Subject: Signals and Systems

Chapter: 04

The Continuous Time Fourier Transform

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Example 4.3

Now let us determine the Fourier transform of the unit impulse

$$x(t) = \delta(t). \tag{4.14}$$

Substituting into eq. (4.9) yields

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t)e^{-j\omega t}\,dt = 1. \tag{4.15}$$

That is, the unit impulse has a Fourier transform consisting of equal contributions at *all* frequencies.

Example 4.4

Consider the rectangular pulse signal

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}, \tag{4.16}$$

Solution:

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(j\omega) = \int_{-T}^{+T} 1.\, e^{-j\omega t}\, dt$$

$$X(j\omega) = -\frac{1}{j\omega}\left[\, e^{-j\omega t}\right]_{-T_1}^{T_1}$$

$$X(j\omega) = -\frac{1}{j\omega}\left[\, e^{-j\omega T_1} - \; e^{-j\omega(-T_1)}\,\right]$$

Cont.

Adjusting the signs.

$$X(j\omega) = \frac{1}{j\omega}\left[-e^{-j\omega T_1} + e^{+j\omega T_1}\right]$$

Rewriting above equation as:

$$X(j\omega) = \frac{1}{j\omega}\left[e^{j\omega T_1} - e^{-j\omega T_1}\right]$$

Multiplying and dividing above equation by 2.

$$X(j\omega) = \frac{2}{2j\omega}\left[e^{j\omega T_1} - e^{-j\omega T_1}\right]$$

$$X(j\omega) = \frac{2}{\omega} \cdot \frac{1}{2j}\left[e^{j\omega T_1} - e^{-j\omega T_1}\right]$$

By Euler's formula:

$$X(j\omega) = \frac{2}{\omega}\sin \omega T_1$$

Rewriting above equation as:

$$X(j\omega) = 2\,\frac{\sin \omega T_1}{\omega}$$

as sketched in Figure 4.8 (b).

Cont.

