*c h a p t e r*

# 3

# Vectors

W e often need to work with physical quantities that have both numerical and directional properties. As noted in Section 2.1, quantities of this nature are represented by vectors. This chapter is primarily concerned with vector algebra and with some general properties of vector quantities. We discuss the addition and subtraction of vector quantities, together with some common applications to physical situations.

Vector quantities are used throughout this text, and it is therefore imperative that you master both their graphical and their algebraic properties.



**Figure 3.1** Designation of points in a cartesian coordinate system. Every point is labeled with coordinates $(x, y)$.

## 3.1  COORDINATE SYSTEMS

Many aspects of physics deal in some form or other with locations in space. In Chapter 2, for example, we saw that the mathematical description of an object's motion requires a method for describing the object's position at various times. This description is accomplished with the use of coordinates, and in Chapter 2 we used the cartesian coordinate system, in which horizontal and vertical axes intersect at a point taken to be the origin (Fig. 3.1). Cartesian coordinates are also called *rectangular coordinates.*

Sometimes it is more convenient to represent a point in a plane by its *plane polar coordinates* $(r, \theta)$, as shown in Figure 3.2a. In this *polar coordinate system, r* is the distance from the origin to the point having cartesian coordinates $(x, y)$, and $\theta$ is the angle between $r$ and a fixed axis. This fixed axis is usually the positive $x$ axis, and $\theta$ is usually measured counterclockwise from it. From the right triangle in Figure 3.2b, we find that $\sin \theta = y/r$ and that $\cos \theta = x/r$. (A review of trigonometric functions is given in Appendix B.4.) Therefore, starting with the plane polar coordinates of any point, we can obtain the cartesian coordinates, using the equations

$$x = r \cos \theta \tag{3.1}$$

$$y = r \sin \theta \tag{3.2}$$
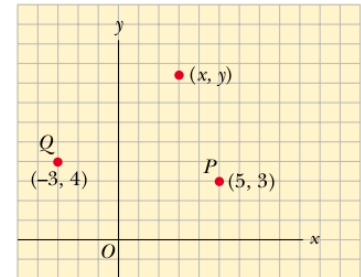
Furthermore, the definitions of trigonometry tell us that

$$\tan \theta = \frac{y}{x} \tag{3.3}$$

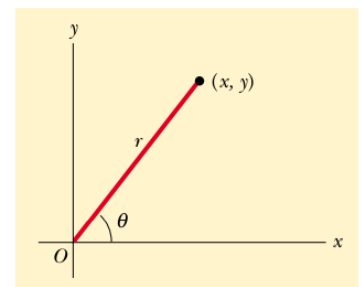$$r = \sqrt{x^2 + y^2} \tag{3.4}$$

These four expressions relating the coordinates $(x, y)$ to the coordinates $(r, \theta)$ apply only when $\theta$ is defined, as shown in Figure 3.2a—in other words, when positive $\theta$ is an angle measured *counterclockwise* from the positive $x$ axis. (Some scientific calculators perform conversions between cartesian and polar coordinates based on these standard conventions.) If the reference axis for the polar angle $\theta$ is chosen to be one other than the positive $x$ axis or if the sense of increasing $\theta$ is chosen differently, then the expressions relating the two sets of coordinates will change.



(a)

$$\sin \theta = \frac{y}{r}$$

$$\cos \theta = \frac{x}{r}$$

$$\tan \theta = \frac{y}{x}$$



(b)

**Figure 3.2** (a) The plane polar coordinates of a point are represented by the distance $r$ and the angle $\theta$, where $\theta$ is measured counterclockwise from the positive $x$ axis. (b) The right triangle used to relate $(x, y)$ to $(r, \theta)$.
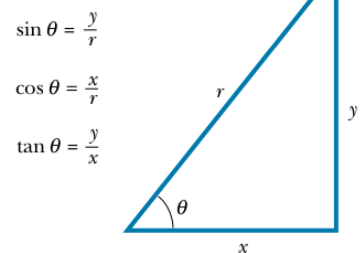
## Quick Quiz 3.1

Would the honeybee at the beginning of the chapter use cartesian or polar coordinates when specifying the location of the flower? Why? What is the honeybee using as an origin of coordinates?

You may want to read *Talking Apes and Dancing Bees* (1997) by Betsy Wyckoff.

**EXAMPLE 3.1**    **Polar Coordinates**

The cartesian coordinates of a point in the $xy$ plane are $(x, y) = (-3.50, -2.50)$ m, as shown in Figure 3.3. Find the polar coordinates of this point.
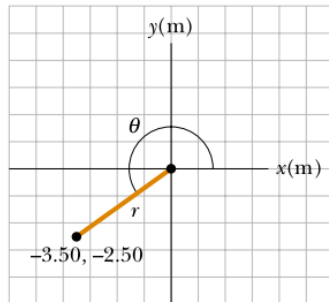


**Figure 3.3**   Finding polar coordinates when cartesian coordinates are given.

**Solution**

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3.50 \text{ m})^2 + (-2.50 \text{ m})^2} = \boxed{4.30 \text{ m}}$$

$$\tan \theta = \frac{y}{x} = \frac{-2.50 \text{ m}}{-3.50 \text{ m}} = 0.714$$

$$\theta = \boxed{216°}$$

Note that you must use the signs of $x$ and $y$ to find that the point lies in the third quadrant of the coordinate system. That is, $\theta = 216°$ and not 35.5°.

**3.2    VECTOR AND SCALAR QUANTITIES**
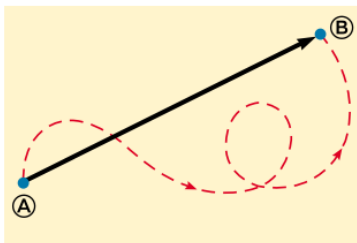
As noted in Chapter 2, some physical quantities are scalar quantities whereas others are vector quantities. When you want to know the temperature outside so that you will know how to dress, the only information you need is a number and the unit "degrees C" or "degrees F." Temperature is therefore an example of a **scalar quantity,** which is defined as a quantity that is completely specified by a number and appropriate units. That is,

> A scalar quantity is specified by a single value with an appropriate unit and has no direction.

Other examples of scalar quantities are volume, mass, and time intervals. The rules of ordinary arithmetic are used to manipulate scalar quantities.

If you are getting ready to pilot a small plane and need to know the wind velocity, you must know both the speed of the wind and its direction. Because direction is part of the information it gives, velocity is a **vector quantity,** which is defined as a physical quantity that is completely specified by a number and appropriate units plus a direction. That is,

> A vector quantity has both magnitude and direction.

Another example of a vector quantity is displacement, as you know from Chapter 2. Suppose a particle moves from some point Ⓐ to some point Ⓑ along a straight path, as shown in Figure 3.4. We represent this displacement by drawing an arrow from Ⓐ to Ⓑ, with the tip of the arrow pointing away from the starting point. The direction of the arrowhead represents the direction of the displacement, and the length of the arrow represents the magnitude of the displacement. If the particle travels along some other path from Ⓐ to Ⓑ, such as the broken line in Figure 3.4, its displacement is still the arrow drawn from Ⓐ to Ⓑ.



**Figure 3.4**   As a particle moves from Ⓐ to Ⓑ along an arbitrary path represented by the broken line, its displacement is a vector quantity shown by the arrow drawn from Ⓐ to Ⓑ.

