Subject: Signals and Systems

Topic: Fourier Analysis

Text Book: Signals & Systems By: Alan V. Oppenheim, Alan S. Willsky with S. Hamid Nawab, 2nd Edition

Determination of Fourier Series Representation of a Continuous Time Periodic Signal

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t} \quad (1)$$

Assuming that a given periodic signal can be represented with the series of eq. (1), we need a procedure for determining the coefficients a_k. Multiplying both sides of eq. (1) by $e^{-jn\omega_0 t}$, we obtain

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}. \quad (2)$$

Integrating both sides from 0 to $T = 2\pi/\omega_0$, we have

$$\int_0^T x(t)e^{-jn\omega_0 t}\,dt = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}\,dt.$$

Here, T is the fundamental period of $x(t)$, and consequently, we are integrating over one period. Interchanging the order of integration and summation yields

$$\int_0^T x(t)e^{-jn\omega_0 t}\,dt = \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{j(k-n)\omega_0 t}\,dt\right]. \quad (3)$$

Cont.

The evaluation of the bracketed integral is straightforward. Rewriting this integral using Euler's formula, we obtain

$$\int_0^T e^{j(k-n)\omega_0 t}\,dt = \int_0^T \cos(k-n)\omega_0 t\,dt + j\int_0^T \sin(k-n)\omega_0 t\,dt. \qquad (4)$$

For $k \neq n$, $\cos(k-n)\omega_0 t$ and $\sin(k-n)\omega_0 t$ are periodic sinusoids with fundamental period $(T/|k-n|)$. Therefore, in eq. (4), we are integrating over an interval (of length T) that is an integral number of periods of these signals. Since the integral may be viewed as measuring the total area under the functions over the interval, we see that for $k \neq n$, both of the integrals on the right-hand side of eq. (4) are zero. For $k = n$, the integrand on the left-hand side of eq. (4) equals 1, and thus, the integral equals T. In sum, we then have

$$\int_0^T e^{j(k-n)\omega_0 t}\,dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases},$$

Cont.

and consequently, the right-hand side of eq. (3) reduces to Ta_n. Therefore,

$$a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt, \qquad (5)$$

which provides the equation for determining the coefficients. Furthermore, note that in evaluating eq. (4), the only fact that we used concerning the interval of integration was that we were integrating over an interval of length T, which is an integral number of periods of $\cos(k-n)\omega_0 t$ and $\sin(k-n)\omega_0 t$. Therefore, we will obtain the same result if we integrate over any interval of length T. That is, if we denote integration over *any* interval of length T by $\int_T$, we have

$$\int_T e^{j(k-n)\omega_0 t}\,dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases},$$

and consequently,

$$a_n = \frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}\,dt. \qquad (6)$$

Cont.

To summarize, if $x(t)$ has a Fourier series representation [i.e., if it can be expressed as a linear combination of harmonically related complex exponentials in the form of eq. (1)], then the coefficients are given by eq. (6). This pair of equations, then, defines the Fourier series of a periodic continuous-time signal:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}, \qquad (7)$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\,dt. \qquad (8)$$

Here, we have written equivalent expressions for the Fourier series in terms of the fundamnetal frequency ω_0 and the fundamental period T. Equation (7) is referred to as the *synthesis* equation and eq. (8) as the *analysis* equation. The set of coefficients $\{a_k\}$ are often called the *Fourier series coefficients* or the *spectral coefficients* of $x(t)$. These complex coefficients measure the portion of the signal $x(t)$ that is at each harmonic of the fundamental component. The coefficient a_0 is the dc or constant component of $x(t)$ and is given by eq. ( ) with $k = 0$. That is.

$$a_0 = \frac{1}{T}\int_T x(t)\,dt, \qquad (9)$$

Example: 01

Consider the signal

$$x(t) = \sin \omega_0 t,$$

whose fundamental frequency is ω_0. One approach to determining the Fourier series coefficients for this signal is to apply eq. (7). For this simple case, however, it is easier to expand the sinusoidal signal as a linear combination of complex exponentials and identify the Fourier series coefficients by inspection. Specifically, we can express $\sin \omega_0 t$ as

$$\sin \omega_0 t = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}.$$

Comparing the right-hand sides of this equation

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

We get,

$$a_1 = \frac{1}{2j}, \qquad a_{-1} = -\frac{1}{2j},$$

$$a_k = 0, \qquad k \neq +1 \text{ or } -1.$$

Example: 02

Let

$$x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

which has fundamental frequency ω_0. We can again expand $x(t)$ directly in terms of complex exponentials, so that

$$x(t) = 1 + \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}] + [e^{j\omega_0 t} + e^{-j\omega_0 t}] + \frac{1}{2}[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}].$$

Collecting terms, we obtain

$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j(\pi/4)}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j(\pi/4)}\right)e^{-j2\omega_0 t}.$$

Thus, the Fourier series coefficients for this example are

$$a_0 = 1,$$

$$a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j,$$

$$a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j,$$

$$a_2 = \frac{1}{2}e^{j(\pi/4)} = \frac{\sqrt{2}}{4}(1 + j),$$

Cont.

$$a_{-2} = \frac{1}{2}e^{-j(\pi/4)} = \frac{\sqrt{2}}{4}(1 - j),$$

$$a_k = 0, \ |k| > 2.$$

In Figure: 01, we show a bar graph of the magnitude and phase of a_k.

