Subject: Signals and Systems

Chapter: 04

The Continuous Time Fourier Transform

Text Book: Signals & Systems By: Alan V. Oppenheim, Alan S. Willsky with S. Hamid Nawab, 2nd Edition

Development of the Fourier Transform Representation of an Aperiodic Signal

To gain some insight into the nature of the Fourier transform representation, we begin by revisiting the Fourier series representation for the continuous-time periodic square wave examined already. Specifically, over one period,

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

and periodically repeats with period T, as shown in Figure 4.1.

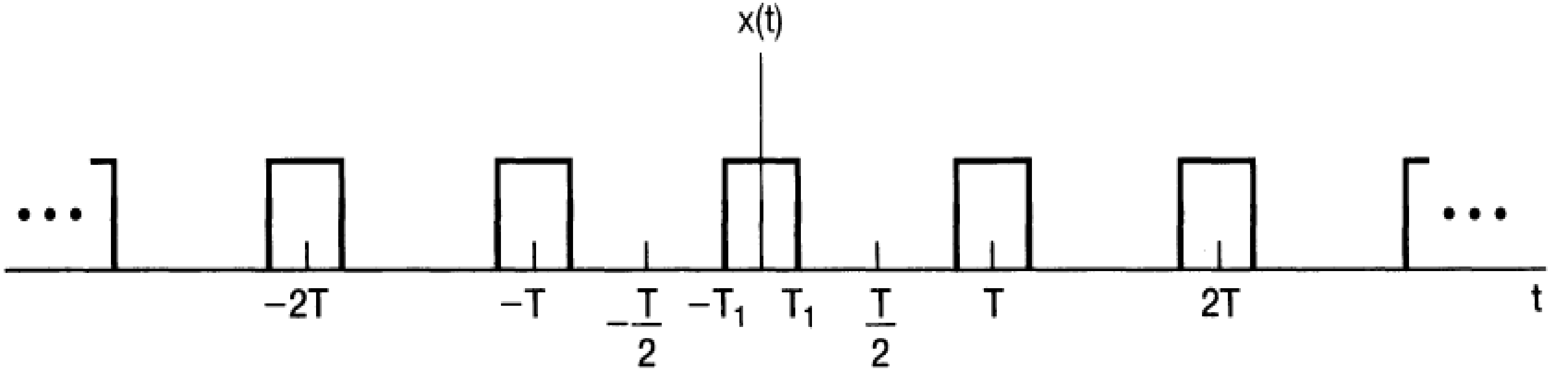

Figure 4.1 A continuous-time periodic square wave.

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As determined already the Fourier series coefficients a_k for this square wave are

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}, \tag{4.1}$$

where $\omega_0 = 2\pi/T$.

An alternative way of interpreting eq. (4.1) is as samples of an envelope function, specifically,

$$Ta_k = \left.\frac{2\sin \omega T_1}{\omega}\right|_{\omega = k\omega_0}. \tag{4.2}$$

That is, with ω thought of as a continuous variable, the function $(2\sin\omega T_1)/\omega$ represents the envelope of Ta_k, and the coefficients a_k are simply equally spaced samples of this envelope. Also, for fixed T_1, the envelope of Ta_k is independent of T.

