CHAPTER 3

Assig Ray 110

Scattring angle in Central field of force.

① ||al Beam of Particles:-

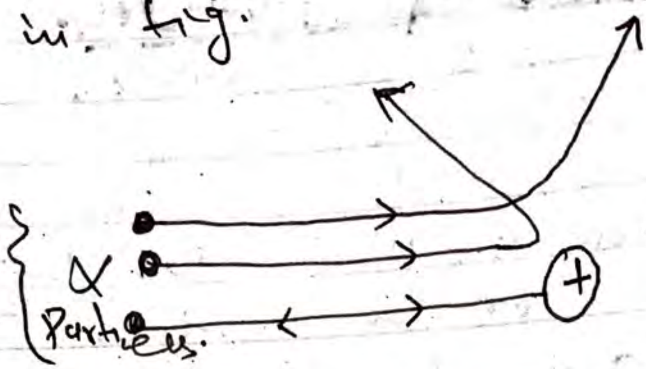
If a group of Particles is moving in such a way That the velocity of all The Particles is ||al to each other Then it is known as ||al Beam of Particles. Similarly we can define a Convergient Beam of Particles or divergent Beam of Particles.

② Intensity of the Beam (I). = flux density:-

Intensity of Beam is defined as the no. of Particles Crossing unit area Placed ⊥ar to the Beam in Unit time. Intensity is uniform if the no. of Particles Crossing unit area Per Unit time at diffrent Crass Sections remain the Same.

③ Angle of Scattring:-

Consider a uniform Beam of α-Particles moving towards +vly charge heavy necleus. This nuclus acts as the Center of force when Particles are faraway ~~fram~~ from the Center of force. There will Be small force of reeplusion on them and their trajactorieres will Be ~~remain Same~~. Starigut lines. as The Particles approach the center of force, They will Be repelled and their Path will diverts from their incident line as shown in fig.

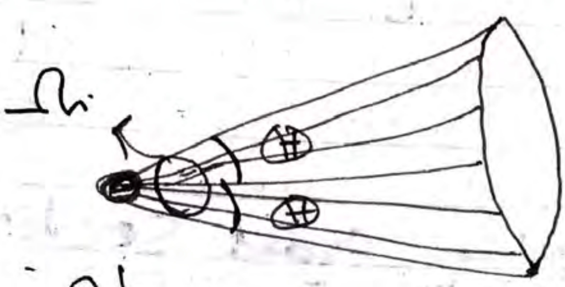


After Passing the center of force, The force acting on Particle will eventually vanishes. So the Particle will again move in a straight line. Genrally the final direction of motion is not the same as the incident direction and the Particle is said to Be scattred.

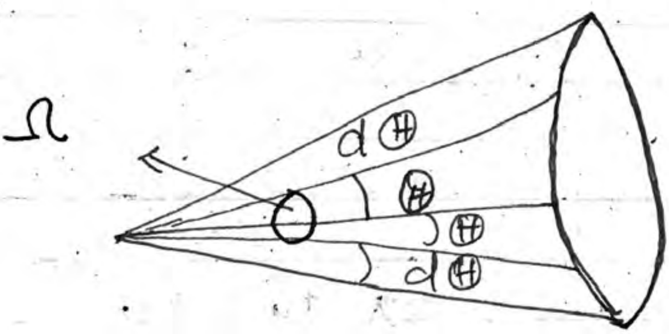
The angle Between inatial and final direction of motion of the Particle is known as angle of scattring and is represented By captial Theata. Θ

In Three dimension all the Particles which are scattred through scattring angle Θ will Be scattring along Solid angle

Ω (omega). as shown in the diagram



It may be noted that those particle which having an angle of scattering between Θ and $\Theta + d\Theta$ will be scattred in solid angle $d\Omega$ as shown in fig.