(a) The number of apples in the basket is one example of a scalar quantity. Can you think of other examples? *(Superstock)* (b) Jennifer pointing to the right. A vector quantity is one that must be specified by both magnitude and direction. *(Photo by Ray Serway)* (c) An anemometer is a device meteorologists use in weather forecasting. The cups spin around and reveal the magnitude of the wind velocity. The pointer indicates the direction. *(Courtesy of Peet Bros.Company, 1308 Doris Avenue, Ocean, NJ 07712)*

In this text, we use a boldface letter, such as **A**, to represent a vector quantity. Another common method for vector notation that you should be aware of is the use of an arrow over a letter, such as $\vec{A}$. The magnitude of the vector **A** is written either $A$ or $|\mathbf{A}|$. The magnitude of a vector has physical units, such as meters for displacement or meters per second for velocity.

## 3.3 SOME PROPERTIES OF VECTORS

### Equality of Two Vectors

For many purposes, two vectors **A** and **B** may be defined to be equal if they have the same magnitude and point in the same direction. That is, **A** = **B** only if $A = B$ *and* if **A** and **B** point in the same direction along parallel lines. For example, all the vectors in Figure 3.5 are equal even though they have different starting points. This property allows us to move a vector to a position parallel to itself in a diagram without affecting the vector.

### Adding Vectors

The rules for adding vectors are conveniently described by geometric methods. To add vector **B** to vector **A**, first draw vector **A**, with its magnitude represented by a convenient scale, on graph paper and then draw vector **B** to the same scale with its tail starting from the tip of **A**, as shown in Figure 3.6. The **resultant vector R =** **A + B** is the vector drawn from the tail of **A** to the tip of **B**. This procedure is known as the **triangle method of addition.**

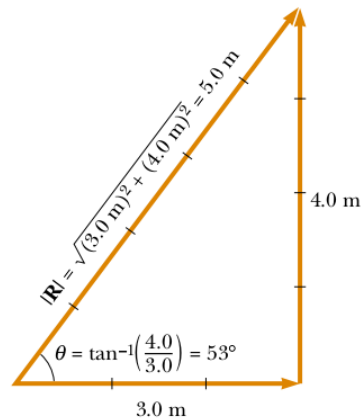For example, if you walked 3.0 m toward the east and then 4.0 m toward the north, as shown in Figure 3.7, you would find yourself 5.0 m from where you
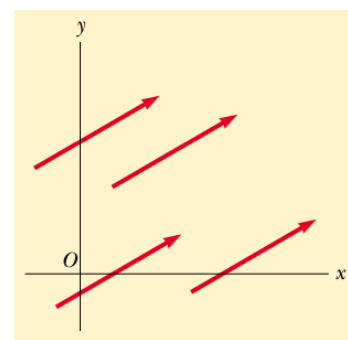


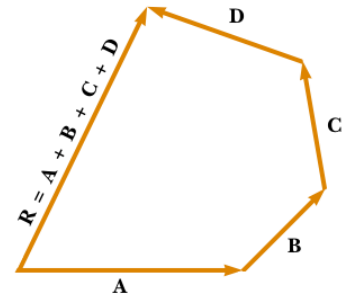**Figure 3.5** These four vectors are equal because they have equal lengths and point in the same direction.
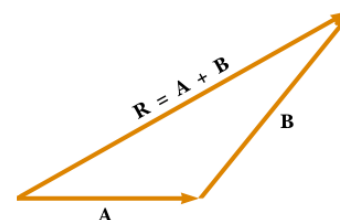


**Figure 3.6** When vector **B** is added to vector **A**, the resultant **R** is the vector that runs from the tail of **A** to the tip of **B**.

**Figure 3.7**   Vector addition. Walking first 3.0 m due east and then 4.0 m due north leaves you $|\mathbf{R}| =$ 5.0 m from your starting point.



**Figure 3.8**   Geometric construction for summing four vectors. The resultant vector $\mathbf{R}$ is by definition the one that completes the polygon.

started, measured at an angle of 53° north of east. Your total displacement is the vector sum of the individual displacements.

A geometric construction can also be used to add more than two vectors. This is shown in Figure 3.8 for the case of four vectors. The resultant vector $\mathbf{R} = \mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}$ is the vector that completes the polygon. In other words, **R is the vector drawn from the tail of the first vector to the tip of the last vector.**

An alternative graphical procedure for adding two vectors, known as the **parallelogram rule of addition,** is shown in Figure 3.9a. In this construction, the tails of the two vectors $\mathbf{A}$ and $\mathbf{B}$ are joined together and the resultant vector $\mathbf{R}$ is the diagonal of a parallelogram formed with $\mathbf{A}$ and $\mathbf{B}$ as two of its four sides.

When two vectors are added, the sum is independent of the order of the addition. (This fact may seem trivial, but as you will see in Chapter 11, the order is important when vectors are multiplied). This can be seen from the geometric construction in Figure 3.9b and is known as the **commutative law of addition:**

Commutative law

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{3.5}$$

When three or more vectors are added, their sum is independent of the way in which the individual vectors are grouped together. A geometric proof of this rule
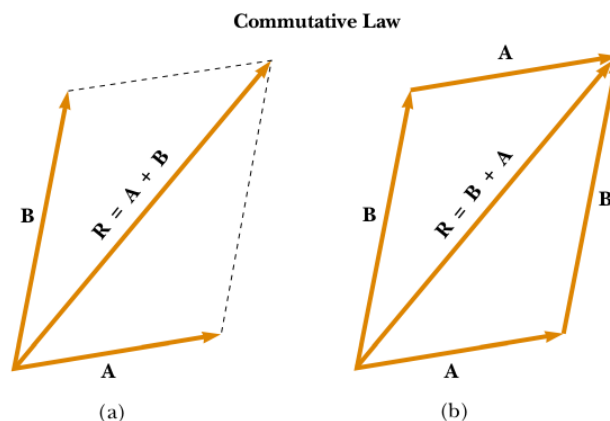
**Commutative Law**



**Figure 3.9**   (a) In this construction, the resultant $\mathbf{R}$ is the diagonal of a parallelogram having sides $\mathbf{A}$ and $\mathbf{B}$. (b) This construction shows that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$—in other words, that vector addition is commutative.

(a)          (b)
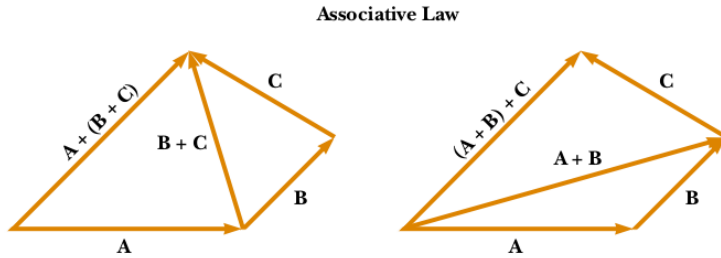
**Associative Law**

for three vectors is given in Figure 3.10. This is called the **associative law of addi-
tion:**

$$A + (B + C) = (A + B) + C \qquad (3.6)$$

Associative law

In summary, **a vector quantity has both magnitude and direction and also
obeys the laws of vector addition** as described in Figures 3.6 to 3.10. When two
or more vectors are added together, *all* of them *must* have the same units. It would
be meaningless to add a velocity vector (for example, 60 km/h to the east) to a dis-
placement vector (for example, 200 km to the north) because they represent dif-
ferent physical quantities. The same rule also applies to scalars. For example, it
would be meaningless to add time intervals to temperatures.