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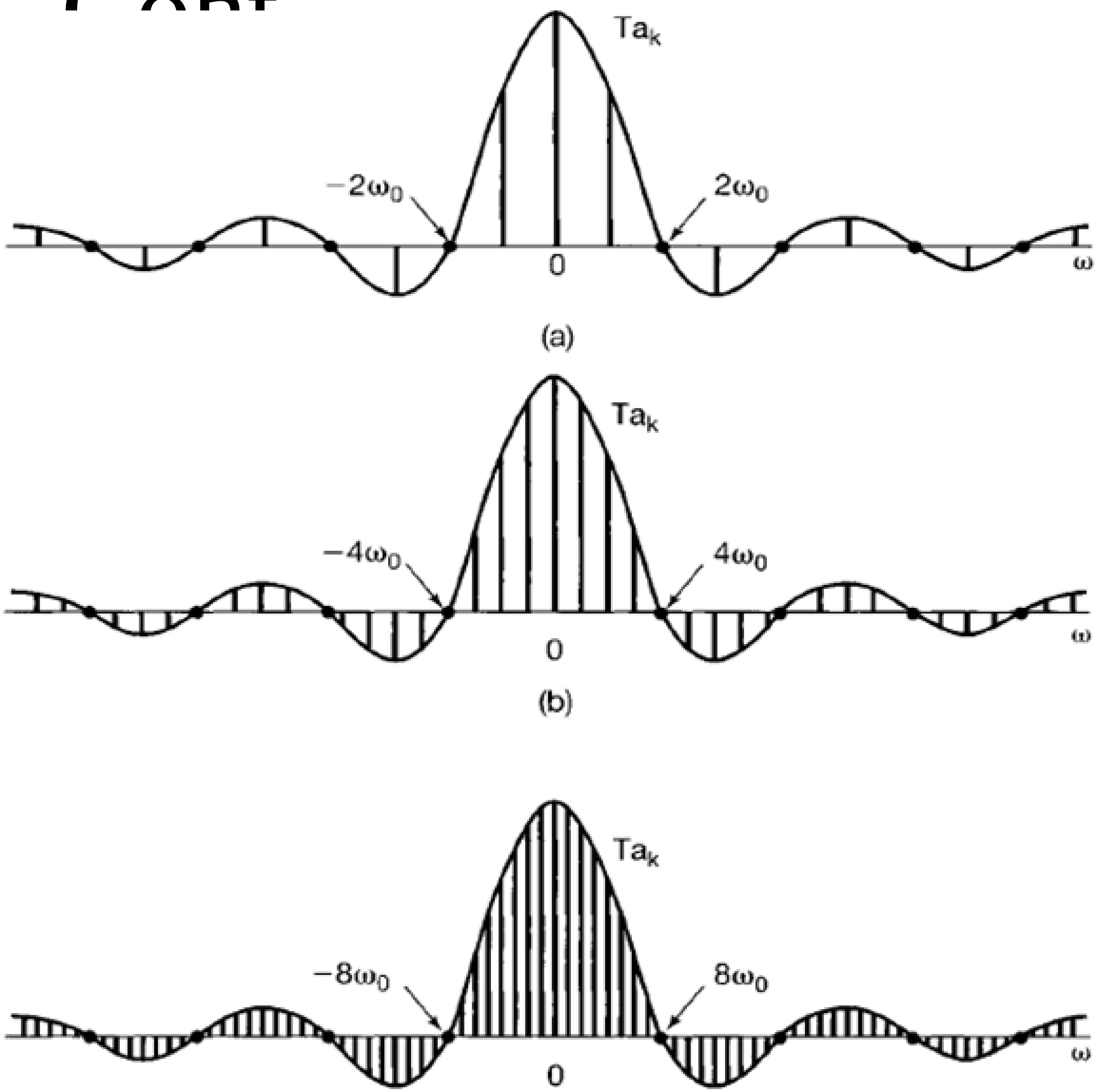

Figure 4.2 The Fourier series coefficients and their envelope for the periodic square wave in Figure 4.1 for several values of T (with T_1 fixed): (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$.

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In Figure 4.2, we again show the Fourier series coefficients for the periodic square wave, but this time as samples of the envelope of Ta_k, as specified in eq. (4.2). From the figure, we see that as T increases, or equivalently, as the fundamental frequency $\omega_0 = 2\pi/T$ decreases, the envelope is sampled with a closer and closer spacing. As T becomes arbitrarily large, the original periodic square wave approaches a rectangular pulse (i.e., all that remains in the time domain is an aperiodic signal corresponding to one period of the square wave). Also, the Fourier series coefficients, multiplied by T, become more and more closely spaced samples of the envelope, so that in some sense (which we will specify shortly) the set of Fourier series coefficients approaches the envelope function as $T \rightarrow \infty$.

This example illustrates the basic idea behind Fourier's development of a representation for aperiodic signals. Specifically, we think of an aperiodic signal as the limit of a periodic signal as the period becomes arbitrarily large, and we examine the limiting behavior of the Fourier series representation for this signal.

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In particular, consider a signal $x(t)$ that is of finite duration. That is, for some number T_1, $x(t) = 0$ if $|t| > T_1$, as illustrated in Figure 4.3(a). From this aperiodic signal, we can construct a periodic signal $\tilde{x}(t)$ for which $x(t)$ is one period, as indicated in Figure 4.3(b). As we choose the period T to be larger, $\tilde{x}(t)$ is identical to $x(t)$ over a longer interval, and as $T \to \infty$, $\tilde{x}(t)$ is equal to $x(t)$ for any finite value of t.