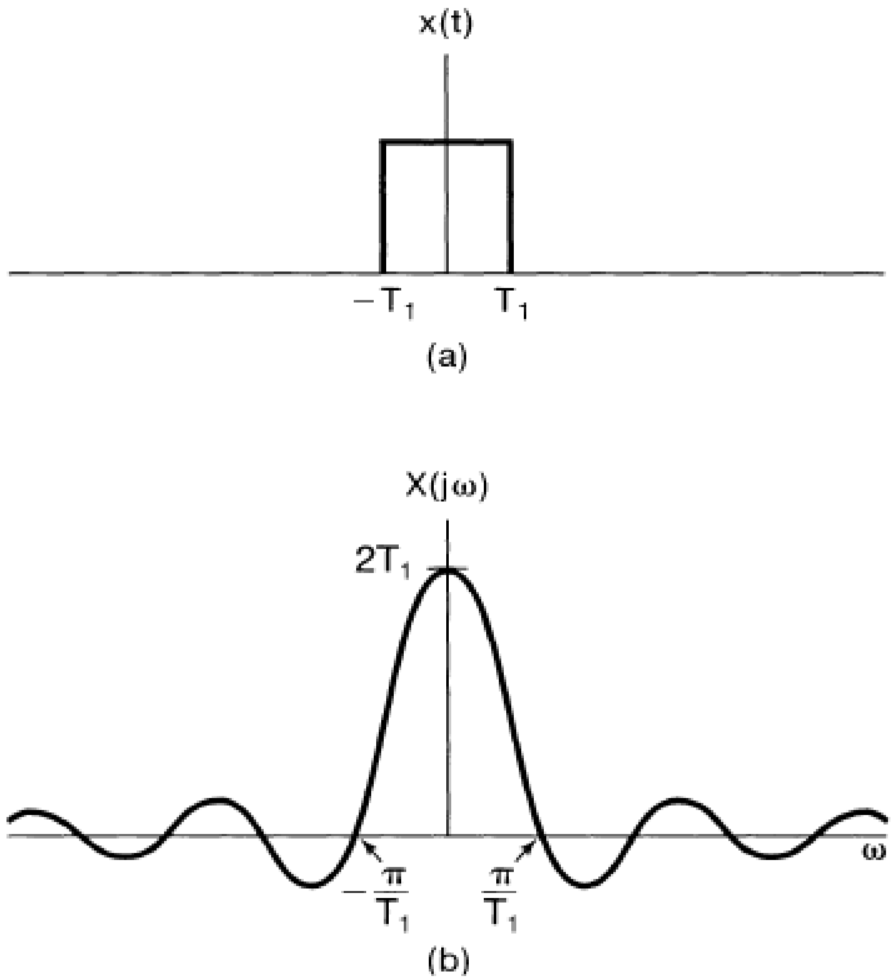

Figure 4.8 (a) The rectangular pulse signal of Example 4.4 and (b) its Fourier transform.

Example 4.5

Consider the signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases} \tag{4.18}$$

This transform is illustrated in Figure 4.9(a). Using the synthesis equation (4.8), we can

Solution:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$x(t) = \frac{1}{2\pi} \int_{-W}^{W} 1 \,.\, e^{j\omega t}\, d\omega$$

$$x(t) = \frac{1}{2\pi} \left(\frac{1}{jt}\right) \left[\, e^{j\omega t} \right]_{-W}^{W}$$

$$x(t) = \frac{1}{2\pi} \left(\frac{1}{jt}\right) \left[e^{jWt} - e^{jt(-W)} \right]$$

Rearranging above equation:

$$x(t) = \frac{1}{\pi t} \left(\frac{1}{2j}\right) \left[e^{jWt} - e^{-jWt} \right]$$

By Euler's formula:

$$x(t) = \frac{1}{\pi t} \sin Wt$$

Rewriting above equation as:

$$x(t) = \frac{\sin Wt}{\pi t}$$

which is depicted in Fig. 4.9 (b).

