

Advanced Microeconomic Theory¹

Guoqiang TIAN
Department of Economics
Texas A&M University
College Station, Texas 77843
(gtian@tamu.edu)

August, 2002/Revised: August 2018

¹This lecture notes are for the purpose of my teaching and convenience of my students in class. Please not distribute it.

Preface

The book of "Advanced Microeconomic Theory" is based on my lecture notes that I have used for more than twenty years. I have added almost double the content of this notes into the book. The Chinese version of the book was already published in 2016. This book is full-fledged and rich in content, and has a wide range of topics, including almost all the typical themes in modern microeconomic theory up to the frontier. It is also an integration of my study, research, and teaching of microeconomic theory over the past 30 years. This book is suitable for the courses of advanced microeconomics for graduate students, and the use of courses about topics of advanced microeconomics. It can also be an important reference for researchers. The English handout was used to teach courses at many universities in and around China, including A&M University, Shanghai University of Finance and Economics, Hong Kong University of Science and Technology, Tsinghua University, and Renmin University of China.

Economics is a seemingly simple, but actually very difficult science to learn, master, thoroughly understand, and truly comprehend. The reason why the economic problems are difficult to solve is that, apart from the most basic objective reality that the individual (whether at the national level or at the corporate, family or individual level) pursues profitability under normal circumstances, the other major objective reality is that, in the vast majority of cases, information among economic people is often asymmetrical, it is easy to pretend: A person said something, but we didn't know whether he tell the truth or lie; even if someone stares at each other, seeming to be listening attentively, but we don't know if they really listened. This will increase the difficulty of understanding and solving problems. If they do not, it will offset the institutional arrangements effect. In this way, how to deal with these two most objective realities, what kind of economic system, incentive mechanism and policy should be used have become the core issues and themes in all areas of economics. At the same time, economics often involves subjective value judgments. Different individuals have different values and different opinions. For example, some people emphasize the efficiency of resource allocation. Some emphasize the equality of resource allocation. Individuals often have different views on economic reforms and policies. It is easy to cause great controversy, making

it difficult to understand and master economics and its logical thought.

In addition, economics is a social discipline with a particularly strong externality. It has a large positive and negative externality (in the popular language, so-called positive energy or negative energy). Unlike doctors, if they do not have good medical skills, they only damage or kill some individuals. The bad application of economics affects all aspects of economic society, affecting society, groups and individuals. Therefore, the correct understanding, study, and master of modern economics, especially the main content of the microeconomics discussed in this book, is not only important for the theoretical innovation of modern economics, but also more important for practical applications. Once mistakes are made and wrong economic policies and systems are formulated, it will affect and endanger not only individuals, but also economic development at the national level as a whole. In this way, in addition to learning economics well, when it comes to proposing policy recommendations, it is particularly not possible to determine your thoughts by your position, so that too much to consider your own interests. Therefore, in addition to being brave to play, you must also have social conscience and responsibility.

Modern economics is an extremely inclusive and open discipline in dynamic development. It has far surpassed the stage of neo-classical economics, and Economic practice in the world can provide rich realities for the innovative development of economic theory. From the author's point of view, as long as rigorous internal logic analysis (not necessarily a mathematical model) is used and rational assumptions (including bounded rationality assumptions) are adopted, such research can belongs to the category of modern economics. Under the framework of the discipline of modern economics, microeconomics is mainly about the theory of how individuals make decisions. It is about the theory of how prices are determined. Therefore, it can be summed up in two heights. It is a theory about how markets operate, and it is also a theory about how the market should be remedied in some cases. It focuses on the study of how limited resources are allocated between different uses to better meet the different needs of humanity. It also constitutes the microfoundation of macroeconomics and almost all other areas of economics. It can help people understand how to fully and properly play the decisive role of the market in resource allocation, and better play the role of the government in maintaining social fairness and justice and providing public services. It can play the dual role of "pointing the way" and "improving techniques".

Knowing the truth, we must know why. It is necessary to know the scope of adaptation of an economic theory, otherwise it will cause great social negative effects once it is used to guide the formulation of economic policies. If we have limited training on the theoretical logic of modern economics and its empirical quantitative analysis, blindly shifting to the study and application of realistic problems without paying attention to its theo-

retical preconditions, many problems will result. If we negate the role of modern economics, we will think that the basic theoretical assumptions of the existing mainstream economics are too strong, too focused on mathematics and rigorousness, and are too far apart from reality. Therefore, it is believed that the real world problems cannot be explained and solved well.

In fact, in most cases, people who have such views do not understand the prerequisites themselves, and thus do not know that theories have their scope of application, and they blindly use them universally. Once wrong, they blame the theory is not good, and even think that the theory is wrong. In fact, just like the theory of any discipline, every rigorous economic theory gives preconditions. It is not valid in all situations. Therefore, unless the theory itself has logical contradictions, there is no right or wrong between them, but only which theory or model is most suitable for an economic institutional environment. If there is no rigour, how can one always draw the result of internal logic? This is as pointed out by Professor Dani Rodrik² of Harvard University. These accusations usually come from laymen or some unorthodox marginalists. Indeed, people who hold such arguments are often those who have limited understanding of economic theory and methods. Therefore, whether it is to do original research or practical application research, it is very necessary to learn economics well, grasp its analytical framework and methods, and pay attention to the rigor of logical analysis. This is the basic purpose of this book.

As a high-level course in microeconomics, the purpose of advanced microeconomics teaching is to reveal the rigorous internal logic behind some common principles and concepts, and to develop students' ability and way of thinking to analyze economic theory problems in a rigorous manner. It's also the purpose to tell students how to grasp and characterize the nature of complex economic behavior and economic phenomena (ie, modelling) to conduct rigorous internal logical economic analysis. This book systematically explains the contents of modern microeconomics from basic theory, benchmark theory, analysis framework, research methods to the latest frontier topics. Therefore, all or part of the chapters can be selected according to the course requirements as advanced microeconomics course materials for economics, finance, statistics, management, applied mathematics, and related disciplines for doctoral, postgraduate and senior undergraduate students. This book can also serve as an important reference for teachers and scholars engaged in economics teaching and research.

The characteristics of this book

While the mainstream of economic profession prior World War II focused mainly on economic thoughts, in great extent, lacking scientific rigor, but

²DANI RODRIK. "Economists vs. Economics", <http://www.project-syndicate.org/commentary/economists-versus-economics-by-dani-rodrik-2015-09>.

nowadays it seems that the main attention is given to techniques and strictness, and the profound economic thoughts behind economics are largely neglected. When studying economics, one should know well not only academic contexts of economics but also its systems so as to master its profound thoughts and wisdom. Combining both, the book attempts to advocate pursuing academics with deep thoughts and for deep thoughts.

In addition, in order to increase the understanding of the source background, development trajectory and inheritance of various economic theories introduced in this book (including the economic theory expressed in advanced mathematics), at the same time also in order to increase readers' interest in learning economic theory and the comprehensiveness of knowledge and to balance scholarship and ideology well, According to the network resources and my own understanding, I comprehensively compiled the biographies of 44 economists who made a pioneering contribution to the development of modern microeconomics.

This book also includes many Chinese scenarios in the introduction of many microeconomic theoretical models. It combines China's national conditions and market-oriented reforms to explain the economic connotation behind the theory and its policy inspiration. It also connects it with ancient Chinese profound philosophical thoughts and Chinese wisdom, hoping to achieve scholarship with ideology and ideology with scholarship, realizing the organic integration of ideology and scholarship. In studying economics, we must not only improve the techniques of economics, but also understand its principles, master its profound thoughts, and become wise men.

Structure of this book

Microeconomics focuses on the analysis of individual economic behaviors to study economic issues. Then based on this, it is further developed into given or designed various institutional arrangements, especially the results of economic operations under the market system. This book is also arranged in accordance with this logic topical chapter. However, before this, this book also provided an introduction to the preliminary knowledge and methods needed to learn advanced microeconomic theory.

The book is divided into upper and lower volumes and seven parts. The upper volume consists of the zeroth and the first to the third parts. It mainly introduces the benchmark model, the benchmark theory, and the analysis framework, methods, and tools that basically do not fail in the market without friction. The lower volume, which consists of the fourth to the sixth parts, shifts from focusing on the discussion of the frictionless free-competition market system to focusing on the issues under which the market economy system will fail. It mainly investigates how to correct and remedy the market in the presence of economic externalities, public goods,

especially information asymmetry and other market failures, in order to solve the problem of effective resource allocation. The content of this book is roughly described as follows:

There are two chapters in the zeroth part, which is the general theory and preliminary knowledge of the book. It mainly introduces the nature and methods of modern economics, the scope of the book, and the preparatory knowledge and methods of mathematics, in order to play the dual role of "pointing the way" and "improving techniques". It begins with an overview of the nature, scope, thought, analytical framework, and research methods of the modern economics discipline, especially the modern microeconomics theory, as well as the interlinkedness of economic thoughts with China's extensive Chinese wisdom. These contents hope to increase people's inclusiveness of various disciplines of economics. This inclusiveness is not only important for the development of various disciplines of economics, such as modern economics and political economics, but also important within the disciplines, such as the benchmark theory and application theory. Secondly, it introduces almost all the commonly used mathematical analysis tools and methods in this book and modern economics. Its length and content are rich. It can be used as a basic textbook or an important reference book for the economic mathematics course of advanced microeconomics/macroeconomics. It can also serve as a manual reference for mathematics needed to learn and study economics.

The first part consists of three chapters, which mainly discuss the internal economic logic of individual decision-making, including consumer theory, producer theory, and individual choice under uncertainty. The individual decision-making model is the microfoundation for the establishment of many theoretical models in economics, and it occupies a central position in the way economists think about problems. At the same time, many choices are made in uncertain situations. People usually need to avoid some uncertainties, for example, by purchasing insurance. So the issue of choice under uncertainty is an extremely important aspect of economics.

The second part consists of four chapters. It mainly discusses game theory and market theory, including game theory, repeated game and reputation mechanism, cooperative game, market theory of various market structure types. Game theory has become a very important subdiscipline in mainstream economics, a core field in microeconomics, and the most basic analytical tool for studying many interactive decision problems in economics. For example, monopolistic competition, especially the discussion of oligopolistic market theory, requires the use of a lot of game theory knowledge and results, so it is discussed together as applications.

The third part consists of five chapters. It mainly discusses the benchmark market theory in the ideal situation of perfect competition - general equilibrium theory and social welfare, including the empirical theory of competitive equilibrium, the normative theory of competitive equilibrium,

economic core, fair allocation, social choice theory, and general equilibrium theory under certainty. General equilibrium theory is one of the most important theories in the history of economic theory development in the past 100 years, and it is also one of the most dazzling achievements in the treasure house of human economic thoughts. It provides an important reference and benchmark for better studying and solving practical problems. This part will describe the nature of competition equilibrium, and discuss how to achieve the fair allocation of resources, so as to further demonstrate the universality, optimality and rationality of the market economy system.

The above three parts mainly describe the benchmark model, the benchmark theory, and the analysis framework, methods, and tools that basically do not fail in the market under ideal conditions. However, in many cases, the market is not omnipotent and often fails. Therefore, the next three parts of this book mainly discuss how to deal with the problems that the market often fails in non-ideal situations that is closer to reality. The fourth part consists of two chapters. It mainly discusses the theory of externalities and public goods, including the typical market failures of externalities and public goods. From the micro and information point of view, the market is still faced with many problems, leading to "market failure." It is important to analyze where the market is failing and how the government should do it. In this part we will see that, in general, these "non-market goods" or "harmful goods" will lead to Pareto inefficient results, leading to market failure. Because of externalities and public goods, the private goods market is generally not a good mechanism for allocating resources.

The fifth part consists of five chapters. It mainly discusses incentive, information and economic mechanism design theory, including principal-agent theory under hidden information, principal-agent theory under moral hazard, general mechanism design under complete information and incomplete information, and dynamic mechanism design, etc. The economic mechanism theory mainly studies whether and how to design a set of mechanisms (game rules or systems) to achieve the set goals under the conditions of free choice, voluntary exchange, incomplete information, and decentralized decision-making. It also requires the ability to compare and judge the pros and cons of a mechanism. The symmetry of information and the compatibility of incentives are the root causes of different performances of different mechanisms.

The sixth part consists of two chapters, which mainly introduce the two hot frontier subfields of modern microeconomic theory: auction theory and matching theory, collectively referred to as market design. This emerging field of market design can be seen as the concrete extension of the fifth part of the general mechanism design theory, and it has its wide application in reality.

Teaching tips

As mentioned earlier, this book is mainly for the use of Ph.D. students in advanced microeconomics series courses, advanced micro topic theory courses, and mathematical economics courses. The diversity of the theme of this book and the relative independence and progressiveness of the content also provide a great deal of choice and free space for the different levels of microeconomics teaching and the most important aspects what teachers think. Its content is applicable to the teaching of advanced microeconomics courses at different levels, and teachers can flexibly select relevant chapters to explain according to teaching needs. In addition to the doctoral microeconomics system teaching and frontier teaching of economics, this book can also be used as a textbook and an important reference for undergraduates and postgraduates. The second chapter about the knowledge of mathematics can be used as a teaching material or an important reference for teaching economic mathematics or mathematical economics. In 2004, the School of Economics at Shanghai University of Finance and Economics started to offer advanced microeconomics courses for senior undergraduates. It also offered courses of advanced microeconomics 1, 2 and 3, as well as frontier micro theories for postgraduate students.

Here are some suggestions for teachers who choose this textbook. (1) For the advanced microeconomics 1 for senior undergraduates and postgraduates, consider the following chapters for the teaching of one semester: the third, fourth, and fifth chapters of individual rational decision-making and the sixth, seventh, and eighth chapters of game theory. (2) For the advanced microeconomics 2 for postgraduate students, consider the following chapters for the teaching of one semester: Chapter 9 on market theory, Chapters 10, 11, 12, and 13 on general equilibrium Theory, Chapters 14, 15 on externalities and public goods, and Chapters 16, 17 on mechanism design. (3) For the advanced microeconomics 3 or frontier topics of microeconomic theory for postgraduate students, according to different focuses, the instructor can choose the 18th, 19th and 20th chapters on the mechanism design theory, or choose the 21st and 22nd chapters on market design. Of course, the selection of chapters in specific teaching needs to be based on the preference of the instructor, research interests, and constraints on the course time. In addition, whether to teach advanced microeconomics 1, 2, 3 or micro frontier topics, students should first prepare for themselves or teachers should roughly teach the contents of the first chapter.

For the undergraduates to have a general understanding of the scope, ideas, analytical framework, and research methods of modern economics, the introduction of Chapter 1 and other chapters are basically sufficient. In fact, every year the one-credit school-wide course which I give at the Shanghai University of Finance and Economics to undergraduates (required) of the School of Economics and other undergraduates (electives)-

"The Ideas and Methods of Economics", is basically corresponds to the content of the first chapter. The course has been listed in the national excellent video open class. The related video is also listed on the homepage of the "Economic Research" website. It has already received 80,000 hits.

Doing exercises is the most reliable and effective method for mastering the content of teaching materials. When I was in college to study for graduate students, I studied many mathematics textbooks translated from the former Soviet Union by myself. For example, the eight volumes of calculus tutorial by Fichtenholz and the "Mathematical Analysis" published by Fudan University Press before the Cultural Revolution, were read by me at least ten times. Demidovich's collection of more than 5,000 mathematics analysis exercises was all done by me several times. This learning experience laid a solid foundation for my math foundation, logical analysis ability, and subsequent study and research of modern economics. Even today, more than 30 years later, I can still remember the basic proof of almost all theorems in mathematical analysis. At the end of each chapter, this book is also accompanied by a certain amount of exercises. Some of these exercises were written by myself. Some of them were adapted from foreign language textbooks. Some were adapted from doctoral qualifications in the economics department of world-class universities, or examples or basic conclusions of original academic papers. Whenever possible, I will indicate its source. We thank the anonymous authors for many exercises in this book.

Research tips

This book can also be used as an important reference book for research. Whether it is to do original research or research on Chinese issues, it is necessary to learn modern economics well, master its basic analytical framework and research methods, and lay a solid foundation for theory and methodology. After completing this book, you can basically master the most advanced microeconomic theory and can engage in original research. Generally speaking, research and innovation in economics can be broadly divided into two categories: The first category is the research and innovation of basic, original, and common theories and tools. These researches and innovations have no borders and are general. The theories introduced in this book basically fall into this category. In this respect, the gap between China's research level and the international level is very large. These gaps are reflected in the originality, the difference in the number of published papers, research methods, and economic thoughts embodied in the articles. They urgently need to catch up. There is a need for a group of people to target the international frontier and do research on pure theory and quantitative methods, not just research on China's economic issues. If China is to become a powerful country, all aspects must rise, including the right to

have an international academic discourse. The second category is a practical issue, that is, applying the basic principles, analytical frameworks, research methods, and analytical tools of modern economics to study the real problems of a country or region, especially the study of China's economic problems. These two are dialectically unified. Do not negate the latter by the former or negate the former by the latter. Both should be parallel and equally emphasised. This is just like the basic research innovation in the natural sciences and the technological innovation research in the business community. They are complementary and they are all very important. They are indispensable.

In addition, it should be pointed out that modern economics, especially microeconomic theory research, mainly provides two types of theories, all of which have strict prerequisites. One is the economic theory that provides the benchmark point or reference system. Most of the theories introduced in the previous part of this book are of the first type, while others are more closely related to reality. They are mainly theories designed to solve practical problems. Most of the theories introduced in the later part of this book are the second types of theories. The first type of theory is mainly based on the theoretical premise of the economic environment of the mature market economy countries. It provides the basic theory in the ideal situation. Although it has an important role in guiding the improvement or reform orientation by "pointing the way", however, it deviates from reality, including that it does not necessarily fully apply to China's actual situation and economic environment that are still in the process of transition. It may not necessarily be copied to guide the formulation of current specific policies. Therefore, it is necessary to revise the benchmark theory and consider the situation that there is friction and is closer to reality, so as to develop a second type of economic theory that solves specific practical problems, and then to draw conclusions and make predictions with intrinsic logic. Thus, these two types of theories are a progressive relationship in the development of disciplines. The basic analytical framework, thoughts, and research methods of modern economics described in this book are not only common to the study of these two types of theories, but also can be used to better use economic theory to study China's economic problems. They can even lay a solid theoretical and methodological foundation for the development of economic theories suitable for studying China's economic problems. This requires the attention of teachers and students when using this book.

Acknowledgements

This book incorporates in different parts some of the original research results I have published over the past 20 years in many top and first-tier international academic journals in economics and mathematics. Here, first of all, I would like to thank the many article collaborators for inspiring my

research. Many of these results were put in the right place in almost each chapter of the book.

