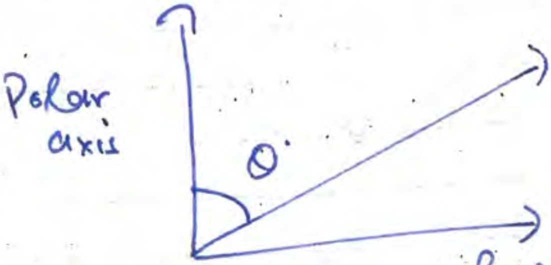
CH(2) CENTRAL FORSSES

Central forcses.

Polar Co-ordinatrs:- (1) The Polar Co-ordinate (r,θ) are special case of spherical co-ordinats in which ϕ is kept constant.



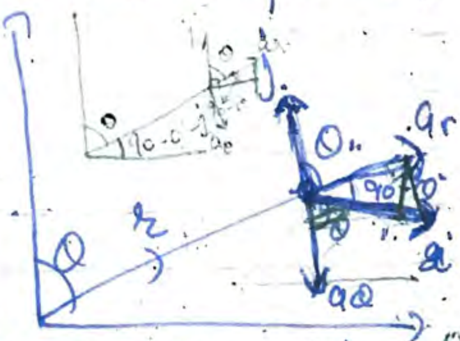
(2) The axis with respect to which θ is measure is the Polar axis.

(3) The Polar co-ordinates are use to consider the motion in a Plane.

Some Expression in terms of Polar Co-ordinats:-

(1) Consider the following digram $\hat{a_r}$ and $\hat{a_\theta}$ are unit vectors, Taking along the increasing direction of r and θ.

(2) $\hat{i}$ and $\hat{j}$ are the usual unit vectors Taking along x & y axis respectivly.



$$\hat{a_r} = |\hat{a_r}| \cos(90-\theta)\hat{i} + |\hat{a_r}| \sin(90-\theta)\hat{j}$$

$|\hat{a_r}| = 1$

$$= \sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\hat{a_r} = \hat{i}\sin\theta + \hat{j}\cos\theta \quad \text{--- (1)}$$

Now

$$\hat{a_\theta} = |\hat{a_\theta}|\cos\theta\, i \cancel{+} |\hat{a_\theta}|\sin\theta\, j$$

$$= \cos\theta\, i - \sin\theta\, j \quad — (2)$$

Now

$$\frac{d}{dt}(\hat{a_r}) = +i\cos\theta\, \theta' - j\sin\theta\, \theta'$$

$$= (i\cos\theta - j\sin\theta)\, \theta'$$

$$\boxed{\frac{d}{dt}\hat{a_r} = \hat{a_\theta}\theta'} \quad — (3)$$

and

$$\frac{d}{dt}(a_\theta) = \frac{d}{dt}(\cos\theta\, i - j\sin\theta)$$

$$= (-\sin\theta\, i \cancel{+} j \overset{\cos}{\cancel{\sin}}\theta)\theta'$$

$$= -(\cos\theta\, i + j\sin\theta)\theta'$$

$$\boxed{\frac{d}{dt}(a_\theta = -(\hat{a_r}\theta')} \quad — (4)$$

Velocity in Polar Co-ordinates:-

we can express

$$\vec{r} = r\hat{a_r}$$

Now

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{a_r} + r\frac{d\hat{a_r}}{dt}$$

Putting value of $\frac{d\hat{a_r}}{dt} \Rightarrow$:

$$\frac{d\vec{r}}{dt} = \dot{r}\hat{a_r} + r(\theta' a_\theta) \quad — (7)$$

This is Expression for velocity in (A)
$\dot{r}$ is Called Radial Component and $r\dot{\theta}$ is the Transverse Component of velocity.

Accleration in terms of Polar Co-ordinates.