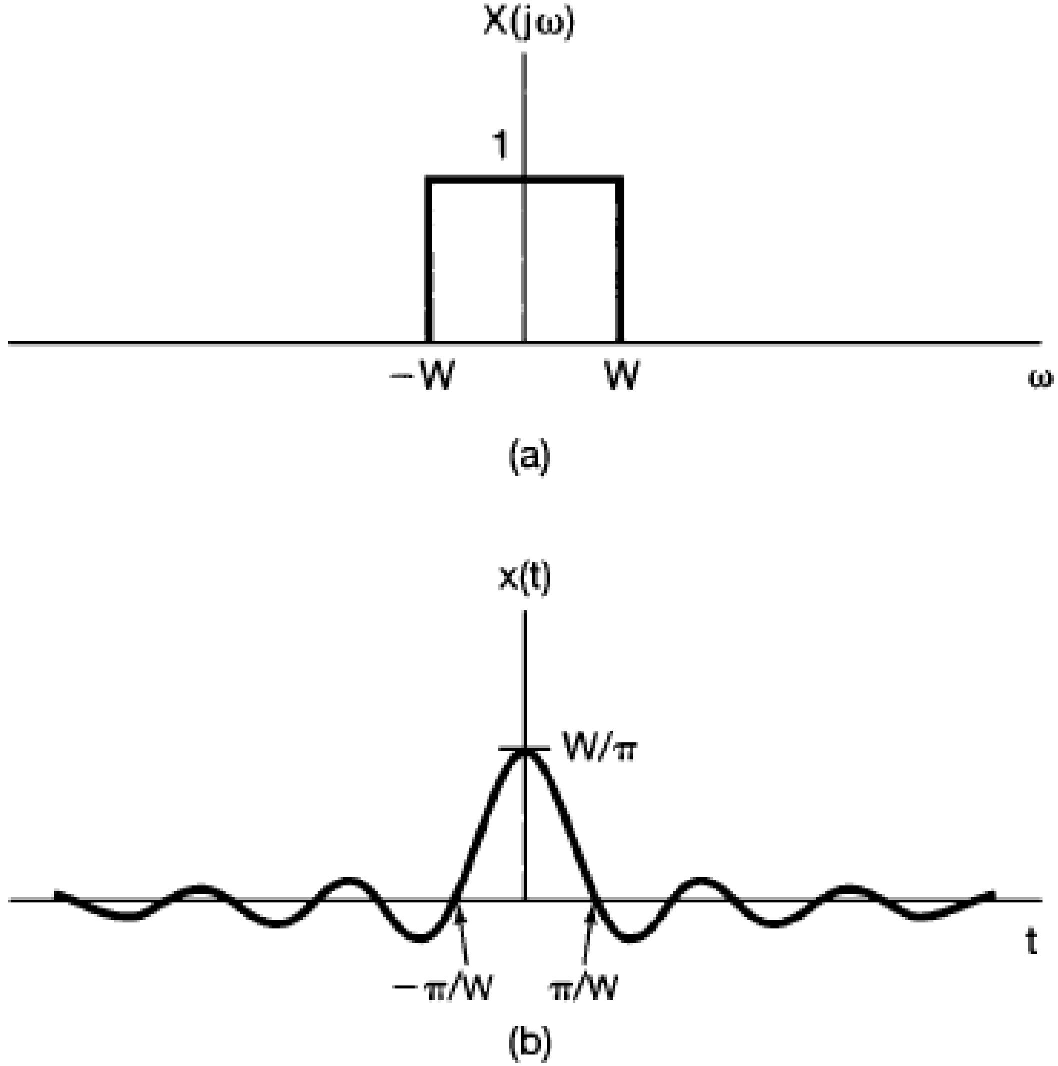

Figure 4.9 Fourier transform pair of Example 4.5: (a) Fourier transform for Example 4.5 and (b) the corresponding time function.

Cont.

Functions of the form given in eqs. (4.17) and (4.19) arise frequently in Fourier analysis and in the study of LTI systems and are referred to as *sinc functions*. A commonly used precise form for the sinc function is

$$\mathrm{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}. \tag{4.20}$$

The sinc function is plotted in Figure 4.10. Both of the signals in eqs. (4.17) and (4.19) can be expressed in terms of the sinc function:

$$\frac{2 \sin \omega T_1}{\omega} = 2T_1 \,\mathrm{sinc}\left(\frac{\omega T_1}{\pi}\right)$$

$$\frac{\sin Wt}{\pi t} = \frac{W}{\pi} \,\mathrm{sinc}\left(\frac{Wt}{\pi}\right).$$

Cont.