## Negative of a Vector

The negative of the vector **A** is defined as the vector that when added to **A** gives
zero for the vector sum. That is, $A + (-A) = 0$. The vectors **A** and $-A$ have the
same magnitude but point in opposite directions.

## Subtracting Vectors

The operation of vector subtraction makes use of the definition of the negative of
a vector. We define the operation $A - B$ as vector $-B$ added to vector **A**:

$$A - B = A + (-B) \qquad (3.7)$$

The geometric construction for subtracting two vectors in this way is illustrated in
Figure 3.11a.

Another way of looking at vector subtraction is to note that the difference
$A - B$ between two vectors **A** and **B** is what you have to add to the second vector
to obtain the first. In this case, the vector $A - B$ points from the tip of the second
vector to the tip of the first, as Figure 3.11b shows.

**Vector Subtraction**



(a)          (b)

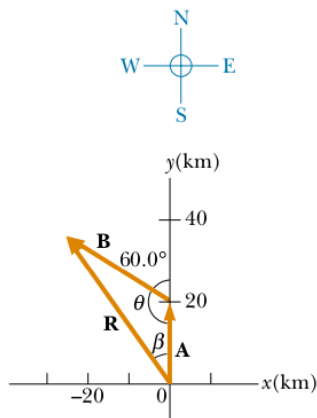**EXAMPLE 3.2**    A Vacation Trip

A car travels 20.0 km due north and then 35.0 km in a direction 60.0° west of north, as shown in Figure 3.12. Find the magnitude and direction of the car's resultant displacement.

**Solution**    In this example, we show two ways to find the resultant of two vectors. We can solve the problem geometrically, using graph paper and a protractor, as shown in Figure 3.12. (In fact, even when you know you are going to be carrying out a calculation, you should sketch the vectors to check your results.) The displacement **R** is the resultant when the two individual displacements **A** and **B** are added.

To solve the problem algebraically, we note that the magnitude of **R** can be obtained from the law of cosines as applied to the triangle (see Appendix B.4). With $\theta = 180° - 60° = 120°$ and $R^2 = A^2 + B^2 - 2AB \cos \theta$, we find that

$$R = \sqrt{A^2 + B^2 - 2AB \cos \theta}$$
$$= \sqrt{(20.0 \text{ km})^2 + (35.0 \text{ km})^2 - 2(20.0 \text{ km})(35.0 \text{ km})\cos 120°}$$
$$= \boxed{48.2 \text{ km}}$$

The direction of **R** measured from the northerly direction can be obtained from the law of sines (Appendix B.4):

$$\frac{\sin \beta}{B} = \frac{\sin \theta}{R}$$

$$\sin \beta = \frac{B}{R} \sin \theta = \frac{35.0 \text{ km}}{48.2 \text{ km}} \sin 120° = 0.629$$
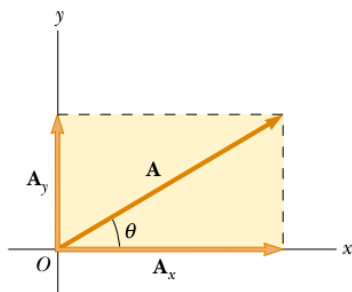
$$\beta = \boxed{38.9°}$$

The resultant displacement of the car is 48.2 km in a direction 38.9° west of north. This result matches what we found graphically.



**Figure 3.12**    Graphical method for finding the resultant displacement vector **R** = **A** + **B**.

## Multiplying a Vector by a Scalar

If vector **A** is multiplied by a positive scalar quantity $m$, then the product $m$**A** is a vector that has the same direction as **A** and magnitude $mA$. If vector **A** is multiplied by a negative scalar quantity $-m$, then the product $-m$**A** is directed opposite **A**. For example, the vector 5**A** is five times as long as **A** and points in the same direction as **A**; the vector $-\frac{1}{3}$**A** is one-third the length of **A** and points in the direction opposite **A**.

**Quick Quiz 3.2**

If vector **B** is added to vector **A**, under what condition does the resultant vector **A** + **B** have magnitude $A + B$? Under what conditions is the resultant vector equal to zero?

## 3.4    COMPONENTS OF A VECTOR AND UNIT VECTORS

The geometric method of adding vectors is not recommended whenever great accuracy is required or in three-dimensional problems. In this section, we describe a method of adding vectors that makes use of the *projections* of vectors along coordinate axes. These projections are called the **components** of the vector. Any vector can be completely described by its components.

Consider a vector **A** lying in the $xy$ plane and making an arbitrary angle $\theta$ with the positive $x$ axis, as shown in Figure 3.13. This vector can be expressed as the



**Figure 3.13**    Any vector **A** lying in the $xy$ plane can be represented by a vector $\mathbf{A}_x$ lying along the $x$ axis and by a vector $\mathbf{A}_y$ lying along the $y$ axis, where $\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y$.

sum of two other vectors $\mathbf{A}_x$ and $\mathbf{A}_y$. From Figure 3.13, we see that the three vectors form a right triangle and that $\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y$. (If you cannot see why this equality holds, go back to Figure 3.9 and review the parallelogram rule.) We shall often refer to the "components of a vector $\mathbf{A}$," written $A_x$ and $A_y$ (without the boldface notation). The component $A_x$ represents the projection of $\mathbf{A}$ along the $x$ axis, and the component $A_y$ represents the projection of $\mathbf{A}$ along the $y$ axis. These components can be positive or negative. The component $A_x$ is positive if $\mathbf{A}_x$ points in the positive $x$ direction and is negative if $\mathbf{A}_x$ points in the negative $x$ direction. The same is true for the component $A_y$.

From Figure 3.13 and the definition of sine and cosine, we see that $\cos \theta = A_x/A$ and that $\sin \theta = A_y/A$. Hence, the components of $\mathbf{A}$ are

$$A_x = A \cos\theta \qquad (3.8)$$

$$A_y = A \sin\theta \qquad (3.9)$$

These components form two sides of a right triangle with a hypotenuse of length $A$. Thus, it follows that the magnitude and direction of $\mathbf{A}$ are related to its components through the expressions

$$A = \sqrt{A_x^2 + A_y^2} \qquad (3.10)$$

Magnitude of $\mathbf{A}$

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right) \qquad (3.11)$$

Direction of $\mathbf{A}$

Note that **the signs of the components $A_x$ and $A_y$ depend on the angle $\theta$.** For example, if $\theta = 120°$, then $A_x$ is negative and $A_y$ is positive. If $\theta = 225°$, then both $A_x$ and $A_y$ are negative. Figure 3.14 summarizes the signs of the components when $\mathbf{A}$ lies in the various quadrants.

When solving problems, you can specify a vector $\mathbf{A}$ either with its components $A_x$ and $A_y$ or with its magnitude and direction $A$ and $\theta$.

| | $y$ | |
|---|---|---|
| $A_x$ negative | | $A_x$ positive |
| $A_y$ positive | | $A_y$ positive |
| $A_x$ negative | | $A_x$ positive $x$ |
| $A_y$ negative | | $A_y$ negative |

**Figure 3.14** The signs of the components of a vector $\mathbf{A}$ depend on the quadrant in which the vector is located.

Can the component of a vector ever be greater than the magnitude of the vector?