Secondly, I would like to express my special thanks to Professor Sun Churen at Southwestern University of Finance and Economics, who received a doctoral degree from the School of Economics at Shanghai University of Finance and Economics under my guidance. He spontaneously translated my English version of "Advanced Microeconomic Theory" into Chinese seven years ago. Without his early translation, I may not have such a big incentive until now to spend many years in this textbook on advanced microeconomics. I would also like to thank Dr. Meng Dawen, who received a doctoral degree from the School of Economics at Shanghai University of Finance and Economics under my guidance, and now is currently an associate professor there. He is my next research collaborator and my previous assistant as a special expert in the National Thousand Talents Program. He first worked with Sun Churen to translate the English lecture papers into Chinese, and provided help for the text translation and writing of the mechanism design theory in Chapters 16-18, including the re-drawing of the book. In addition, I would like to express my gratitude to Associate Professor Lou Guoqiang, a colleague from the Institute for Advanced Research of Shanghai University of Finance and Economics. He is my current assistant as the special expert of the National Thousand Talents Program. He has made great contributions to the writing assistance after the completion of the first draft of the translation, the editing of many chapters, the editorial arrangement, and the organization and coordination. He has saved me a lot of time. Without him, I don't think the publication of this book will be so smooth and it will not be completed yet. At the same time, I would also like to thank my doctoral student at Shanghai University of Finance and my assistant researcher Chen Xudong for his assistance in the writing of the book, and the proofreading work of Qin Guangyan, Li Yuan, and Cheng Ning after the completion of the book.

In addition, I have also received opinions and suggestions about this book from many colleagues at home and abroad. In particular, colleagues from the School of Economics at Shanghai University of Finance and Economics reviewed the relevant chapters after the book was basically formed. They include Du Ninghua (Chapter 1, Chapter 13), Yang Zhe (Chapter 2, 10, 11), Fan Cuihong (Chapter 3, 4), Zheng Bingyong (Chapter 5, 6, 7), Tang Qianfeng (Chapter 8), Wu Shanlin (Chapter 9, 14) and Rong Kang (Chapter 12, 18, 19, and 20). I thank them for their many very specific revise opinion. Cao Xiaoyong (Chapter 3-5, 9, and 21) who received a doctoral degree in economics from A&M University in Texas under my guidance and is currently an associate professor at the University of International Business and Economics, Assistant Professor Xue Shaojie (Chapter 6-8, 12-13, 16-17, 21) of Xiamen University and Long Xinghua (Chapter 1-2, 10-11, 14-15, 18-20, and 22) of Shanghai Finance University, and Associate Pro-

fessor Gong Rukai of Glorious Sun School of Business and Management at Donghua University read and proofread after the basic manuscript was finalized. Thanks here.

I would also like to thank the students and the teaching assistants who have studied my advanced microeconomics courses in various universities and many doctoral students and readers around the world. They gave many useful suggestions for the content of the courses. Many graduated or in-school doctoral students (many of them are my Ph.D. students) from School of Economics at Shanghai University of Finance and Economics and Department of Economics at Texas A&M University in the United States also participated in the preparation of exercises, the harmonization of document formats, and glossary work. They include Wang Guojing (Chapter 3-4), Hu Jun (Chapter 5-7), Long Xinghua (Chapter 10-11), Rong Jianxin (Chapter 15-17), Wang Dazhong (Chapter 12, 14 and 18), Ju Yan (Chapter 7, 8, 21), Fang Guanfu (Chapter 2, 6), Hao Liang (Chapter 2, 3, 4, 7, 9, 15, 18, 19), Lang Youze (Chapter 13, 14, 16, 17, and 20), Tian Yougong (Chapter 2, 5, 13, and 20), Liu Quanlin (Chapter 12, 13, 14, and 15), Dai Darong (Chapter 15-18), Huang Chao (Chapter 13, 14, 15, and 22) and Jiao Zhenhua (Chapter 22), etc.. Hao Liang and Lang Youze participated in the collation of the glossary of the two volumes, as well as the examination of the documents and the harmonization of the formats.

Advanced Microeconomics was included in the development of the Shanghai University of Finance and Economics' excellent course (theoretic economics). The teaching materials are an important part of this. We are grateful to the university for providing material support for the excellent courses, and also thank Professor Chang Jinxiong, the associate dean of the School of Economics for teaching, for the time and energy spent in coordinating the preparation of the teaching materials. The National Natural Science Foundation of China has funded many of my studies. The Shanghai Education Commission, the Graduate School of Shanghai University of Finance and Economics, and the School of Economics have provided funding for the editors of this book. We also express our gratitude here.

Of course, we must also thank the editors of China Renmin University Press, Gao Xiaofei, Zhou Huajuan, Pan Weilin and others for their help and patience in publishing this book.

Finally, this book is also my dedication to the 100th anniversary of the establishment of Shanghai University of Finance and Economics in 2017. I would like to thank Shanghai University of Finance and Economics and its previous leaders. In particular, the former principal, Professor Tan Min, who has given me full confidence. Since 2004, he has appointed me as the Dean of School of Economics at Shanghai University of Finance and Economics, and has participated in the practice of the reform and development of China's economic education, including the introduction of large-scale, organized-system overseas high-level talents and the reform of the curricu-

lum system with advanced international standards. And since then, every year, I have given doctoral students at Shanghai University of Finance and Economics a course on Advanced Microeconomics 2, never stopped. Without this opportunity, it is impossible to have this textbook.

In short, this book is a crystal of a lot of people's wisdom in a sense. Of course, I am responsible for the deficiencies and errors of this book. Because teaching, research, and administrative work at the Shanghai University of Finance and Economics are very busy and coupled with limited levels, the theory discussed in many chapters and the economic ideas behind them are not fully understood, let alone make any academic contributions. The content that was written still looks rough, and I'm not even satisfied with myself. But in any case, the reason why I am willing to spend so much time and energy on doing something that may be a hard but thankless job is to hope to do something for China's economic education reform and cultivate more high-quality economic talents for the society. The good thing is that publishing does not mean the end. In the future, I will continue to update and revise this book.

Guoqiang Tian
In my house: "Xingkong Study"
January 18, 2016

Contents

Preface

I Preliminary Knowledge and Methods	1
1 Nature of Modern Economics	5
1.1 Economics and Modern Economics	5
1.1.1 What is economics about?	5
1.1.2 Four Basic Questions That Must Be Answered	6
1.1.3 What is Modern Economics?	7
1.1.4 The Difference between Economics and Natural Science	8
1.2 Two Categories of Modern Economic Theory	9
1.2.1 Benchmark Theory and Relatively Realistic Theory .	9
1.2.2 Three Roles of Economic Theory	15
1.2.3 Microeconomic Theory	16
1.3 Modern Economics and Market System	17
1.3.1 Market and Market Mechanism	17
1.3.2 Three Functions of Price	20
1.3.3 The Superiority of Market System	22
1.4 Governance Boundaries of Government, Market, and Society	26
1.4.1 Three Dimensions of State Governance: Government, Market, and Society	26
1.4.2 Good State Governance Brings about Good Market Economy	27
1.4.3 The Principle for Defining the Governance Bound- aries between Government and Market and between Government and Society	30
1.5 Three Institutional Arrangements for Comprehensive Gov- ernance	32
1.5.1 Regulatory Governance	32
1.5.2 Incentive Mechanism	33
1.5.3 Social Norms	34

1.5.4	The Complementary Structure of the Three Institutional Arrangements	34
1.6	Ancient Chinese thoughts on Market	37
1.7	The Cornerstone Assumption in Modern Economics	41
1.7.1	Selfishness, Self-love, and Self-interest	42
1.7.2	Practical Rationality of Self-interested Behavior	42
1.7.3	Applicable Boundaries of Self-interest and Altruism	44
1.8	Key Points in Modern Economics	45
1.8.1	Scarcity and Limitation of Resources	46
1.8.2	Information Asymmetry and Decentralized Decision-making	46
1.8.3	Economic Freedom and Voluntary Exchange	48
1.8.4	Constraints and Feasible Options	49
1.8.5	Incentive and Incentive Compatibility	50
1.8.6	Property Rights Incentive	51
1.8.7	Outcome Fairness and Equity in Opportunity	52
1.8.8	Allocative Efficiency of Resources	52
1.9	A Proper Understanding of Modern Economics	53
1.9.1	How to Regard the Scientific Nature of Modern Economics	54
1.9.2	How to Regard the Mathematical Nature of Modern Economics	55
1.9.3	How to Regard Economic Theory Correctly	56
1.9.4	How to Regard the Criticism for the Difficulty to Conduct Experiments in Economics	59
1.10	Basic Analytical Framework of Modern Economics	62
1.10.1	Specifying Economic Environment	64
1.10.2	Making Behavioral Assumptions	66
1.10.3	Setting Economic Institutional Arrangements	67
1.10.4	Determining Equilibrium	69
1.10.5	Making Evaluations	70
1.11	Basic Research Methodology in Modern Economics	71
1.11.1	Establishing Benchmarks	72
1.11.2	Setting Reference Systems	72
1.11.3	Building Studying Platforms	73
1.11.4	Developing Analytical Tools	75
1.11.5	Constructing Rigorous Analytical Models	75
1.11.6	Positive Analysis and Normative Analysis	76
1.12	Practical Role of the Analytic Framework and Methodologies	77
1.13	Basic Requirements for Mastering Modern Economics	79
1.14	Distinguishing Sufficient and Necessary Conditions	80
1.15	The Role of Mathematics and Statistics in Modern Economics	80
1.16	Conversion between Economic and Mathematical Languages	84
1.17	Biographies	84

1.17.1	Adam Smith	84
1.17.2	David Ricardo	86
1.18	Exercises	87
1.19	References	91
2	Knowledge and Methods of Mathematics	97
2.1	Basic Set Theory	97
2.1.1	Set	97
2.1.2	Mapping	98
2.2	Basic Linear Algebra	99
2.2.1	Matrix and Vector	99
2.2.2	Matrix Operations	100
2.2.3	Linear Dependence of Vectors	102
2.2.4	Transpose and Inverse of a Matrix	102
2.2.5	Solving a Linear System	103
2.2.6	Quadratic Form and Matrix	105
2.2.7	Eigenvalues, Eigenvectors and Traces	107
2.3	Basic Topology	108
2.3.1	Topological Space	108
2.3.2	Metric Space	109
2.3.3	Open Sets, Closed Sets and Compact Sets	109
2.3.4	Connectedness of Set	113
2.3.5	Sequence and Convergence	113
2.3.6	Convex Set and Convexity	114
2.4	Single-Valued Function and Its properties	115
2.4.1	The continuity of function	115
2.4.2	Upper Semi-continuity and Lower Semi-continuity	116
2.4.3	Transfer Upper and Lower Continuity	117
2.4.4	Differentiation and Partial Differentiation of Function	117
2.4.5	Mean Value Theorem and Taylor Expansion	119
2.4.6	Homogeneous Functions and Euler's Theorem	121
2.4.7	Implicit Function Theorem	121
2.4.8	Concave and Convex Function	122
2.4.9	Quasi-concave and Quasi-convex Function	123
2.4.10	Separating Hyperplane Theorem	126
2.5	Multi-Valued Function and Its properties	126
2.5.1	Point-to-Set Mappings	126
2.5.2	Upper Hemi-continuous and Lower Hemi-continuous Correspondence	128
2.5.3	The Open and Closed Graphs of Correspondence	131
2.5.4	Transfer Closed-valued Correspondence	132
2.6	Static Optimization	134
2.6.1	Unconstrained Optimization	134
2.6.2	Optimization with Equality Constraints	139

2.6.3	Optimization with Inequality Constraints	142
2.6.4	The Envelope Theorem	144
2.6.5	Maximum Theorems	145
2.6.6	Continuous Selection Theorems	147
2.6.7	Fixed Point Theorems	147
2.6.8	Variation Inequality	150
2.6.9	FKKM Theorems	151
2.7	Dynamic Optimization	152
2.7.1	Variational Method	153
2.7.2	Optimum Control	156
2.7.3	Dynamic Programming	158
2.8	Differential Equations	161
2.8.1	Existence and Uniqueness Theorem of Solutions for Ordinary Differential Equations	163
2.8.2	Some Common Ordinary Differential Equations with Explicit Solutions	164
2.8.3	Higher Order Linear Equations with Constant Coef- ficients	166
2.8.4	System of Ordinary Differential Equations	169
2.8.5	Simultaneous Differential Equations and Types of E- quilibrium Stability	172
2.8.6	The Stability of Dynamical System	174
2.9	Difference Equations	175
2.9.1	First Difference Equations	176
2.9.2	Second Order Difference Equation	178
2.9.3	Difference Equation of Order n	179
2.9.4	The Stability of n th Order Difference Equations	180
2.9.5	Difference Equations with Constant Coefficients	181
2.10	Basic Probability	182
2.10.1	Probability and Conditional Probability	182
2.10.2	Mathematical Expectation and Variance	182
2.10.3	Continuous Distributions	183
2.10.4	Common Probability Distributions	184
2.11	Stochastic Dominance and Affiliation	186
2.11.1	Hazard Rates	186
2.11.2	Stochastic Dominance	187
2.11.3	Hazard Rate Dominance	190
2.11.4	Reverse Hazard Rate Dominance	190
2.11.5	Order Statistic	191
2.11.6	Affiliation	192
2.12	Biographies	194
2.12.1	Friedrich August Hayek	194
2.12.2	Joseph Alois Schumpeter	195
2.13	Exercises	198

2.14 Reference	207
V Information, Incentives, and Mechanism Design	211
16 Principal-Agent Theory: Hidden Information	217
16.1 Introduction	217
16.2 Basic Settings	218
16.2.1 Economic Environment (Technology, Preferences, and Information)	219
16.2.2 Outcomes Space and Contracting Variables	220
16.2.3 Information Structure and Timing	220
16.2.4 Complete Information Optimal Contract (Benchmark)	220
16.2.5 Incentive Feasible Contracts	222
16.2.6 Monotonicity Constraints	224
16.2.7 Information Rents	224
16.2.8 The Optimal Contracts under Asymmetric Information	225
16.2.9 The Rent Extraction-Efficiency Trade-Off	226
16.2.10 Shutdown Policy	229
16.3 Application	230
16.3.1 Regulation	230
16.3.2 Financial Contracts	232
16.4 The Revelation Principle	232
16.5 Extensions to the Basic Model	234
16.5.1 Finite Types	235
16.5.2 Continuum Types	237
16.5.3 Bunching and Ironing	240
16.6 Ex Ante Participation Constraints	243
16.6.1 Risk Neutrality	243
16.6.2 Risk Aversion	245
16.6.3 Risk-Averse Principal	247
16.7 Extension of Classical Model	249
16.7.1 Network Externalities	249
16.7.2 Countervailing Incentives	253
16.8 Adverse Selection in Competitive Market	262
16.8.1 Discriminating Mechanism of Competitive Market un- der Asymmetric Information	263
16.8.2 Signal transmission under asymmetric information .	266
16.9 Further Extension	267
16.10 Biographies	268
16.10.1 Ludwig Mises	268
16.10.2 Leonid Hurwicz	269
16.11 Exercises	271
16.12 Reference	285

17 Principal-Agent Theory: Moral Hazard	289
17.1 Introduction	289
17.2 Basic Model	292
17.2.1 Model Setting	292
17.2.2 Benchmark Case	293
17.2.3 Incentive Feasible Contracts	295
17.2.4 Risk Neutrality and First-Best Implementation	296
17.3 Second-best Contract under Limited Liability and Risk Aversion	298
17.3.1 Second-best Contract under Limited Liability	298
17.3.2 Second-best Contract under Risk Aversion	301
17.3.3 Basic Trade-off: Risk Aversion and Incentive	303
17.4 Contract Theory at Work	304
17.4.1 Efficiency Wage	304
17.4.2 Sharecropping	305
17.4.3 Financial Contracts: Credit Rationing	308
17.5 Extensions to the Basic Model	310
17.5.1 More than Two Outcomes, Two Actions	310
17.5.2 Two Levels of Performance, Continuous Actions	312
17.5.3 Continuous Performance, Continuous Actions	313
17.6 Relative Performance Incentive of Multi-Agents	317
17.6.1 Tournament Model	317
17.6.2 Tournament Contract under Risk Aversion	318
17.6.3 Comparison of Tournament and Individual Performance Contract	320
17.7 Performance Incentive of Multi-Task	321
17.8 Implicit Incentive and Career Concern	324
17.9 Incentive Intensity and Efficiency-Tradeoff between Distortions	325
17.10A Mixed Model of Moral Hazard and Adverse Selection	327
17.10.1 Optimal Wage Contract with Unobservable Efforts Only	327
17.10.2 Optimal Wage Contract with Unobservable Efforts and Risk Aversion	330
17.10.3 Optimal Wage Contract with Unobservable Efforts and Cost	335
17.11 Character biography	338
17.11.1 Oskar Lange	338
17.11.2 James Mirrlees	339
17.12 Exercises	340
17.13 Reference	352

18 General Mechanism Design: Contracts with Multi-Agents	357
18.1 Introduction	357
18.2 Basic Settings	368
18.2.1 Economic Environments	368
18.2.2 Social Goal	369
18.2.3 Economic Mechanism	369
18.2.4 Solution Concept of Self-Interested Behavior	371
18.2.5 Implementation and Incentive Compatibility	371
18.3 Examples	372
18.4 Dominant Strategy and Truthful Revelation Mechanisms	374
18.5 Gibbard-Satterthwaite Impossibility Theorem	378
18.6 Hurwicz Impossibility Theorem	383
18.7 Vickrey-Clark-Groves Mechanisms	385
18.7.1 Vickrey-Clark-Groves Mechanisms for Discrete Public Good	385
18.7.2 Vickrey-Clark-Groves Mechanisms with Continuous Public Goods	389
18.7.3 Uniqueness of VCG for Efficient Decision	393
18.7.4 Balanced VCG Mechanisms	394
18.8 Nash Implementation	396
18.8.1 Nash Equilibrium and General Mechanism Design	396
18.8.2 Characterization of Nash Implementation	397
18.9 Better Mechanism Design	402
18.9.1 Groves-Ledyard Mechanism	403
18.9.2 Walker's Mechanism	404
18.9.3 Tian's Mechanism	406
18.10 Refining Nash execution, Approximate Nash execution, Nash and strong Nash double execution	408
18.10.1 Refining Nash execution	409
18.10.2 Approximate Nash execution	409
18.10.3 Nash and strong Nash double execution	410
18.11 Information Efficiency in Mechanism Design	410
18.11.1 Information Efficiency in Mechanism Design	410
18.11.2 Informational Efficiency and Uniqueness of Competitive Market Mechanism	416
18.12 Biographies	421
18.12.1 Michael Spence	421
18.12.2 Joseph E. Stiglize	422
18.13 Exercises	423
18.14 Reference	436

19 Incomplete Information and Bayesian-Nash Implementation	445
19.1 Introduction	445
19.2 Basic Analytical Framework	446
19.2.1 Model	446
19.2.2 Bayesian Incentive-Compatibility and Bayesian Implementation	448
19.3 Truthful Implementation of Pareto Efficient Outcomes	450
19.4 Characterization of DIC and BIC under Linear Mode	452
19.5 Impossibility of BIC Pareto Efficient Optimal Contract	455
19.5.1 Participation Constraints	455
19.5.2 Myerson-Satterthwaite Impossibility Theorem	457
19.6 The Revenue Equivalence Theorem in Auctions	460
19.7 Cremer-McLean Full Surplus Extraction Theorem	463
19.8 Characterization of Bayesian Implementability	465
19.9 Characterization of Ex-post Implementability	472
19.10 Biographies	478
19.10.1 Jean Jacques Laffont	478
19.10.2 Roger B. Myerson	479
19.11 Exercises	481
19.12 Reference	489
20 Dynamic Mechanism Design	493
20.1 Introduction	493
20.2 Dynamic Contracts under Full Commitment	493
20.2.1 Constant Types	494
20.2.2 Independent Dynamic Types	495
20.2.3 Correlated Types	497
20.3 Dynamic Contracts under Different Commitment Power	501
20.3.1 Contracting with Full Commitment	503
20.3.2 Dynamic Contracting with No Commitment	503
20.3.3 Dynamic Contracts with Partial Commitment	508
20.4 Sequential Screening	510
20.4.1 An Example of Sequential Screening	510
20.4.2 Sequential Screening under Incomplete Information	512
20.5 Efficient Budget-Balanced Dynamic Mechanism	517
20.5.1 Efficient Budget-Balanced Static Mechanism	518
20.5.2 Incentive Problem in Dynamic Environments	519
20.5.3 Efficient Budget-Balanced Dynamic Mechanism	521
20.6 Biographies	522
20.6.1 Jean Tirole	522
20.6.2 Thomas Schelling	524
20.7 exercises	525
20.8 References	531

Part I

Preliminary Knowledge and Methods

In order to enable readers to grasp the contents in this book more effectively, learn the modern economics, understand its profound economic thoughts and the theoretical models as well as their proofs of economic problems rigorously, this part introduces the preliminary knowledge and methods of economics and mathematics.