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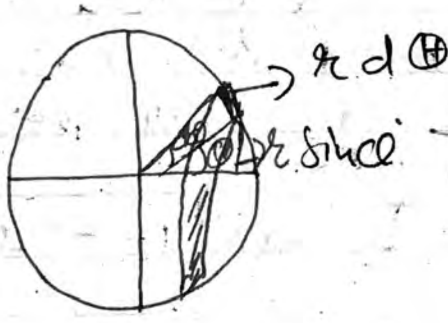
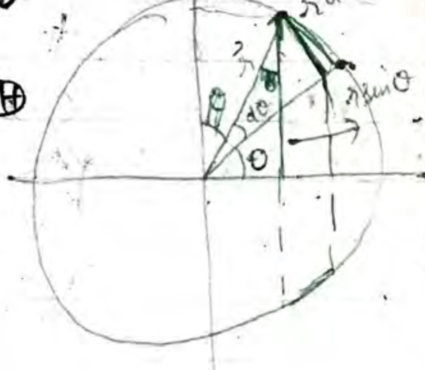




As

$$\text{Area of strip} = 2\pi r \sin\Theta \cdot r\,d\Theta$$
$$= 2\pi r^2 \sin\Theta\, d\Theta$$

and

$$d\Omega = \frac{2\pi r^2 \sin\Theta\, d\Theta}{r^2}$$

$d\Omega$ (Solid angle) $= 2\pi \sin\Theta\, d\Theta$

$d\Omega = 2\pi \sin\theta\, d\theta$

$n = I\, \sigma(\Theta)\, 2\pi \sin\theta\, d\theta$

Its unit is steradian. It has SI unit. It is the angle that enclosed a surface on the sphere equal to the square of the radius of the sphere.

(4) Scattring Crass Section:

Scattring Crass Section is denoted By $\sigma(\Omega)$ or $\sigma(\Theta)$ and defind as

$$\sigma(\Omega)\, d\Omega = \cancel{\text{no. of}}$$

$$= \frac{\text{no. of Particle scattred into a solid angle } d\Omega \text{ in unit tme}}{\text{Intensity of Beam}}$$

where $d\Omega$ is Element of ~~Beam~~ ~~and~~ Solid angle and is Equal to

$$d\Omega = 2\pi \sin\Theta\, d\Theta$$

from this Expression the no. of Particles Scattred Per unit time in Solid angle ~~dw~~ $d\Omega$ will Be

$$n = \text{Intensity}\; \sigma(\Omega)\, d\Omega$$

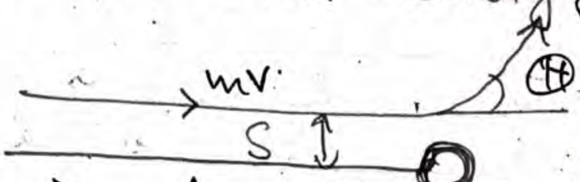
$$n = I \cdot \sigma(\Omega) \cdot 2\pi \sin\Theta\, d\Theta$$

$$= I\, \sigma(\Theta)\, 2\pi \sin\Theta\, d\Theta$$

Impact Parameter:

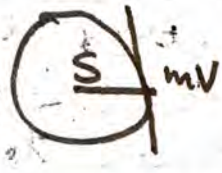
Impact Parameter is defind as the ⊥ar distance between the Center of force and dirction of incident velocity of Particles for a Certain Energy.

If the impact



Parameter is less, The Particle will Be Scattred By a ~~longer~~ larger angle, i.e when S decreases ds will Be −ve and Then Θ increases and $d\Theta$ is +ve. Thus the $\frac{d\Theta}{ds}$ will Be −ve quantity we will Consider its absolute value and will not Consider the its absolute value. If V is the incident speed of ~~light~~ Particles, whose impact Parameter is S. Then the angular momentum Can Be Expressed as