Suppose you are working a physics problem that requires resolving a vector into its components. In many applications it is convenient to express the components in a coordinate system having axes that are not horizontal and vertical but are still perpendicular to each other. If you choose reference axes or an angle other than the axes and angle shown in Figure 3.13, the components must be modified accordingly. Suppose a vector $\mathbf{B}$ makes an angle $\theta'$ with the $x'$ axis defined in Figure 3.15. The components of $\mathbf{B}$ along the $x'$ and $y'$ axes are $B_{x'} = B \cos \theta'$ and $B_{y'} = B \sin \theta'$, as specified by Equations 3.8 and 3.9. The magnitude and direction of $\mathbf{B}$ are obtained from expressions equivalent to Equations 3.10 and 3.11. Thus, we can express the components of a vector in *any* coordinate system that is convenient for a particular situation.

## Unit Vectors

Vector quantities often are expressed in terms of unit vectors. **A unit vector is a dimensionless vector having a magnitude of exactly 1.** Unit vectors are used to specify a given direction and have no other physical significance. They are used solely as a convenience in describing a direction in space. We shall use the symbols
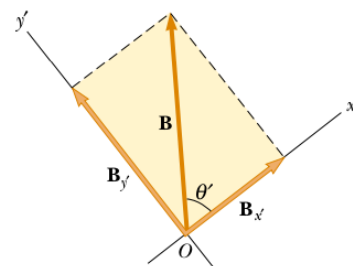
**Figure 3.15** The component vectors of $\mathbf{B}$ in a coordinate system that is tilted.

$\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ to represent unit vectors pointing in the positive $x$, $y$, and $z$ directions, respectively. The unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ form a set of mutually perpendicular vectors in a right-handed coordinate system, as shown in Figure 3.16a. The magnitude of each unit vector equals 1; that is, $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$.

Consider a vector $\mathbf{A}$ lying in the $xy$ plane, as shown in Figure 3.16b. The product of the component $A_x$ and the unit vector $\mathbf{i}$ is the vector $A_x\mathbf{i}$, which lies on the $x$ axis and has magnitude $|A_x|$. (The vector $A_x\mathbf{i}$ is an alternative representation of vector $\mathbf{A}_x$.) Likewise, $A_y\mathbf{j}$ is a vector of magnitude $|A_y|$ lying on the $y$ axis. (Again, vector $A_y\mathbf{j}$ is an alternative representation of vector $\mathbf{A}_y$.) Thus, the unit–vector notation for the vector $\mathbf{A}$ is

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} \tag{3.12}$$

For example, consider a point lying in the $xy$ plane and having cartesian coordinates $(x, y)$, as in Figure 3.17. The point can be specified by the **position vector r,** which in unit–vector form is given by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} \tag{3.13}$$

This notation tells us that the components of $\mathbf{r}$ are the lengths $x$ and $y$.

Now let us see how to use components to add vectors when the geometric method is not sufficiently accurate. Suppose we wish to add vector $\mathbf{B}$ to vector $\mathbf{A}$, where vector $\mathbf{B}$ has components $B_x$ and $B_y$. All we do is add the $x$ and $y$ components separately. The resultant vector $\mathbf{R} = \mathbf{A} + \mathbf{B}$ is therefore
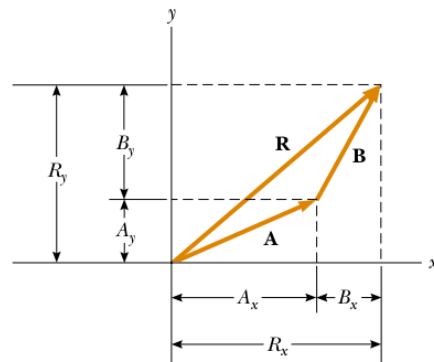
$$\mathbf{R} = (A_x\mathbf{i} + A_y\mathbf{j}) + (B_x\mathbf{i} + B_y\mathbf{j})$$

or

$$\mathbf{R} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} \tag{3.14}$$

Because $\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j}$, we see that the components of the resultant vector are

$$R_x = A_x + B_x$$
$$R_y = A_y + B_y \tag{3.15}$$

Position vector

(a)

(b)
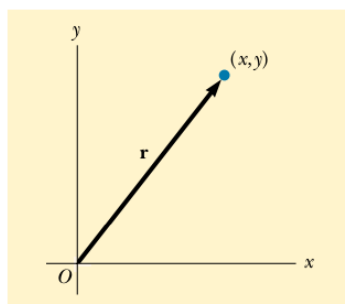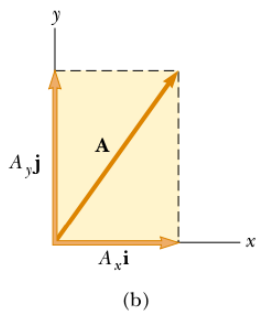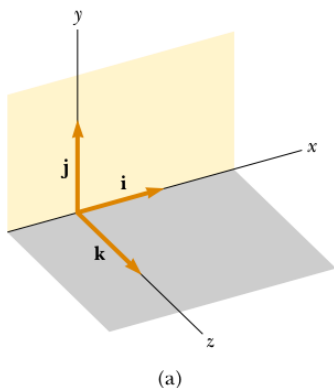
**Figure 3.16** (a) The unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are directed along the $x$, $y$, and $z$ axes, respectively. (b) Vector $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j}$ lying in the $xy$ plane has components $A_x$ and $A_y$.

**Figure 3.17** The point whose cartesian coordinates are $(x, y)$ can be represented by the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.

**Figure 3.18** This geometric construction for the sum of two vectors shows the relationship between the components of the resultant $\mathbf{R}$ and the components of the individual vectors.

We obtain the magnitude of **R** and the angle it makes with the $x$ axis from its components, using the relationships

$$R = \sqrt{R_x{}^2 + R_y{}^2} = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2} \tag{3.16}$$

$$\tan \theta = \frac{R_y}{R_x} = \frac{A_y + B_y}{A_x + B_x} \tag{3.17}$$

We can check this addition by components with a geometric construction, as shown in Figure 3.18. Remember that you must note the *signs* of the components when using either the algebraic or the geometric method.

At times, we need to consider situations involving motion in three component directions. The extension of our methods to three-dimensional vectors is straightforward. If **A** and **B** both have $x$, $y$, and $z$ components, we express them in the form

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \tag{3.18}$$

$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k} \tag{3.19}$$

The sum of **A** and **B** is

$$\mathbf{R} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k} \tag{3.20}$$

Note that Equation 3.20 differs from Equation 3.14: in Equation 3.20, the resultant vector also has a $z$ component $R_z = A_z + B_z$.

> **QuickLab**
>
> Write an expression for the vector describing the displacement of a fly that moves from one corner of the floor of the room that you are in to the opposite corner of the room, near the ceiling.

## Quick Quiz 3.4

If one component of a vector is not zero, can the magnitude of the vector be zero? Explain.

## Quick Quiz 3.5

If **A** + **B** = 0, what can you say about the components of the two vectors?

---

### Problem-Solving Hints

**Adding Vectors**

When you need to add two or more vectors, use this step-by-step procedure:

- Select a coordinate system that is convenient. (Try to reduce the number of components you need to find by choosing axes that line up with as many vectors as possible.)
- Draw a labeled sketch of the vectors described in the problem.
- Find the $x$ and $y$ components of all vectors and the resultant components (the algebraic sum of the components) in the $x$ and $y$ directions.
- If necessary, use the Pythagorean theorem to find the magnitude of the resultant vector and select a suitable trigonometric function to find the angle that the resultant vector makes with the $x$ axis.

