left[1 - 2\frac{(1+i)}{2}\sqrt{\frac{2\mu_1\sigma_2}{\mu_2\omega\epsilon_1}}\, E_{10}\right]^{1/2} \left[1 - 2\frac{(1+i)}{2}\sqrt{\frac{2\mu_1\sigma_2}{\mu_2\omega\epsilon_1}}\right]^{-1/2}$$

$$= \left[1 - (1+i)\sqrt{\quad}\right]^{1/2} \left[1 - (1+i)\sqrt{\quad}\right]^{1/2}$$

Again using binomial

$$\left[1 - \frac{1+i}{2}\sqrt{}\ \right]\left[1 - \frac{1+i}{2}\sqrt{}\ \right]$$

$$= \left[1 - \frac{1+i}{2}\sqrt{\frac{2\mu_1\sigma_2}{\mu_2\epsilon_2\omega}}\right]^2$$

using binomial

$$E_{20} = \left[1 - (1+i)\sqrt{2\mu_1\sigma_2/\mu_2\epsilon_2\omega}\right]E_{10}$$

$$E_{20}^* = \left(1 - (1-i)\sqrt{\frac{2\mu_1\sigma_2}{\mu_2\epsilon_2\omega}}\right)E_{10}^*$$

Reflection co-efficient

$$R = \frac{E_{20}E_{20}^*}{|E_{10}|^2} = \frac{|E_{20}|^2}{|E_{10}|^2}$$

putting values

$$= \left[1 - (1+i)\sqrt{\frac{2\mu_1\sigma_2}{\mu_2\epsilon_2\omega}}\right]E_{10}\left[1 - (1-i)\sqrt{\frac{2\mu_1\sigma_2}{\mu_2\epsilon_2\omega}}\right]E_{10}^*$$

$$= \frac{\left(1 - 2\sqrt{\frac{\mu_2}{\mu_1}\omega\epsilon_2}\right)|E_{10}|^2}{|E_{10}|^2} \qquad \because E_{10}E_{10}^* = |E_{10}|^2$$

$$= \left(1 - 2\sqrt{\frac{\mu_2}{\mu_1}\omega\epsilon_2}\right)$$

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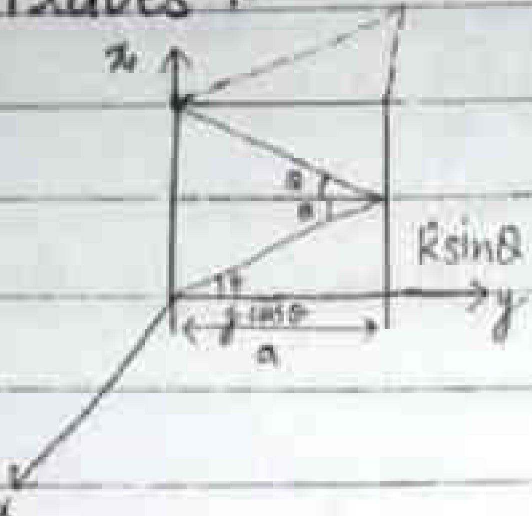
Propagation between Parallel Conducting Plates :-



Equation of electric field is

$$\vec{E} = \vec{E}_0 e^{i(k \cdot r - \omega t)} \quad \text{—— ①}$$

Now we define $\vec{r}$ and $\vec{k}$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{k} = \hat{j}\cos\theta + \hat{k}\sin\theta$$

$$\vec{k} \cdot \vec{r} = k(y\cos\theta + z\sin\theta).$$

put in eq. ①

$$\vec{E}_1 = \hat{i}E_1 e^{i[k(y\cos\theta + z\sin\theta) - \omega t]}$$

and,

$$\vec{E}_2 = \hat{i}E_2 e^{i[k(-y\cos\theta + z\sin\theta) - \omega t]}$$

Total electric field will be

$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

$$\vec{E} = \hat{i}\left[E_1 e^{i[(ky\cos\theta + kz\sin\theta) - \omega t]} + E_2 e^{i[(-ky\cos\theta + kz\sin\theta) - \omega t]}\right]$$