$$l = mVS$$



$$l = S\sqrt{m^2V^2}$$

$$l = S\sqrt{2m\frac{1}{2}mV^2} \quad \text{where } E = \frac{1}{2}mV^2$$

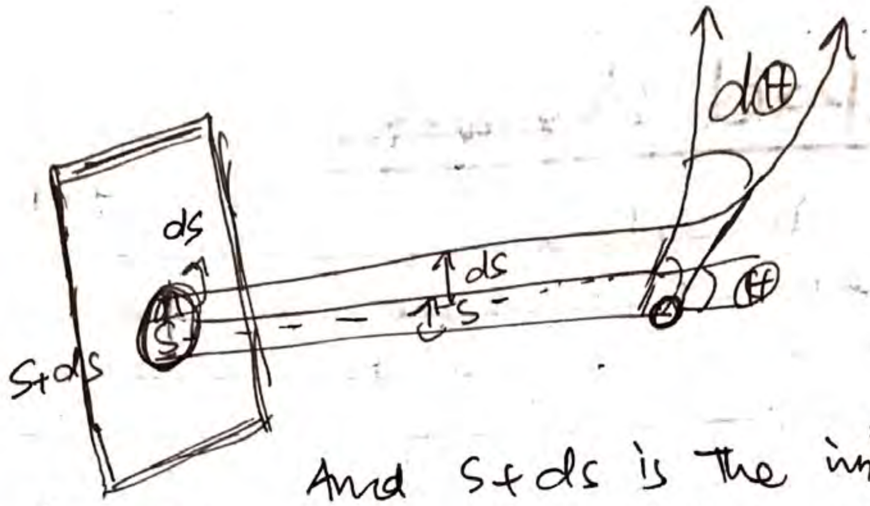
$$l = S\sqrt{2mE}$$

$L = r \times P$
$= S \times mV$
$= SmV$

Rutherford's formula for Scattring Cross Section:-

If S is The impact Parameter. Then Θ is the Scattring angle

(Area of Ring $= (2\pi S)\,ds$



And S+ds is the impact Parameter. Then angle of scattering will be $\Theta - d\Theta$. Thus the Particle having Impact Parameter Between S & S+ds having an angle of scattering Between Θ and $\Theta - d\Theta$. These Particles are incident from an annular ring, whose inner radius is S and outer radius is S+ds, as shown in fig. These incident Particles will Be scattered in solid angle $d\Omega$. The number of These Particles will Be.

$$= 2\pi S\, ds\, I \longleftarrow (A)$$

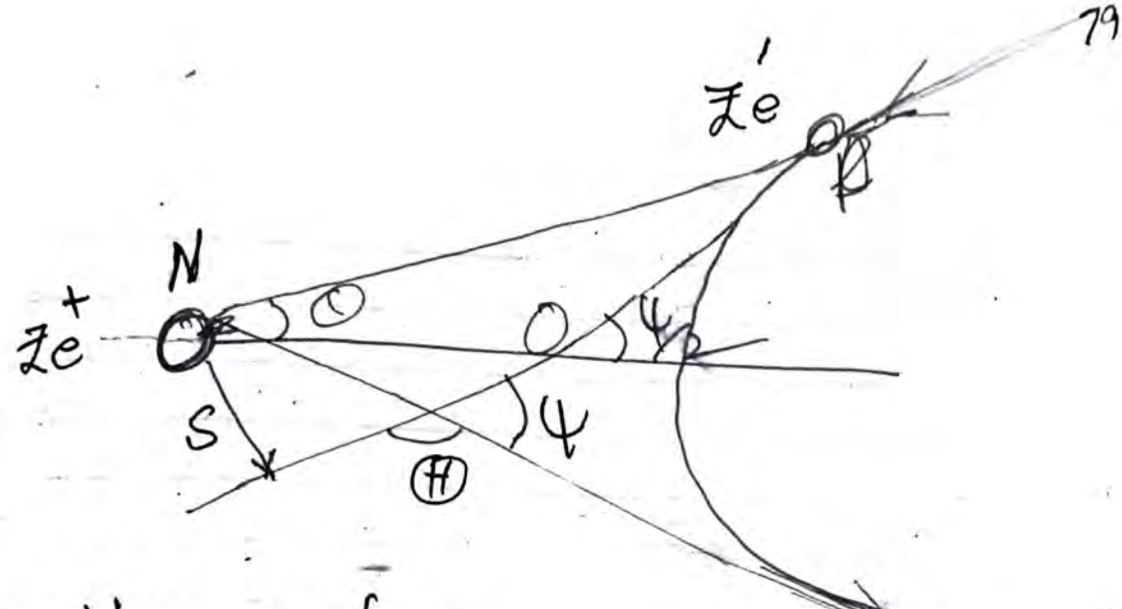
where I is the Intensity of Incident Beam. we Know that the no. of Particles scattered in Between scattering angle Θ and $\Theta + d\Theta$ (i.e Solid angle $d\Omega$) is equal to

$$= 2\pi I\, \sigma(\Theta) \sin\Theta\, d\Theta \quad (B)$$

where $\sigma(\Theta)$ is scattering Cross section. Comparing eq (A) & (B), we get.

$$\sigma(\Theta) = \frac{S}{\sin\Theta}\left(\frac{ds}{d\Theta}\right)$$

Now we shall find the Expression for scattering angle Θ as a function of Parameter S. which can Be ⇒



⇒ obtained from the orbit equation we will illustrate the procedure by considering the scattering of charged particle by a central field of force in terms of Coloumb,s force.