### EXAMPLE 3.3 ▸ The Sum of Two Vectors

Find the sum of two vectors **A** and **B** lying in the $xy$ plane and given by

$$\mathbf{A} = (2.0\mathbf{i} + 2.0\mathbf{j}) \text{ m} \quad \text{and} \quad \mathbf{B} = (2.0\mathbf{i} - 4.0\mathbf{j}) \text{ m}$$

**Solution** Comparing this expression for **A** with the general expression $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j}$, we see that $A_x = 2.0$ m and that $A_y = 2.0$ m. Likewise, $B_x = 2.0$ m and $B_y = -4.0$ m. We obtain the resultant vector **R**, using Equation 3.14:

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = (2.0 + 2.0)\mathbf{i} \text{ m} + (2.0 - 4.0)\mathbf{j} \text{ m}$$
$$= (4.0\mathbf{i} - 2.0\mathbf{j}) \text{ m}$$

or

$$R_x = 4.0 \text{ m} \qquad R_y = -2.0 \text{ m}$$

The magnitude of **R** is given by Equation 3.16:

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(4.0 \text{ m})^2 + (-2.0 \text{ m})^2} = \sqrt{20} \text{ m}$$
$$= \boxed{4.5 \text{ m}}$$

We can find the direction of **R** from Equation 3.17:

$$\tan\theta = \frac{R_y}{R_x} = \frac{-2.0 \text{ m}}{4.0 \text{ m}} = -0.50$$

Your calculator likely gives the answer $-27°$ for $\theta = \tan^{-1}(-0.50)$. This answer is correct if we interpret it to mean 27° clockwise from the $x$ axis. Our standard form has been to quote the angles measured counterclockwise from the $+x$ axis, and that angle for this vector is $\theta = \boxed{333°}$.

### EXAMPLE 3.4 ▸ The Resultant Displacement

A particle undergoes three consecutive displacements: $\mathbf{d}_1 = (15\mathbf{i} + 30\mathbf{j} + 12\mathbf{k})$ cm, $\mathbf{d}_2 = (23\mathbf{i} - 14\mathbf{j} - 5.0\mathbf{k})$ cm, and $\mathbf{d}_3 = (-13\mathbf{i} + 15\mathbf{j})$ cm. Find the components of the resultant displacement and its magnitude.

**Solution** Rather than looking at a sketch on flat paper, visualize the problem as follows: Start with your fingertip at the front left corner of your horizontal desktop. Move your fingertip 15 cm to the right, then 30 cm toward the far side of the desk, then 12 cm vertically upward, then 23 cm to the right, then 14 cm horizontally toward the front edge of the desk, then 5.0 cm vertically toward the desk, then 13 cm to the left, and (finally!) 15 cm toward the back of the desk. The

mathematical calculation keeps track of this motion along the three perpendicular axes:

$$\mathbf{R} = \mathbf{d}_1 + \mathbf{d}_2 + \mathbf{d}_3$$
$$= (15 + 23 - 13)\mathbf{i} \text{ cm} + (30 - 14 + 15)\mathbf{j} \text{ cm}$$
$$\quad + (12 - 5.0 + 0)\mathbf{k} \text{ cm}$$
$$= (25\mathbf{i} + 31\mathbf{j} + 7.0\mathbf{k}) \text{ cm}$$

The resultant displacement has components $R_x = 25$ cm, $R_y = 31$ cm, and $R_z = 7.0$ cm. Its magnitude is

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$
$$= \sqrt{(25 \text{ cm})^2 + (31 \text{ cm})^2 + (7.0 \text{ cm})^2} = \boxed{40 \text{ cm}}$$

### EXAMPLE 3.5 ▸ Taking a Hike

A hiker begins a trip by first walking 25.0 km southeast from her car. She stops and sets up her tent for the night. On the second day, she walks 40.0 km in a direction 60.0° north of east, at which point she discovers a forest ranger's tower. (a) Determine the components of the hiker's displacement for each day.

**Solution** If we denote the displacement vectors on the first and second days by **A** and **B**, respectively, and use the car as the origin of coordinates, we obtain the vectors shown in Figure 3.19. Displacement **A** has a magnitude of 25.0 km and is directed 45.0° below the positive $x$ axis. From Equations 3.8 and 3.9, its components are

$$A_x = A\cos(-45.0°) = (25.0 \text{ km})(0.707) = \boxed{17.7 \text{ km}}$$

$$A_y = A\sin(-45.0°) = -(25.0 \text{ km})(0.707) = \boxed{-17.7 \text{ km}}$$
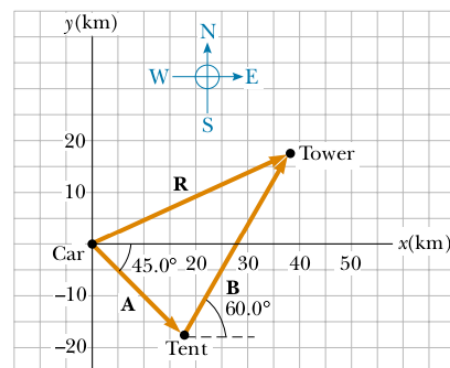


**Figure 3.19** The total displacement of the hiker is the vector **R = A + B**.

The negative value of $A_y$ indicates that the hiker walks in the negative $y$ direction on the first day. The signs of $A_x$ and $A_y$ also are evident from Figure 3.19.

The second displacement **B** has a magnitude of 40.0 km and is 60.0° north of east. Its components are

$$B_x = B\cos 60.0° = (40.0 \text{ km})(0.500) = \boxed{20.0 \text{ km}}$$

$$B_y = B\sin 60.0° = (40.0 \text{ km})(0.866) = \boxed{34.6 \text{ km}}$$

(b) Determine the components of the hiker's resultant displacement **R** for the trip. Find an expression for **R** in terms of unit vectors.

**Solution**   The resultant displacement for the trip **R** = **A** + **B** has components given by Equation 3.15:

$$R_x = A_x + B_x = 17.7 \text{ km} + 20.0 \text{ km} = \boxed{37.7 \text{ km}}$$

$$R_y = A_y + B_y = -17.7 \text{ km} + 34.6 \text{ km} = \boxed{16.9 \text{ km}}$$

In unit–vector form, we can write the total displacement as

$$\boxed{\mathbf{R} = (37.7\mathbf{i} + 16.9\mathbf{j}) \text{ km}}$$

***Exercise***   Determine the magnitude and direction of the total displacement.

***Answer***   41.3 km, 24.1° north of east from the car.

---

## EXAMPLE 3.6 ▶ Let's Fly Away!

A commuter airplane takes the route shown in Figure 3.20. First, it flies from the origin of the coordinate system shown to city A, located 175 km in a direction 30.0° north of east. Next, it flies 153 km 20.0° west of north to city B. Finally, it flies 195 km due west to city C. Find the location of city C relative to the origin.