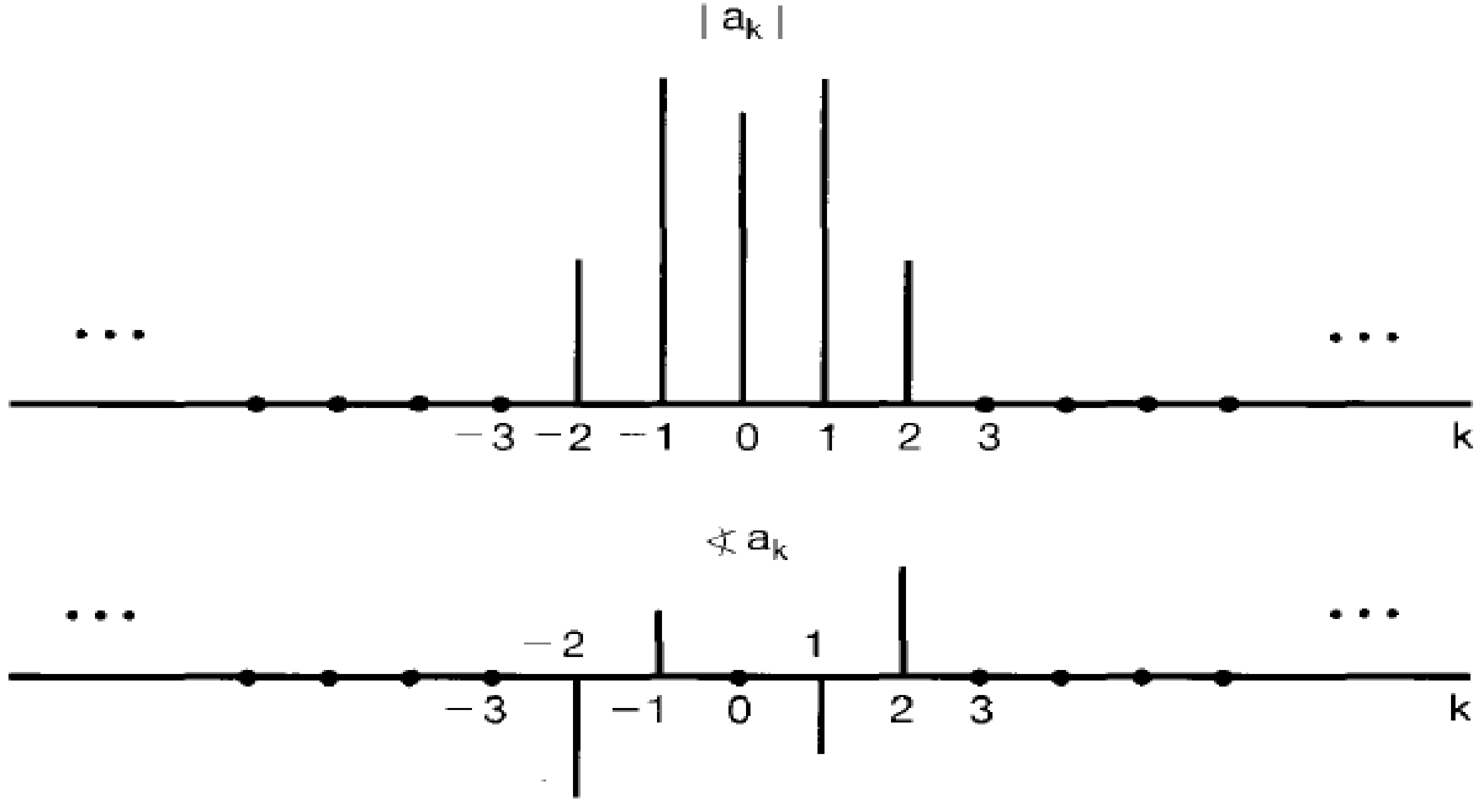

Figure: 01 Plots of the magnitude and phase of the Fourier coefficients of the signal considered in Example: 02.

Properties of Continuous Time Fourier Series

1. Linearity

Let $x(t)$ and $y(t)$ denote two periodic signals with period T and which have Fourier series coefficients denoted by a_k and b_k, respectively. That is,

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k,$$

$$y(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} b_k.$$

Since $x(t)$ and $y(t)$ have the same period T, it easily follows that any linear combination of the two signals will also be periodic with period T. Furthermore, the Fourier series coefficients c_k of the linear combination of $x(t)$ and $y(t)$, $z(t) = Ax(t) + By(t)$, are given by the same linear combination of the Fourier series coefficients for $x(t)$ and $y(t)$. That is,

$$z(t) = Ax(t) + By(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} c_k = Aa_k + Bb_k. \tag{10}$$