So,

$$\vec{E} = \hat{i}E\left[e^{i[(ky\cos\theta + kz\sin\theta) - \omega t]} - e^{i[(-ky\cos\theta + kz\sin\theta) - \omega t]}\right] \quad \because E_1 = -E_2 = E$$

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$$\vec{E} = \hat{i}E\left(e^{iky\cos\theta} - e^{-iky\cos\theta}\right)e^{i(kz\sin\theta-\omega t)}$$

$$\vec{E} = \hat{i}E\left(2i\sin(ky\cos\theta)\right)e^{i(kz\sin\theta-\omega t)}$$

To vanish electric field we may have

$$\sin ky\cos\theta = 0$$

when $y = a$

$$\sin ka\cos\theta = 0$$

$$ka\cos\theta = n\pi$$

As we know,

$$k = \frac{\omega}{c}$$ $\omega/k = c$

or $k = \frac{1}{\sqrt{\epsilon\mu}}$

$$c = \frac{\omega}{k}$$

$$c = \omega\sqrt{\epsilon\mu}$$

Along z-axis

$$c = V_p = \frac{\omega}{k} = \frac{\omega}{k\sin\theta}$$

$$\omega = V_p k\sin\theta \quad \text{--- 2}$$

Along y-axis

$$c = V_p = \frac{\omega}{k} = \frac{\omega}{k\cos\theta}$$

$$\omega = V_p\cos\theta k \quad \text{--- 3}$$

From 2 and 3 we have

$$V_p = \frac{\omega}{k\sin\theta} = \frac{\omega}{\frac{\omega}{c}\sin\theta} = \frac{c}{\sin\theta}$$

Now we define wavelength,

$$k = 2\pi\lambda$$

$$\lambda = \frac{k}{2} \quad \lambda_0 = \frac{2\pi}{k}$$

putting value of k,

$$\lambda = \frac{2\pi c}{\omega}$$

Along z axis

$$\lambda_g = \frac{2\pi}{k\sin\theta} = \frac{\lambda_0}{\sin\theta} \Rightarrow k\sin\theta = \frac{2\pi}{\lambda_g}$$

Along y-axis,

$$\lambda_c = \frac{2\pi}{k\cos\theta} = \frac{\lambda_0}{\cos\theta} \Rightarrow k\cos\theta = \frac{2\pi}{\lambda_c}$$

we write electric field in terms of wavelength,

$$\vec{E} = \hat{i}E(\sin ky\cos\theta)e^{i(kz\sin\theta - \omega t)}$$

$$\vec{E} = \hat{i}E\sin\left(\frac{2\pi y}{\lambda_c}\right)e^{i\left(\frac{2\pi z}{\lambda_g} - \omega t\right)}$$

$\because ka\cos\theta = n\pi$

Putting value of $k\cos\theta$

$$n\pi = \frac{2\pi}{\lambda_c}a$$

So, π cancelling we get.

$$\frac{n}{2} = \frac{a}{\lambda_c}$$

$$2a = n\lambda_c$$

$$\lambda_c = \frac{2a}{n}$$

when $n = 2$ then, λ_c has largest wavelength

$$\lambda_c = \frac{2a}{2}$$

$$\lambda_c = a$$

here 'a' is the distance b/w two conducting plates and λ_c is the cut off wavelength.

Phase velocity $= V_p = \dfrac{c}{\sin\theta}$

This is the velocity of constant phase.

