CHAPTER 43

GRATINGS AND SPECTRA

In Chapter 41 we discussed the interference pattern produced when monochromatic light is incident on a double slit: a pattern of bright and dark bands (interference fringes) is produced. Each of the slits is regarded as a source of diffracted waves. In this chapter we extend our discussion to cases in which the number of diffraction sources is larger (often much larger) than two. We consider multiple arrays of slits in a plane and also three-dimensional arrays of atoms in a solid (for which we use x rays rather than visible light).

43-1 MULTIPLE SLITS

Just as Thomas Young used his double-slit interference technique for the first measurement of the wavelength of light, we continue today to use a variety of interference techniques for precise measurement of wavelengths. For example, from 1960 to 1983, the standard meter was defined in terms of the wavelength of the light emitted by atoms of krypton (see Section 1-4). To enable its use as a standard, it was necessary to determine the value of the wavelength to nine significant figures! Double slit interference

In principle, we can use a double-slit interference pattern to measure the wavelength. If we know the slit separation d, then by locating the bright fringes on a screen and using Eq. 41-1 ($d \sin\theta = m\lambda$) we can find the wavelength. Often it is more convenient to determine the separation between adjacent maxima, in which case we can use Eq. 41-4 ($\Delta y = \lambda D/d$) to find λ. The precision of this method is limited by our ability to make an accurate determination of the location or spacing of the fringes. Figure 43-1a shows a typical double-slit interference pattern; you can see that trying to measure from the midpoint of one fringe to the midpoint of the next one involves an uncertainty in locating the midpoints.

The interference pattern in Fig. 43-1b, which was obtained using five slits, offers a slight improvement; note that the bright fringes are narrower, meaning that we will be able to do a slightly better job of measuring the fringe spacing and thus determine the wavelength more precisely. The separation d between adjacent slits has the same value for both cases, and you can see by comparing the two interference patterns that the spacing of the fringes is determined

(a) (b)

FIGURE 43-1. The diffraction pattern for a grating with (a) two slits and (b) five slits. Note that, in the case of the five-slit grating, the fringes are sharper (narrower), and secondary maxima of low intensity appear between the bright principal maxima.

by the slit separation d, independently of the number of slits. As we increase the number of slits, the bright fringes continue to become narrower, and the precision of our wavelength measurement continues to improve. Such a multiple-slit arrangement, in which the number of slits N may be as large as 10^4, is called a *diffraction grating*.

From Fig. 43-1*b* you can see a second effect of increasing the number of slits—faint *secondary maxima* (three in Fig. 43-1*b*) appear between adjacent bright fringes. As the number of slits increases, the number of secondary maxima increases but their brightness decreases; when N is large we can neglect the secondary maxima completely.

In this chapter we consider the interference patterns for diffraction gratings. We will consider how the sharpness of the fringes depends on the number of slits in the grating, and we will explain the origin of the secondary maxima. As we did in Chapter 41, we consider only the Fraunhofer arrangement, in which there is an assumed infinite distance between the light source and the slits as well as between the slits and the screen. Equivalently, parallel light is incident on the slits, and parallel light emerges from the slits (perhaps to be focused by a lens) to form an image on the screen.

Figure 43-2 shows a five-slit grating (compare Fig. 41-6). A principal maximum occurs when the path difference between rays from any pair of adjacent slits, which is given by $d \sin\theta$, is equal to an integral number of wavelengths, or

$$d \sin\theta = m\lambda \qquad m = 0, \pm 1, \pm 2, \ldots, \tag{43-1}$$

where m is called the *order number*. Equation 43-1 is identical with Eq. 41-1 for the maxima of the double slit. Note that if light passing through any pair of *adjacent* slits is in phase at a particular point on the screen, then light passing through any pair of slits, even nonadjacent ones, is also in phase at that point. For a given slit separation d, the locations of the principal maxima are determined by the wavelength, according to Eq. 43-1. The locations of the principal

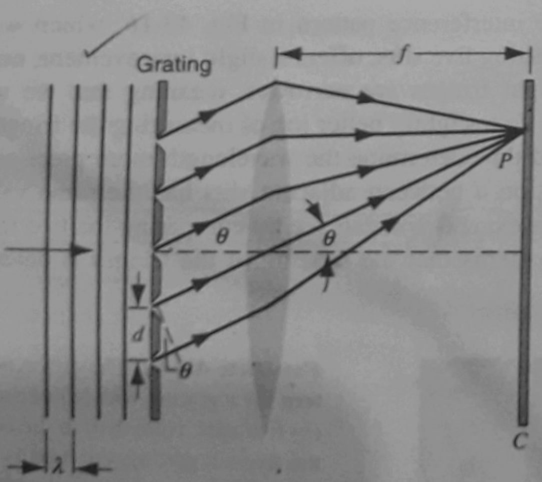

FIGURE 43-2. An idealized diffraction grating containing five slits. The slit width a is assumed to be much smaller than λ, although this condition may not be realized in practice. Also, the focal length f will in practice be much greater than d; the figure distorts these dimensions for clarity.