Let a +vely charged particle (α-particle) having charge $Z'e$ be inatially moving along the direction PO in diagram. It apporaches a relativly heavier nuclus scattring at Pt N. having charge equal to Ze. There will be a force of repulsion between these two charges which is governed by Coloumb,s law and will be equal to:

$$F = \frac{1}{4\pi\epsilon_0}\frac{ZZ'e^2}{r^2} \quad \text{But } \frac{1}{4\pi\epsilon_0}ZZ'e^2 = K$$

$$\Rightarrow F = \frac{K}{r^2}, \quad \text{But } r = \frac{1}{u}$$

$$\Rightarrow F = Ku^2$$

Basically it is two body problem, but it is converted to a single body problem by considring the origion at Pt-N. which is stationary. The mass of particle will be considred as its reduced mass. Let (r, θ) Be The Polar Co-ordinates of Particle at an instant t.

The Equation of motion which may Be used for The determination of orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{-m}{l^2u^2} f\left(\frac{1}{u}\right) \quad (1)$$

Now Putting $F\left(\frac{1}{u}\right) = ku^2$ in above equation, we get

$$\frac{d^2u}{d\theta^2} + u = \frac{-m}{l^2\cancel{u^2}} (k\cancel{u^2})$$

$$\frac{d^2u}{d\theta^2} + u = \frac{-mk}{l^2} \quad (2)$$

This eq. (2) has a general Solution of the following form.

$$u = C\cos(\theta - \phi) + \frac{mk}{l^2} \quad (3)$$

The Polar Co-ordinates are choosen in such a way. $\phi = 0$.

$$\therefore u = C\cos\theta + \frac{mk}{l^2} \quad (4)$$

where C is a Constant

From this Eq. Constant C Comes out to Be. Using following Eq.

$$C = \sqrt{\frac{m^2k^2}{l^4} + \frac{2mE}{l^2}} \quad (5)$$

and $e = \sqrt{1 + \frac{2El^2}{mk^2}} \quad (6)$

we know that

$$l = S\sqrt{2mE}$$

Putting this value of l in eq. (6), we get

$$e = \sqrt{1 + \frac{2ES^2 2mE}{m K^2}}$$

$$e = \sqrt{1 + \frac{(2ES)^2}{K^2}}$$

$$e = \sqrt{1 + \left(\frac{2ES}{K}\right)^2} \quad \text{—— (7)}$$

In this case as E is the K.E of particle which is +ve. Thus $e > 1$. So the conic section will be a Hyperbola.

Refering from fig.

$$\Theta + \Psi = \pi \quad \text{—— (8)}$$

where Θ is scattering angle and Ψ is the angle between two asymptotes of hyperbola. $\Psi/2$ is the angle which one of the asymptotes makes with the polar axis. If $r = \infty$, then the polar angle θ will be equal to $\Psi/2$ or $\theta = \Psi/2$

and

$$\Psi/2 = \frac{\pi}{2} - \frac{\Theta}{2}$$

and

$$\cos\Psi/2 = \cos\left(\frac{\pi}{2} - \frac{\Theta}{2}\right)$$

But $\Psi/2 = \theta \Rightarrow$

$$\cos\phi = \cos\left(\frac{\pi}{2} - \frac{\Theta}{2}\right)$$

$$\cos\phi = \sin\frac{\Theta}{2}$$

and $\sin\frac{\Theta}{2} = \pm\frac{1}{e}$ — (9)

As Eq. of Conic is

$$r = \frac{P}{\pm 1 + e\cos\theta}$$

$$\pm 1 + e\cos\theta = \frac{P}{r}$$

if $r = \infty$

$$\Rightarrow \cos\theta = \pm\frac{1}{e}$$

Squaring Both sides of Eq (9) we get.

$$e^2 = \frac{1}{\sin^2\frac{\Theta}{2}} = \csc^2\frac{\Theta}{2}$$

or

$$e^2 = 1 + \cot^2\frac{\Theta}{2}$$

OR

$$\cot\frac{\Theta}{2} = \sqrt{e^2 - 1}$$

Putting value of e^2, we get

$$\cot\frac{\Theta}{2} = \sqrt{\cancel{1} + \left(\frac{2ES}{K}\right)^2 - \cancel{1}}$$

$$\cot\frac{\Theta}{2} = \frac{2ES}{K}$$

$$\Rightarrow \boxed{S = \frac{K}{2E}\cot\frac{\Theta}{2}}$$ ⇒ which is relation Between S & Θ

→ (10)