As we know

$$\lambda_c = \frac{\lambda_0}{\cos\theta} \quad \text{or} \quad \frac{1}{\lambda_c} = \frac{\cos\theta}{\lambda_0} \qquad \text{—— (4)}$$

$$\lambda_g = \frac{\lambda_0}{\sin\theta} \quad \text{or} \quad \frac{1}{\lambda_g} = \frac{\sin\theta}{\lambda_0} \qquad \text{—— (5)}$$

squaring & adding (4) & (5)

$$\frac{1}{\lambda_c^2} + \frac{1}{\lambda_g^2} = \frac{\cos^2\theta}{\lambda_0^2} + \frac{\sin^2\theta}{\lambda_0^2}$$

$$\frac{1}{\lambda_c^2} + \frac{1}{\lambda_g^2} = \frac{1}{\lambda_0^2}$$

$\downarrow$ max wavelength $\downarrow$ min wavelength.

if $n=1$

then, $\lambda_c = 2a \Rightarrow \lambda_c > 2a$

If the value of λ_0 increase then ω would decrease. When there comes a point where λ_g becomes negative, that limit is called cut off wavelength.

$$\vec{E} = E' \sin\left(\frac{2\pi y}{\lambda_c}\right) e^{i\left(\frac{2\pi}{\lambda_g}x - \omega t\right)} \quad \text{--- (7)}$$

diff w.r.t x.

$$\frac{\partial E}{\partial x} = \frac{i2\pi}{\lambda_g} E' \sin\left(\frac{2\pi y}{\lambda_c}\right) e^{i\left(\frac{2\pi}{\lambda_g}x - \omega t\right)}$$

diff w.r.t y.

$$\frac{\partial E}{\partial y} = \frac{2\pi}{\lambda_c} E' \cos\left(\frac{2\pi y}{\lambda_c}\right) e^{i\left(\frac{2\pi}{\lambda_g}x - \omega t\right)}$$

As we know

$$\text{curl}\,\vec{E} = -\frac{\partial B}{\partial t} = i\omega\vec{B}$$

$$\vec{B} = B(r)e^{-i\omega t}$$

$$\frac{\partial B}{\partial t} = -i\omega\vec{B}$$

So,

$$\text{curl}\,\vec{E} = i\omega\vec{B}$$

Taking determinant,

$$\left[\frac{2\pi}{\lambda_g\omega} E'\sin\left(\frac{2\pi y}{\lambda_c}\right) e^{i\left[\frac{2\pi}{\lambda_g}z-\omega t\right]} \hat{j} + i\,\frac{2\pi}{\lambda_c\omega} E'\cos\left(2\pi y/\lambda_c\right) e^{i\left(\frac{2\pi}{\lambda_g}z-\omega t\right)}\right]$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E & 0 & 0 \end{vmatrix} = i\omega B$$

$$\frac{\partial E}{\partial z}\hat{j} - \frac{\partial E}{\partial y}\hat{k} = i\omega\vec{B} \quad \text{— ⑥}$$

velocity of wave guide is the group velocity.

putting values in eq ⑥

$$i\omega\vec{B} = i\frac{2\pi}{\lambda_g}E'\sin\frac{2\pi y}{\lambda_c} e^{i\left(\frac{2\pi}{\lambda_g}z-\omega t\right)}\hat{j} - \frac{2\pi}{\lambda_c}E'\cos\left(\frac{2\pi}{\lambda_c}y\right)e^{i\left(\frac{2\pi}{\lambda_g}z-\omega t\right)}\hat{k}$$

mag and sing by $i\omega$

$$\vec{B} = \frac{2\pi}{\lambda_g\omega}E'\sin\left(\frac{2\pi}{\lambda_c}y\right)e^{i\left(\frac{2\pi}{\lambda_g}z-\omega t\right)}\hat{j} + \frac{i\,2\pi}{\lambda_c\omega}E'\cos\left(\frac{2\pi}{\lambda_c}y\right)e^{i\left(\frac{2\pi}{\lambda_g}z-\omega t\right)}\hat{k} \quad ⑦$$