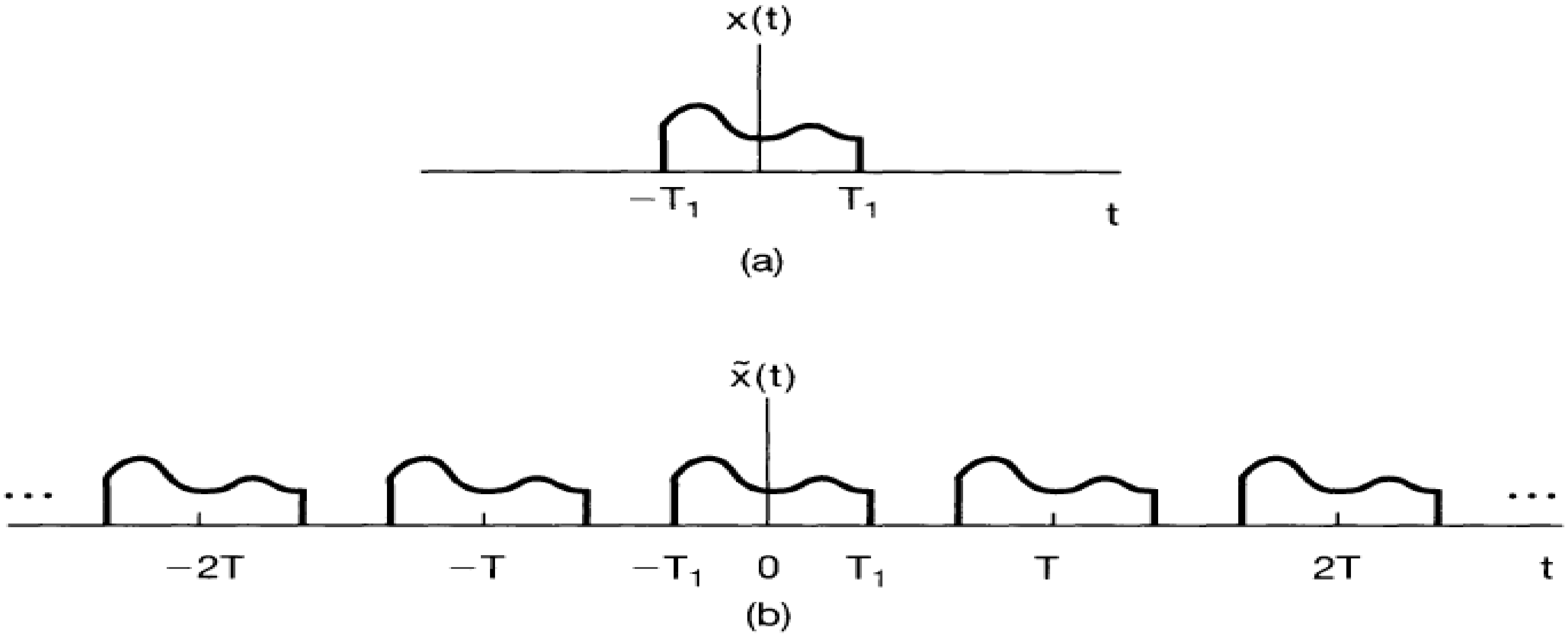

Figure 4.3 (a) Aperiodic signal $x(t)$; (b) periodic signal $\tilde{x}(t)$, constructed to be equal to $x(t)$ over one period.

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Let us now examine the effect of this on the Fourier series representation of $\tilde{x}(t)$.

For interval $-T/2 \le t \le T/2$, we have

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \tag{4.3}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt, \tag{4.4}$$

where $\omega_0 = 2\pi/T$. Since $\tilde{x}(t) = x(t)$ for $|t| < T/2$, and also, since $x(t) = 0$ outside this interval, eq. (4.4) can be rewritten as

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_{-\infty}^{+\infty} x(t) e^{-jk\omega_0 t} dt.$$

Therefore, defining the envelope $X(j\omega)$ of Ta_k as

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt, \tag{4.5}$$

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we have, for the coefficients a_k,

$$a_k = \frac{1}{T}X(jk\omega_0). \tag{4.6}$$

Combining eqs. (4.6) and (4.3), we can express $\tilde{x}(t)$ in terms of $X(j\omega)$ as

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T}X(jk\omega_0)e^{jk\omega_0 t},$$

or equivalently, since $2\pi/T = \omega_0$,

$$\tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0. \tag{4.7}$$

Cont.