Now

$$\frac{dS}{d\Theta} = \frac{-K}{2E}\csc^2\frac{\Theta}{2}\cdot\frac{1}{2}$$

we can neglect -ve sign

$$\left|\frac{ds}{d\Theta}\right| = \frac{K}{4E}\operatorname{Cosec}^2\frac{\Theta}{2}$$

we know That

$$\sigma(\Theta) = \frac{S}{\sin\Theta}\left(\frac{ds}{d\Theta}\right)$$

$$\sigma(\Theta) = \frac{K\cot\frac{\Theta}{2}}{\sin\Theta\,(2E)}\frac{K}{4E}\operatorname{Cosec}^2\frac{\Theta}{2}$$

$$= \frac{1}{2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}}\,\frac{K^2}{8E^2}\,\frac{\cos\Theta/2}{\sin\frac{\Theta}{2}}\operatorname{Cosec}^2\frac{\Theta}{2}$$

$$\boxed{\sigma(\Theta) = \left(\frac{K}{4E}\right)^2\operatorname{Cosec}^4\frac{\Theta}{2}} \quad \text{---} (11)$$

Eq (11) is formula for Scattring Cross Section. This Eq. gives the formula of Rutherford formula for Scattring Cross Section originally derived By Rutherford for the Scattring of α- Particles By atomic nuclie.

4. Angle of Scattring ~~for~~ in diffrent Systems.

(1) Equivalent One Body system:-

In this Case the nuclus which is the target Particle has Been taken at the origin. The Initial and final Position of the incident →

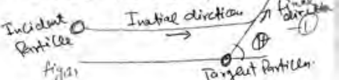


fig (1)

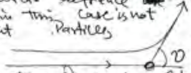
fig (2)

⇒ Particles and its direction of motion has been measured with respect to the target Particles. This is how ~~a~~ a two Body Problem has been reduced to single Body Problem. In this ~~System~~ ~~Case~~ Equivalent One Body ~~Problems~~, we ~~thus~~ have assumed that target Particle remain stationary so the Scattering angle in this Case is angle Between the initial direction and final direction of the incident Particle As shown in fig (1).

It is however not ~~...~~ always the Case, The target Particle may recoil from its initial Position as the result of interaction Between two Particles. In this Case Scattering angle Θ is the angle Between the initial and final direction of relitive vector ($\vec{r}$) Between The two Particles

② Labratory frame of refrence:-

The scattering angle in The labratory frame of refrence is The angle Between Inatial and final direction of motion of the incident Particle as seen By an observer located some where in The labratory. In other words refrence ~~or~~ or The origion in this case is not the target Particles



But another Pt in the labratory. The two scattering angle Θ & θ will Be equal if the target Particle remain Stationary through out the scattering Process

③ Angle of scattring in Center of mass system::

In this case we define the angle of scattering with respect to center of mass.

Θ is the angle Between Initial direction of motion and final direction of motion of the incident Particles, with respect to Center of mass.

Q. Show that:-

① The angle of scattering θ in one Body ~~Problem~~ is the Same as Θ in Center of mass. system?

② Drive The relation Between θ and Θ.

<u>Solution</u>. In a scattering Problem, Usually one of the Particle is initially at rest, we shall take it The Particle ② and Called it Target Particle. ⇒

($\vec{r_i}$ Position vector with refrence to Center of mass)

⇒ Particle ① which is approaching The target with Initial velocity $\vec{v}_i$ is The incident Particle. The two Particles are located by r_1 and r_2 with respect to origin O. we shall call r_1 & r_2 as the Co-ordinates the Co-ordinates in the laboratory system and so is the Coordinate R.

$\vec{r}$ is the Position vector of m_1 with respect m_2, which is considered to Be at rest equivalent one Body Problem. In other words $\vec{r}_1$ is the Co-ordinate in Equivalent one Body Problem. Similarly $\vec{r}_{1C}$ and $\vec{r}_{2C}$ are Co-ordinates at in the Center of masses it self is considered to Be at rest in this system and is taken as the origion

we know the Position of Center of mass is given By;

$$R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} \quad (1)$$

and from fig

$$\vec{r}_{1C} = \vec{r}_1 - \vec{R} \quad (2)$$

and $$\vec{r}_{2C} = \vec{r}_2 - \vec{R} \quad (3)$$

The Position of m_1 with respect to m_2 is given By the relative Position of vector $\vec{r}$, such that;

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \quad (4)$$

Using eq (1) & (4), It can Be shown;