Poynting vector $= S = \vec{E}\times\vec{H}$

Energy Density $= W = \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})$

Ignoring imaginary part of $\vec{B}$

$$W = \frac{1}{4}\,\mathrm{Re}\,(\vec{E}^*\cdot\vec{D} + \vec{B}^*\cdot\vec{H})$$

$$W = \frac{1}{4}\,\mathrm{Re}\,\left(E^*\cdot\epsilon_0 E + B^*\cdot\frac{\vec{B}}{\mu_0}\right)$$

$$W = \frac{1}{4}\mathrm{Re}\left[\epsilon_0 E^{*'}\left(\sin\frac{2\pi}{\lambda_c}y\right) e^{-i\left(\frac{2\pi}{\lambda_g}z-\omega t\right)}\right]\cdot\left[E'\sin\left(\frac{2\pi y}{\lambda_c}\right)e^{i\left(\frac{2\pi}{\lambda_g}z-\omega t\right)}\right]$$

$$+ \frac{1}{4\mu_0}\left[\frac{2\pi}{\lambda_g\omega}E^{*'}\sin\left(\frac{2\pi}{\lambda_c}y\right)e^{-i\left(\frac{2\pi}{\lambda_g}z-\omega t\right)}\hat{j} - iE^{*'}\frac{2\pi}{\lambda_c\omega}\cos\left(\frac{2\pi y}{\lambda_c}\right)e^{-i\left(\frac{2\pi}{\lambda_g}z-\omega t\right)}\hat{k}\right].$$

$$W = \frac{1}{4}\mathrm{Re}\left[\varepsilon_0 E E' \sin^2\left(\frac{2\pi y}{\lambda_c}\right)\right] + \frac{1}{\mu_0}\left[E E^{*\prime}\left(\frac{2\pi}{\lambda_g \omega}\right)^2 \sin^2\left(\frac{2\pi y}{\lambda_c}\right)\right]$$

$$+ \frac{1}{\mu_0}\left[E E^{*\prime}(2\pi/\lambda_c \omega)^2 \cos^2(2\pi y/\lambda_c)\right]$$

$\because j \times j = 1$

$\because k \times k = 1$

$$W = \frac{1}{4} E E^{*\prime}\left[\varepsilon_0 (a/2) + 1/\mu_0 (2\pi/\lambda_g \omega)^2 (a/2) + 1/\mu_0 (2\pi/\lambda_c \omega)^2 (a/2)\right]$$

$$W = \frac{1}{4} E^* E (a/2)\left[\varepsilon_0 + \frac{1}{\mu_0}\left(\frac{2\pi}{\lambda_g \omega}\right)^2 + \frac{1}{\mu_0}\left(\frac{2\pi}{\lambda_c \omega}\right)^2\right]$$

Integrating

$$\int_0^a W\,dy = \frac{1}{4} E E^{*\prime} \frac{a}{2}\left[\varepsilon_0 + \frac{1}{\mu_0}\left(\frac{2\pi}{\lambda_g \omega}\right)^2 + \frac{1}{\mu_0}\left(\frac{2\pi}{\lambda_c \omega}\right)^2\right]$$

$$= \frac{1}{4} E E^{*\prime} \frac{a}{2}\left[\varepsilon_0 + \frac{1}{\mu_0}\left(\frac{2\pi}{\omega}\right)^2\left\{\frac{1}{\lambda_g^2} + \frac{1}{\lambda_c^2}\right\}\right]$$

$$\int_0^a W\,dy = \frac{1}{4} E E^{*\prime} \frac{a}{2}\left[\varepsilon_0 + \frac{1}{\mu_0}\left(\frac{2\pi}{\omega}\right)^2 \frac{1}{\lambda_0^2}\right]$$

$$\lambda_0 = \frac{2\pi}{k} = \frac{2\pi c}{\omega} \Rightarrow \frac{1}{\lambda_0} = \frac{\omega}{2\pi c} \Rightarrow \frac{1}{\lambda_0^2} = \frac{\omega^2}{(2\pi c)^2}$$

$$= \frac{1}{4} E E^{*\prime} \frac{a}{2}\left[\varepsilon_0 + \frac{1}{\mu_0}\left(\frac{2\pi}{\omega}\right)^2 \times \frac{\omega^2}{(2\pi c)^2}\right]$$

$$= \frac{1}{4} E E^{*\prime} \frac{a}{2}\left\{\varepsilon_0 + \frac{1}{\mu_0 c^2}\right\}$$