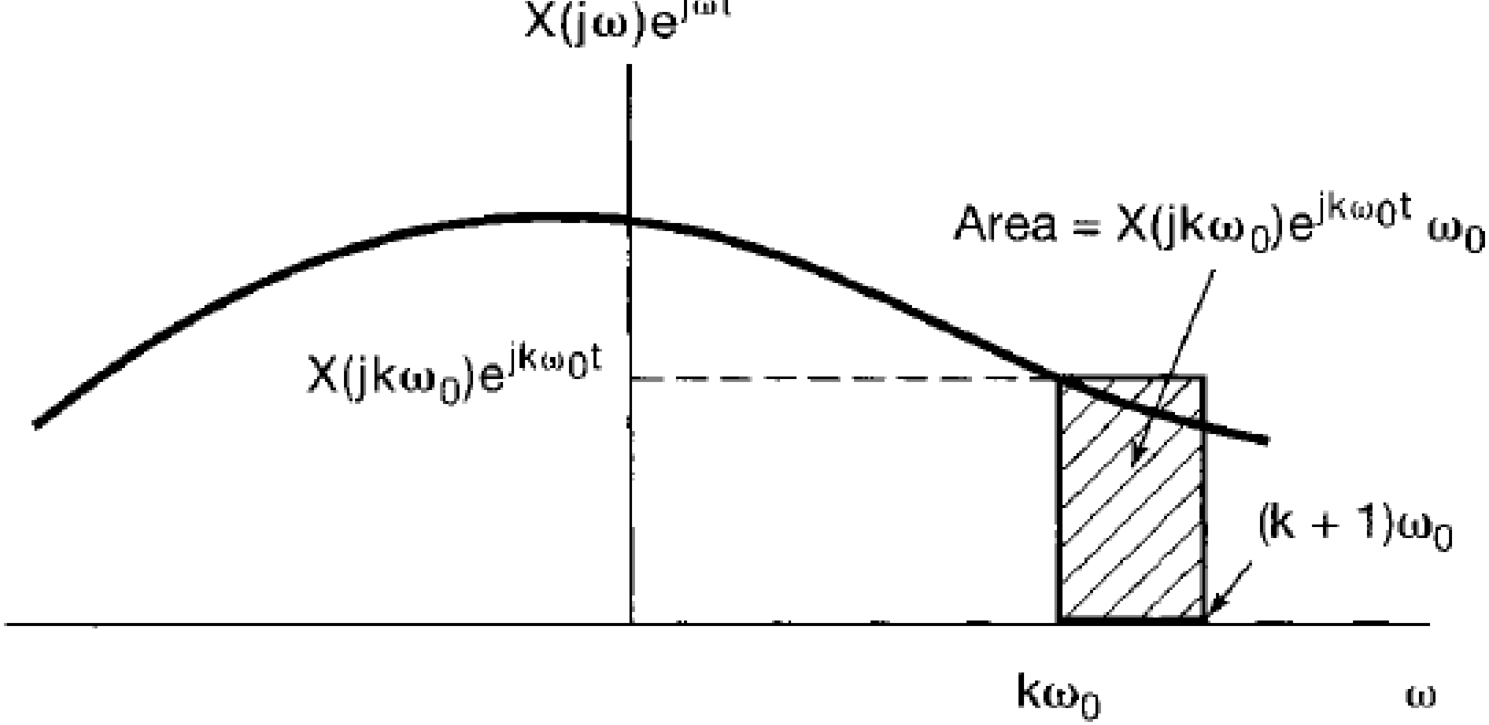

Figure 4.4 Graphical interpretation of eq. (4.7).