Chapter 1 briefly introduces the essence, category, thoughts and methods of modern economics discipline, as well as the compatibility of economic thought with China's extensive and profound Chinese intellectual wisdom. We will discuss the ideas and methods of modern economics, especially those that are involved in this book. While the mainstream of economic profession prior World War II focused mainly on economic thoughts, in great extent, lacking scientific rigor, but nowadays it seems that the main attention is given to techniques and strictness, and the profound economic thoughts behind economics are largely neglected. When studying economics, one should know well not only academic contexts of economics but also its systems so as to master its profound thoughts and wisdom. Combining both, the book attempts to advocate pursuing academics with deep thoughts and for deep thoughts.

Chapter 2 introduces the basic knowledge and results of mathematics required for studying modern economics in general and advanced microeconomic theory in particular. They are used for rigorous analysis of various economic problems and especially provide necessary mathematical knowledge and tools for models, axioms, and science of microeconomic theory to derive and prove many theoretical results in microeconomics and the boundaries within which they are true.

Chapter 1

Nature of Modern Economics

In this chapter, we first introduce the nature and methodology of modern economics in general and the scope, preliminary knowledge, thoughts, and methods particularly involved in the book. We will introduce the basic terminologies, core assumptions, standard analytical framework, methodologies and techniques used in modern economics, discuss its research object of market system, its connection with ancient Chinese economic thoughts, as well as some key points one should pay attention to.

The methodologies and techniques for studying modern economics include: providing benchmark, establishing reference system, setting up studying platforms, developing analytical tools, making empirical and normative analysis, learning the basic requirements of economic theory, understanding the role of economic theory, clarifying necessary and sufficient conditions for a statement, understanding the role of mathematics and statistics in economics, and mastering the conversion between economic and mathematical languages.

1.1 Economics and Modern Economics

1.1.1 What is economics about?

To learn well economics, one must firstly know its definition and understand its connotation, scope, and concerns.

Economics is the social science that studies how to make decisions in face of resource scarcity and/or information asymmetry. Specifically, it studies economic behavior, economic phenomena, and how rational economic agents, including individuals, families, firms, institutions, organizations, and government agencies, make optimal trade-off choices subject to the constraint of limited resources.

It is actually because of the fundamental inconsistency and conflict between resource scarcity and individuals' unlimited desires (or wants) that

economics could come into being. The core idea is that individuals, who are under the basic constraint of limited resources (limited information, capital, time, capacity and freedom) and driven by unlimited desires, must make trade-off choices in resource allocation to make the best use of limited resources to maximize the satisfaction of their needs.

Economics occupies the top position among social sciences. As a discipline of social science, it studies the problem of social choices based on inherent logic analysis and scientific viewpoints and establishes itself via systematic exploration of the matter of choice. Such exploration not only involves the building of theory but also provides analytical tools for the test of economic data.

1.1.2 Four Basic Questions That Must Be Answered

For any economic system, regardless of whether it is the planned economy wherein the government plays a decisive role, the free economy wherein the market plays a decisive role, or the semi-market and semi-planned mixed economy wherein the state-owned economy plays a leading role, when it comes to the allocation of resources, all face the following four basic questions:

- (1) What should be produced and in what quantity?
- (2) How should the product be produced?
- (3) For whom should it be produced and how should it be distributed?
- (4) Who makes the decision on production?

These questions must be answered in all economic systems, but different economic institutional arrangements provide different answers. Whether an institutional arrangement can effectively resolve these problems depends on whether it can properly deal with the issues induced by information and incentives.

As such, two basic economic institutional arrangements have been used in the real world:

- (1) The institutional arrangement of planned economies: All the four questions are answered by the government who determines most economic activities and monopolizes decision-making processes and all industries; it makes decisions on market access, product catalog, infrastructure investment allocation, individual job assignment, product price, employment wage, and among others, and the risk is borne by the government.

- (2) The institutional arrangement of market economies: Most economic activities are organized through free market; the decisions on what product to produce, how to produce and for whom to produce are mainly made by decentralized firms and consumers, and the risk is borne by individuals.

While almost every real-world economic system is somewhere in between these two extremes, the key is which extreme is in the dominant position. The fundamental flaw of the planned economic system is that it cannot effectively resolve the problems induced by information and incentives, which in turn results in inefficient allocation of resources, whereas free-market economic system can be a good solution in these respects. This is the fundamental reason why countries that once adopted planned economic system inevitably failed and why China carried out market-oriented reforms and wishes to bring market to the decisive role in resource allocation.

1.1.3 What is Modern Economics?

Modern economics, which has developed mainly since the 1940s and was built on the basic recognition of individuals' pursuit of self-interest, systematically studies individuals' economic behavior and social economic phenomena by adopting scientific methods for rigorous reasoning and utilizing mathematical tools – specifically, it makes historical and empirical observations of the real world, elevates the observations towards the formation of theory through rigorous logical analysis, and then again test the theory in the continuing real world – thus making itself a branch of science equipped with scientific analytical framework and research methods. Such systematic study involves formal theory and also provides analytical tools for testing economic data.

Social economic issues cannot be studied by simply doing social experiments, so the approach requires theoretical analysis based on inherent logical inference, historical comparisons for drawing experience and lessons, and tools of statistics and econometrics for quantitative analysis or empirical test, the three of which are indispensable. When conducting economic analysis or giving policy suggestions in the realm of modern economics, the analysis often combines theory, history, and statistics, presenting not only theoretical analysis of inherent logic and comparative analysis from the historical perspective but also empirical and quantitative analysis with the help of statistical tools. Indeed, all knowledge is presented as history, all science exhibits as logics, and all judgment is understood in the sense of statistics. That's why Joseph Schumpeter (see 2.12.2 for his biography) thought that the difference between an economic "scientist" and a general economist lies in whether they adopt the three elements when conducting

economic analysis: the first element is theory for inherent logical analysis, the second one is history for historical analysis, and the third one is statistics for empirical analysis with data.¹

For theoretical innovations and practical applications, it is of crucial importance to correctly understand and master the general knowledge of modern economics and the content of this book in particular. It is useful for studying and analyzing economic problems, interpreting economic phenomena and individuals' economic behavior, setting up goals and specifying the direction for improvements. More importantly, with the help of comparative analysis from the historical perspective and quantitative analysis based on data, we can draw conclusions of inherent logic and make relatively accurate predictions through rigorous inference and analysis.

Modern economics is referred to as the "crown" of social sciences due to its extremely general analytical framework, research methods and analytical tools. Its basic ideas, analytical framework and research methodologies are powerful enough for studying economic problems and phenomena occurred in different countries, regions, customs and cultures, and can be applied to almost all social sciences. It can even be helpful for realizing good leadership, management and work so that it is jokingly called "economics imperialism" or "omnipotent" discipline by Gary S. Becker (1930-2014, see 13.7.2 for his biography).

1.1.4 The Difference between Economics and Natural Science

There are three major differences between modern economics and natural science:

(1) Economics studies human behavior and needs to impose some associated assumptions, while natural science in general does not involve the behavior of human being (of course, such distinction is not absolute; for example, biology and medicine sometimes involve human behavior. However, these involvements are not from the perspective of utilitarianism, while economics considers human behavior mainly from the perspective of utilitarianism). Once individuals are involved, the information is very much asymmetric and easy to disguise because their behavior is unpredictable without appropriated mechanisms, making it very difficult and complicated to deal with.

(2) In the discussion and study of economic problems, descriptive empirical analysis and normative analysis of value judgment are both need-

¹In his inaugural speech titled "Science and Ideology" when assuming the position of President of the American Economic Association in 1949, Schumpeter pointed out that "Science is knowledge processed by special skills. Economic analysis, i.e., scientific economics, involves skills of history, statistics, and economic theory." See Schumpeter, Joseph A. (1984). "Science and Ideology" in Daniel M. Hausman, eds., *The Philosophy of Economics*, Cambridge: Cambridge University Press, 260-275.

ed. As people have different values and self-interests, controversies often emerge, while natural science generally makes descriptive empirical analysis only and the conclusions can be verified through practice.

(3) Society cannot be simply taken for experiment or test of most conclusions in economics because policies have broad impact and large externalities, while this is not a problem for almost all branches of natural science.

These three differences make the study of economics more complex and difficult, and a more detailed discussion will be carried out in the sequel.

1.2 Two Categories of Modern Economic Theory

Modern economic theory is an axiomatic way to study economic problems. Similar to mathematics, it relies on logical deductions from presupposed assumptions. It consists of assumptions/conditions, analytical frameworks and models, and some conclusions (interpretation and/or prediction). These conclusions are strictly derived from the assumptions and analytical frameworks and models used, so it is an analytical method with inherent logic. This analyzing method is very helpful for clearly explaining the problem and can avoid many unnecessary complexities. Modern economics is to explain and evaluate observed economic phenomena and make predictions based on economic theory.

1.2.1 Benchmark Theory and Relatively Realistic Theory

The modern economic theory can be divided into two categories according to its function. One is benchmark economic theory that provides a benchmark or a reference system², relatively far away from the reality and dealing with ideal situations. The first half of the book mainly discusses such benchmark theories. The other one is relatively more realistic economic theories that aim to solve practical issues so that assumptions are closer to reality, which are usually modifications to the benchmark theory. The second half of the book devotes to discuss such relatively more realistic and practical microeconomic theories. As such, both of the two types of theories are very important and can be used to draw logical conclusions and make predictions. Besides, there is a progressive and complementary relationship of development and extension between these two. The second category of realistic theories is developed from the continuing revision of the first category of the benchmark theories, thus making the theoretical system of modern economics complete and close to the real world.

The benchmark theories are built on the economic environment of mature market economies and ideal situations. Their great significance should

²We will come back to discuss the role of the benchmark and the reference system in more detail.

not be underestimated, misunderstood or denied. They must not be neglected and are absolutely indispensable, and demonstrate their importance in at least two aspects. First, though the theoretical results of this category cannot be realized in practice, one cannot deny the fact that they play an extremely important role in guiding, orientating, and providing benchmarks. When we study and solve a problem, we need to figure out first what to do and whether it should be done and then proceed to the question of how to do. Benchmark theories answer what to do, or provide the direction and goals of improvements towards the ideal situation. Though sometimes only a relatively better result can be achieved when doing anything, we can approach closer and closer towards the best by checking the benchmark or reference system. This is why we claim that only through learning from and comparing with the best can we become better. Therefore, the benchmark theory provides necessary standards for judging what is better and whether it is the right direction, without which what we do may be poles apart from our goals. So, who can say that the benchmark theory is not important and deny its critical significance? Second, it lays down the necessary foundations for developing the other category of realistic theories.

Any theory, conclusion, or statement can only be considered relatively, otherwise there is no way to analyze or evaluate. This is true for both physics which is natural science and economics which is social science, so benchmark theories are demanded. For example, a world with friction is relative to a world with no friction, information asymmetry is relative to information symmetry, monopoly is relative to competition, technological progress and institutional changes are relative to technological and institutional fixedness, and so on. Therefore, we must firstly develop the benchmark theory under rather ideal situations. It's like the basic laws and principles in physics that only hold under the ideal situation without friction but don't exist in reality, but still no one can deny their importance because they provide indispensable benchmarks for solving physics problems in reality. Similarly, to study real economic behavior and phenomenon that includes "friction", we should firstly be clear about the ideal situation without "friction" and use it as a benchmark and reference system. Modern economics develops at such a fast pace, which is unimaginable without the support of these economic theories under ideal conditions as the benchmark and reference system.

As an important component of modern economics, neoclassical economics assumes the regularity conditions of complete economic information, zero transaction cost, and convexities of consumer preference and production sets and hence falls into the first category of benchmark economic theory that provides a benchmark and reference system. Neoclassical economics considers ideal situations; though it doesn't involve goals set by individuals, it proves that as long as individuals are self-interested, market

of free competition will naturally lead to efficient allocation of resources – which can be regarded as a rigorous statement of the “invisible hand” as proposed by Adam Smith (172-1790, see 1.17.1 for his biography) – and thus shall be set as the reference system for us to determine the direction and goals for reforms so as to improve the economic, political, and social environment, establish the competitive market system, and let market play a decisive role in the allocation of resources.

Some considers ideal reference system is far away from the real economy, so they deny neoclassical economics and then further deny the instructive role of modern economics in economic reform, which is a big misunderstanding as they do not realize that the great gap between reality and the benchmark/reference system only shows that nation, like China, needs market-oriented reforms to continuously improve the efficiency in resource allocation. This kind of opinion that denies benchmark theory is like the denial of physics by a junior high who has just learned several formulas of Newton’s three laws and criticized it for it is totally different in the real world. In fact, they did not grasp the role of benchmark theory. Try thinking about that without the benchmark theory in physics about free fall and uniform motion, how do we know the magnitude of frictional force so we can build a house stably and rightly? And how do we know how much frictional force should be overcome to solve problems concerning the taking-off and landing of airplanes or launching satellites? Without benchmark theory, applied physics cannot possibly be developed. The study of economics follows the same logic, so we have directions, structure, goals, and some fundamentals in mind when doing things in practice, which is especially true for China’s reform.

To have reforms for those transitional economies like China, goals must be established so it must also require a benchmark and reference system for the reform to orientate itself. It is indeed so. For the social economic development of a country, it is needless to say more about the importance of theoretical discussion, rational thinking, and theoretical innovation, but as the direction and goals of the reform should be determined in the first place, what plays the fundamental, decisive, and key role in this is the basic system that determines national policies and strategies. If basic systems of politics, economy, society, and culture that concerns a country’s path of development and long-term stability are not determined, economic theories of the best kind may help accomplish nothing but even the opposite. In economics, there is not a kind of economic theory that is always right for all development stages but there is the fittest one for certain institutional environments.

For market-oriented reform, it is natural and necessary to set neoclassical economic theory – especially such economic theory of the first category as general equilibrium theory that demonstrates market as the optimal economic system– as benchmark and competitive market as reference system

for the orientation of the reform so that results of the reform will be continuously improved towards the better outcome. According to the economic environment defined by such benchmark, we need to carry out reforms of deregulation and delegation for liberalization, privatization, and marketization or reforms against government monopoly of resources and control of market access. At the same time, as we are aware of that under certain circumstances market may fail, the general equilibrium theory strictly defines the applicable range of market mechanism and shows the circumstances under which market may fail so for us to know the areas in which it will be better for the government to make rules and institutions or provide public service; in such way, this theory plays a big role in defining the scope of market.

Therefore, the study of economic problems, implementation of reforms, and especially the determination of the general reform direction must start from the benchmark of economics, while reforms that go against the common sense in economics will end up with nothing but failure. The benchmark and reference system strictly present the premises on which market will lead to efficient allocation and become good market economy, and such premises show the direction of the reform. The third part of this book lays stress on the Arrow-Debreu general equilibrium theory (Kenneth Joseph Arrow, 1921-, see 10.8.2 for his biography; Gerald Debreu, 1921-2004, see 11.9.2 for his biography) and Lucas's macroeconomic theory of rational expectation (referred to as neoclassical macroeconomics), the two of which are both standard theories of neoclassical economics and rigorously demonstrate that market of free competition leads to efficient allocation of resources.

Needless to say, there are various benchmarks and reference systems with different value judgements and goals that are also likely to lead to very different outcomes. For example, when students regard "pass" as their benchmark, they may more often than not just fail because the test of questions is a random variable to the students. Just as Confucius and Sun Zi's statement, those who aim at the superior get the medium, those who aim at the medium get the inferior, and those who aim at the inferior will only lose, which illustrates the extreme importance of the choice of benchmark. Meanwhile, **because many benchmark theories are basic theories under ideal conditions, though they can be instructive for the improvement or the orientation of the reform, we should also bear in mind that they are actually far from the reality and cannot be simply adopted to solve real problems in practice.** That is to say, the goal doesn't determine the process, and a well-trained economist will not mechanically apply economic theories in the first category. In reality, there are many vulgar economists who do not analyze the dynamics in the transition, consider only developed countries but not developing countries, and neglect the objective law in special development stages. However, the target institution, especially

the path and schedule for accomplishing the target institution, should not always be evaded by substituting development for reform, otherwise the composite force of theories that promote market-oriented reform will be destroyed and the transitional status is likely to become permanent.

It should be pointed out that the benchmark economic environment is a rather ideal situation which must be a given factor, otherwise any question cannot be discussed if everything is changing. So, in neoclassical theories and many other economic theories, economic environment including basic institutions and production technologies must be exogenously given. Of course, there are many theories in modern economics that are specialized in the study of institutional evolution or transition, and technological progress so they cannot be given. But even so, we still need to clarify the given situation in which the institution is given and technology is not developing. Since the market system under the ideal situation is the goal that we pursue, we may as well set it as given exogenous institutional arrangement so as to well investigate its desirable property. However, it does not mean that modern economics only studies situations where the institution is given. Such narrow understanding of the category of modern economics has given rise to many debates which mistake modern economics for benchmark economic theory of the first category and thus take neoclassical economics that mainly provides benchmark economic theories as the unique component of modern economics. Since neoclassical theory considers ideal situations that depart from the real world, they further think that modern economics is unchanged and thus deny it, which is a great misunderstanding.

The second category of theories is mainly relatively more realistic economic theories that aim to solve practical problems, which are built on presupposed assumptions closer to reality and are revisions of the benchmark theory. According to their functions, they can further be divided into two kinds: the first kind provides the analytical framework, method, and tools for solving practical problems, such as game theory, mechanism design theory, principal-agent theory, auction theory, matching theory, etc.; the second kind is to give specific policy suggestions, such as Keynes theory and rational expectation theory in macroeconomics.

Modern economics consisting of the two categories of theories is a greatly inclusive and open discipline in dynamic development, which far exceeds the scope of neoclassical economics. Through relaxation of the presupposed assumptions of benchmark theories and standardized axiomatic formulation of descriptive theories, modern economics has continuously developed the second category of economic theories, giving itself great insight, explanatory power, and predictability; meanwhile. In my opinion, **as long as the study involves rigorous inherent logical analysis (not necessarily using mathematical models) and rationality assumption (bounded rationality assumption included), it falls in the category**

of modern economics.