That $$\vec{r}_1 = R + \left(\frac{m_2}{m_1+m_2}\right)\vec{r} \quad (5)$$

and $$\vec{r}_2 = R - \left(\frac{m_1}{m_1+m_2}\right)\vec{r} \quad (6)$$

Putting r_1 in (2) we get;

$$\vec{r}_{1C} = \cancel{R} + \frac{m_2}{(m_1+m_2)}\vec{r} - \cancel{R}$$

$$\boxed{\vec{r}_{1C} = \left(\frac{m_2}{m_1+m_2}\right)\vec{r}} \quad (7)$$

or $\vec{r}_{1C} = \frac{\mu}{m_1} r$ as $\mu = \frac{m_1 m_2}{m_1+m_2}$

Similarly.

$$\vec{r_{2c}} = \left(\frac{-m_1}{m_1+m_2}\right)\vec{r}$$

or

$$\vec{r_{2c}} = \frac{-\mu}{m_2}\vec{r} \quad (8)$$

and from (8) we get

$$\boxed{\vec{V_{1C}} = \frac{\mu}{m_1}\vec{v}} \quad (9)$$

and

$$\boxed{\vec{V_{2C}} = \frac{-\mu}{m_2}\vec{v}} \quad (10)$$

So the direction of velocity of Incident Particle in Center of mass System is Same as the direction of velocity of Particle (1) relative to Particle (2) (i.e Equivalent one Body Problem). As the target Particle due to which Particle (1) is Scattered is Same in the two Cases. So The angle of Scattring (θ) in The Center of mass System is equal to the angle of Scattring in Equivalent one Body System.

Q find The relation Between ϑ & θ

The angle of Scattring ϑ in Labratory frame of refrence is The angle Between V_i and V_f. where

V_i = Inatial velocity of Particle one.

Now we shall Express $v_{c.m}$ in terms of $\vec{v_{1f}}$.
Expressing Eq (e) in terms of final velocity.

$$\vec{V}_{c.m} = \frac{\mu}{m_2} v_{1f}$$

Putting the value of v_{1f} from Eq (ii) in

$$v_{c.m} = \frac{\mu}{m_2}\left(v_{1f}^{c} + V_{c.m}\right)$$

$$v_{c.m} = \frac{\mu}{m_2}\left(v_{1f}^{c} + V_{c.m}\right)$$

$$V_{c.m}\left(1 - \frac{\mu}{m_2}\right) = \frac{\mu}{m_2} v_{1f}^{c}$$

$$\frac{\mu}{m_1} V_{c.m} = \frac{\mu}{m_2} v_{1f}^{c}$$

$$V_{c.m} = \frac{m_1}{m_2} v_{1f}^{c}$$

$$1 - \frac{\mu}{m_2} = \frac{m_2 + ...}{...}$$

$$1 - \frac{m_1 m_2}{m_1 + m_2}$$

$$\frac{m_1 + m_2 m_1}{(m_1 + m_2)}$$

$$= \frac{\mu}{m_1}$$

Now $\tan\vartheta$ will Be;

$$\tan\vartheta = \frac{\sin\Theta}{\cos\Theta + \frac{m_1}{m_2}\frac{v_{1f}^{c}}{v_{1f}^{c}}}$$

or

$$\tan\vartheta = \frac{\sin\Theta}{\cos\Theta + \frac{m_1}{m_2}}$$

Ist Case. If m_2 is much greater ($m_2 >>> m_1$). Then m_1. Then $\frac{m_1}{m_2} << 6$.
Then It Can Be neglected and
Then

$$\tan\vartheta = \frac{\sin\Theta}{\cos\Theta} = \tan\Theta$$

$$\boxed{\vartheta = \Theta}$$

2nd Case: If mass are Equal

as $m_1 = m_2$

$$\tan\vartheta = \frac{\sin\Theta}{\cos\Theta + 1} = \frac{2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}}{2\cos^2\frac{\Theta}{2}}$$

$$\tan\vartheta = \tan\frac{\Theta}{2}$$

$$\boxed{\vartheta = \frac{\Theta}{2}}$$

Probable Question:-

(1) Define Scattering Cross section (Co-efficient of Scattering) and drive Rutherford formula for Scattering Cross Section.

(2) what is the angle of Scattring in the lab frame of refrence in Centre of mass System, and in One Body Problem, Show That.

(I) Angle of Scattring in C.M.S is Same as angle of scattering in Equivalent One Body Problem

(II) Drive the relation Between ϑ and Θ