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As $T \rightarrow \infty$, $\tilde{x}(t)$ approaches $x(t)$, and consequently, in the limit eq. (4.7) becomes a representation of $x(t)$. Furthermore, $\omega_0 \rightarrow 0$ as $T \rightarrow \infty$, and the right-hand side of eq. (4.7) passes to an integral. This can be seen by considering the graphical interpretation of the equation, illustrated in Figure 4.4. Each term in the summation on the right-hand side is the area of a rectangle of height $X(jk\omega_0)e^{jk\omega_0 t}$ and width ω_0. (Here, t is regarded as fixed.) As $\omega_0 \rightarrow 0$, the summation converges to the integral of $X(j\omega)e^{j\omega t}$. Therefore, using the fact that $\tilde{x}(t) \rightarrow x(t)$ as $T \rightarrow \infty$, we see that eqs. (4.7) and (4.5) respectively become

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega \tag{4.8}$$

and

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt. \tag{4.9}$$

Cont.

Equations (4.8) and (4.9) are referred to as the *Fourier transform pair*, with the function $X(j\omega)$ referred to as the *Fourier Transform* or *Fourier integral* of $x(t)$ and eq. (4.8) as the *inverse Fourier transform* equation. The *synthesis* equation (4.8) plays a role for aperiodic signals similar to that of eq. (3.38) for periodic signals, since both represent a signal as a linear combination of complex exponentials. For periodic signals, these complex exponentials have amplitudes $\{a_k\}$, as given by eq. (3.39), and occur at a discrete set of harmonically related frequencies $k\omega_0$, $k = 0, \pm 1, \pm 2, \ldots$. For aperiodic signals, the complex exponentials occur at a continuum of frequencies and, according to the synthesis equation (4.8), have "amplitude" $X(j\omega)(d\omega/2\pi)$. In analogy with the terminology used for the Fourier series coefficients of a periodic signal, the transform $X(j\omega)$ of an aperiodic signal $x(t)$ is commonly referred to as the *spectrum* of $x(t)$, as it provides us with the information needed for describing $x(t)$ as a linear combination (specifically, an integral) of sinusoidal signals at different frequencies.

Relation between α_k and $X(j\omega)$

Based on the above development, or equivalently on a comparison of eq. (4.9) and eq. (3.39), we also note that the Fourier coefficients a_k of a periodic signal $\tilde{x}(t)$ can be expressed in terms of equally spaced *samples* of the Fourier transform of one period of $\tilde{x}(t)$. Specifically, suppose that $\tilde{x}(t)$ is a periodic signal with period T and Fourier coefficients a_k. Let $x(t)$ be a finite-duration signal that is equal to $\tilde{x}(t)$ over exactly one period—say, for $s \leq t \leq s+T$ for some value of s—and that is zero otherwise. Then, since eq. (3.39) allows us to compute the Fourier coefficients of $\tilde{x}(t)$ by integrating over any period, we can write

$$a_k = \frac{1}{T}\int_{s}^{s+T} \tilde{x}(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{s}^{s+T} x(t)e^{-jk\omega_0 t}dt.$$

Cont.

Since $x(t)$ is zero outside the range $s \leq t \leq s + T$ we can equivalently write

$$a_k = \frac{1}{T} \int_{-\infty}^{+\infty} x(t) e^{-jk\omega_0 t} dt.$$

Comparing with eq. (4.9) we conclude that

$$a_k = \frac{1}{T} X(j\omega) \Big|_{\omega = k\omega_0}, \tag{4.10}$$

where $X(j\omega)$ is the Fourier transform of $x(t)$. Equation 4.10 states that the Fourier coefficients of $\tilde{x}(t)$ are proportional to samples of the Fourier transform of one period of $\tilde{x}(t)$.

Example 4.1

Consider the signal

$$x(t) = e^{-at}u(t) \quad a > 0.$$

From eq. (4.9),

$$X(j\omega) = \int_0^{\infty} e^{-at}e^{-j\omega t}dt = -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty}.$$

That is,

$$X(j\omega) = \frac{1}{a+j\omega}, \quad a > 0.$$

Cont.

Since this Fourier transform is complex valued, to plot it as a function of ω, we express $X(j\omega)$ in terms of its magnitude and phase:

$$|X(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}, \quad \sphericalangle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right).$$

Each of these components is sketched in Figure 4.5.