As we Know That

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= \frac{d}{dt}\left(\dot{r}\,\hat{a}_r + r\dot{\theta}\,\hat{a}_\theta\right)$$

$$= \ddot{r}\,\hat{a}_r + \dot{r}\frac{d\hat{a}_r}{dt} + \dot{r}\dot{\theta}\,\hat{a}_\theta + r\ddot{\theta}\,\hat{a}_\theta + r\dot{\theta}\frac{d}{dt}(\hat{a}_\theta)$$

$$= \ddot{r}\,\hat{a}_r + \dot{r}(\dot{\theta}\,\hat{a}_\theta) + \dot{r}\dot{\theta}\,\hat{a}_\theta + r\ddot{\theta}\,\hat{a}_\theta + r\dot{\theta}(-\dot{\theta}\,\hat{a}_r)$$

$$= \ddot{r}\,\hat{a}_r + 2\dot{r}\dot{\theta}\,\hat{a}_\theta + r\ddot{\theta}\,\hat{a}_\theta - r\dot{\theta}^2\,\hat{a}_r$$

$$= (\ddot{r} - r\dot{\theta}^2)\,\hat{a}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{a}_\theta$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{a}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{a}_\theta \quad \text{— (B)}$$

from equation (B)

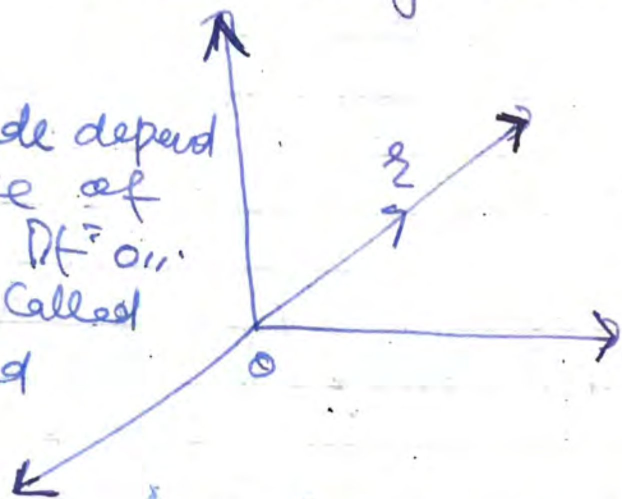
Radial Component of $\vec{a} = \ddot{r} - r\dot{\theta}^2$

Transverse " " " $= 2\dot{r}\dot{\theta} + r\ddot{\theta}$

Central forces:

Suppose that a force acting on a Particle of mass "m", Such That

(a) It is always directed towards a fix Pt. O or directed away from fixed Pt.



(b) Its magnitude depend upon the distance of the Particle from Pt "O". This force is called central force and the Pt "O" is called Centre of force. Symbolically This force is expressed as

$$\vec{F} = f(r)\, \hat{a}_r$$
$$= f(r)\, \frac{\vec{r}}{r}$$

as $\hat{a}_r = \frac{\vec{r}}{|\vec{r}|}$ (unit vector)

$\hat{a}_r$ is unit vector along the direction of r. This force is attractive if $f(r) < 0$ and it will be repulsive if $f(r) > 0$

∴ **Properties of Central force:-**

(1) A motion under the effect of central force take Placed in a Plane:-

Let the Central force be given By

$$F = f(r)\, \hat{a}_r$$

If r is the Position vector of Particle moving under central force at any time t. Also r and $\vec{F}$ are acting along the same direction. Then

$$r \times F = 0$$

or

or $\vec{r} \times ma = 0$

$\vec{r} \times m \frac{dv}{dt} = 0$

$m(r \times \frac{dv}{dt}) = 0$

$m \frac{d}{dt}(\vec{r} \times \bar{v}) = 0$

$\frac{d}{dt}(\vec{r} \times \bar{v})$

$\frac{dr}{dt} \times v + r \times \frac{dv}{dt}$

$= v \times v + r \times \frac{dv}{dt}$

$\Rightarrow v \times v = 0$

$r \times \frac{dv}{dt}$

$\Rightarrow \bar{r} \times \bar{v} = \text{Constant}$

or $\vec{r} \times v = h$

As $\vec{r}$ is always at right angle to a Constant vector, h, so it must lie in a Plane.