**Solution**   It is convenient to choose the coordinate system shown in Figure 3.20, where the $x$ axis points to the east and the $y$ axis points to the north. Let us denote the three consecutive displacements by the vectors **a**, **b**, and **c**. Displacement **a** has a magnitude of 175 km and the components

$$a_x = a\cos(30.0°) = (175 \text{ km})(0.866) = 152 \text{ km}$$

$$a_y = a\sin(30.0°) = (175 \text{ km})(0.500) = 87.5 \text{ km}$$
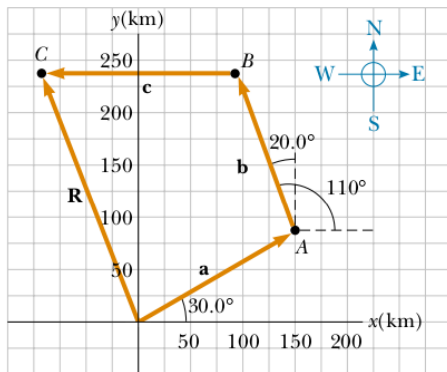


**Figure 3.20**   The airplane starts at the origin, flies first to city A, then to city B, and finally to city C.

Displacement **b**, whose magnitude is 153 km, has the components

$$b_x = b\cos(110°) = (153 \text{ km})(-0.342) = -52.3 \text{ km}$$

$$b_y = b\sin(110°) = (153 \text{ km})(0.940) = 144 \text{ km}$$

Finally, displacement **c**, whose magnitude is 195 km, has the components

$$c_x = c\cos(180°) = (195 \text{ km})(-1) = -195 \text{ km}$$

$$c_y = c\sin(180°) = 0$$

Therefore, the components of the position vector **R** from the starting point to city C are

$$R_x = a_x + b_x + c_x = 152 \text{ km} - 52.3 \text{ km} - 195 \text{ km}$$

$$= \boxed{-95.3 \text{ km}}$$

$$R_y = a_y + b_y + c_y = 87.5 \text{ km} + 144 \text{ km} + 0$$

$$= \boxed{232 \text{ km}}$$

In unit–vector notation, $\boxed{\mathbf{R} = (-95.3\mathbf{i} + 232\mathbf{j}) \text{ km.}}$ That is, the airplane can reach city C from the starting point by first traveling 95.3 km due west and then by traveling 232 km due north.

***Exercise***   Find the magnitude and direction of **R**.

***Answer***   251 km, 22.3° west of north.
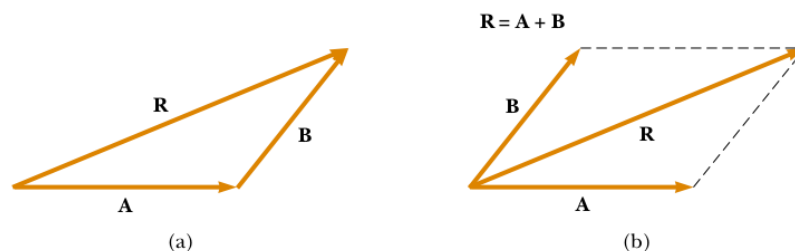
***Figure 3.21***    (a) Vector addition by the triangle method. (b) Vector addition by the parallelogram rule.

## SUMMARY

**Scalar quantities** are those that have only magnitude and no associated direction. **Vector quantities** have both magnitude and direction and obey the laws of vector addition.

We can add two vectors **A** and **B** graphically, using either the triangle method or the parallelogram rule. In the triangle method (Fig. 3.21a), the resultant vector **R** = **A** + **B** runs from the tail of **A** to the tip of **B**. In the parallelogram method (Fig. 3.21b), **R** is the diagonal of a parallelogram having **A** and **B** as two of its sides. You should be able to add or subtract vectors, using these graphical methods.

The $x$ component $A_x$ of the vector **A** is equal to the projection of **A** along the $x$ axis of a coordinate system, as shown in Figure 3.22, where $A_x = A \cos \theta$. The $y$ component $A_y$ of **A** is the projection of **A** along the $y$ axis, where $A_y = A \sin \theta$. Be sure you can determine which trigonometric functions you should use in all situations, especially when $\theta$ is defined as something other than the counterclockwise angle from the positive $x$ axis.

If a vector **A** has an $x$ component $A_x$ and a $y$ component $A_y$, the vector can be expressed in unit–vector form as **A** = $A_x$**i** + $A_y$**j**. In this notation, **i** is a unit vector pointing in the positive $x$ direction, and **j** is a unit vector pointing in the positive $y$ direction. Because **i** and **j** are unit vectors, $|\mathbf{i}| = |\mathbf{j}| = 1$.

We can find the resultant of two or more vectors by resolving all vectors into their $x$ and $y$ components, adding their resultant $x$ and $y$ components, and then using the Pythagorean theorem to find the magnitude of the resultant vector. We can find the angle that the resultant vector makes with respect to the $x$ axis by using a suitable trigonometric function.
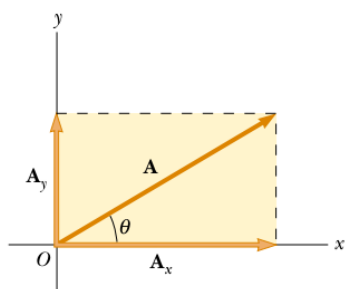


***Figure 3.22***    The addition of the two vectors $A_x$ and $A_y$ gives vector **A**. Note that $\mathbf{A}_x = A_x\mathbf{i}$ and $\mathbf{A}_y = A_y\mathbf{j}$, where $A_x$ and $A_y$ are the *components* of vector **A**.

## QUESTIONS

1. Two vectors have unequal magnitudes. Can their sum be zero? Explain.
2. Can the magnitude of a particle's displacement be greater than the distance traveled? Explain.
3. The magnitudes of two vectors **A** and **B** are $A = 5$ units and $B = 2$ units. Find the largest and smallest values possible for the resultant vector **R** = **A** + **B**.
4. Vector **A** lies in the $xy$ plane. For what orientations of vector **A** will both of its components be negative? For what orientations will its components have opposite signs?
5. If the component of vector **A** along the direction of vector

    **B** is zero, what can you conclude about these two vectors?
6. Can the magnitude of a vector have a negative value? Explain.
7. Which of the following are vectors and which are not: force, temperature, volume, ratings of a television show, height, velocity, age?
8. Under what circumstances would a nonzero vector lying in the $xy$ plane ever have components that are equal in magnitude?
9. Is it possible to add a vector quantity to a scalar quantity? Explain.

## PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging    ☐ = full solution available in the *Student Solutions Manual and Study Guide*
**WEB** = solution posted at **http://www.saunderscollege.com/physics/**    🖥 = Computer useful in solving problem    🌐 = Interactive Physics
☐ = paired numerical/symbolic problems

### Section 3.1    Coordinate Systems

**WEB** **1.** The polar coordinates of a point are $r = 5.50$ m and $\theta = 240°$. What are the cartesian coordinates of this point?

**2.** Two points in the *xy* plane have cartesian coordinates $(2.00, -4.00)$ m and $(-3.00, 3.00)$ m. Determine (a) the distance between these points and (b) their polar coordinates.

**3.** If the cartesian coordinates of a point are given by $(2, y)$ and its polar coordinates are $(r, 30°)$, determine $y$ and $r$.

**4.** Two points in a plane have polar coordinates (2.50 m, 30.0°) and (3.80 m, 120.0°). Determine (a) the cartesian coordinates of these points and (b) the distance between them.

**5.** A fly lands on one wall of a room. The lower left-hand corner of the wall is selected as the origin of a two-dimensional cartesian coordinate system. If the fly is located at the point having coordinates (2.00, 1.00) m, (a) how far is it from the corner of the room? (b) what is its location in polar coordinates?

**6.** If the polar coordinates of the point $(x, y)$ are $(r, \theta)$, determine the polar coordinates for the points (a) $(-x, y)$, (b) $(-2x, -2y)$, and (c) $(3x, -3y)$.