Note that if a is complex rather than real, then $x(t)$ is absolutely integrable as long as $\Re e\{a\} > 0$, and in this case the preceding calculation yields the same form for $X(j\omega)$. That is,

$$X(j\omega) = \frac{1}{a + j\omega}, \quad \Re e\{a\} > 0.$$

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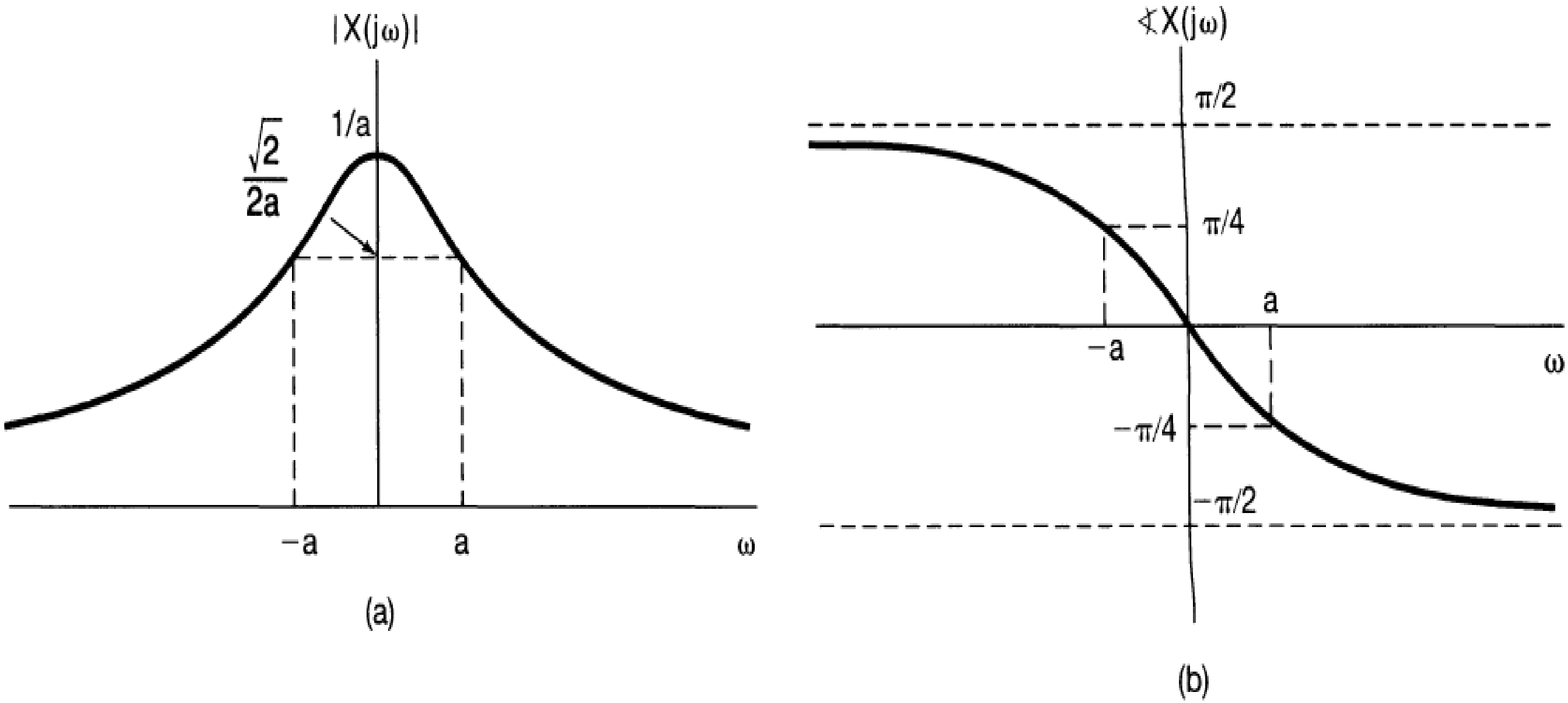

Figure 4.5 Fourier transform of the signal $x(t) = e^{-at}u(t), a > 0$, considered in Example 4.1.

Example 4.2

Let

$$x(t) = e^{-a|t|}, \quad a > 0.$$

This signal is sketched in Figure 4.6. The Fourier transform of the signal is

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{+\infty} e^{-a|t|} e^{-j\omega t}\, dt = \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt \\ &= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} \\ &= \frac{2a}{a^2 + \omega^2}. \end{aligned}$$

In this case $X(j\omega)$ is real, and it is illustrated in Figure 4.7.