Modern economics originated from classical economics, which was developed based on the integration of Adam Smith's theory by Thomas Robert Malthus (1766-1834, see 4.6.1 for his biography) and David Ricardo (1772-1823, see 1.17.2 for his biography), including not only benchmark theories like neoclassical marginal analysis economics established by Alfred Marshall (1842-1924, see 3.11.1 for his biography) and Arrow-Debreu general equilibrium theory but also many more realistic economic theories. For example, the new institutional economics of Douglass C. North (1920-2015, see 15.5.1 for his biography) and mechanism design theory of Leonid Hurwicz (1917-2008, see 16.10.2 for his biography) both have been revolutionary development of the neoclassical theory: while neoclassical theory sees institution as given, North and Hurwicz internalized institution, saw it as changeable, shapeable, and designable, and thus formulated various institutional arrangements for different objective environments; they both have become very important components of modern economics. For another example, the development and innovation of modern political economics have, to a large degree, borrowed the analytical methods and tools of the second category of economic theory, the use of which, however, should not be generalized.

It is important to note that because theories of the second category aim to provide analytical framework, methods, and tools for solving practical problems and give specific policy suggestions mostly based on mature modern market system, its application should be handled with caution so as to avoid troubles. In fact, for no matter original theory or theory that provides analytical tools, and no matter the first category of theory that provides benchmarks or reference systems or the second category of theory that aims to solve practical economic problems, every rigorous economic theory in modern economics has self-consistent inherent logic system and thus must have boundaries and scopes within which they are applicable. Therefore, large quantities of mathematical tools are often needed, which incurs a common criticism on modern economics' overemphasis on details and increasing involvement of mathematics, statistics, and models, making economic questions even more obscure and difficult to understand.

As a matter of fact, the reason that modern economics uses so much mathematics and statistics is because in the study of economic and social problems and the formulation of economic policies, experiments cannot simply be done upon society; otherwise, the cost may be huge. That is to say, the theory, once adopted, will have very big externality, so if it is used without knowing boundary conditions, there will be big negative externality. Though policy makers and the public do not need to know details or premises of rigorous theoretical analysis, economists who propose policy suggestions must know. If the premise is not considered in the policy suggestion and application, there can be big problems and even disastrous

results, so mathematics is needed to rigorously define the boundary conditions and applicable scope. Meanwhile, the application of a theory or formulation of a policy will often than not need tools of statistics and econometrics for quantitative analysis or empirical test.

In addition, in most cases, as real society cannot be simply used for experiment, the larger perspective of history is needed for vertical and horizontal comparisons. What's more, many unnecessary disputes can also be avoided in the discussion of questions. In many academic debates from time to time, the two sides of the debate do not have a clear or unified definition, or accurate connotation and denotation, of key terms in economics, and no formal and definite conclusions supported by rigorous mathematical logic can be drawn without scientific quantitative analytical tools, so the two sides of the debate do not even match point by point and cannot convince each other. Hurwicz believed that the biggest problem of traditional economic theories was arbitrary explanation of concepts, while the greatest significance of the axiomatic method lies in its definite and clear-cut formulation of the theory, providing a commensurable research paradigm and analytical framework for discussion and criticism.

Thus, as the basic theoretical foundation for market economic system, modern economics lays much stress on the introduction of research methodology and analytical framework of natural sciences to study social economic behavior and phenomena, on the inherent logic from assumption, derivation, and conclusion, on mathematics and mathematical models as basic analytical tools, and on empirical research based on mathematical statistics and econometrics, showing strong colors of practicality, empiricism, and natural science. Modern economics is very different from other humanities and social sciences with distinct ideologies and values.

1.2.2 Three Roles of Economic Theory

Economic theory has at least three roles.

The first role is to provide a benchmark and a reference system to set up goals to catch up with or create so as to point a direction for improving. Through reform and innovation guided by theory, the economy in the real world is impelled increasingly closer to the ideal state.

The second role is that it can be used to learn and understand the real economic world, and to explain economic phenomena and economic behavior so as to solve real problems, which is the major content of modern economics.

The third role is that it can be used to make logically inherent inferences and predictions. Practice is the sole criteria for testing truth but not the sole criteria for predicting truth. In many cases, problems may still arise if only historic examination and existing data are used for economic prediction, so theoretic analysis with inherent logic is needed. Through

the logical analysis of economic theory, we can make logically inherent inferences and predictions on the possible outcomes under given economic environments, behavior of economic agents and economic institutional arrangements. This will guide us to solve economic problems in reality in a better way. As long as the presupposed assumptions in the theoretic model are roughly met, we can obtain scientific conclusions and make basically correct predictions and inferences accordingly, so we may know the outcomes without experiments. For instance, the theoretic inference that planned economy is unfeasible proposed in 1920s by Friedrich August Hayek (1899-1992, see 2.12.1 for his biography) has this kind of insight. A good theory can deduce the logically inherent result without experimenting, which can solve the problem that economics cannot experiment on real society to a great extent. What we need to do is to check whether the assumptions made on economic environments and behavior are reasonable (experimental economics that is popular in recent years is mainly engaged in fundamental theoretical research such as testing individual behavioral assumptions). For example, we are not allowed to issue currency recklessly just for the sake of studying the relationship between inflation and unemployment. Like astronomers and biologists, most of the time economists can only utilize existing data and phenomena to test and revise the theory.

Of course, we should not exaggerate the role of economic theory and expect it to solve key and fundamental problems. It's self-evident that theoretical discussion, rational thinking, and theoretical innovation are important for the social and economic development of a country, but what is fundamental, key, and decisive is the basic system that determines the country's fundamental policy. If the basic system that concerns the direction of the country and long-term prosperity in politics, economy, society, and culture is not determined, the best kind of economic theory cannot help too much and may even lead to the opposite of our wish. There is no such an economic theory that is always right and fits every development stage, but a kind that fits certain institutional environment the best.

1.2.3 Microeconomic Theory

A notable feature of microeconomic theory is to set up theoretical hypothesis or modeling for economic activities of self-interested individuals, especially in market economy, and conduct rigorous analysis and examine how the market works on such basis.

The whole microeconomics runs through a main theme – price or pricing: which factors affect pricing? whether enterprises have pricing power? how to have pricing advantages? and how to make optimal pricing? Therefore, we need to study the demand, supply, characteristics and functions of market and the pricing in all kinds of markets and various economic environments so as to maximize profits or utilities. As a result, microeconomics

is also called price theory.

Microeconomics is the core of economics and the theoretical foundation of all branches of modern economics. It enables us to use simplified assumptions for in-depth analysis of complex problems so as to find clues from the complex, making the complex problems become relatively simple. It can help us to extract the most useful information from things unrelated and think of various issues using the method of economics so as to make explanations and predictions that conform to reality. Other branches, such as macroeconomics, finance, applied econometrics, etc., all rely on the support of microeconomic theory.

1.3 Modern Economics and Market System

A main purpose of modern economics is to study the objective laws of market and individuals' (such as consumers and firms) behavior in the market. Specifically speaking, it studies how to realize harmony among self-interested individuals in the market, how the market allocates social resources, and how to achieve economic stability and sustainable growth, etc. China's economic reform over the past three decades is mainly about market-oriented reform, and it has also been proposed in recent years that market should be allowed to play a decisive role while government should play a better role (but not to do more) in the allocation of resources. Therefore, no matter for the purpose of better learning of this book or for the research and resolution of practical problems in China's economic reform and development, it is a must to have a general understanding of the function and advantage of modern market mechanism.

1.3.1 Market and Market Mechanism

Here we will have a brief introduction of the operation and basic functions of market and how market coordinates individuals' economic activities without needing excessive participation or intervention of the government.

Market: Market is a trade mode where buyer and seller conduct voluntary exchanges. It refers not only to the place where buyer and seller conduct exchanges but also to any form of trading activities, such as auction and bargaining mechanisms.

When learning microeconomics, it should be noted that the buyer and the seller are both important in any trade in the market. For a buyer of any good, there is a corresponding seller. The final result of the market process is determined by the rivalry of relative forces of sellers and buyers in the market. There are three forms of competition for such rivalry: consumer-producer competition, consumer-consumer competition, and

producer-producer competition. In the market, the position of consumers who bargain with producers is constrained by the three forms of competition. Every form of competition is like a disciplinary mechanism leading the market and exerts different influences upon different markets.

Market mechanism: Market mechanism, or price mechanism, is an economic mechanism where individuals make decentralized decisions guided by price, which is usually a rather narrow definition of market mechanism. Market mechanism or market system is the set of all systems and mechanisms closely related with market (including the system of market laws and regulations). As a form of economic organization featuring decentralized decision-making, voluntary cooperation, and voluntary exchange of products and services, it is one of the greatest inventions in human history and by far the most successful means for human beings to solve their own economic problems. The establishment of market mechanism is not a conscious, purposeful human design, but a natural process of evolution. In the opinion of Hayek, market order is a spontaneous extended order of economy which evolves through long-term choices and trials and errors. The emergence, development and further extension of modern economics are mainly based on the study of market system. At first glance, the operation of market is amazing and beyond comprehension. In the market system, decisions on resource allocation are independently made by producers and consumers who pursue their own interests under the guidance of market price without the imposition of any command or order. Market system unknowingly solves the four basic questions that no economic system can evade: what to produce, how to produce, for whom to produce, and who makes the decision.

Under market system, firms and individuals make the decision on voluntary exchange and cooperation. Consumers seek the maximal satisfaction of their demand while firms pursue profits. In order to maximize profits, firms must have meticulous plans for the most effective use of resources. That is to say, for resources with similar utility or effect, they will choose the ones of the lowest possible cost. The best use of things originally has different meanings from the standpoints of firms and society, respectively, but price links them up, which, as a result, harmonizes the interest of firms and that of the whole society and leads to efficient allocation of resources. **The price level reflects the supply and demand of resources in society and the degree of scarcity of resources.** For example, in the case of inadequate timber supply and ample steel supply in society, timber will be expensive while steel will be cheap; to reduce expenses and make more profits, firms will try to use more steel and less timber. In doing so, firms do not take the interests of society into consideration, but the outcome is totally in line with social interests, which is precisely the underlying role of resource price that does the trick. Resource price coordinates the interest of firms and that of the whole society and solves the problem of how

to produce. Price system also guides firms to make production decisions in the interest of society. It is the group of consumers that determine what to produce, while the only consideration of firms is to produce the product of high price. Yet in the market system, the price level exactly reflects the social needs. For instance, poor harvests and the corresponding rising grain price will encourage farmers to produce more grain. As such, profit-pursuing producers "come to the rescue" under guidance, and the problem of what to produce is solved. Moreover, the market system also addresses the problem of how to distribute products among consumers. If a consumer really needs a shirt, he or she will offer a higher price than others. Profit-pursuing producers will definitely sell the shirt to the consumer who offers the highest price. Thus, the problem of for whom to produce is addressed. All these decisions are made by producers and consumers in a decentralized manner – thus, the problem of who makes the decision is also solved.

As such, market mechanism easily coordinates the seemingly incompatible individual interest and social interest. As early as two hundred years ago, Adam Smith, Father of Modern Economics, saw the harmony and wonder of market mechanism in his masterpiece *The Wealth of Nations* (Adam Smith, 1776). He regarded the competitive market mechanism as an "invisible hand". Under the guidance of the invisible hand, individuals pursuing their own interests unintentionally head for a common goal and thus achieve the maximization of social welfare:

"... every individual necessarily labours to render the annual revenue of the society as great as he can. He generally, indeed, neither intends to promote the public interest, nor knows how much he is promoting it ... he intends only his own gain, and he is in this, as in many other cases, led by an invisible hand to promote an end which was no part of his intention. Nor is it always the worse for the society that it was no part of it. By pursuing his own interest he frequently promotes that of the society more effectually than when he really intends to promote it."

Smith closely studied how market system combines self-interestedness of individuals with social interests and the division and cooperation of labor. The core of Smith's thoughts is that if division of labor and exchange of goods are totally voluntary, then exchange will only occur when people realize the result of exchange is mutually beneficial to both parties of exchange, otherwise no one will exchange. As long as there are benefits, people driven by self-interests will voluntarily cooperate. External pressure is not the necessary condition for cooperation. Even if there are language barriers, as long as mutual benefits exist, exchange can still be carried on as normal. At usual times, market mechanism works so perfectly that people cannot feel its existence. With the metaphor of "invisible hand", Smith pointed out the importance of voluntary cooperation and voluntary exchange in economic activities of the market. However, the thought on the welfare brought about by market system for the people, either at the

times of Smith or today, was not and still have not been fully recognized by all. The Arrow-Debreu general equilibrium theory had formal statement of Smith's "invisible hand", which rigorously demonstrated how market of free competition could lead to the maximization of social welfare and proved the social optimality of market in resource allocation.

1.3.2 Three Functions of Price

As discussed above, the normal operation of market system is realized via the price mechanism. As analyzed by the Nobel laureate in economics, Milton Friedman (1912-2006, see 4.6.2 for his biography), when organizing constantly changing economic activities that involve hundreds of millions of individuals, price fulfills three functions of:

- (1) Transmitting information: price transmits production and consumption information in the most efficient way;
- (2) Providing incentive: price provides incentives for people to carry out consumption and production in an optimal way;
- (3) Determining income distribution: resource endowment, price, and the efficiency of economic activities determine the income distribution.

In fact, as early as the Han dynasty of China, Sima Qian (a Chinese historian of the Han dynasty who is considered the father of Chinese historiography) had already noticed and summarized the law of commodity price fluctuation, saying that all goods "will return to being cheap when it is expensive to the extreme and will become expensive when it is cheap to the extreme", so for the sake of becoming rich, individuals shall make good use of this law to "pursue interests along with opportunity".

Function 1 of Price: Transmitting Information

Price guides the decision-making of participants and transmits the information of changes in supply and demand. When the demand for a certain commodity increases, sellers will notice the increase of sales and thus place more orders with wholesalers who will place more orders with manufacturers so the price will go up, and the manufacturers will then invest more factors of production to produce this commodity. In this way, the message of increasing demand for this commodity is received by all related parties.

The price system transmits information in a highly efficient way and it only transmits information to those who need it. Meanwhile, the price system not only transmits information but also produces a certain incentive mechanism to ensure the smooth transmission of information so that information will not be detained in the hands of those who do not need it. Those who transmit information are internally motivated to look for those who

are in need of information while those who need information are internally motivated to acquire information. For example, ready-to-wear apparel manufacturers are always hoping to get the best kind of cloth and constantly looking for new suppliers. Meanwhile, cotton cloth manufacturers are also always reaching out to clients to attract them with good quality and cheap price of their products by various means of publicity. Those who are not involved in such activities will surely be uninterested and indifferent to prices and supply and demand of cotton cloth. The general mechanism design theory as discussed in Chapter 18 of this book will demonstrate that competitive market mechanism is the most efficient in the utilization of information for it requires the least information and lowest transaction cost. In 1970s, Hurwicz et al. already proved that under the neoclassical economic environment of pure exchange, no other economic mechanisms can achieve efficient resource allocation using less information than does the competitive market mechanism.

Function 2 of Price: Providing Incentive

Price can also provide incentive so that individuals will react to changes of supply and demand. When the demand of a commodity decreases, an economic society should provide certain incentive so that manufacturers of the commodity will increase production. One of the advantages of the market price system is that prices not only transmit information but also provides incentive for individuals to react to the information voluntarily for the sake of their own interest, so consumers are motivated to consume in an optimal way while producers are motivated to conduct production in the most efficient way. The incentive function of price is closely related to the third function of price: determining income distribution. As long as the increased gain brought by increased production exceed the increased cost, producers will continue to increase production until the two are even and maximal profits are realized.

Function 3 of Price: Determining Income Distribution

In a market economy, an individual's income depends upon the resource endowment he owns (e.g., assets, labor) and the outcomes of economic activities he's engaged in. When it comes to income distribution, people always want to separate price's function of income distribution out from the other functions, in the hope of more equal income distribution without affecting the other two functions of transmitting information and providing incentive. However, the three functions are closely related and all indispensable. Once price no longer influences income, its functions of transmitting information and providing incentive will no longer exist either. If one's income does not depend upon the price of labor or commodities he offers to others, why would he bother to acquire the information of price and market demand and supply and react to such information? If one gets the same income no matter how he works, then who would like to do a good job? If no benefits are given for creation and inventions, then who

are willing to make efforts in this regard? If price has no impact on income distribution, it will also lose the other two functions.

1.3.3 The Superiority of Market System

Modern market system is a sophisticated and delicate economic mechanism that has emerged, gradually taken shape, and been constantly improved in the long-term evolution of human society. The fundamental and decisive role of market mechanism in resource allocation is the key to market economy's capacity for optimal resource allocation. The optimality here has the same meaning as the Pareto optimality (efficiency) proposed by Vilfredo Pareto (1848–1923, see his biography in 11.9.1) which will be discussed in Chapter 11 in more detail. It means that, under existing resource constraints, there is no such a scheme of resource allocation that can make some participants better off without hurting the welfare of all others. Even though Pareto optimality fails to consider the issue of social fairness and justice, it provides a basic criterion of whether the resource is wasted or not in terms of social benefit for an economic system and evaluates social economic effects in terms of feasibility. According to it, if an allocation is not efficient, there is room for improvement.

Two fundamental theorems of welfare economics, which we will make effort to discuss in the part of general equilibrium theory, provide a rigorous formal expression of Adam Smith's conclusion. As the formal expression of his 'invisible hand', the theorems prove that market of free competition can maximize social welfare and achieve the market optimality in terms of resource allocation. The first fundamental theorem of welfare economics proves that when individuals pursue self-interest, and if economic agents have unlimited and locally non-satiable desire, the competitive market system can achieve Pareto-efficient allocation under general economic environment (private goods, complete information, no externality, divisible goods). The second fundamental theorem of welfare economics, on the other hand, proves that under the neo-classical economic environment, any Pareto-efficient allocation can be achieved by reallocation of initial endowments and competitive market equilibrium without the need to introduce other economic systems to replace the market. Precise statements and mathematical proofs of these theorems will be given in Chapter 11.

The economic core theorem and the limit theorem on the core in Chapter 12, from another perspective, prove that market system can benefit the social stability and is optimal and unique in terms of resource allocation, being the result of natural selection with objective inherent logic in economic activities. Moreover, for competitive market mechanism to lead to optimal allocation of resources, the modern market economic system can well solve the problem of social stability and orderliness. According to the basic connotation of the economic core theorem, when the allocation of social re-

sources has the core property, there will not be any small groups of agents that are unsatisfied with the allocation and want to improve their welfare by controlling and utilizing their own resources. In this sense, there will be no powers or groups that constitute threat to society, and hence society will be relatively stable. As the economic core theorem reveals, when a market achieves competitive equilibrium, equilibrium allocation caused by market equilibrium will have the core property under some regularity conditions, such as monotonicity, continuity, and convexity (diminishing marginal rate of substitution) of preference. The limit theorem on the core tells us that, under the greatest reality of individuals' pursuit of self-interest, as long as individuals are given economic freedom (i.e. the freedom to cooperate and exchange voluntarily) and perfect competition, the outcome will be identical to that of competitive market equilibrium without giving any institutional arrangements in advance. Thus, market system is not an invention but an inherent economic rule and spontaneous order, which is objective as the law of nature. The policy implication of this conclusion is that the market should be let make the decision when competitive market mechanism is able to obtain optimal allocation. Only under the circumstance where the competitive market is incapable will other mechanisms be designed to make up for market failures.

