

Plasma Physics - II

21-02-2019.

Thursday.

Hydromagnetic Waves:-

We shall take into account both electrons and ions.

- i) Alfvén wave.
- ii) Magnetosonic wave.

Alfvén wave:-

low frequency waves. It means

$$\omega \ll \omega_{ce} = \frac{eB_0}{m_e}$$

$$\omega \ll \omega_{ci} = \frac{eB_0}{m_i}$$

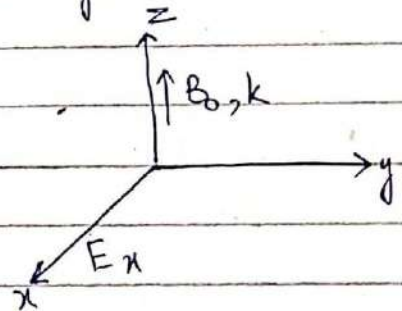
→ Ions are low frequency waves because their inertia is high.

Suppose that Electric field is along x-axis, magnetic field is along z-axis & \vec{k} is along \hat{z} axis.

$$\vec{E} = E_x \hat{x}$$

$$\vec{B} = B_0 \hat{z}$$

$$\vec{k} = k \hat{z}$$



Maxwell's Equation:

$$\nabla \times \vec{B} = \frac{4\pi}{c^2} \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \quad (1)$$

This is electromagnetic wave so, \vec{B} , \vec{E} are perturbed

$$\nabla \times \vec{E}_1 = -\dot{\vec{B}}_1 \quad (2)$$

Differentiating eq. (1) w.r.t "time".

$$\nabla \times \dot{\vec{B}}_1 = \frac{4\pi}{c^2} \dot{\vec{J}}_1 + \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \quad (3)$$

From eq. (2) value of B_1 substituting in eq. (3)

$$-c^2 [\nabla \times (\nabla \times \vec{E}_1)] = 4\pi \dot{\vec{J}}_1 + \ddot{\vec{E}}_1$$

$$-c^2 [\nabla \cdot \vec{E}_1 - \nabla^2 E_1] = \frac{4\pi}{\epsilon_0} \vec{J}_1 + \frac{1}{\epsilon_0} \ddot{\vec{E}}_1$$

$\nabla \cdot \vec{E}_1 = 0$ because \vec{E} , \vec{E}_1 , \vec{k} are perpendicular to each other in transverse wave.

$$c^2 \nabla^2 E_1 = \frac{4\pi}{\epsilon_0} \vec{J}_1 + \frac{1}{\epsilon_0} \ddot{\vec{E}}_1$$

$$c^2 (-k^2 E_1) = \frac{4\pi}{\epsilon_0} (-i\omega \vec{J}_1) + \frac{1}{\epsilon_0} (-\omega^2) E_1$$

$$(\omega^2 - c^2 k^2) E_1 = -4\pi i\omega \vec{J}_1$$

Writing the x-component of above equation,

$$(\omega^2 - c^2 k^2) E_x = -i4\pi\omega n_0 e (V_{ix} - V_{ex}) \quad \because \vec{J} = n_e (\vec{V}_i - \vec{V}_e)$$

$$(\omega^2 - c^2 k^2) E_x = -i4\pi\omega n_0 e (V_{ix} - V_{ex}) \quad \text{--- (4)}$$

For the velocity we have to write the equation of motion,

Linearized form of Equation of motion for ions:

$$m_i n_0 \frac{\partial V_{ix}}{\partial t} = n_0 e [E_x + V_{iy} B_0]$$

X-component:

$$m_i \frac{\partial V_{ix}}{\partial t} = e [E_{ix} + V_{iy} B_0]$$

By applying fourier transformation, we get

$$-i\omega V_{ix} = \frac{e}{m_i} (E_{ix} + V_{iy} B_0)$$

$$V_{ix} = \frac{+ie}{m_i \omega} [E_{ix} + V_{iy} B_0] \quad \text{--- (5)}$$

Y-component:

$$V_{iy} = \frac{ie}{m_i \omega} (0 - V_{ix} B_0)$$

$$V_{iy} = \frac{-ie B_0}{m_i \omega} V_{ix}$$

$$V_{iy} = \frac{-i\omega c_i}{\omega} V_{ix} \quad \text{--- (6)}$$

Substituting eq. (6) into eq. (5)

$$V_{ix} = \frac{ie}{m_i \omega} \left(E_{ix} + \left(-\frac{i \omega c_i}{\omega} V_{ix} \right) B_0 \right)$$

$$V_{ix} = \frac{ie}{m_i \omega} E_{ix} + \frac{\omega c_i e}{m_i \omega^2} V_{ix} B_0$$

$$\therefore \frac{e B_0}{m_i} = \omega c_i$$

$$V_{ix} = \frac{ie}{m_i \omega} E_{ix} + \frac{\omega c_i^2}{\omega^2} V_{ix}$$

$$\left(1 - \frac{\omega c_i^2}{\omega^2} \right) V_{ix} = \frac{ie}{m_i \omega} E_{ix}$$

$$V_{ix} = \frac{ie}{m_i \omega} E_{ix} \left(1 - \frac{\omega c_i^2}{\omega^2} \right)^{-1} \quad \text{--- (7)}$$

Similarly for the electrons,

$$m_e \frac{\partial V_{ey}}{\partial t} = -\gamma_e e (E_x + V_{ey} \times B_0)$$

x-component:

$$m_e \frac{\partial V_{ex}}{\partial t} = -e (E_x + V_{ey} B_0)$$

$$-i \omega V_{ex} = \frac{-e}{m_e} (E_x + V_{ey} B_0)$$

$$V_{ex} = \frac{-ie}{m_e \omega} (E_x + V_{ey} B_0) \quad \text{--- (A)}$$

y-component:

$$V_{ey} = \frac{-ie}{m_e \omega} (0 - V_{ex} B_0)$$

$$V_{ey} = \frac{ie}{m_e \omega} V_{ex} B_0 \Rightarrow V_{ey} = \frac{i \omega c_e}{\omega} V_{ex} \quad \text{--- (B)}$$

Put eq. (B) in eq. (A)

$$V_{ex} = \frac{-ie}{m_e \omega} \left(E_x + \frac{i \omega c_e}{\omega} V_{ex} B_0 \right)$$

$$V_{ex} = \frac{-ie}{m_e \omega} E_x + \frac{\omega c_e^2}{\omega^2} V_{ex}$$

$$\left(1 - \frac{\omega_{ce}^2}{\omega^2}\right) V_{ex} = \frac{-ie}{m_e \omega} E_x$$

$$V_{ex} = \frac{-ie}{m_e \omega} E_x \left(1 - \frac{\omega_{ce}^2}{\omega^2}\right)^{-1} \quad \text{--- (8)}$$

Substituting eq. (7) ^{eq. (8)} into eq. (4).

$$(\omega^2 - c^2 k^2) E_x = -i 4\pi \omega n_0 e \left[\frac{ie}{m_i \omega} \left(1 - \frac{\omega_{ci}^2}{\omega^2}\right)^{-1} + \frac{ie}{m_e \omega} \left(1 - \frac{\omega_{ce}^2}{\omega^2}\right)^{-1} \right] E_x$$

$$(\omega^2 - c^2 k^2) = \frac{4\pi n_0 e^2}{m_i} \left(1 - \frac{\omega_{ci}^2}{\omega^2}\right)^{-1} + \frac{4\pi n_0 e^2}{m_e} \left(1 - \frac{\omega_{ce}^2}{\omega^2}\right)^{-1}$$

$$(\omega^2 - c^2 k^2) = \omega_{pi}^2 \left(1 - \frac{\omega_{ci}^2}{\omega^2}\right)^{-1} + \omega_{pe}^2 \left(1 - \frac{\omega_{ce}^2}{\omega^2}\right)^{-1}$$

$$(\omega^2 - c^2 k^2) = \omega_{pi}^2 \left(\frac{\omega^2 - \omega_{ci}^2}{\omega^2}\right)^{-1} + \omega_{pe}^2 \left(\frac{\omega^2 - \omega_{ce}^2}{\omega^2}\right)^{-1} \quad \text{--- (9)}$$