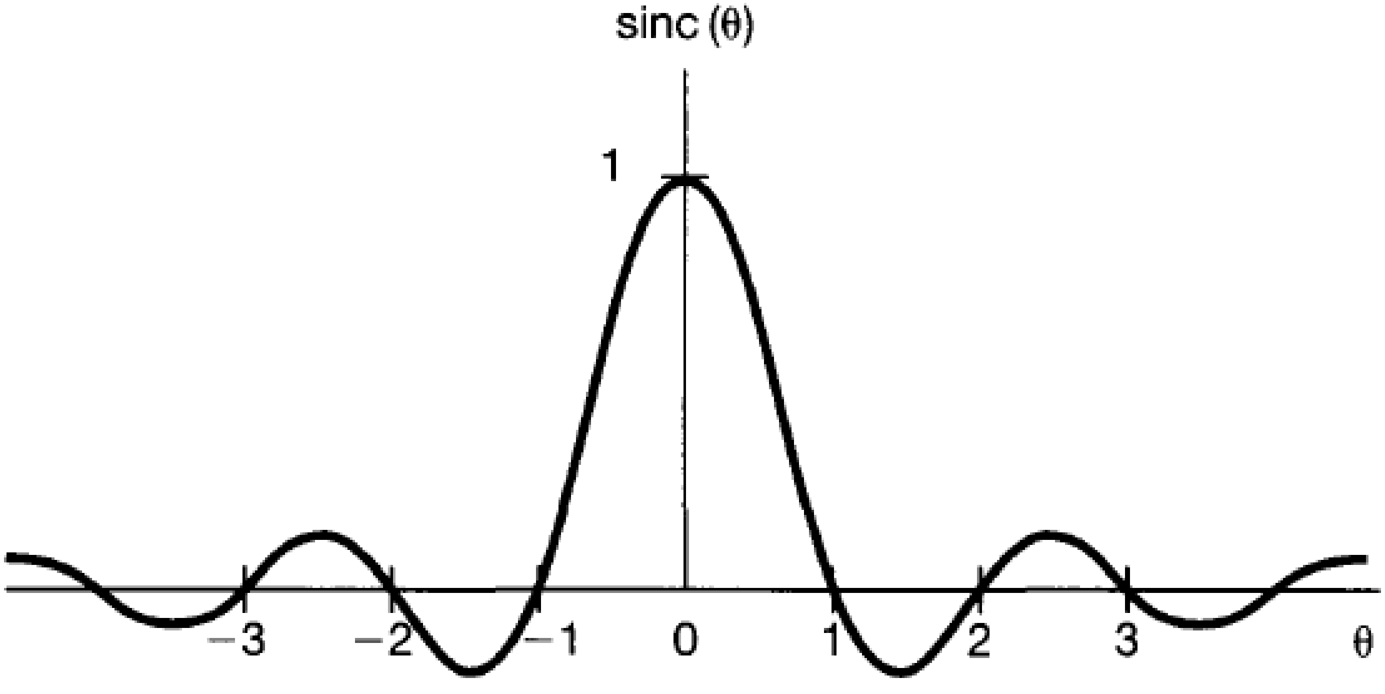

Figure 4.10 The sinc function.

Cont.

Finally, we can gain insight into one other property of the Fourier transform by examining Figure 4.9, which we have redrawn as Figure 4.11 for several different values of W. From this figure, we see that as W increases, $X(j\omega)$ becomes broader, while the main peak of $x(t)$ at $t = 0$ becomes higher and the width of the first lobe of this signal (i.e., the part of the signal for $|t| < \pi/W$) becomes narrower. In fact, in the limit as $W \to \infty$, $X(j\omega) = 1$ for all ω, and consequently, from Example 4.3, we see that $x(t)$ in eq. (4.19) converges to an impulse as $W \to \infty$. The behavior depicted in Figure 4.11 is an example of the inverse relationship that exists between the time and frequency domains, and we can see a similar effect in Figure 4.8, where an increase in T_1 broadens $x(t)$ but makes $X(j\omega)$ narrower.

Cont.