Even though the market mechanism cannot perfectly solve the problem of social equity featured by the large gap between the rich and the poor, as long as the government can try the best to provide fair opportunities and equitable resource for every individual and let the market play its role instead of filling in for the market, fair and efficient resource allocation will be reached by the market, as the fairness theorem in Chapter 12 enlightens us.

The above statements and proof about the optimality, uniqueness, and fairness of the modern competitive free market system in resource allocation and its contribution to social stability are all important contents of the general equilibrium theory. Joseph Schumpeter discussed the optimality of the market mechanism from the perspective of how the dynamic game between competition and monopoly leads to innovation-driven growth. His innovation theory told us that the valuable competition was not price competition, but the competition in new commodities, new technologies, new markets, new supply sources, and new combinations. Therefore, the root of long-term vitality of market economy was innovation and creativity, which stemmed from entrepreneurship and entrepreneurs' constant, creative destruction of the market equilibrium, referred to as 'creative destruction' by him. It basically depends on profit-seeking entrepreneurs and private economy to cultivate the soil for innovation and to encourage and protect innovation.

Competition and monopoly, like supply and demand, can form an astonishing unity of opposites through the power of market, thus revealing

the beauty and power of the market system. If there is no competition, there will be no motivation of innovation either, like what happens in the state-owned enterprises of state monopoly. On the contrary, since profit will decrease with the increase in the intensity of competition, profit-seeking private enterprises will often have great incentive to innovate to develop new products and set the price of new products above the competitive equilibrium price to gain high profits. Other enterprises, however, will soon come to share the profits by developing similar products. Thus, the market competition reduces the profits and urges enterprises to innovate, and the innovation of enterprises then brings monopoly profits, which are considerable enough to attract other enterprises to join the competition. In this way, market competition leads to the decrease of profits, so enterprises have to obtain new monopoly profits through innovation; thus, a repeated dynamic cycle of competition–innovation–monopoly–competition takes shape, in which market competition tends to achieve equilibrium but innovation breaks the equilibrium. Market constantly goes through such games to inspire enterprises to continuously pursue innovation. Through such a game process, market economy will keep its long-term vitality, social welfare will increase, and economy will develop, demonstrating the grace and power of the market system. So, in order to encourage innovation, the government should enforce the the law of intellectual property rights protection; meanwhile, in order to encourage competition and form externalities of technological innovation, the anti-monopoly law should be enacted. Also, the protection for intellectual property rights will not last forever but will be limited within a certain number of years so that they will not become fixed or perpetual oligopoly or monopoly.

Innovation means to break rules and regulations, which inevitably contains high risks. High-tech innovation in particular means high risks and extremely low possibility of success for venture capital investment, but once it succeeds, it will bring considerable returns and then attract more investment. However, it is impossible for state-owned enterprises to take such high risks due to the fact that it naturally lacks the incentive mechanism to take risks. On the contrary, private enterprises dare the most to take the risk out of strong motivation to pursue self-interest, so they are the most creative and innovative entities. Therefore, entrepreneurial innovation (not fundamental scientific research) mainly takes place in private business. Actually, Sima Qian, in the history of Chinese thoughts, had also affirmed that competition and survival of the fittest were a natural tendency. He believed that one did not rely on certain trades to get rich, while wealth was not exclusively occupied by certain people; a capable person would accumulate wealth, whereas those incapable would lose their properties.

Furthermore, the competitive market mechanism is not only optimal and unique in social stability maintenance and efficient resource allocation

but also efficient in the transmission of information. In the 1970s, Hurwicz et al. proved that, under the environment of neoclassical pure exchange economies, there is no economic mechanism which can lead to efficient allocation of resources using less information than the competitive market mechanism. In 1982, Jordan further proved that, in pure exchange economies, market mechanism is the only one which achieves the efficient allocation using the least information. Guoqiang Tian (2006) proved that this conclusion is true not only in pure exchange economies but also in economies of production, and meanwhile market mechanism is unique. Hence, here comes an important inference: in no matter command planned economy, state-owned economy, or mixed economy, information needed to realize efficient allocation of resources is surely more than that in a competitive market mechanism, so those economies are not information-efficient, meaning that they cost more to achieve the optimal allocation of resources. This conclusion provides an important theoretical explanation for the issue why China needs market-oriented economic reform and the privatization of state-owned economy. The uniqueness result of information efficiency will be discussed in Chapter 18.

What should be specially noted is, with the emergence of innovation in internet finance, the deviation between the real economic situation and the ideal state will be decreased. The innovation will push the real market economy toward the ideal state of market economy described by Adam Smith, Hayek, Kenneth Joseph Arrow, Gerard Debrue, and Ronald H. Coase (1910-2013, see 14.6.2 for his biography). According to no matter the role of competitive market as the 'invisible hand' described by Adam Smith, or the general equilibrium in the perfectly competitive market by Arrow and Debrue and the theory about the zero transaction cost in the perfectly competitive market by Coase which will be discussed with details in this book, or the statement that competition benefits innovation from Schumpeter, the market can all be proved to be the optimal. The basic conclusion of these theories is that the perfectly competitive market leads to Pareto efficient resource allocation and maximal social welfare. The perfectly competitive market mainly provides a reference system or an ultimate goal, which means that the more competitive the market is, the better; the more symmetric the information is, the better. This perfectly competitive market, however, does not exist in reality because the communication cost and transaction cost, including the important financing cost, cannot be zero.

With internet as the medium for finance, the transaction cost is becoming smaller and smaller. Due to the innovation and development of internet finance, to some extent, the perfectly competitive market, as the first kind of economic theory, is not just an ideal state but tends to approach the reality increasingly closer with the disruptive innovation of internet finance. Internet finance will greatly reduce the cost of information communication in reality so as to make market economic activities closer to the ideal state

of perfect competition and thus more efficient. China is blessed to get on the train of the information era through the reform and opening-up policy and the market-oriented reform.

1.4 Governance Boundaries of Government, Market, and Society

In fact, the theoretical conclusion about market optimality relies on a hidden key assumption of the important and decisive role of institution. That is to say, there is a perfect governance structure to regulate government, market, and society, on the premise of which the market economy under discussion will be a unified, open, competitive, and orderly free market economy.

1.4.1 Three Dimensions of State Governance: Government, Market, and Society

Market mechanism may give people the wrong impression that, in a market economy, one seems to be allowed to do whatever he likes to pursue self-interest, which, however, is actually not true. There is no completely laissez-faire market economy that is totally independent of the government in the world. A well-functioning market requires proper placement and effective integration of government, market, and society, which form a three-dimensional structure of state governance. The completely laissez-faire market that is totally independent of the government is not omnipotent. As we shall discuss in Parts IV and V of this book, market often fails under many circumstances such as monopoly, unfair income distribution, polarization of rich and poor, externality, unemployment, inadequate supply of public goods, and information asymmetry, thus resulting in inefficient allocation of resources and various social problems.

So, in transitional development and deepening reform, attention shall be given to two kinds of logic, namely, development and governance, and the inherent dialectical relationship between them shall be correctly understood. The logic of development focuses on the improvement of a nation's hard power while that of governance stresses the construction of soft power. Of course, governance should be all-dimensional from different aspects including governance systems of government and market, social equity and justice, culture, values, etc. How the relationship between government and market and between government and society is handled often determines the effect of state governance. If they cannot be well balanced so any side is given any preference, there may be a series of serious problems and crises including excessive gap between the rich and the poor, unfair opportunities, etc., preventing an inclusive market economy and a tolerant

1.4. GOVERNANCE BOUNDARIES OF GOVERNMENT, MARKET, AND SOCIETY²⁷

and harmonious society from coming into being. In this way, in the logic of governance, there is good kind and bad kind of governance that will lead to good or bad market economy and good or bad social norms. Therefore, governance cannot be simply taken as being equal to rule, control, or regulation, or regarded as the opposition of development, making it difficult to attend to both governance and development at the same time. The very reason that China is now confronted with extremely complicated situation in the reform with extraordinary achievements on the one hand and severe problems on the other is because of the main focus on the logic of economic development and the neglect of the logic of governance to a large extent during the past three decades.

The necessary condition of an efficient market and a normal society is a limited but appropriately positioned efficient government, making the appropriate position of government fatally important. Under the current semi-market and semi-planned dual system, the government holds much more power than necessary but fails to fulfill its duties of maintenance and service; without rational definition of the governance boundaries between government and market and between government and society, the phenomena of over-playing, under-playing, and mis-playing of government role prevails, leading to "over-emphasis on government versus under-emphasis on market, over-emphasis on enriching the state versus under-emphasis on enriching the people, and over-emphasis on development versus under-emphasis on service."

1.4.2 Good State Governance Brings about Good Market Economy

Market economy can be classified into "good market economy" and "bad market economy". Whether it is good or bad depends on the system of state governance and whether the governance boundaries among government, market, and society are clearly and reasonably defined. In a good market economy, the government allows the market to fully play its role; however, in case of market failures, the government can also provide good remedy, which does not mean that the government should directly intervene in economic activities but asks the government to enact proper rules or systems to make the market efficient or resolve the problem of market failures so as to achieve the result of incentive-compatibility between individual interest and social interest (one of the most successful examples of system design is the enactment of the basic constitution of the U.S. when it was just founded, which made the U.S. become the most powerful country in the world through only more than 100 years). A good, tolerant, and efficient modern market economy should protect the private interest of individuals to the best through systems or laws and meanwhile limit and counterbalance the government and its public powers as much as possible; in this sense, it

is a contract economy and rule-of-law economy constrained by commodity exchange contracts, market operation rules, and reputation. Under the constraint of individuals' pursuit of self-interests, resources, and information asymmetries in an economic society, to build a powerful country with enriching people, first of all, individuals shall be endowed with private rights, the core of which includes the basic right to survival, freedom of choice for one to pursue happiness, and private property right; through their participation in full competition, voluntary cooperation, and voluntary exchange of market mechanism and their pursuit of self-interest, efficient allocation of resources and maximization of social welfare can be realized. Thus, modern market economy is established upon the basis of the rule of law that works in two ways: first, it restricts arbitrary government intervention in market economic activities, which is of fundamental importance; second, it further supports and promotes the market in ways including definition and protection of property rights, enforcement of contracts and laws, maintenance of fair market competition, etc., so as to let market play the fundamental and decisive role in resource allocation and give full play to the three basic functions of price, namely transmitting information, providing incentives and determining income distribution. What's more, a good market requires good social norms, for one's pursuit of self-interest should be on the premise of respect for others' pursuit of self-interest, and self-interest and fair competition should run parallel to one another with no contradiction. The spirit of compromise and respect for other's standard of values judgment are the premise for normal proceeding of exchanges.

On the other hand, in a bad market economy, for lack of adequate ruling and governance capacity in the economic and social transformation, the government is unable to provide necessary and sufficient public goods and services to make up for market failures; instead, because of over-playing of government role in economic activities, public powers are not effectively counterbalanced, property rights of state-owned enterprises are not clearly defined, and the government is involved in numerous rent-seeking and corruption phenomena so that equity and justice in social and economic areas are greatly impaired. This breeds the so-called "State Capture", which refers to the phenomenon that by providing personal interests for government officials, economic agents interfere in decisions on laws, rules and regulations and thus, without going through fair competition, they convert their personal preferences to the basis of game rules of the whole market economy which become a great deal of policy arrangements that produce high monopoly profits for specific individuals at the expense of enormous social costs and the decrease of government credibility. As a result, inefficient balance in public choice continues in the long run. The behavior of scrambling for social and governmental resources by means of unfair rent-seeking instead of fair competition will not only cause market failures but more importantly gradually become bad social norms in the long run,

1.4. GOVERNANCE BOUNDARIES OF GOVERNMENT, MARKET, AND SOCIETY²⁹

bringing about distortion of social resource allocation and values, moral decline, absence of good faith, “false, big, and empty” in words and deeds, frivolity of society, and increased factors of instability, which finally result in enormous explicit and implicit transaction costs. Some sociologists call such social status “social corruption”, meaning that social cells of the social organism are dead and suffering functional failures.

Thus, among the three private rights of individuals as mentioned above, the definition and protection of property rights are of fatal importance. Currently, one important reason for prevalent corruption is the unclarified definition of public property rights and over-playing of public powers, making it possible for public powers to be engaged in rent-seeking activities, which also makes the radical reason of corruption. With respect to anti-corruption, the best strategy is to “subdue the enemy without fighting” as suggested in *The Art of War* by Sun Tzu, while anti-corruption action after occurrence of corruption can only be the second best at most, or maybe even the worst, considering that corrupt officials would damage the reputation of the government and induce very bad social influence, which history has already told us. The fundamental way out for anti-corruption is to further promote market-oriented reform and privatization: by rationally defining governance boundaries of the government through formal institutions, officials will have no chance for corruption; through the rule of law, officials will not be tolerated to corrupt; through accountability and social supervision, officials will not dare to corrupt.

Therefore, in the three-dimensional framework of government, market, and society, government as an institutional arrangement with strong positive and negative externalities plays a vital role. It can make the market efficient, become the impetus for economic development, help construct a harmonious society, and realize scientific development. On the other hand, it may also make the market inefficient, lead to various social contradictions, become huge resistance for the harmonious development of society and economy and exert bad social influence. Almost all countries in the world adopt market economy, but a majority of them did not achieve sound and rapid development. Among many reasons, the most fundamental one is the lack of reasonable and clear definition of the governance boundaries between government, market, and society so there is over-playing, under-playing and mis-playing of government role. Only when the government loosens its omnipresent “visible hand” and the functions and governance scope of the government are scientifically and reasonably defined can it be expected to reasonably define governance boundaries between government, market and society.

1.4.3 The Principle for Defining the Governance Boundaries between Government and Market and between Government and Society

How to reasonably define the governance boundaries among government, market, and society? The answer is to let the market do whatever it can do well while the government does not participate in economic activities directly (however, it is necessary for the government to maintain market order and guarantee strict implementation of contracts and rules); as for those that the market cannot do, or when it is not appropriate for the market to be involved considering other factors such as national security, the government can then directly participate in economic activities. That is to say, when considering the construction of a harmonious society and the harmonious development of economy, or when transforming government functions and innovating the management mode, consideration shall be given to the boundaries of market, government, and society. For instance, the government should exit from competitive sectors, although the government may not survive for long time even if it does not exit. Only in case of market failure should the government play its role in solving problems by itself or along with the market. However, the basic guideline is that the government should not directly intervene in economic activities but enact proper rules and systems to solve problems of market inefficiency or market failures. Due to constraints of individuals' pursuit of self-interest and information asymmetry, direct intervention in economic activities (e.g. large amounts of state-owned enterprises and arbitrary restriction of market access and interference with commodity prices) often do not generate good results. In this respect, mechanism design theory, the special study of designing rules and institutional incentives, can play a huge role in making the market efficient and solving the problem of market failures.³ In Hurwicz's opinion, "law-making by the U.S. Congress or other legislative bodies equals to designing new mechanisms".

Under modern market economy, the basic and only functions of government can be summarized as "maintenance" and "service", that is, making the fundamental rules to ensure national security, social stability, and economic order and providing public goods and services. Just as Hayek pointed out, government has two basic functions: firstly, the government must be responsible for law enforcement and defense against enemies; secondly, the government must provide services that market is unable to provide or unable to fully provide. Meanwhile, he also stated that "it is indeed most important that we keep clearly apart these altogether different tasks of government and do not confer upon it in its service functions the

³See Leonid Hurwicz, Stanley Reiter: *Designing Economic Mechanisms*, translated by Guoqiang Tian, et al. P2, Shanghai: Truth and Wisdom Press, SDX Joint Publishing House, Shanghai People's Publishing House, 2014.

1.4. GOVERNANCE BOUNDARIES OF GOVERNMENT, MARKET, AND SOCIETY 31

authority which we concede to it in the enforcement of the law and defence against enemies.”⁴ This requires that in addition to undertaking necessary functions, the government should separate its powers to market and society. Abraham Lincoln, one of the greatest presidents in the American history gave a clear and incisive definition of the functions of the government:

*“The legitimate object of government is to do for a community of people, whatever they need to have done, but cannot do, at all, or cannot, so well do, for themselves-in their separate, and individual capacities. In all that the people can individually do as well for themselves, government ought not to interfere.”*⁵

Meanwhile, good, tolerant, and efficient modern market economy and state governance mode need an independent autonomous civil society with a strong ability of interest coordination as an auxiliary non-institutional arrangement. Otherwise, the explicit and implicit transaction costs of economic activities would be huge and it will be hard to establish the most basic relationship of trust in society.

In summary, a reasonable and clear definition of governance boundaries between government, market, and society is a prerequisite for establishing a good and efficient market economic system and achieving scientific development featured by efficiency, fairness, and harmony. Of course, the transition to an efficient modern market system is often a long process. Due to various constraints, governance boundaries of government, market, and society cannot be clearly defined in one leap, but a series of transitional institutional arrangements are often needed. However, with the deepening of transition, some transitional institutional arrangements may decline in efficiency, and may even degenerate into invalid institutional arrangements or negative institutional arrangements. If governance boundaries of government, market, and society cannot be timely and appropriately clarified while some temporary, transitional institutional arrangements (such as government-led economic development) are fixed as permanent and ultimate institutional arrangements, it is then impossible to achieve an efficient market and a harmonious society. With the development of modern economics, its analytical framework and research methods play an irreplaceable role in the study of how to reasonably and clearly define governance boundaries among government, market, and society and how to conduct comprehensive governance.

⁴See Von Hayek F. *Law, Legislation, and Liberty* (Vol. 3). Chicago, IL: The University of Chicago Press, 1979, pp42.

⁵Abraham Lincoln's Quotes on Government, *Wisdom of Politics*, P56, Beijing: Police Education Press, 1992.

1.5 Three Institutional Arrangements for Comprehensive Governance

As discussed above, in order to establish a well-functioning market and an efficient modern market system, it is necessary to coordinate and integrate the relationship of the three basic coordination mechanisms, namely, government, market, and society, so as to regulate and guide individuals' economic behavior and conduct comprehensive governance. Government, market, and society correspond exactly to the three basic elements of governance, incentive, and social norms in an economy. Mandatory public governance and formal institutional arrangement like incentive market mechanism as two elements of comprehensive governance which are overlapping, through long-term accumulation, may help guide and mould normative informal institutional arrangements, enhance the predictability and certainty of social and economic activities, and greatly reduce transaction costs. The informal institutional arrangement mentioned here is culture. For example, for enterprises, the first-class enterprise does branding, the second-class enterprise develops technologies, and the third-class enterprise produces, in which it is corporate culture that plays the key role. Similarly, it is the position of the government that plays the key role in the establishment of good or bad social norms and market economy.