2. Time Shifting

When a time shift is applied to a periodic signal $x(t)$, the period T of the signal is preserved. The Fourier series coefficients b_k of the resulting signal $y(t) = x(t - t_0)$ may be expressed as

$$b_k = \frac{1}{T}\int_T x(t - t_0)e^{-jk\omega_0 t}dt. \tag{11}$$

Letting $\tau = t - t_0$ in the integral, and noting that the new variable τ will also range over an interval of duration T, we obtain

$$\begin{aligned} \frac{1}{T}\int_T x(\tau)e^{-jk\omega_0(\tau+t_0)}d\tau &= e^{-jk\omega_0 t_0}\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0\tau}d\tau \\ &= e^{-jk\omega_0 t_0}a_k = e^{-jk(2\pi/T)t_0}a_k, \end{aligned} \tag{12}$$

where a_k is the kth Fourier series coefficient of $x(t)$. That is, if

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k,$$

then

$$x(t - t_0) \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0}a_k = e^{-jk(2\pi/T)t_0}a_k.$$

One consequence of this property is that, when a periodic signal is shifted in time, the *magnitudes* of its Fourier series coefficients remain unaltered. That is, $|b_k| = |a_k|$.

3. Time Reversal

The period T of a periodic signal $x(t)$ also remains unchanged when the signal undergoes time reversal. To determine the Fourier series coefficients of $y(t) = x(-t)$, let us consider the effect of time reversal on the synthesis equation (7):

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk2\pi t/T}. \tag{13}$$

Making the substitution $k = -m$, we obtain

$$y(t) = x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm2\pi t/T}. \tag{14}$$

We observe that the right-hand side of this equation has the form of a Fourier series synthesis equation for $x(-t)$, where the Fourier series coefficients b_k are

$$b_k = a_{-k}.$$

Cont.

That is, if

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k,$$

then

$$x(-t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_{-k}.$$

In other words time reversal applied to a continuous-time signal results in a time reversal of the corresponding sequence of Fourier series coefficients. An interesting consequence of the time-reversal property is that if $x(t)$ is even—that is, if $x(-t) = x(t)$—then its Fourier series coefficients are also even—i.e., $a_{-k} = a_k$. Similarly, if $x(t)$ is odd, so that $x(-t) = -x(t)$, then so are its Fourier series coefficients—i.e., $a_{-k} = -a_k$.

4. Time Scaling

Time scaling is an operation that in general changes the period of the underlying signal. Specifically, if $x(t)$ is periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$, then $x(\alpha t)$, where α is a positive real number, is periodic with period T/α and fundamental frequency $\alpha\omega_0$. Since the time-scaling operation applies directly to each of the harmonic components of $x(t)$, we may easily conclude that the Fourier coefficients for each of those components remain the same. That is, if $x(t)$ has the Fourier series representation in eq. (7), then

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha\omega_0)t}$$

is the Fourier series representation of $x(\alpha t)$. We emphasize that, while the Fourier coefficients have not changed, the Fourier series representation *has* changed because of the change in the fundamental frequency.

5. Multiplication

Suppose that $x(t)$ and $y(t)$ are both periodic with period T and that

$$x(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} a_k,$$

$$y(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} b_k.$$

Since the product $x(t)y(t)$ is also periodic with period T, we can expand it in a Fourier series with Fourier series coefficients h_k expressed in terms of those for $x(t)$ and $y(t)$. The result is

$$x(t)y(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}. \tag{15}$$

6. Conjugation and Conjugate Symmetry

Taking the complex conjugate of a periodic signal $x(t)$ has the effect of complex conjugation *and* time reversal on the corresponding Fourier series coefficients. That is, if

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k,$$

then

$$x^*(t) \overset{\mathcal{FS}}{\longleftrightarrow} a^*_{-k}. \quad (16)$$

This property is easily proved by applying complex conjugation to both sides of eq. (7) and replacing the summation variable k by its negative.

Some interesting consequences of this property may be derived for $x(t)$ real—that is, when $x(t) = x^*(t)$. In particular, in this case, we see from eq. (16) that the Fourier series coefficients will be *conjugate symmetric*, i.e.,

$$a_{-k} = a_k^* \quad (17)$$

7. Parseval's Relation for Continuous-Time Periodic Signals

Parseval's relation for continuous-time periodic signals is

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2, \tag{18}$$

where the a_k are the Fourier series coefficients of $x(t)$ and T is the period of the signal.

Note that the left-hand side of eq. (3.67) is the average power (i.e., energy per unit time) in one period of the periodic signal $x(t)$. Also,

$$\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt = |a_k|^2, \tag{19}$$

so that $|a_k|^2$ is the average power in the kth harmonic component of $x(t)$. Thus, what Parseval's relation states is that the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

TABLE 1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l}x(t)\\y(t)\end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting	$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution	$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$

Cont.

Property	Periodic Signal	Fourier Series Coefficients
Integration	$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} a_k = a_{-k}^* \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x(t)$ real and even	a_k real and even
Real and Odd Signals	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Text Book: Signals & Systems By: Alan V. Oppenheim, Alan S. Willsky with S. Hamid Nawab, 2nd Edition