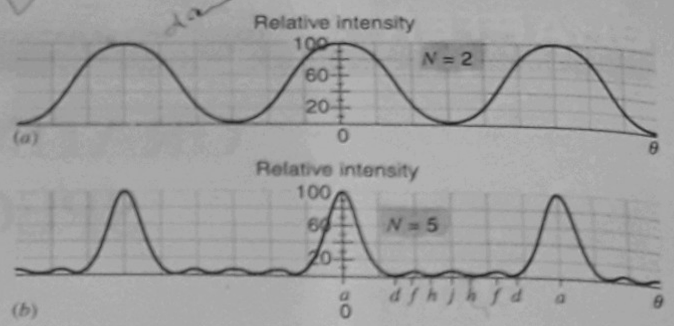

FIGURE 43-3. Calculated intensity patterns for (*a*) a two-slit and (*b*) a five-slit grating, having the same values of d and λ. Note the sharpening of the principal maxima and the appearance of faint secondary maxima in (*b*); compare with Fig. 43-1. The letters in (*b*) refer to Fig. 43-6. This calculation does not include diffraction effects due to the slit width; that is, we assume we are near the central region of Fig. 43-1 where the principal maxima have essentially equal intensities.

a/λ :-

maxima are independent of the number of slits N, which, as we shall see, determines the width or sharpness of the principal maxima. For slits of width a, the relative intensities of the principal maxima within the diffraction envelope are determined by the ratio a/λ, which does not affect their locations. A comparison of the intensity patterns for two-slit and five-slit gratings is given in Fig. 43-3.

Width of the Maxima

The sharpening of the principal maxima as N is increased can be understood by a graphical argument, using phasors. Figures 43-4*a* and 43-4*b* show conditions at the central principal maximum for a two-slit and a five-slit grating. The small arrows represent the amplitudes of the wave disturbances from each slit arriving at the screen at the position of the central maximum, for which $m = 0$, and thus $\theta = 0$, in Eq. 43-1.

On either side of the central maximum there is a minimum of zero intensity, which lies at an angle $\delta\theta_0$ off the central axis, as shown in Fig. 43-5. Figures 43-4*c* and 43-4*d* show the phasors at this point. The phase difference $\Delta\phi$ between waves from adjacent slits, which is zero at the central principal maximum, must be chosen so that the array of phasors just closes on itself, yielding zero resultant intensity. For $N = 2$, $\Delta\phi = 2\pi/2$ $(= 180°)$; for $N = 5$, $\Delta\phi = 2\pi/5$ $(= 72°)$. In the general case it is given by

$$\Delta\phi = \frac{2\pi}{N}. \tag{43-2}$$

$N\Delta\phi = 2\pi$

This phase difference for adjacent waves corresponds to a path difference ΔL given by

$$\frac{\text{phase difference}}{2\pi} = \frac{\text{path difference}}{\lambda},$$

$\frac{\Delta\phi}{2\pi} = \frac{\Delta L}{\lambda}$

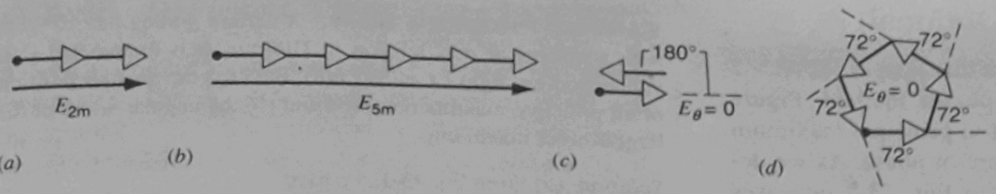

FIGURE 43-4. (*a*, *b*) The conditions at the central maximum for a two-slit and a five-slit grating, respectively. (*c*, *d*) The corresponding conditions at a minimum of zero intensity that lies on either side of this central maximum.

or

$$\Delta L = \left(\frac{\lambda}{2\pi}\right)\Delta\phi = \left(\frac{\lambda}{2\pi}\right)\left(\frac{2\pi}{N}\right) = \frac{\lambda}{N}. \quad (43\text{-}3)$$

However, the path difference ΔL at the first minimum (see Figs. 43-2 and 43-5) is also given by $d \sin \delta\theta_0$, so that we can write

$$d \sin \delta\theta_0 = \frac{\lambda}{N},$$

or

$$\sin \delta\theta_0 = \frac{\lambda}{Nd}. \quad (43\text{-}4)$$

Since $N >> 1$ for actual gratings, $\sin \delta\theta_0$ is ordinarily quite small (that is, the lines are sharp), and to a good approximation we may replace $\sin \delta\theta_0$ by $\delta\theta_0$, expressed in radians, or

$$\delta\theta_0 = \frac{\lambda}{Nd}. \quad (43\text{-}5)$$

This equation shows specifically that if we increase N for a given λ and d, then $\delta\theta_0$ decreases, which means that the central principal maximum becomes sharper.