To put it in another way, the three basic institutional arrangements for comprehensive governance are employed to "enlighten with reason, guide with interests, and persuade with emotions", the three of which are realized and implemented mainly by government, market, and society, respectively, on the level of state governance. To "enlighten with reason" is to work through the stimulus of legal principles and reasons; to "guide with interests" is to link up economic activities with income through the stimulus of rewards and penalties, thus becoming the incentive mechanism; to "persuade with emotions" is to work through the stimulus of emotions and shared beliefs, as sometimes relationships, friendships, and emotions, especially shared beliefs and concepts, can help solve big problems, which as a kind of social culture will greatly reduce the transaction cost.

1.5.1 Regulatory Governance

Regulatory governance, as the basic institutional arrangement and management rule, is mandatory. The basic criterion for whether such rules and regulations shall be formulated is whether it is easy for clear definition (according to the possibility of information transparency and symmetry) and whether the costs of information acquisition, supervision, and enforcement are too high. If a regulation is too costly in terms of supervision, it will not be feasible for implementation. Protection of property rights, contract implementation, and proper supervision all call for relevant rules, which thus

1.5. THREE INSTITUTIONAL ARRANGEMENTS FOR COMPREHENSIVE GOVERNANCE³³

requires a third party to oversee the enforcement of the rules. The third party is then the government. In order to maintain market order, the role of government is inevitable. As government is also an economic agent, it has huge influence of being both the referee and the player. This requires procedures and rules for the behavior of the government that should be clearly and thoroughly formulated. Regulations on other economic agents and market should be quite the opposite. Due to information asymmetry, regulations in this regard should not be detailed so as not to interfere too much with the freedom of choice of economic agents and cause high costs.

Institutional arrangements that “enlighten with reason” and deter people like sticks are similar to the thoughts of Legalism in ancient China. However, one big problem of the Legalism was it aimed to govern economic agents only but exerted no restriction on the government. Besides, this kind of institutional arrangement only considered the cruelty in people’s fight for powers and profits but neglected the influence of blood relations upon people’s emotion and behavior. The employment of the heavy hand of Legalism without considering other institutional arrangements will often lead to coercive powers and hinders the formation of modern market economy.

1.5.2 Incentive Mechanism

Incentive mechanism like market mechanism is inducing, which is of the widest application and also a big concern of this book. Due to information asymmetry and high cost of information acquisition, the specific operation rules should arouse individuals’ enthusiasm through inducing incentive mechanisms such as market to realize incentive compatibility and make people work hard for their own sake that will benefit others and society at the same time. Reputation and integrity under the market incentive mechanism are also one kind of punishment incentive mechanism. Integrity is essential in doing business, which does not mean that business owners perform integrity out of their own will, but means they have no other choice because otherwise, they will be eliminated from the market. Besides, integrity can save economic costs and lower transaction costs.

Institutional arrangements that “guide with interests” are similar to the thoughts of Taoism in ancient China. Individuals are all self-interested and have limited ideological realm, the fact of which should always be fully recognized and respected. It is not difficult to stop “talking” about interests, but in reality, it is hard not to “value” interests. Taoism believed that “the law of nature is beneficial but not harmful” and emphasized compliance with natural tendency and governance by non-interference; however, they neglected two necessary conditions for governing by non-interference, namely, the basic institution and the role of government.

1.5.3 Social Norms

Social norms are a kind of institutional arrangement that is implemented by neither forces nor incentives. Solving problems with mandatory laws and inducing incentives in the long term will become a kind of social norms, beliefs, and culture that requires neither enforcement nor incentive, such as corporate culture, folk custom, religious faith, ideology, and concepts. This is the way of the minimum transition cost. Especially when people shared the same concepts, problems will be much easier to be resolved and the work efficiency will be greatly improved. Otherwise, even if one problem is solved through mandatory command, inducing incentive mechanism, or relations, there will always be new problems to be solved in the same way, which will cause large implementation cost. Chapter 7 of this book will discuss the important role of social norms in stimulating voluntary cooperation among strangers.

Even so, social norms that preach morality and “persuade with emotions” are still largely limited by the reality of people’s ideological height. Relying on improvement of humanity, social norms lack the power of constraint and have limited scope of governance. This kind of institutional arrangements are similar to the thought of Confucianism in ancient China. The thought of “rule of morality” in Confucianism over-emphasized ethical relations among people but deliberately overlooked the relation of economic interests. It was successful in managing a family but biased in the governance of a nation. Benevolence and morality can be dominant, or at least considerably important in a family or a small group but may not be very effective in the face of strangers. Benevolence is highly personal, the effect of which will weaken as with the enlargement of the realm. Those who rely on others’ benevolence to acquire necessities of life cannot be satisfied in most cases. As there are differences between beggars and ordinary people, though people may all need help from others, they cannot count on mere benevolence for everything. Therefore, the experience of managing a family cannot be simply popularized to all economic and social activities, otherwise there may be problems and even disastrous results. Especially under the environment of modern market economy and the circumstance of limited ideological height of people, reliance on only internal ethical norms and absence of external laws and incentive mechanisms will make market economy slide towards a bad situation.

1.5.4 The Complementary Structure of the Three Institutional Arrangements

The three kinds of institutional arrangements all have advantages and disadvantages. They also have different functions, ranges of application, and limitations. As emphasis on only one of them will have serious negative

1.5. THREE INSTITUTIONAL ARRANGEMENTS FOR COMPREHENSIVE GOVERNANCE³⁵

consequences, it is required for the three of them to play their own roles and complement each other. Actually, the ancient people had summarized this truth: if one makes friends who offer interests, the friendship will fall apart when the interests are exhausted; if one makes friends who have influence, the friendship will fall apart when the influence fails; if one makes friends with powers, the friendship will fall apart when their powers are lost; if one makes friends out of love, one will be hurt when the love ends; only when one makes friends with true heart can the friendship last forever. Making friends with true heart is the best but very difficult, for even couples of husbands and wives may not be able to do that.

It should be pointed out that, even though the book mainly discusses the problem of economic incentives, the regulatory governance (or institution) is still the most basic and fundamental in the three kinds of arrangements for it establishes the most basic institutional environment, has strong positive and negative externalities, and determines whether the role of the government is appropriate or not, thereby determining the effect of incentive mechanism design and the formation of good or bad social norms. In addition, for the formulation of institutional arrangements of both regulatory governance and incentive mechanism, the principle should not mean to and also basically can not change the self-interested nature of human beings. Instead, it should make use of people's unchangeable self-interestedness to guide them to do something beneficial to the society. The design of an institution should conform to the self-interested nature of people rather than trying to change it. People's self-interestedness cannot be simply judged as good or evil, but the key lies in what kinds of institution are used and towards what direction it is guided. Different institutional arrangements will result in individuals' different responses to incentives and different trade-off choices, thereby leading to very different consequences. As Deng Xiaoping put forward, "good institutions can make bad people unable to have wilful acts, while bad institutions can make good people unable to do good things and even go to the opposite side."⁶ Putting it in plain language, bad institutions can turn people into demons, but good institutions can even turn demons into people.

Good regulatory governance, which is not strict control, is more likely to lead to good incentive mechanisms and good social norms, which shows the importance and long-term nature of institution and state governance system. The construction of institution only for the next few years, or even 30 or 50 years, is far from enough. The fact that there was no dynasty in the Chinese history that had lasted for more than 200 years before it collapsed, declined, or broke off like the Western and Eastern Han and the Northern and Southern Song has fully revealed the importance of systems

⁶*Selected Works of Deng Xiaoping (Second Version)*. Beijing: People's Publishing House. Volumn 2, 333.

that can maintain long-term peace and stability. Therefore, an institution should be intended for at least 200 years and even longer. It is important to notice that dynastic changes brought great damage to the Chinese nation in term of both accumulation of property and growth of the population. What China needs are long-term peace and stability featured by stability of the political regime, harmony of the society, and prosperity of the people. In addition, the importance of culture cannot be ignored. Culture is a basic and strategic key link with the function of value guidance and humanity shaping and plays an important role in the harmony between man and man, man and society, man and nature, and man and self. Therefore, the core of China's reform in the next step is the modernization of governance system and governance capability of the state.

Therefore, the use of "emotion, reason, and interest" should be synthesized and vary with people, cause, place, and time to analyze and solve problems case by case. The criterion for deciding which to use is determined by the importance of regulations, the degree of information symmetry, and the cost of supervision and law enforcement. All in all, the three systems all have their boundary conditions. To "enlighten with reason" should depend on the availability of information symmetry and difficulty of legal supervision. The law will be meaningless if its cost of supervision and execution is too high to enforce it.

To sum up this and the previous sections, a well-functioning market needs government, market, and society all in the right place so that the three-dimensional structure of state governance can be effectively interconnected and integrated. Defining the boundaries between government and market and between government and society involves two levels. The first level is defining the boundaries. We should firstly know the reasonable boundaries between them. The necessary condition of efficient market and normal society is a limited and well-positioned effective government, so the reasonable position of government is of vital importance. The principle here is that the market should be allowed to do what it can do, while the government should do what the market cannot do or is not good at, and so the function of government can be generalized as maintenance and service. The second level is the identification of priorities. So, which is the key? The answer is institution. After knowing the boundaries of the three, we then need to sort them out. Who is the one to sort them out? The answer is government. Government, market, and society respectively correspond to the three kinds of basic arrangements, namely, governance, incentive, and social norm of an economy. Then, who is the one to regulate the position of government? It must be the rule of law. Regulatory governance (or institution) is the most important and foundational arrangement, which lays down the most basic institutional environment, has great positive or negative externalities, determines whether the position of government is appropriate, and hence determines the effect of the incentive mechanism design

and the formation of good or bad social norms. However, is government willing to limit its power? Normally speaking, of course not. Therefore, a government or a country as a whole needs to further divide their powers and separate the responsibilities and powers of administrative department, law making department, and judicial department.

Therefore, the underlying governance system is of determinant importance. Only when the governance boundaries between government and market and between government and society are reasonably defined through comprehensive governance by the three dimensions of institution, the rule of law, and civil society that can regulate, restrict, and supervise governmental power, can problems of efficiency, social equity and justice be resolved, phenomenon of corruption and bribery be eradicated, and healthy relations among government, market, society, enterprises, and individuals be built, which means relations of benign interaction between all of them. When benign interaction is realized, the government can then strengthen the efficiency of the market by continuous enactment and enforcement of laws and regulations so as to truly promote the rejuvenation and long-term peace and stability of the Chinese nation.

1.6 Ancient Chinese thoughts on Market

Many people regard that the concepts of market economy and commodity price determined by market are from the Western world. However, it is not true. As early as the prehistoric period, China has advocated simple free market economy and believed that price is determined by the market. There was a great number of particularly insightful ideas of market economy and incentive-compatible, dialectical strategies of state governance in ancient China. Almost all fundamental ideas, core assumptions and basic conclusions of modern economics, such as behavior assumption of pursuing self-interest, economic freedom, governance by the invisible hand, social division of labor, the intrinsic relationship between national prosperity and individual wealth and between development and stability, and the relationship between government and market had all been discussed by ancient Chinese philosophers. Some examples are given as follows.

As early as over 3,000 years ago, JIANG Shang (also known as JIANG Ziya and JIANG Taigong, an ancient Chinese strategist and adviser) believed that “averting risks and pursuing interests” is the innate nature of human beings, that is to say, “All people resent death and enjoy life, welcome virtue and chase profits.” He put forward the people-centered thought of dialectical unity between wealth of the people and stability and strength of the state and the fundamental law of national governance by stating that “the state is not the property of one man but of all people. The man who shares interests with all men will win the state”, and provided the funda-

mental strategy of state governance, that is, the government should take the common interests, risks, welfare and livelihood of the populace as its own, so as to obtain an incentive-compatible result that the populace shares the same interests and risks with the government. JIANG Shang also gave an incisive answer to the relationship and priority of wealth of the state and wealth of the people, that is, "A kingly state makes its people rich, a ruling state makes lower-rank officials wealthy, a barely surviving state makes higher-rank officials affluent, and an unprincipled, ill-governed state only makes itself prosperous." His advice was accepted by King Wen of Zhou Dynasty then, who ordered to open the granary to help the poor and reduce taxes to enrich the people. The Western Zhou thus became a growing power.

Over 2,600 years ago, GUAN Zhong (a Legalist chancellor and reformer of the State of Qi in ancient China) had deep insights on many economic thoughts. The core of his economic thought was "the theory of self-interest". In *Guan Zi: Jinzang* (On Maintaining Restraint), he explained social economic activities with individuals pursuing interests very vividly and profoundly: "All men pursue interests and avert harm. When doing business, merchants hasten on the way day and night and make light of traveling from afar because, for them, interests are on their way. When fishermen go fishing in the sea, though the sea is hundreds of meters deep, they sail against the current for hundreds of miles day and night because, for them, interests are in water. So as long as there is interest, people would climb the mountain regardless of its height and go to sea regardless of its depth. Therefore, if those who know well how to govern the origin of interest, people will naturally admire the state and settle. The governor does not need to push them to go or lead them to come. Without being bothered or disturbed, people will get rich in a natural course. It's like a bird incubating eggs, the process of which is invisible and silent but the result is noticeable when it's done." This was basically a very vivid demonstration of Adam Smith's "invisible hand", only more than 2,000 years earlier than the latter. In his book *Guan Zi*, Guan Zhong presented the law of demand by stating that "The devaluation comes from the excess, while the value from the scarcity", and also drew the basic conclusion that people's wealth leads to national stability, security, prosperity and power by saying that "Only at times of plenty will people observe the etiquette. Only when they are well-clad and fed will they have a sense of honor and shame." He further pointed out that "State governance must start with enriching the people. When the people become well-off, the state will be easy to govern. If they are in poverty, the state will be hard to govern." ... "Usually, an orderly state is rolling in prosperity while a disorderly one is deep in poverty. So a king versed in ruling a state must give priority to making people wealthy over governance itself." Besides, comprehensive governance is another essential point in Guan Zhong's thought of state management. For example,

with respect to vassal kings, Guan Zhong suggested “restraining them with interest, associating with them with trust, admonishing them with military power” so that vassal kings “wo not dare to defy the king and will accept his interest, trust his benevolence, and fear his military force.” It does not require much effort to find out that there are certain corresponding relations between “restraining them with interest, associating with them with trust, admonishing them with military power” mentioned here and the three institutional arrangements we have discussed above.

About 2,400 years ago, SUN Tzu (a military general, strategist and philosopher in ancient China), in the first chapter, “Laying Plans”, of his book entitled *The Art of War* discussed military strategies and tactics which coincide, to a large extent, with the basic analytical framework of modern economics and can be fully adopted in the context of accomplishing an endeavor. It can be the rule of accomplishing big goals, making right decisions, and winning competitions in governing a country and managing an enterprise or organization. Meanwhile, he also gave the basic conclusion of information economics: it is possible to achieve the optimal outcome (“the best is first best”) only under complete information; under information asymmetry, we can at most obtain suboptimal outcome (“the best is second best”). Hence we have the well-known saying: “If you know the enemy and yourself, you wo not endanger yourself in a hundred battles. If you know yourself but not the enemy, for every victory gained you will also suffer a defeat. If you know neither the enemy nor yourself, you will succumb in every battle.”

In the same period, a more remarkable fact was that LAO Tzu (a famous philosopher of ancient China, the founder of Taoism) presented the supreme law of comprehensive governance: “Govern the state with fairness, use tactics of surprise in war, and win the world by non-intervention.” (Chapter 57, *Tao Te Ching*) This is the essential way of governing a state or administering an organization, which can be abstracted in common parlance as being righteous in deed, flexible in practice and minimal in intervention so for the government to realize governing by non-intervention. Lao Tzu considered “Tao” as the invisible inner law of nature, while “Te” (meaning the inherent character, integrity, virtue) as the concrete embodiment of “Tao”. He deemed that the governance of state and people should follow the Way of Heaven (referring to the objective laws governing nature or the manifestations of heavenly will), the Virtue of Earth and the Principle of Non-intervention by saying that “Man follows the Earth; Earth follows the Heaven; Heaven follows the Tao; The Tao follows what it is.” (Chapter 25, *Tao Te Ching*) In addition, he also pointed out that “Difficult tasks always stem (and should be tackled) from easy parts, and great undertakings always start with small beginnings.” (Chapter 63, *Tao Te Ching*) That is to say, for whatever we do, success lies in details. All these statements mentioned above demonstrate that Lao Tzu’s thought of non-intervention

does not mean “doing nothing” as is commonly regarded, which is a wrong interpretation of Lao Tzu’s real intention. The non-intervention discussed by Lao Tzu is a relative concept, which requires non-intervention in major aspects but action and care in specific aspects. In other words, we should never lose sight of the general goal and begin by tackling small practical problems at hand to take action.

Over 2,300 years ago, SHANG Yang (an important Chinese statesman) of the State of Qin used the example of hare to expound the utmost importance of establishing private property rights and how the clear definition of property rights can “determine ownership and set disputes” and help establish the market order. He came to this conclusion 2,300 years earlier than Coase. In *Shang Jun Shu (The Book of Lord Shang)*, Shang Yang wrote that “when a hare is running, one hundred men chase after it; this is not because the hare can be divided into one hundred shares, but its ownership is not determined yet. On the other side, hare traders fill the market, but thieves dare not take one because the ownership has been determined. For this reason, when ownership is not determined, sages like Yao, Shun, Yu, and Tang would also chase after it; when ownership is definite, even a greedy thief would not dare to take it.” The hare is chased because people are motivated to strive for its ownership, and even sages would do the same. On the contrary, ownership of a captured hare on market is determined, so others cannot take it as they will.

About 2,100 years ago, SIMA Qian (a Chinese historian of the Han dynasty who is considered the father of Chinese historiography) made an incredible statement in his work *Shiji: Huozhi Liezhuan (Records of the Grand Historian: Biographies of Merchants)*, “Jostling and joyous, the whole world comes after interest; racing and rioting, after interest the whole world goes,” which succeeded Guan Zhong’s thought of self-interest; besides, he also demonstrated the economic thought of achieving social welfare through social division of labor based on self-interest, which is similar to that of Adam Smith. SIMA Qian investigated the development of social and economic life and realized the importance of social division of labor. He wrote that “All these goods are what people want, which are the necessities for life.” Thus, “People rely on the farmer for food, the planter for wood, the craftsman for utensil, and the businessman for circulation.” Moreover, he believed that the whole social economy composed of agriculture, forestry, industry, and commerce should develop in a natural course without the constraint of administrative orders.

Also in the *Biographies of Merchants*, SIMA Qian continued to write that “Are there any orders and instructions to mobilize or assemble the people periodically? Each man makes his efforts to satisfy his own needs based on his own abilities. People seek for purchasing the cheap and selling the expensive. With diligence and commitment, each man delights in his own business, like water flowing downwards ceaselessly. People gather

1.7. THE CORNERSTONE ASSUMPTION IN MODERN ECONOMICS 41

together on their own initiative and produce various goods without any orders. Is not it the proof of compliance with the law of nature?"

Confucius affirmed that the pursuit of personal material interests on the premise of social ethics is justifiable. He said, "It is a shame for one to be poor and obscure when the moral way is holding its sway in a state; it is a shame for one to be rich and honored when chaos reigns in a state." (*The Analects: Tai Bo*) By saying so, Confucius encouraged people to pursue decent material wealth. *The Analects* also recorded Confucius's compliments on his disciple Zigong (Duanmu Ci), who was a merchant. In *The Analects: Xianjin*, it recorded that "Confucius said, 'Hui, who is nearly attained to perfect virtue, is often in want. Ci, who does not acquiesce in his destiny and has taken to trading, is often accurate in judgments.' " Here, Confucius compared Yan Hui, his favorite disciple, with Zigong: the former was almost perfect in morality but often lived in poverty, which did not seem to be the right way of living, while the latter who did not follow the arrangement of destiny and went into business turned out excellent in predicting the market.