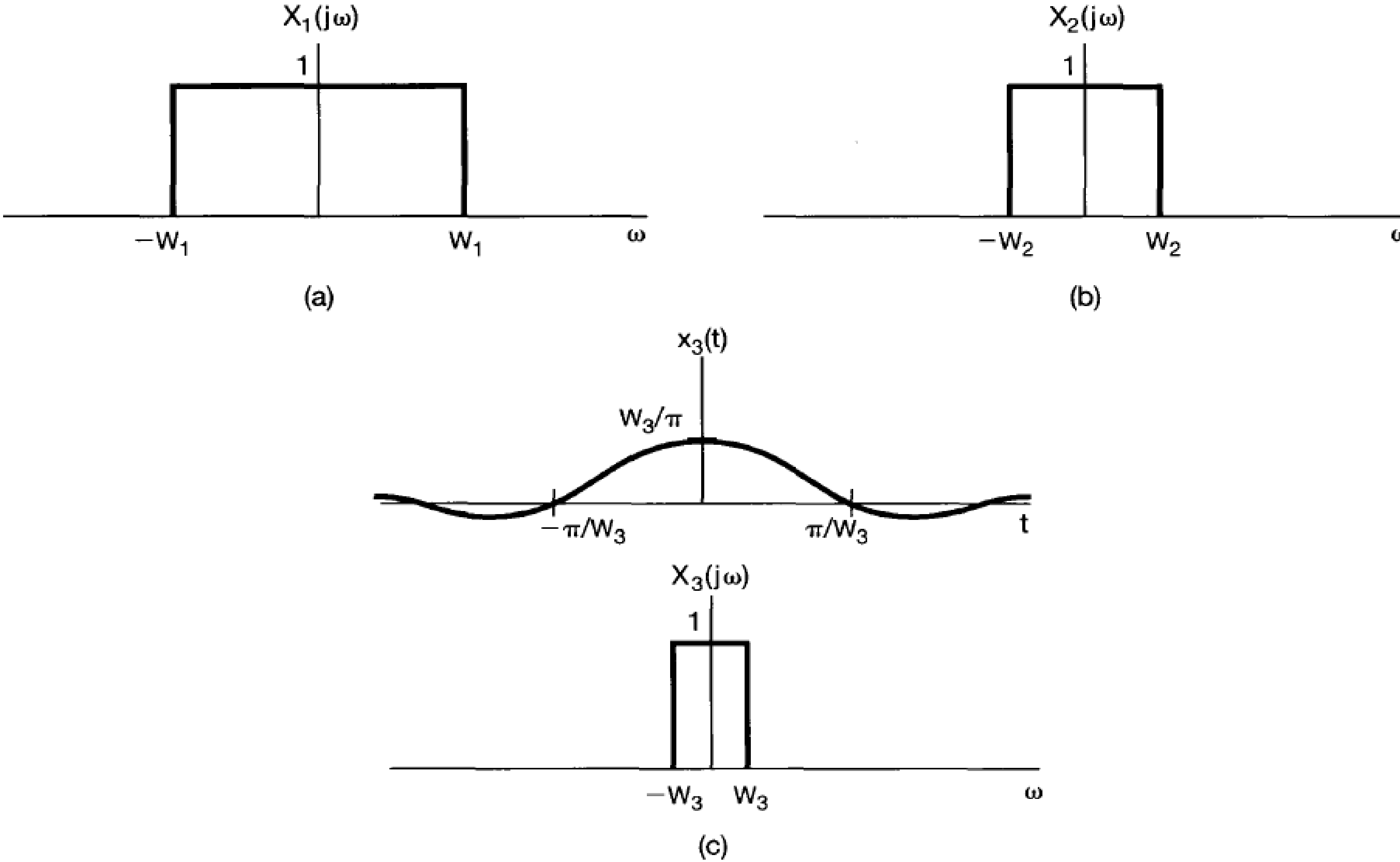

Figure 4.11 Fourier transform pair of Figure 4.9 for several different values of W.

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Property	Aperiodic signal	Fourier transform
	$x(t)$	$X(j\omega)$
	$y(t)$	$Y(j\omega)$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega - \theta))d\theta$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$

Cont.

Property	Aperiodic signal	Fourier transform
Differentiation in Frequency	$tx(t)$	$j\frac{u}{d\omega}X(j\omega)$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$

Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Systems Characterized By Linear Constant-Coefficient Differential Equations

As we have discussed on several occasions, a particularly important and useful class of continuous-time LTI systems is those for which the input and output satisfy a linear constant-coefficient differential equation of the form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \quad (4.72)$$

In this section, we consider the question of determining the frequency response of such an LTI system.

There are two closely related ways in which to determine the frequency response $H(j\omega)$ for an LTI system described by the differential equation (4.72).
Specifically, if $x(t) = e^{j\omega t}$, then the output must be $y(t) = H(j\omega)e^{j\omega t}$. Substituting these expressions into the differential equation (4.72) and performing some algebra, we can then solve for $H(j\omega)$.