(ii) Angular moment will Be Constant:-

we know that $\bar{r} \times \bar{v} = \text{Constant} = h$

multply Both side By m, we get

$m(\vec{r} \times \vec{v}) = m\vec{h}$

$\vec{r} \times m\vec{v} = m\vec{h}$

$\bar{r} \times \bar{P} = m\bar{h}$

But $r \times P = L$

$\Rightarrow L = \text{Constant}$

Magnitude of L, in terms of Polar Coordinats:-

$\vec{L} = \vec{r} \times \vec{P}$

$= \vec{r} \times m\bar{v}$

$= m\vec{r} \times (r'\vec{a_r} + r\theta'\hat{a_\theta})$

$= m r \vec{a_r} \times (r'\hat{a_r} + r\theta'\hat{a_\theta})$

$= m r^2 \theta' \vec{a_r} \times \vec{a_\theta}$

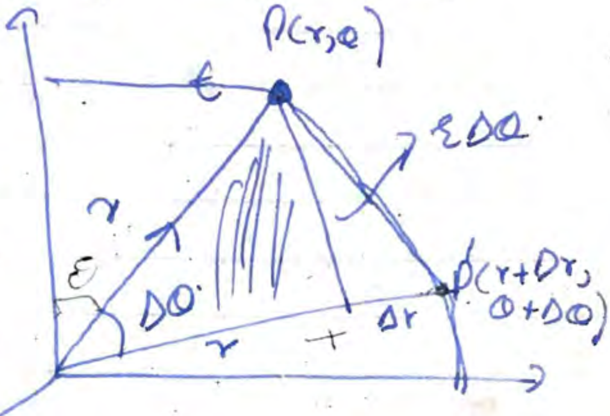
$= m r^2 \theta' \vec{k}$

$\vec{a_r} \times \vec{a_\theta} =$ another unit vector

$$|\vec{L}| = mr^2\dot{\theta} = \text{Constant}$$
$$= mr^2\theta = \text{constant}$$

(iii) Areal velocity is Constant:-

Consider a Position of Particle in the fig at time t, and $(t+\Delta t)$.



Let the Polar Co-ordinates at these Pts be $P(r, \theta)$ and $P'(r+\Delta r, \theta+\Delta\theta)$, the area swept during this time interval ~~By the Position vector~~ By the Position vector represented by shaded Portion of fig.

Areal velocity is defind as follows,

$$\text{Areal velocity} = \underset{\Delta t \to 0}{\mathrm{lt}} \frac{\Delta A}{\Delta t}$$

ΔA is Area swept in time Δt for small value of $\Delta\theta$. The area ΔA is approximatly equal to the area of Triangle with Base $r+\Delta r$ and altitude $\varepsilon\Delta\theta$ as shown in fig.

$$\text{Area} = \frac{1}{2}(\text{Base})(\text{Altitude})$$

$$\Delta A = \frac{1}{2}(r+\Delta r)\varepsilon\Delta\theta$$

$$= \frac{1}{2}r^2\Delta\theta + \frac{1}{2}r\Delta r\Delta\theta$$

As $\lim_{\Delta t \to 0}$ $\Rightarrow$ $\Delta r \Delta \theta \lll 0$, This can Be neglected so that the rate at which area is swept will Be;

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{1}{2} r^2 \frac{\Delta \theta}{\Delta t}$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} \Rightarrow$$ This is the areal velocity, we know that angular momentum

$$L = \frac{1}{2} m r^2 \dot{\theta}$$

$$r^2 \dot{\theta} = \frac{2L}{2m} = \text{Constant}$$

This areal velocity is Constant which is One of the Property.

(iv) The Central force is Conservative:

we know That the Central force is express as

$$\vec{F} = f(r)\, a\vec{r}$$

$$= f(r) \frac{\vec{r}}{r} \qquad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$r = (x^2 + y^2 + z^2)^{1/2}$

$$= \frac{F(r)}{r}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

So

$$F_x = \frac{\partial}{\partial y}\left(\frac{f(r)}{r} z\right) - \frac{\partial}{\partial z}\left(\frac{F(r)}{r} y\right)$$

$$\frac{\partial}{\partial y}\left(\frac{f(r)}{r}\right) z = \frac{\partial}{\partial r}\frac{f(r)}{r}\left(\frac{\partial r}{\partial y}\right) z - \frac{\partial}{\partial r}\left(\frac{f(r)}{r}\frac{\partial r}{\partial z}\right) y$$