### Section 3.2    Vector and Scalar Quantities

### Section 3.3    Some Properties of Vectors

**7.** An airplane flies 200 km due west from city A to city B and then 300 km in the direction 30.0° north of west from city B to city C. (a) In straight-line distance, how far is city C from city A? (b) Relative to city A, in what direction is city C?

**8.** A pedestrian moves 6.00 km east and then 13.0 km north. Using the graphical method, find the magnitude and direction of the resultant displacement vector.

**9.** A surveyor measures the distance across a straight river by the following method: Starting directly across from a tree on the opposite bank, she walks 100 m along the riverbank to establish a baseline. Then she sights across to the tree. The angle from her baseline to the tree is 35.0°. How wide is the river?

**10.** A plane flies from base camp to lake A, a distance of 280 km at a direction 20.0° north of east. After dropping off supplies, it flies to lake B, which is 190 km and 30.0° west of north from lake A. Graphically determine the distance and direction from lake B to the base camp.

**11.** Vector **A** has a magnitude of 8.00 units and makes an angle of 45.0° with the positive *x* axis. Vector **B** also has a magnitude of 8.00 units and is directed along the neg-ative *x* axis. Using graphical methods, find (a) the vector sum **A** + **B** and (b) the vector difference **A** − **B**.

**12.** A force $\mathbf{F}_1$ of magnitude 6.00 units acts at the origin in a direction 30.0° above the positive *x* axis. A second force $\mathbf{F}_2$ of magnitude 5.00 units acts at the origin in the direction of the positive *y* axis. Find graphically the magnitude and direction of the resultant force $\mathbf{F}_1 + \mathbf{F}_2$.

**WEB** **13.** A person walks along a circular path of radius 5.00 m. If the person walks around one half of the circle, find (a) the magnitude of the displacement vector and (b) how far the person walked. (c) What is the magnitude of the displacement if the person walks all the way around the circle?

**14.** A dog searching for a bone walks 3.50 m south, then 8.20 m at an angle 30.0° north of east, and finally 15.0 m west. Using graphical techniques, find the dog's resultant displacement vector.

**WEB** **15.** Each of the displacement vectors **A** and **B** shown in Figure P3.15 has a magnitude of 3.00 m. Find graphically (a) **A** + **B**, (b) **A** − **B**, (c) **B** − **A**, (d) **A** − 2**B**. Report all angles counterclockwise from the positive *x* axis.
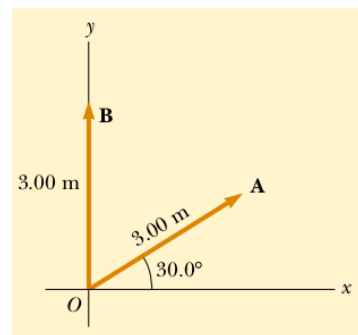


**Figure P3.15**    Problems 15 and 39.

**16.** Arbitrarily define the "instantaneous vector height" of a person as the displacement vector from the point halfway between the feet to the top of the head. Make an order-of-magnitude estimate of the total vector height of all the people in a city of population 100 000 (a) at 10 a.m. on a Tuesday and (b) at 5 a.m. on a Saturday. Explain your reasoning.

**17.** A roller coaster moves 200 ft horizontally and then rises 135 ft at an angle of 30.0° above the horizontal. It then travels 135 ft at an angle of 40.0° downward. What is its displacement from its starting point? Use graphical techniques.

**18.** The driver of a car drives 3.00 km north, 2.00 km north-east (45.0° east of north), 4.00 km west, and then

3.00 km southeast (45.0° east of south). Where does he end up relative to his starting point? Work out your answer graphically. Check by using components. (The car is not near the North Pole or the South Pole.)

19. Fox Mulder is trapped in a maze. To find his way out, he walks 10.0 m, makes a 90.0° right turn, walks 5.00 m, makes another 90.0° right turn, and walks 7.00 m. What is his displacement from his initial position?

**Section 3.4    Components of a Vector and Unit Vectors**

20. Find the horizontal and vertical components of the 100-m displacement of a superhero who flies from the top of a tall building following the path shown in Figure P3.20.
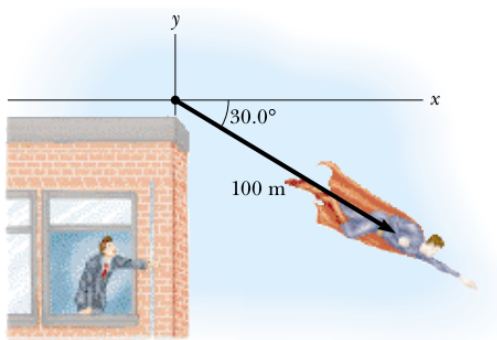


**Figure P3.20**

21. A person walks 25.0° north of east for 3.10 km. How far would she have to walk due north and due east to arrive at the same location?

22. While exploring a cave, a spelunker starts at the entrance and moves the following distances: She goes 75.0 m north, 250 m east, 125 m at an angle 30.0° north of east, and 150 m south. Find the resultant displacement from the cave entrance.

23. In the assembly operation illustrated in Figure P3.23, a robot first lifts an object upward along an arc that forms one quarter of a circle having a radius of 4.80 cm and
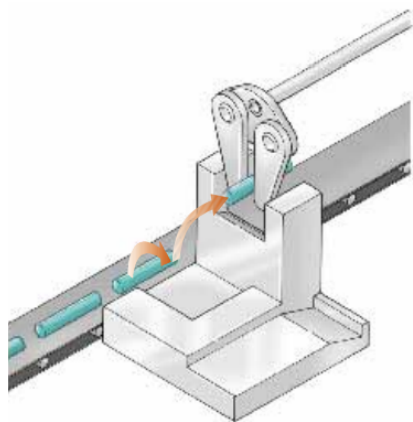


**Figure P3.23**

lying in an east–west vertical plane. The robot then moves the object upward along a second arc that forms one quarter of a circle having a radius of 3.70 cm and lying in a north–south vertical plane. Find (a) the magnitude of the total displacement of the object and (b) the angle the total displacement makes with the vertical.

24. Vector **B** has $x$, $y$, and $z$ components of 4.00, 6.00, and 3.00 units, respectively. Calculate the magnitude of **B** and the angles that **B** makes with the coordinate axes.

WEB 25. A vector has an $x$ component of $-25.0$ units and a $y$ component of 40.0 units. Find the magnitude and direction of this vector.

26. A map suggests that Atlanta is 730 mi in a direction 5.00° north of east from Dallas. The same map shows that Chicago is 560 mi in a direction 21.0° west of north from Atlanta. Assuming that the Earth is flat, use this information to find the displacement from Dallas to Chicago.

27. A displacement vector lying in the $xy$ plane has a magnitude of 50.0 m and is directed at an angle of 120° to the positive $x$ axis. Find the $x$ and $y$ components of this vector and express the vector in unit–vector notation.

28. If **A** $= 2.00\mathbf{i} + 6.00\mathbf{j}$ and **B** $= 3.00\mathbf{i} - 2.00\mathbf{j}$, (a) sketch the vector sum **C** $=$ **A** $+$ **B** and the vector difference **D** $=$ **A** $-$ **B**. (b) Find solutions for **C** and **D**, first in terms of unit vectors and then in terms of polar coordinates, with angles measured with respect to the $+x$ axis.

29. Find the magnitude and direction of the resultant of three displacements having $x$ and $y$ components (3.00, 2.00) m, $(-5.00, 3.00)$ m, and (6.00, 1.00) m.