Applying low frequency condition,

$$\omega^2 \ll \omega_{ce}^2 \quad \& \quad \omega^2 \ll \omega_{ci}^2$$

$$(\omega^2 - c^2 k^2) = \omega_{pi}^2 \left(-\frac{\omega_{ci}^2}{\omega^2}\right)^{-1} + \omega_{pe}^2 \left(-\frac{\omega_{ce}^2}{\omega^2}\right)^{-1}$$

$$\omega^2 - c^2 k^2 = - \left[\frac{\omega_{pi}^2 \omega^2}{\omega_{ci}^2} + \omega_{pe}^2 \frac{\omega^2}{\omega_{ce}^2} \right]$$

$$\omega^2 - c^2 k^2 = -\omega^2 \left(\omega_{pi}^2 \cdot \frac{1}{\omega_{ci}^2} + \omega_{pe}^2 \cdot \frac{1}{\omega_{ce}^2} \right)$$

$$\omega^2 - c^2 k^2 = -\omega^2 \left(\frac{4\pi n_0 e^2 \cdot m_i^2}{m_i \cdot e^2 B^2} + \frac{4\pi n_0 e^2 \cdot m_e^2}{m_e \cdot e^2 B^2} \right)$$

$$\omega^2 - c^2 k^2 = -\omega^2 \left[\frac{4\pi n_0 m_i}{B^2} + \frac{4\pi n_0 m_e}{B^2} \right]$$

$$\omega^2 - c^2 k^2 = -\omega^2 \frac{4\pi}{B^2} (n_0 m_i + n_0 m_e)$$

So,

$$\because \beta_i = n_0 m_i$$

$$\because \beta_e = n_0 m_e$$

$$\omega^2 - c^2 k^2 = -\omega^2 \frac{4\pi P}{B_0^2}$$

$$\because P = P_i + P_e$$

= n_{om} + n_{me} .

$$\omega^2 + \omega^2 \frac{4\pi P}{B_0^2} = c^2 k^2$$

$$\omega^2 \left(1 + \frac{4\pi P}{B_0^2} \right) = c^2 k^2 \quad \text{--- (10)}$$

$$V_A^2 = \left(\frac{4\pi P}{B_0^2 c^2} \right)^{-1} \text{ is Alfvén velocity } \Rightarrow V_A^2 = \frac{B_0^2 c^2}{4\pi P}$$

Multiplying and dividing eq. (10) by c^2 .

$$\omega^2 \left(1 + \frac{4\pi P c^2}{B_0^2 c^2} \right) = c^2 k^2$$

$$\omega^2 \left(1 + \frac{V_A^2}{c^2} \right) = c^2 k^2$$

$$\frac{\omega^2}{k^2} = \frac{c^2}{1 + \frac{V_A^2}{c^2}}$$

$$\frac{\omega^2}{k^2} = \frac{V_A^2}{V_A^2 + c^2}$$

$$\frac{\omega^2}{k^2} = \frac{c^2 V_A^2}{V_A^2 + c^2} \quad \because V_A \ll c, \text{ so we can write } V_A^2 + c^2 = c^2.$$

$$\frac{\omega^2}{k^2} = \frac{c^2 V_A^2}{c^2} = V_A^2$$

It is the dispersion relation of Alfvén wave.

$$\frac{\omega^2}{k^2} = V_A^2 = \frac{B_0^2 c^2}{4\pi P_0}$$

On the L.H.S, we have wave character & on R.H.S we have particle character. It means wave & particles are stuck together. Particle & wave are move together. Example: Guitar.