To obtain the result for *any* principal maximum, we consider the geometry of Fig. 43-5, in which the mth principal maximum occurs at an angle θ. We move away from the maximum through an angular displacement $\delta\theta$ to arrive at the next minimum; we take this angle $\delta\theta$ to be a measure of the angular width of the maximum. At the maximum, the path difference between rays from adjacent slits is $m\lambda$ (see Eq. 43-1). At the next minimum, the path difference between rays from adjacent slits is $m\lambda + \lambda/N$, the additional path length of λ/N being given by Eq. 43-3. For example, consider the case of $N = 10$. The additional path length between adjacent slits at the minimum is 0.1λ. The path difference between slits 1 and 6 is therefore $5(m\lambda + 0.1\lambda) = 5m\lambda + 0.5\lambda$; the path lengths differ by a half-integral number of wavelengths, so the rays interfere destructively. The same is true for slits 2 and 7, slits 3 and 8, and so forth. If the additional path difference is λ/N, then rays from the lower $N/2$ slits undergo pairwise destructive interference with rays from the upper $N/2$ slits.

At the angle $\theta + \delta\theta$, the path difference between rays from adjacent slits is

$$\begin{aligned} d \sin(\theta + \delta\theta) &= d(\sin\theta\cos\delta\theta + \cos\theta\sin\delta\theta) \\ &\approx d\sin\theta + (d\cos\theta)\delta\theta, \end{aligned}$$

where we assume $\delta\theta$ is small, which allows us to approximate $\cos\delta\theta \approx 1$ and $\sin\delta\theta \approx \delta\theta$. Setting this path difference equal to $m\lambda + \lambda/N$, its value at the minimum, we obtain

$$d\sin\theta + (d\cos\theta)\delta\theta = m\lambda + \frac{\lambda}{N}$$

or, using Eq. 43-1,

$$(d\cos\theta)\delta\theta = \frac{\lambda}{N}.$$

Solving for $\delta\theta$ gives

$$\delta\theta = \frac{\lambda}{Nd\cos\theta}. \quad (43\text{-}6)$$

This result gives the angular width* for the principal maximum that occurs at the angle θ, corresponding to the particular order m. Note that Eq. 43-6 reduces to Eq. 43-5 for the central maximum ($\theta = 0$). For a given N, d, and λ, the central maximum is the narrowest ($\cos\theta = 1$); the widths increase as we go to larger θ (and therefore to larger orders m). Equation 43-6 shows that $\delta\theta$ becomes smaller (the maxima become sharper) as the product Nd increases. This product (the number of slits times the distance between slits) gives the total width of the grating. Thus the peaks become sharper as the width of the grating increases.

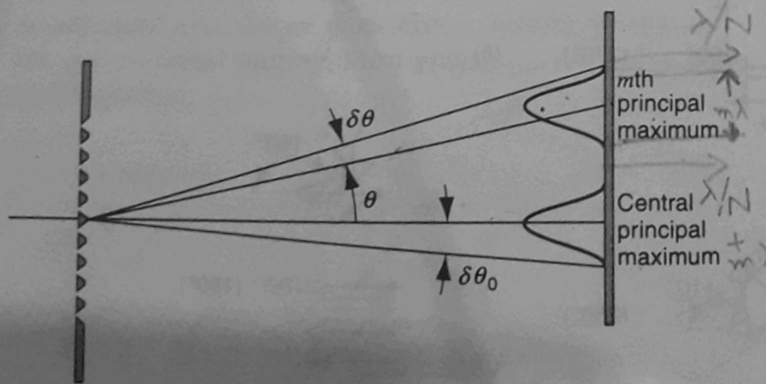

FIGURE 43-5. A principal maximum lies at the position given by the angle θ, and the first minimum occurs at the angle $\delta\theta$ from that maximum. The angle $\delta\theta$ can be taken as a measure of the width or sharpness of the maximum. The width of the central maximum is given by the angle $\delta\theta_0$.

* As defined by Eq. 43-6, the width is the angular interval from the peak to the first minimum. The usual definition of the width of a peak is the full interval covered by the peak at half its maximum height (see, for example, Fig. 42-12). These two measures of the width are roughly equal, and we take Eq. 43-6 to represent a measure of the width of the peak.