Cont.

Consider an LTI system characterized by eq. (4.72). From the convolution property,

$$Y(j\omega) = H(j\omega)X(j\omega),$$

or equivalently,

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}, \tag{4.73}$$

where $X(j\omega)$, $Y(j\omega)$, and $H(j\omega)$ are the Fourier transforms of the input $x(t)$, output $y(t)$, and impulse response $h(t)$, respectively. Next, consider applying the Fourier transform to both sides of eq. (4.72) to obtain

$$\mathfrak{F}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathfrak{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}. \tag{4.74}$$

From the linearity property, eq. (4.26), this becomes

$$\sum_{k=0}^{N} a_k \mathfrak{F}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k \mathfrak{F}\left\{\frac{d^k x(t)}{dt^k}\right\}, \tag{4.75}$$

Cont.

and from the differentiation property, eq. (4.31),

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega),$$

or equivalently,

$$Y(j\omega)\left[\sum_{k=0}^{N} a_k (j\omega)^k\right] = X(j\omega)\left[\sum_{k=0}^{M} b_k (j\omega)^k\right].$$

Thus, from eq. (4.73),

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}. \tag{4.76}$$

Example 4.24

Consider a stable LTI system characterized by the differential equation

$$\frac{dy(t)}{dt} + ay(t) = x(t), \tag{4.77}$$

with $a > 0$. From eq. (4.76), the frequency response is

$$H(j\omega) = \frac{1}{j\omega + a}. \tag{4.78}$$

Comparing this with the result of Example 4.1, we see that eq. (4.78) is the Fourier transform of $e^{-at}u(t)$. The impulse response of the system is then recognized as

$$h(t) = e^{-at}u(t).$$

Example 4.25

Consider a stable LTI system that is characterized by the differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t).$$

From eq. (4.76), the frequency response is

$$H(j\omega) = \frac{(j\omega) + 2}{(j\omega)^2 + 4(j\omega) + 3}. \tag{4.79}$$

As a first step, we factor the denominator of the right-hand side of eq. (4.79) into a product of lower order terms:

$$H(j\omega) = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)}. \tag{4.80}$$

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Cont.

By partial fraction:

$$\frac{j\omega + 2}{(j\omega + 1)\,(j\omega + 3)} = \frac{A}{(j\omega + 1)} + \frac{B}{(j\omega + 3)}$$

$$\frac{j\omega + 2}{(j\omega + 1)\,(j\omega + 3)} = \frac{A\,(j\omega + 3) + B(j\omega + 1)}{(j\omega + 1)\,(j\omega + 3)}$$

$$j\omega + 2 = A\,(j\omega + 3) + B(j\omega + 1)$$

$$j\omega + 2 = A\,(j\omega) + 3A + B\,(j\omega) + B$$

Comparing $j\omega$ and constants:

$$j\omega = A\,(j\omega) + B\,(j\omega)$$

and

$$2 = 3A + B$$

Then:

$$A = \frac{1}{2} \text{ and } B = \frac{1}{2}$$

Cont.

Then, using the method of partial-fraction expansion, we find that

$$H(j\omega) = \frac{\frac{1}{2}}{j\omega + 1} + \frac{\frac{1}{2}}{j\omega + 3}.$$

The inverse transform of each term can be recognized from Example 4.24, with the result that

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t).$$

The procedure used in Example 4.25 to obtain the inverse Fourier transform is generally useful in inverting transforms that are ratios of polynomials in $j\omega$. In particular, we can use eq. (4.76) to determine the frequency response of any LTI system described by a linear constant-coefficient differential equation and then can calculate the impulse response by performing a partial-fraction expansion that puts the frequency response into a form in which the inverse transform of each term can be recognized by inspection. In addition, if the Fourier transform $X(j\omega)$ of the input to such a system is also a ratio of polynomials in $j\omega$, then so is $Y(j\omega) = H(j\omega)X(j\omega)$.

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