In addition, ancient Chinese thought is also sparkling with wisdom in the philosophy of government governance, the importance of economic freedom, and the order of priority of several basic institutional arrangements. Sima Qian gave a very incisive conclusion in the *Biographies of Merchants*, "The master way is to follow the natural course and not to intervene, to guide with interests comes second, to teach with moral comes third, to rule by regulations comes next, and to compete for profits comes last."

These ancient Chinese economic thoughts are extremely profound. What Adam Smith discussed had already been addressed by ancient Chinese philosophers much earlier. Yet as those ancient Chinese statements were just summaries of experience, they did not form rigorous scientific systems, provide boundary conditions and scopes for conclusions, or make logically inherent analyses. As a result, little is known to the outside world.

In remaining parts of this chapter, we will have a rough discussion on core assumptions, key points, analytical frameworks, and research methodology of modern economics to help you understand the rigorous analysis of the content covered by this book.

1.7 The Cornerstone Assumption in Modern Economics

Every subject of social sciences imposes assumptions on individual behavior as the logical starting point of its theoretical system, so here we will expound this in more detail. As discussed above, the essential distinction of social science and natural science is that the former studies the behavior of

people and needs to make assumptions on people's behavior whereas the latter studies the natural environments and objects. Economics is a very special subject for it not only studies and explains economic phenomena and does empirical analyses but also studies individual behavior so as to make better predictions and value judgments.

1.7.1 Selfishness, Self-love, and Self-interest

When talking about people's behavior, three words are often involved: self-love, selfish, and self-interest, the three of which are related with and also different from each other. Self-love means one's esteem and affection for oneself, which can be positive as it encourages people to keep themselves pure on the one hand and can also be negative for it may lead to massive ego and even self-harm or self-deceit. Self-love can also generate self-interestedness and even selfishness.

Selfish means benefiting oneself on the premise of harming others. Being selfish makes one greedy; being greedy makes one ambitious; being ambitious makes one vain and arrogant; being vain makes people lose themselves, while being arrogant makes people ruthless and offensive.

Self-interest means benefiting oneself at the price of benefiting others. So, self-interest makes one reasonable and rational but selfishness only breeds greed. In other words, it is altruism for the sake of self-interest. To pursue and obtain interest, people have to choose altruistic behavior. This is the self-interest assumption adopted in the study of economics. When self-love and self-interest complement one another, people will have clear self-knowledge; when self-love is related with selfishness, people will go through moral decline. So, when people are dominated by self-love, it does not necessarily mean they will disregard others because self-love may also be combined with self-interest.

1.7.2 Practical Rationality of Self-interested Behavior

In this book, especially in the proof that market economic mechanism leads to optimal resource allocation, a key assumption is that individual behavior is driven by self-interest, which is actually the most basic assumption in economics. This is not only an assumption but also the biggest reality in the current stage of social economic development.

This assumption also applies to the handling of relations among countries, units, families, and individuals, being an objective reality or constraint that must be considered when studying and solving political, social, and economic problems. For example, when dealing with the relation between countries, as a citizen, one must protect the interests of his own country and speak and act from the standpoint of the country, and may be

1.7. THE CORNERSTONE ASSUMPTION IN MODERN ECONOMICS 43

sentenced to penalties if he divulges state secrets; when dealing with the relation between enterprises, as an employee, one must protect the interests of his organization and if he leaks firm secrets to competitors, he may also be sentenced according to the consequence. The self-interest assumption is often questioned by this saying: why are there families if individuals are rationally self-interested and pursue personal interests? In fact, when the question comes to family, individuals act in the interest of their own families. That is to say, under normal circumstances, individuals care about their own families instead of others'. The discussion on relation between individuals also follows the same reasoning. In practice, many people have misunderstandings of this assumption and interpret it in a simple and narrow sense as an assumption about individual persons in every case.

It is necessary to assume that individuals are self-interested because it conforms to the basic reality, and more importantly, the risk is minimal: even if self-interest assumption is wrong, it will not lead to serious consequences; on the contrary, if we adopt altruistic behavior assumption, once it proves wrong, the consequences will be much more serious than the former situation. In fact, the rules of game adopted under the self-interest assumption also apply to altruistic individuals in most cases, and institutional arrangements or game rules and individuals' trade-off under altruistic behavior assumption are much simpler. However, once altruistic behavior assumption proves wrong, the consequences will be much more severe than those of incorrect self-interest assumption and may even be disastrous. Actually the assumption is also very important for making right judgment on individual behavior in daily life. Just imagine the costs incurred if a selfish, cunning person is mistaken for a simple, selfless, and honest person and even trusted with important responsibilities, you will understand the serious consequences of the wrong assumption of altruistic behavior.

Acknowledging that individuals are self-interested shows a realistic and responsible attitude towards solving social problems. This is why we need disciplines of the party and laws of the state to prevent opportunists from taking advantage of loopholes in the institutions under altruism assumption. On the contrary, if altruism is used as the premise to solve social and economic problems, the consequence can be disastrous. For example, in the organization of production, if we deny individuals' self-interest as we did before the reform and only motivate individuals by emphasizing contribution to the country and group, the result would be that everyone wants to take advantage of the institutions to get an equal share and hopes to benefit from others' contribution to constructing a wonderful communist society.

1.7.3 Applicable Boundaries of Self-interest and Altruism

It should be specially noted that although the self-interest assumption holds true in most cases, it also has its application boundaries. Under abnormal circumstances like natural or man-made disasters, war, earthquake, and others' crisis, people will often demonstrate their altruistic and selfless character, make sacrifices to fight for the country, and help others in crisis. This is another form of rationality, i.e., altruism, with which people are willing to sacrifice their lives (even animals have the instinct), otherwise extreme individualism or egoism may arise. For example, when the Japanese imperialism invaded China and the Chinese nation was under the threat of destruction, people stood up against the enemy and sacrificed their lives for the national interest. After the Wenchuan Earthquake of China in 2008, people all over the country donated money and labor to those in the disaster area. However, under normal and peaceful environment, when engaged in economic activities, individuals often pursue their own interests. All these demonstrate that self-interest and altruism are not at all contrary to each other but only natural responses to different situations and environments.

Thus, we can see self-interest and altruism as relative terms. In fact, such duality can also be seen in animals. For example, when the wild goats were chased to the edge of a cliff, the old goats would sacrifice themselves by making the first jump, so that the younger goats or goatlings could jump on them and have a chance of running away. Adam Smith not only wrote the foundational work of *The Wealth of Nations*, but also wrote *The Theory of Moral Sentiments* to argue that people should have sympathy and a sense of justice. These two works complement each other in the academic thought system of Adam Smith. It is indeed so. Under the reality of individual's self-love and self-interest, morality should be a kind of balance, an equilibrium outcome, and a convention realized through social division of labor and cooperation. Under the guidance of proper institutions, people voluntarily divide the work and cooperate so as to build a harmonious, civilized, stable, and orderly society. It is against human nature to regard self-interest as an opposition of moral, or is biased and wrong to regard self-interest as being equal to selfishness. On the contrary, the organic combination of moral ethics and self-interest can actually promote social civilization and individual's decency. The biggest advantage of modern market relies on its utilization of the power of self-interest to counteract the weakness of benevolence so that those obscure hard-workers can also be satisfied. Therefore, we should not neglect the role of benevolence and moral in the formation of market system. Social progress cannot be relied on those who always want to hurt the interest of others.

Anyway, self-interested individuals can be benevolent, altruistic, and moral. "Self-interest" should not be "at others' cost". There are limitation-

s and boundaries for self-interest and altruism, but selfishness that benefits oneself at others' cost is the origin of evil and greed. Rationally self-interested behavior will conform to social norms as a necessary constraint. We agree to educate individuals to pursue personal interest without violating public order and to protect public interest built upon the basis of individual rationality, but disagree with the economic idealism that bases policies on the ignorance of personal interest and violation of reasonable personal interest in the name of defending collective interest. In short, the self-interested behavior under constraints of laws and regulations must be distinguished from selfishness in violation of laws and regulations that hurt others' interests, the former of which should be protected while the latter should be opposed.

Even with self-interested behavior, there is a difference in the extent. Under ideal situations, the less self-interested people are, the better. However, it is also impossible to eliminate self-interest. We can safely say that self-interest is the logical starting point for economics. If all human beings are unselfish and always considerate to others, then economics involving human behavior will turn out useless so that industrial engineering or input-output analysis may be enough. It is out of the consideration that self-interest is an objective reality and people tend to pursue their own interest when involving in economic activities that China carries out reform and opening-up and makes the transition from planned economy to market economy. As a matter of fact, the incredible achievements China has made through reform in the past three decades have much to do with the recognition of self-interest and adoption of market system.

1.8 Key Points in Modern Economics

Economists usually base the study of economic issues on some key points, constraints, basic axioms or principles:

- (1) Scarcity of resources;
- (2) Information asymmetry and decentralized decision-making:
individuals prefer decentralized decision-making;
- (3) Economic freedom: voluntary cooperation and voluntary exchange;
- (4) Decision-making under constraints;
- (5) Incentive compatibility: the system or economic mechanism should solve the problem of interest conflicts among individuals or economic units;
- (6) Well-defined property rights;
- (7) Equity in opportunity;

(8) Allocative efficiency of resources.

Relaxing any of these assumptions and constraints may result in different conclusions. The consideration and application of these assumptions, constraints and principles are also useful for people to deal with daily affairs. Although they seem to be simple, it is not easy to thoroughly understand and skillfully use them in reality. In the following, we will discuss these key points, conditions, axioms, and principles respectively.

1.8.1 Scarcity and Limitation of Resources

Economics stems from the fact that resources are limited in the world (at least the mass of earth is finite). As long as an individual is self-interested and his material desires are infinite (the more he possesses, the better), it is impossible to realize distribution according to wants, and the problem of how to use limited resources to meet the wants has to be addressed. Hence we need economics.

1.8.2 Information Asymmetry and Decentralized Decision-making

In addition to the basic objective reality of self-interested nature of individuals, another fundamental objective reality is that in most cases, information is asymmetric among economic agents so that the effect of institutional arrangements adopted may be inadvertently neutralized. These are why economic problems are difficult to solve. For example, people's words can be high-sounding, but it's hard to tell whether they truly mean it; listeners seems to concentrate, but you do not know if the message is well-received. This is what we mean when we say that people may "say one thing but mean another", "people's heart is unpredictable", and "people are the most difficult to deal with". The fundamental reason for such phenomena is incomplete and asymmetric information. Information asymmetry together with the self-interested nature of individuals can often lead to conflicts of interests among economic agents. If no proper governance institution is in place to mediate, to obtain various limited resources in nowadays society, it has become a normality that people say "false, big, empty" words that do not conform with their action and put forward slogans that are divorced from reality. Such phenomena can be summed up and depicted in one word, that is, "fake". Everyone is faking with only the difference in extent. This is why in modern society, due to dishonesty and fraud, many people are reluctant to communicate with other people but willing to spend more time with animals, thinking that animals will not lie to them or hurt them. This is the main reason that social sciences, especially economics, are much more complex and difficult to study than natural sciences. Also due to information asymmetry, centralized decision-making is often inefficient,

while decentralized decision-making, such as the use of market mechanism, is required to solve economic problems.

Only when complete information is acquired can the outcome be the first best. As stated in information economics, it is possible to achieve the optimal outcome only under the circumstance of complete information. However, information symmetry is often difficult to obtain, so incentive mechanism is needed to induce truthful information. However, information acquisition incurs costs; as such, we can only obtain suboptimal outcome. This is the basic result of the principal-agent theory, optimal contract theory, and optimal mechanism design theory that will be discussed in Part V of this book. In most cases, information is asymmetric, so there will be market failures and agency problems. Yet whatever the approach, the outcome will always be suboptimal due to information asymmetry. Without reasonable institutional arrangements, there will be incentive distortion where inducing information inevitably incurs costs and prices; therefore, the outcome cannot be optimal (first best). Information symmetry is particularly important, without which many misunderstandings may arise. By communicating with others, you let others understand you (signaling) and also get to know others (screening) so as to obtain information symmetry, clear up misunderstandings and reach consensus, which is the fundamental premise of obtaining a good outcome.

Excessive intervention of government in economic activities and the over-playing of government role will lead to low efficiency, which, in root, is caused by information asymmetry. There are many problems in information acquisition and discrimination of the government. If decision-makers are able to have all related information, centralized decision-making featuring direct control would not be problematic, and it would be a simple question of optimal decision-making. However, it is impossible for decision-makers to have all related information at hand. That is why people prefer decentralized decision-making. That is also why economists stress that the incentive mechanism, a decentralized decision-making method featuring indirect control, should be used to motivate (stimulate) people to do as decision-makers desire, or to achieve the goals decision-makers aspire for. We will focus on the issue of information and incentive in Part V.

It is worth mentioning that centralized decision-making also has its advantages in some aspects, especially in the decision-making regarding major changes. For instance, centralized decision-making is more efficient when a country, unit or an enterprise is laying down the vision, orientation, and strategies or making big decisions. However, such major changes might bring about huge success or severe mistakes. For example, the decision of adopting the reform and opening-up policy has led to rapid development of the Chinese economy and unprecedented achievements. In contrast, the decision of "Cultural Revolution" almost pushed the Chinese economy to the verge of collapse. One solution to this problem is to give

public opinion full respect and select outstanding leaders.

1.8.3 Economic Freedom and Voluntary Exchange

Because of economic agents' pursuit of their own interests and the factor of information asymmetry, institutional arrangements of the mandatory "stick" style that are supposed to enlighten the people with reason are often not efficient. So, we need to give people more freedom of economic choice, which is the most important kind of right among the three private rights as mentioned before (right to survival, freedom of choice for one to pursue happiness, and private property right). Thus, we should mobilize economic agents with free economic options based on voluntary cooperation and exchange through inducing incentive mechanism such as market. Therefore, the freedom of economic choice (i.e., "deregulation") plays a vital role in market mechanism with decentralized decision-making (i.e., "decentralization"), being a prerequisite for the normal operation of market mechanism and also a fundamental precondition to ensure optimal allocation of resources under competitive market mechanism.

In fact, the Economic Core Theorem to be discussed in Chapter 12 reveals that once full economic freedom is given and free competition, voluntary cooperation and exchange are allowed, even without any institutional arrangement in advance, the outcome of resource allocation driven by the self-interested behavior of individuals will be consistent with the equilibrium result of a perfectly competitive market. The essence of the Economic Core Theorem can be summarized as follows: under the rationality assumption, i.e., the assumption that the ideological level of individuals is not high, as long as freedom and competition are given while institutional arrangement is not considered, the economic core thus obtained is competitive market equilibrium.

China's reform and opening up over the past 30 years have proved this theorem in practice. When analyzing the reasons for China's remarkable economic achievements, despite any other crucial issues, the critical key is to give people more freedom of economic choice. Reform practice from rural to urban areas indicates that wherever there are looser policies and a greater degree of economic freedom provided for producers and consumers, there will be higher levels of economic efficiency. China's miraculous economic growth actually stems from the government's delegation of powers to the market, while its imperfect market today has actually been a result of excessive government intervention and inadequate or inappropriate government regulation and institutional arrangements.

1.8.4 Constraints and Feasible Options

Doing things under constraints is one of the most fundamental principles in economics, as the saying goes that people have to bow under the eaves. Everything has its objective constraints, i.e., individuals make trade-off choice under existing constraints, which is one of the basic principles in economics. People's choice is determined by objective constraints and subjective preferences. Constraints include material constraint, information constraint, and incentive constraint, which make it difficult for economic agents to achieve their goals. In economics, one embodiment of the basic idea of constraints is the budget constraint line of consumer theory as we will discuss in Chapter 3, which states that an individual's budget is constrained by the price of the commodity and his income. For an enterprise, constraints include available technologies and price of input, under which, the goal of maximum profit requires managers to determine optimal prices for products, the quantity of production, technologies to be adopted, quantity of each kind of input, response to competitor's decision-making, etc. The development of a person or even a country is faced with various restrictions and constraints, including political, social, cultural, environmental, and resource constraints. If we do not make the constraint condition clear, it is hard to have things done.

When introducing a reform measure or an institutional arrangement, it is a must to consider feasibility and meet the objective constraints; meanwhile, the implementation risk is expected to be reduced to the minimum so that social, political and economic turmoil wo not be caused. Feasibility means that it is necessary to consider the various constraint conditions to be confronted when doing things, otherwise it is not practical. Feasibility, therefore, is a necessary condition that is used to judge whether a reform measure or an institutional arrangement is beneficial to economic development and the smooth transition of economic system. In a country's economic transformation, an institutional arrangement is feasible because it conforms to the institutional environment of the particular stage of development of the country. Again, in the case of China, the reform must adapt to China's national conditions and fully consider the various constraint conditions, including the limited ideological level of people and participation constraint, etc.

Participation constraint is very important when considering incentive mechanism design, which means that an economic agent can benefit, or at least shall not suffer losses, from economic activities, otherwise he will not participate, or even oppose the rules or policies to be implemented. Individuals who pursue the maximization of self-interest will not automatically accept an institutional arrangement but will make a choice between acceptance and refusal. Only under an institutional arrangement where the individual's benefit is not less than his retained benefit (when he does not

accept the arrangement) will the individual be willing to work, produce, trade, distribute, and consume. If a reform measure or an institutional arrangement does not meet the participation constraint, individuals may give up. If everyone is reluctant to accept the reform measure or institutional arrangement, it cannot be successfully implemented. Mandatory reform may arouse opposition and cause social instability so that development will not be possible. Therefore, participation constraint is closely related to social stability and is a basic judgment of social stability in development.

1.8.5 Incentive and Incentive Compatibility

Incentive is one of the core concepts in economics. Each individual has his own self-interest; to obtain interests from some activity, one must also pay the price or cost. By the comparison between benefits and costs, individuals may be willing (have incentive) to get something done or well done, or be reluctant or unwilling to get it done or well done, and thus will have rational incentive response to game rules. However, it often leads to incentive-incompatible conflicts of interests among individuals or between individuals and society and brings about chaos. The thought of incentive compatibility has been discussed by Adam Smith in *The Theory of Moral Sentiments*, as he stated that "...in the great chess-board of human society, every single piece has a principle of motion of its own, altogether different from that which the legislature might choose to impress upon it. If those two principles coincide and act in the same direction, the game of human society will go on easily and harmoniously, and is very likely to be happy and successful. If they are opposite or different, the game will go on miserably, and the society must be at all times in the highest degree of disorder."⁷ The reason is that under given institutional arrangements or game rules, individuals will make optimal choices according to their own interests, but such choice will not automatically satisfy interests or goals of others and society, and meanwhile information asymmetry makes it difficult to implement social optimum by command. A good institutional arrangement or rule is able to guide self-interested individuals to act subjectively for themselves but objectively for others, making individuals' social and economic behavior beneficial to the country and the people as well as to themselves. This is the core content of modern economics.