$\because c = 1/\sqrt{\varepsilon_0 \mu_0}$

$c^2 = \frac{1}{\varepsilon_0 \mu_0}$

$\frac{1}{c^2} = \varepsilon_0 \mu_0$

$$= \frac{1}{4} E E^{*\prime} \frac{a}{2}\left[\varepsilon_0 + \frac{1}{\mu_0}\mu_0 \varepsilon_0\right]$$

$$= \frac{1}{4} E E^{*\prime} \frac{a}{2}(2\varepsilon_0)$$

$$\int W\,dy = \frac{1}{4} E E^{*\prime} a\, \varepsilon_0$$

This is the value of energy density.

Poynting vector is

$$\vec{S} = \vec{E} \times \vec{H}$$

$$\vec{S_z} = E_x \times H_y$$

If we take real term we will put $\frac{1}{2}$ as,

$$S_z = \frac{1}{2} \mathrm{Re}\, E_x^* \times H_y$$

from eq (7) and (8) put in above eq.

$\because B = \mu_0 H$

$\Rightarrow \frac{B}{\mu_0} = H$

$$S_z = \frac{1}{2} \mathrm{Re}\left[E'^* \sin\left(\frac{2\pi y}{\lambda_c}\right) \cdot \frac{1}{\mu_0} E' \frac{2\pi}{\lambda_g \omega} \sin\frac{2\pi y}{\lambda_c}\right]$$

$$S_z = \frac{1}{2} E'^* E' \frac{2\pi}{\lambda_g \omega \mu_0} \sin^2 \frac{2\pi y}{\lambda_c}$$

$$\int_0^a S_z\, dy = \frac{1}{2} E'^* E' \frac{2\pi}{\lambda_g \omega \mu_0} \int_0^a \sin^2 \frac{2\pi y}{\lambda_c}\, dy$$

$$= \frac{1}{2} E'^* E' \frac{2\pi}{\lambda_g \omega \mu_0} \frac{a}{2}$$

$$= \frac{1}{2} E'^* E' \frac{\pi a}{\lambda_g \omega \mu_0}$$

Taking ratio of poynting vector and energy density

$$V_g = \frac{\int_0^a S_z\, dy}{\int_0^a W\, dy} = \frac{\frac{1}{2} E'^* E' \pi a / \lambda_g \omega \mu_0}{\frac{1}{4} E'^* E' \varepsilon_0 a}$$

$$= \frac{2\pi}{\lambda_g \omega \mu_0 \varepsilon_0} = \frac{2\pi}{\lambda_g \omega} c^2$$

$$V_g = \frac{2\pi c^2}{\lambda_g \omega} = \frac{2\pi}{\lambda_g}\frac{c}{\omega}\cdot c$$

$$= \frac{2\pi}{\lambda_g k} c$$

$$V_g = \frac{\lambda_0}{\lambda_g} c$$

So, phase velocity relation would be,

$$V_p = \frac{c}{\sin\theta} \qquad \text{as, } V_g = \frac{\lambda_0 c}{\lambda_g}$$

$$V_g = \frac{\lambda_0 c}{\lambda_0 / \sin\theta} \qquad \because \lambda_g = \frac{\lambda_0}{\sin\theta}$$

$$V_g = c\sin\theta$$

multiplying we get,

$$V_g V_p = c\sin\theta \times \frac{c}{\sin\theta}$$

$$V_g V_p = c^2 .$$