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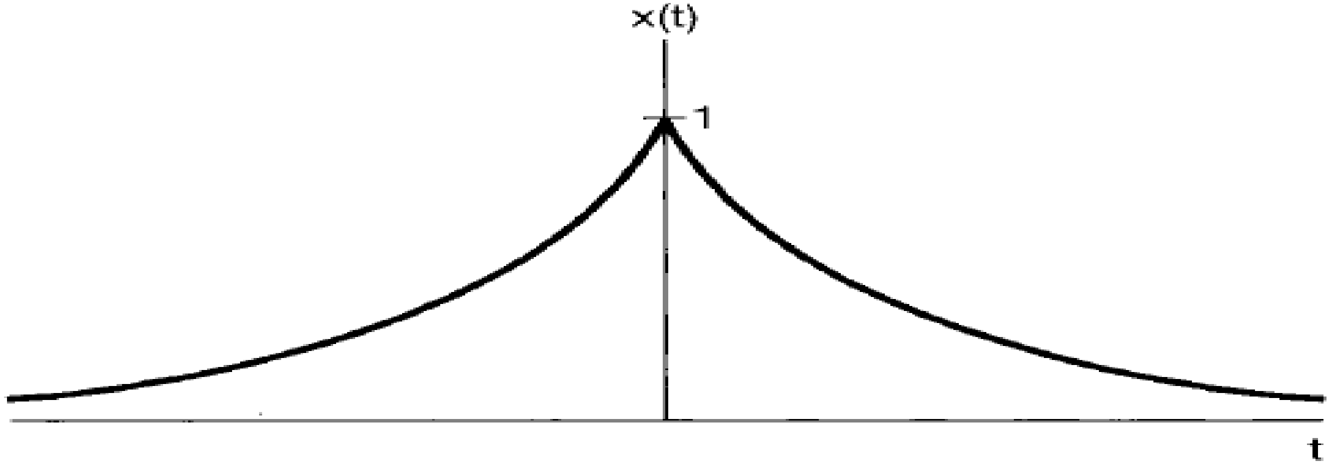

Figure 4.6 Signal $x(t) = e^{-a|t|}$ of Example 4.2.

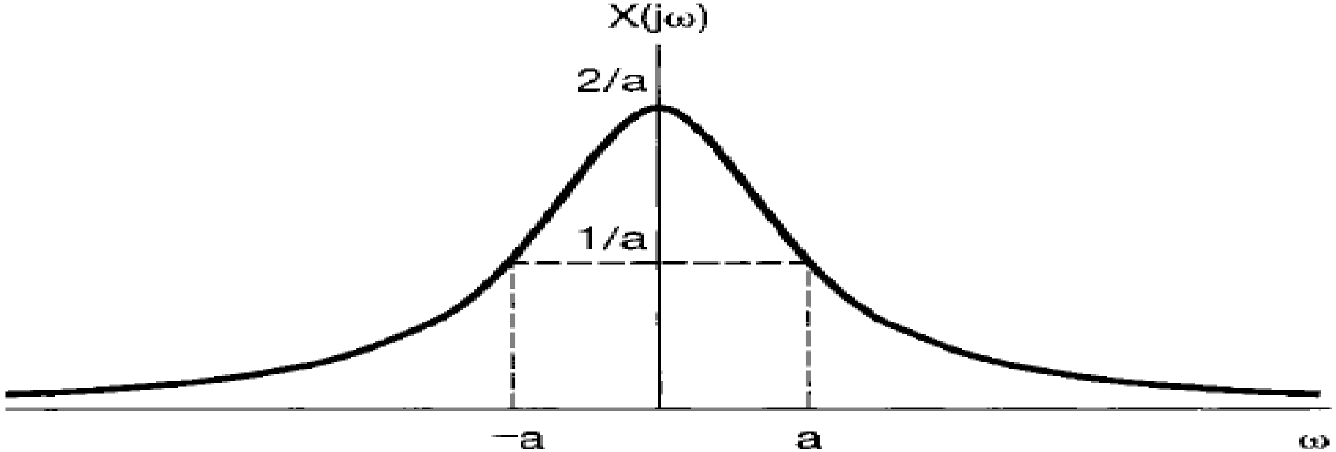

Figure 4.7 Fourier transform of the signal considered in Example 4.2 and depicted in Figure 4.6.

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