30. Vector **A** has $x$ and $y$ components of $-8.70$ cm and 15.0 cm, respectively; vector **B** has $x$ and $y$ components of 13.2 cm and $-6.60$ cm, respectively. If **A** $-$ **B** $+$ $3$**C** $= 0$, what are the components of **C**?

31. Consider two vectors **A** $= 3\mathbf{i} - 2\mathbf{j}$ and **B** $= -\mathbf{i} - 4\mathbf{j}$. Calculate (a) **A** $+$ **B**, (b) **A** $-$ **B**, (c) $|\mathbf{A} + \mathbf{B}|$, (d) $|\mathbf{A} - \mathbf{B}|$, (e) the directions of **A** $+$ **B** and **A** $-$ **B**.

32. A boy runs 3.00 blocks north, 4.00 blocks northeast, and 5.00 blocks west. Determine the length and direction of the displacement vector that goes from the starting point to his final position.

33. Obtain expressions in component form for the position vectors having polar coordinates (a) 12.8 m, 150°; (b) 3.30 cm, 60.0°; (c) 22.0 in., 215°.

34. Consider the displacement vectors **A** $= (3\mathbf{i} + 3\mathbf{j})$ m, **B** $= (\mathbf{i} - 4\mathbf{j})$ m, and **C** $= (-2\mathbf{i} + 5\mathbf{j})$ m. Use the component method to determine (a) the magnitude and direction of the vector **D** $=$ **A** $+$ **B** $+$ **C** and (b) the magnitude and direction of **E** $= -$**A** $-$ **B** $+$ **C**.

35. A particle undergoes the following consecutive displacements: 3.50 m south, 8.20 m northeast, and 15.0 m west. What is the resultant displacement?

36. In a game of American football, a quarterback takes the ball from the line of scrimmage, runs backward for 10.0 yards, and then sideways parallel to the line of scrimmage for 15.0 yards. At this point, he throws a forward

pass 50.0 yards straight downfield perpendicular to the line of scrimmage. What is the magnitude of the football's resultant displacement?

37. The helicopter view in Figure P3.37 shows two people pulling on a stubborn mule. Find (a) the single force that is equivalent to the two forces shown and (b) the force that a third person would have to exert on the mule to make the resultant force equal to zero. The forces are measured in units of newtons.
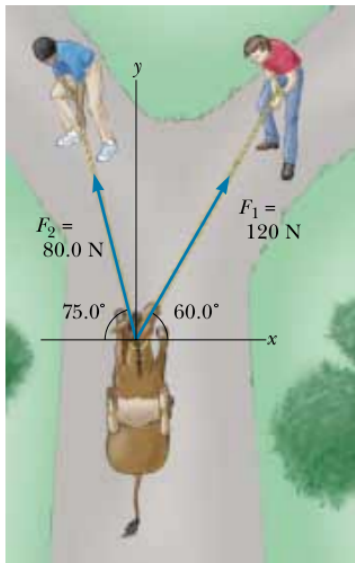


**Figure P3.37**

38. A novice golfer on the green takes three strokes to sink the ball. The successive displacements are 4.00 m to the north, 2.00 m northeast, and 1.00 m 30.0° west of south. Starting at the same initial point, an expert golfer could make the hole in what single displacement?

39. Find the $x$ and $y$ components of the vectors **A** and **B** shown in Figure P3.15; then derive an expression for the resultant vector **A** + **B** in unit–vector notation.

40. You are standing on the ground at the origin of a coordinate system. An airplane flies over you with constant velocity parallel to the $x$ axis and at a constant height of $7.60 \times 10^3$ m. At $t = 0$, the airplane is directly above you, so that the vector from you to it is given by $\mathbf{P}_0 = (7.60 \times 10^3 \text{ m})\mathbf{j}$. At $t = 30.0$ s, the position vector leading from you to the airplane is $\mathbf{P}_{30} = (8.04 \times 10^3 \text{ m})\mathbf{i} + (7.60 \times 10^3 \text{ m})\mathbf{j}$. Determine the magnitude and orientation of the airplane's position vector at $t = 45.0$ s.

41. A particle undergoes two displacements. The first has a magnitude of 150 cm and makes an angle of 120° with the positive $x$ axis. The *resultant* displacement has a magnitude of 140 cm and is directed at an angle of 35.0° to the positive $x$ axis. Find the magnitude and direction of the second displacement.

42. Vectors **A** and **B** have equal magnitudes of 5.00. If the sum of **A** and **B** is the vector 6.00 **j**, determine the angle between **A** and **B**.

43. The vector **A** has $x$, $y$, and $z$ components of 8.00, 12.0, and − 4.00 units, respectively. (a) Write a vector expression for **A** in unit–vector notation. (b) Obtain a unit–vector expression for a vector **B** one-fourth the length of **A** pointing in the same direction as **A**. (c) Obtain a unit–vector expression for a vector **C** three times the length of **A** pointing in the direction opposite the direction of **A**.

44. Instructions for finding a buried treasure include the following: Go 75.0 paces at 240°, turn to 135° and walk 125 paces, then travel 100 paces at 160°. The angles are measured counterclockwise from an axis pointing to the east, the $+ x$ direction. Determine the resultant displacement from the starting point.

45. Given the displacement vectors $\mathbf{A} = (3\mathbf{i} − 4\mathbf{j} + 4\mathbf{k})$ m and $\mathbf{B} = (2\mathbf{i} + 3\mathbf{j} − 7\mathbf{k})$ m, find the magnitudes of the vectors (a) $\mathbf{C} = \mathbf{A} + \mathbf{B}$ and (b) $\mathbf{D} = 2\mathbf{A} − \mathbf{B}$, also expressing each in terms of its $x$, $y$, and $z$ components.

46. A radar station locates a sinking ship at range 17.3 km and bearing 136° clockwise from north. From the same station a rescue plane is at horizontal range 19.6 km, 153° clockwise from north, with elevation 2.20 km. (a) Write the vector displacement from plane to ship, letting **i** represent east, **j** north, and **k** up. (b) How far apart are the plane and ship?

47. As it passes over Grand Bahama Island, the eye of a hurricane is moving in a direction 60.0° north of west with a speed of 41.0 km/h. Three hours later, the course of the hurricane suddenly shifts due north and its speed slows to 25.0 km/h. How far from Grand Bahama is the eye 4.50 h after it passes over the island?

48. (a) Vector **E** has magnitude 17.0 cm and is directed 27.0° counterclockwise from the $+ x$ axis. Express it in unit–vector notation. (b) Vector **F** has magnitude 17.0 cm and is directed 27.0° counterclockwise from the $+ y$ axis. Express it in unit–vector notation. (c) Vector **G** has magnitude 17.0 cm and is directed 27.0° clockwise from the $+ y$ axis. Express it in unit–vector notation.

49. Vector **A** has a negative $x$ component 3.00 units in length and a positive $y$ component 2.00 units in length. (a) Determine an expression for **A** in unit–vector notation. (b) Determine the magnitude and direction of **A**. (c) What vector **B**, when added to vector **A**, gives a resultant vector with no $x$ component and a negative $y$ component 4.00 units in length?

50. An airplane starting from airport A flies 300 km east, then 350 km at 30.0° west of north, and then 150 km north to arrive finally at airport B. (a) The next day, another plane flies directly from airport A to airport B in a straight line. In what direction should the pilot travel in this direct flight? (b) How far will the pilot travel in this direct flight? Assume there is no wind during these flights.