$$= \frac{\partial}{\partial r} \frac{f(r)}{r} \left(z \frac{\partial r}{\partial y} - y \frac{\partial r}{\partial z} \right)$$

$$= \frac{\partial}{\partial r} \frac{f(r)}{r} \left(z \frac{\partial r}{\partial y} - y \frac{\partial r}{\partial z} \right)$$

$$= \frac{\partial}{\partial r} \frac{f(r)}{r} \left(z \frac{\partial}{\partial y} (x^2+y^2+z^2)^{1/2} - y \frac{\partial}{\partial z} (x^2+y^2+z^2)^{1/2} \right)$$

$$= \frac{\partial}{\partial r} \frac{f(r)}{r} \left[z \cdot \frac{1}{\cancel{2}} \frac{\cancel{2}y}{\sqrt{x^2+y^2+z^2}} - y \cdot \frac{1}{\cancel{2}} \frac{\cancel{2}z}{\sqrt{x^2+y^2+z^2}} \right]$$

$$= \frac{\partial}{\partial r} \frac{f(r)}{r} \left[\frac{zy}{\sqrt{x^2+y^2+z^2}} - \frac{yz}{\sqrt{x^2+y^2+z^2}} \right)$$

$$= \frac{\partial}{\partial r} \left(\frac{f(r)}{r} (0) \right)$$

$$= 0$$

Similarly we can prove

$f_y = 0, \; f_z = 0$

Thus Curl $F = 0$, Hence The Central force will Be Conservative.

Equations of motion for Particle in a Central field of force:-

we know That acceleration $\vec{a}$ in Polar Coordinates is

$$\vec{a} = (r'' - r\theta'^2)\vec{a_r} + (2r'\theta' + r\theta'')\vec{a_\theta}$$

The motion of a Particle in a central field of force Taken Place in a Plane - Choosing this Plane as xy - Plane and Co-ordinates, The equations of motion, are obtained as Below.

from 2nd law of motion we know That

$$\vec{F} = m\vec{a} \quad -(1)$$

if this force is a Central force Then

$$\vec{F} = f(r)\, \vec{a_r} \quad -(2)$$

from (1) & (2) we get,

$$f(r)\, \vec{a_r} = m\vec{a}$$

Substitute the value of $\vec{a}$ in terms of Polar Co-ordinates

$$f(r)\, \vec{a_r} = m(\ddot{r} - r\dot{\theta}^2)\vec{a_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{a_\theta}$$

Equating Radial & Transverse Components.

$$m(r'' - r\dot{\theta}^2) = f(r) \quad -(3)$$

and

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \quad -(4)$$

from (3) & (4) equations, are the Basic equations of motion for Particle under the center field of force.

It can Be seen That the Central force has no transverse Components Considering eq (3) which show That

$$f(r) = m(r'' - r\dot{\theta}^2)$$

Then angular momentum is Constant, as we know That

$$L = mr^2\dot{\theta}$$

(4) multiplying Both sides By r to we get

$$m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}) = 0.$$

$$m\frac{d}{dt}(r^2\dot{\theta}) = 0 \qquad m\frac{d}{dt}(r^2\dot{\theta}) = 0$$

$$mr^2\dot{\theta} = \text{Constant.} \qquad m \neq 0$$

$$L = mr^2\dot{\theta} = \text{Constant} \qquad \frac{d}{dt}(r^2\dot{\theta}) = r^2\dot{\theta}$$

which is Proof

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Question (Important). (P.U) 2016 Paper

Using The Basic equation of motion for the Central force drive the following equation:

(i) $$\frac{d^2r}{d\theta^2} - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2 - r = \frac{mr^4}{l^2} f(r) \quad \text{—— (A)}$$

This is an equation in terms of derivatives of r with respect to θ. It can Be drived as follows. we know The Basic equation of motion in Central field of force.

$$m(\ddot{r} - r\dot{\theta}^2) = f(r) \quad \text{—— (1)}$$

we also know That angular momentum

$$L = mr^2\dot{\theta}$$

$$\dot{\theta} = \frac{L}{mr^2} \quad \text{—— (2)}$$

Substitute the value of $\dot{\theta}$ in eq (1) we get