Everything an individual does involves interests and costs (benefits and costs), making incentive an ubiquitous issue to deal with in daily work and life. As long as the benefits and costs are not equal, there will be different incentive reactions. To pursue profits, an enterprise has the incentive to use resources in the most efficient way and provide incentives to guide employees to make the greatest effort. Outside the enterprise, changes of

⁷Adam Smith: *The Theory of Moral Sentiments*.

profits provide an incentive for resource holders to change their ways of using the resource; inside the enterprise, incentive influences the way of using the resource and the effort employees put in work. To make management efficient, one must be clear about the role of incentive in organizations and how to construct incentive to guide subordinates to exert the greatest effort in work.

Since the interests of individuals, society, and economic organizations cannot be completely the same, how can self-interest, mutual benefits, and social interests be organically combined? It then requires incentive compatibility, with which the reform measures and institutional arrangements adopted can greatly mobilize people's enthusiasm for production and work. Therefore, to implement some goal of one's own or of the society, proper game rules shall be given, under which, when individuals involved pursue self-interest, the expected goal can also be achieved, which is what we call incentive compatibility. That is to say, it unifies the self-interest of individuals and mutual benefits between individuals so that when pursuing self-interest, each individual can help attain the goal intended by the society or other individuals. We will focus on the issue of how to achieve incentive compatibility in Part V of the book.

1.8.6 Property Rights Incentive

Property rights are an important component of market economy. In the previous section about the ancient Chinese thoughts on market, we've talked about that over 2,300 years ago, Shang Yang of the State of Qin used the example of hare to expound the utmost importance of establishing private property rights and how the clear definition of property rights can "determine ownership and set disputes" and help establish the market order.

Property rights include ownership, right of use, and decision right of property. A clear definition of property rights will help clearly define the attribution of profits, thus providing incentives for property owners to consume and produce in the most effective way, to provide quality products and good service, to build reputation and credibility, and to maintain their own commodities, housing, and equipment. If property rights are not clearly defined, the enterprises' enthusiasm will be harmed, giving rise to incentive distortion and moral risk. For example, unclear property rights in state-owned enterprises will surely cause low efficiency, bring about ubiquitous corruption like rent-seeking or interest transfer, squeeze private economy, be detrimental to innovation, and lead to unfair competition. In the market mechanism, incentives are given to individuals mainly in forms of property possession and profit acquisition. The Coase Theorem to be discussed in Chapter 14 of the book is a benchmark theorem in property rights theory, which claims that when there is neither transaction cost nor income effect, as long as property rights are clearly defined, an efficient al-

location of resources can be achieved through voluntary coordination and cooperation.

1.8.7 Outcome Fairness and Equity in Opportunity

“Outcome fairness” is a goal that an ideal society wants to achieve. However, for human society with self-interested behavior, this “outcome fairness” often brings low efficiency. So, in what sense can fairness be consistent with economic efficiency? The answer is that if people use “equity in opportunity” as the value judgment standard, fairness and efficiency can be consistent. “Equity in opportunity” means that there should not be any barrier to hinder individuals from pursuing their goals in their capacity, and there should be a starting point of equal competition for every individual. The Outcome Fairness Theorem to be introduced in Chapter 12 tells us that as long as each individual’s initial endowment is of equal value, through the operation of competitive market, allocation of resources that is not only efficient but also fair can be achieved even if individuals pursue self-interest. A concept similar to “equity in opportunity” is “individual equality” (also known as, “All men are created equal”), which means that, although people are born with different values, genders, physical conditions, cultural backgrounds, capacities, ways of life, “individual equality” requires due respect for such individual differences.

As individuals have different preferences, the seemingly equal distribution of milk and bread may not satisfy everyone. Therefore, in addition to equal allocation that defines fairness as absolute equalitarianism, the concept of fairness in other senses is also used in the discussion of economic issues. For example, equitable allocation to be introduced in Chapter 12 considers both subjective and objective factors, which means that everyone is satisfied with their shares.

1.8.8 Allocative Efficiency of Resources

Whether social resources are efficiently allocated is a basic criterion to evaluate the quality of an economic system. In economics, efficient allocation of resources usually refers to Pareto efficiency or optimality, which means that there does not exist other resource allocation schemes such that at least someone is better off without hurting others. As such, it requires not only efficient consumption and production but also production of products that can best meet needs of consumers.

It may be remarked that, when it comes to economic efficiency, we should distinguish between three types of efficiency: individual firm’s production efficiency, industrial production efficiency, and social resource allocation efficiency. By saying that an individual firm’s production is efficient,

we mean that with a given input, the output is maximized, or, with a given output, the input is minimized. Industry is the sum of all firms that produce a category of commodity and the efficiency of which can be similarly defined. It should be noted that the efficiency of an individual firm does not equal to the efficiency of the whole industry. The reason is that if the production materials of firms with outdated technologies are given to those firms with advanced technologies, there will be more output for the whole industry. At the same time, even if production of the whole industry is efficient, the allocation of social resources may not be (Pareto) efficient.

The concept of Pareto efficient allocation of resources is applicable to any economic institution. It provides a basic criterion of value judgment for an economic institution from the viewpoint of social benefit and makes evaluation on the economic effect from the perspective of feasibility. It can be applied to planned economy, market economy, and mixed economy. The first welfare theorem to be introduced in Chapter 11 proves that when individuals pursue self-interest, a fully competitive market will lead to efficient allocation of resources.

1.9 A Proper Understanding of Modern Economics

A proper and profound understanding of modern economics can help people correctly use basic principles and analytical methods of economics to study various kinds of economic problems under different economic environments, behavioral assumptions, and institutional arrangements. The different schools and theories of modern economics per se show the speciality, universality, and generality of the analytical framework and methodologies of modern economics. Under different economic environments, different assumptions and specific models will surely be required. Only in this way will the theory developed be able to explain different economic phenomena and individuals' economic behavior, and more importantly, make logically inherent analyses, draw conclusions of inherent logic, or make scientific predictions and reasoning under various economic environments that are close to the theoretical assumptions. However, because of the complexity of economic environments, for the sake of semantic and logic clarity, modern economics uses various rigorous mathematical tools to build economic models so as to develop economic theory. Rigorous mathematical tools are often difficult to master, which results in frequent misunderstandings of modern economics. In addition to the common misunderstandings of benchmark theories as discussed before, there are misunderstandings about modern economics concerning the following several aspects.

1.9.1 How to Regard the Scientific Nature of Modern Economics

One of the main misunderstandings is that economics is not science and includes many seemingly conflicting or contradicting theories. Criticisms are often heard that there exist too many different economic theories coming from different economic schools in modern economics, making it difficult to figure out which is right and which is wrong. As a matter of fact, all the changes are revolving on the same theme, but people who hold such opinion fail to understand that all kinds of economic theories are not divorced from the two basic categories as discussed before. It is because of the complexity of reality, different economic, political, social, and cultural environments of different countries and regions, varying thoughts and preferences of people, and possibly different economic goals that various theoretical economic models and economic institutional arrangements should be developed accordingly.

It is easy to understand that different economic theories or models should be developed for different economic, social, and political environments, but difficult for many to comprehend why different economic theories are developed under the same economic environment. Thus, some people, while disavowing modern economics and its scientific characteristics, scorch economists by saying that 100 economists will have 101 opinions. They do not realize that it's like that we need maps for different purposes such as traffic, travel, military, etc., though there is only one earth; under the same given economic environment, we will need to develop different economic theories and provide different economic institutional arrangements to realize different goals.

The fact that different opinions of economists will arise for the same problem just shows the precision and perfection of modern economics because when premise and environment changes, the conclusion should change accordingly, which is especially true in the case of the second category of economic theory that aims to solve practical problems. Besides, as different people will also have different subjective judgements of values, there is few universally correct conclusion that satisfies everyone and is feasible in all situations, otherwise we would not need case-by-case analysis according to time and place and adjust to changing circumstances. It is similar to the philosophical thoughts of using military forces in war and using medicine to treat diseases: medicines vary as diseases vary; when considering and solving economic problems, case-by-case analysis shall be done according to time, place, people, and occasion. The difference is that as economics has great externalities as we've discussed before, while a charlatan may kill individual persons with wrong medicines, the wrong prescription of economic policy will influence a larger group of people and even a country.

Though there are different economic theories and models, either benchmark economic theory that provides benchmark or reference system or the

second category of economic theory that aims to solve practical problems is definitely not different “Economics”. It is often heard that the specific national condition of China calls for the development of “Chinese economics”. Well, there are numerous buildings in the world, and even the buildings designed by the same person are quite different, so does that mean we need different “architecture sciences”? The answer is definitely no, because the basic principles and methods for construction are of no essential difference. The same logic is true for the study on economic issues, where the same analytical framework and research methods are adopted for no matter Chinese or foreign economic issues. Owing to the different economic, political, and social environment of China from those of other countries, there are “Chinese issues”, “Chinese path” and “Chinese characteristics” and economics about the Chinese economy, but there is no such difference between the so-called “Chinese Economics” and “Western Economics”.

The basic analytical framework and methodologies of modern economics, just like those of mathematics, physics, chemistry, engineering, etc., are not bounded by regions or countries, and there are no framework and methodologies independent of other countries. The basic principles, methodologies, and analytical framework can be used to study a variety of economic issues under any economic environment and institutional arrangement and to study the economic behavior and phenomena in a specific area and time period. The analytical framework and methodologies to be introduced later can be used to study and conduct comparative analysis on almost every economic phenomenon and issue. Thus, various economic problems under China’s actual economic environment can also be analyzed by the analytical framework of modern economics. In fact, this is exactly in which the power and glamour of the analytical framework of modern economics lies: its essence and core require that the economic, political and social environment conditions at a specific time and place must be considered and clearly defined when doing research. Modern economics can be used to study economic issues and phenomena under human behavior as manifested in different countries and regions, customs and cultures. Its basic analytical framework and methodologies can also be used to study other social phenomena and human decision making. It is proven that because of the universality and generality of the analytical framework and methodologies of modern economics, in the past few decades, many analytical methods and theories have been extended to other disciplines like political science, sociology, and humanities.

1.9.2 How to Regard the Mathematical Nature of Modern Economics

Besides the criticism that the assumptions of economic theory do not conform with reality and science, another common criticism is that modern

economics pays too much attention to details and involves an increasing amount of mathematics, statistics, and models, making questions even more obscure and incomprehensible. However, the reason that modern economics uses so much mathematics and statistics is for the sake of rigor and quantifiability of empirical studies. Though decision-makers in the administration and the general public do not need to understand details or premises of the rigorous theoretical analysis, it is a must for economists who put forward policy suggestions to know. As economic theory will generate great externalities once adopted, blind application without considering the premise may bring about big problems and even disastrous consequences. That's why mathematics should be employed to rigorously define the boundary condition of the theory. Meanwhile, the application of a theory or enactment of a policy often requires tools of statistics and econometrics to carry out quantitative analysis or empirical test. Also, because the real economic society is too complicated, the use of mathematical models in economic theories can help abstract and depict the real economic world for people to deeply understand and comprehend problems to be solved in reality. We will further discuss the role of mathematics in modern economics later in this chapter.

1.9.3 How to Regard Economic Theory Correctly

Each theory or model in economics, for either benchmark economic theory or the second category of economic theory that aims to solve practical problems, is composed of a set of presupposed assumptions on economic environments, behavior patterns, and institutional arrangements and conclusions based on these assumptions. Considering the complexity of economic environments and the diversity of individual preferences in reality, the more general the presupposed assumption of a theory is, the more helpful and powerful the theory is. If the presupposed assumption of a theory is too restrictive, the theory will lack generality and thus be less useful in reality. As economics is used to serve the society and the government, the theory must be of some breadth. So, a necessary requirement for a good theory is its generality: the more general it is, the larger explanatory capacity it will have and the more useful it will be. General equilibrium theory that studies competitive market is such a theory. It proves that competitive equilibrium exists and leads to optimal resource allocation under the condition of very general preference relations and production technologies.

Even so, theories of social sciences, especially those of modern economics, like all theorems in mathematics, have their boundary conditions. As discussed before, because of the great externalities of economic theories, when discussing or applying an economic theory, we need to pay attention to its presupposed assumption and application scope, for conclusions of any economic theory are not absolute but only hold true when the pre-

supposed assumption is satisfied. Whether this point is recognized when discussing economic issues is a basic judgment of whether an economist is well-trained. As economic issues are closely related to daily life, even ordinary people can give certain opinions about economic issues such as inflation, business climate, balance or imbalance of supply and demand, unemployment, stock market, and housing market. For this reason, many people do not regard economics as a science. Indeed, "economics" would not be a science if it did not take into account any constraints or base inherent logical analysis on accurate data and rigorous theories, while people doing so are not real economists. A well-trained economist always discusses issues based on certain economic theories and is fully aware of the boundary condition for the relationship among economic variables and the inherent logic of the conclusions. It's very important to fully understand the boundary condition of economic theories, otherwise one will not be able to distinguish between the theory and the reality but instead be liable to go to two extremes: either simply applying the theory to reality regardless of constraints in reality, or totally denying the value of modern economic theory.

The first extreme viewpoint overestimates the role of theory and abuses theory. For example, some people disregard the real objective constraints facing China, blindly or mechanically apply the two categories of theories in modern economics to solve China's problem, and indiscriminately copy models to study China's problem, thinking that the inclusion of mathematical models will make good papers and good theories. Conclusions and suggestions obtained by simply copying economic theories or models, in the case of no matter benchmark economic theory or the second category of economic theory that is more close to reality, for general application to the reality of China without taking full consideration of various constraints and boundary conditions created by China's real situation and economic institutional environment will often lead to big problems. In fact, no matter how general the theory and behavioral assumption are, they have application boundaries and limitations and thus should not be used indiscriminately. Especially for those theories developed based on an ideal state far from reality and mainly for purposes of establishing reference system, benchmark, and goal, we should not apply them directly; otherwise, we might get wrong conclusions. Without the sense of social responsibility or good training in economics, a person may overestimate the role of theory and blindly apply or misuse economic theory in real economy without taking the premise into account, which could bring severe consequences, affect social and economic development, and lead to huge negative externalities. For instance, the conclusion of first welfare theorem that competitive market leads to efficient allocation of resources is subject to a series of preconditions and its abuse will give rise to severe policy errors and serious damage to the real economy.

Another extreme viewpoint totally denies the role of theory. People holding such point of view underestimate and even deny the practical significance of modern economics, including its behavioral assumption, analytical framework, basic principles, and research methodology. They believe that modern economics, together with its analytical framework and research methodology, is exotic concept and complete Westernization not suitable for the national condition of China; to solve China's problems, a set of analytical framework and economics of China's own should be innovated. In fact, just as the benchmark theory of modern economics, there is no discipline in the world all of whose assumptions and principles coincide with reality perfectly (like concepts of free fall without air resistance and fluid motion without friction in physics). We should not deny the scientific feature and usefulness of a discipline, and modern economics as well, just based on this. We learn modern economics to acquire not only its basic principles and usefulness but also the way of thinking, asking, and solving questions. As discussed before, the value of benchmark economic theories lies not in direct explanation of the reality but in provision of research platform and reference system for developing new theories to explain the real world. With these methods, people can be enlightened on how to solve the problems in reality. In addition, as discussed in the previous section, a theory applicable to one country or region may not be applicable to another country or region due to different environments. Instead of applying the theory mechanically and indiscriminately, we need to modify the original theory to develop new theories according to the local economic environment and individual behavior pattern.

Surely, currently there is another extreme viewpoint held by some influential economists, who do not think that market failures may occur under any circumstance and that market has externalities. They thus deny the practical significance of modern economic theory, believing that modern economics is strongly hypothetical while these assumptions are not needed because market does not have boundaries.

Some people often claim that they have toppled an existing theory or conclusion. As some conditions of the theory are not in line with the reality, they think that the theory is incorrect and has been toppled. Generally speaking, their logic is not scientific or even wrong. Assumptions, even those in the second category of theories that aim to solve practical problems, cannot fully coincide with reality or cover every possible case. A theory may be applicable to the economic environment of one place but inapplicable to that of another nation or region. But, as long as there is no inherent error, we cannot say that the theory is wrong and needs to be overturned. We may only state that it is not applicable at a certain place or a certain time.

Instead of overall negation for a new start, the innovation of economics in China should be marginal innovation or combinatorial innovation based

on the cornerstone of economic theories. Technology and application innovations are often re-combination and development of existing technologies on the cornerstone of basic research, just as new prescription of various herbs in the traditional Chinese medicine. Just as natural sciences, economic theories of vitality are surely developed based on previous theoretical results through comparisons and extensions. People can criticize a theory for being too limited or unrealistic, but what economists should do is to relax or modify the presupposed assumption of the theory and adjust the model so as to improve or develop the original theory. We cannot claim that the new theory topples the original one, and a proper way of putting it may be that the new theory improves or extends the old theory so it can be applied to more general or different economic environments.

Besides, another common mistake is trying to draw a general theoretical conclusion just based on some specific examples, which is a mistake in methodology. Of course, here we do not deny the unique role of history, culture, and thought of each country in the establishment of their own discourse of economics.

1.9.4 How to Regard the Criticism for the Difficulty to Conduct Experiments in Economics

Many people criticize economics for not being an experimental science and thus negate the scientific feature of economics. Such viewpoint is a misunderstanding. First of all, as with the rapid development of experimental economics in recent years, economics is becoming an experimental science. Experimental economics tests individual's behavior and rationality of behavioral assumption through experiments and thus becomes an important tool to examine whether an economic theory fits the objective reality. Theorists have also obtained important information from experiments so as to promote the advancement of theories (There are discussions of many economists on how to understand experiments in economics on the website of Al Roth). What's more, experiments in economics have already walked out of laboratory toward society (See relevant discussion by John List).

Indeed, from the empirical perspective, in real economic activities, experiments in economics are of indispensable advantage to verify policies and systems, especially for the need of institutional transition. After constant exploration by early scholars and the systematic synthesis of methodologies and tools of experiments in economics by Vernon Smith, winner of the 2002 Nobel Memorial Prize in Economic Sciences, modern experimental economics as an important empirical tool has received increasing attention in market mechanism design. When external environment changes rapidly and new technologies spring up like mushrooms, reform becomes an inevitable choice, but meanwhile, strategic risk and social cost of various policy suggestions and proposals have to be considered with caution.

Therefore, it naturally becomes a difficult point and key in the institutional reform to find a way to comprehensively and thoroughly examine in advance problems that might arise out of new proposals on institution. China has taken various measures including “special economic zone policy”, “pioneering pilot scheme”, “typical example as the lead”, etc., in the process of reform and opening-up. Economic experiments are consistent with such measures in the guiding thought of lowering risk and cost of the reform, but there are still major differences in methodology between economic experiments and pilot experiments for experience accumulation. In comparison with the pilot method, firstly, the economic experiment is liable to focus on a single research question so that each experiment examines effects of only one policy and characteristics of only one mechanism. Secondly, the economic experiment employs rather normal techniques and tools. There are multiple influential factors in real economic activities, while the methodology of economic experiments is able to control factors irrelevant to the research question so as to focus on the effect of one specific factor on certain economic phenomenon. Last, in comparison with the pilot method, it is less costly to conduct economic experiments.

Besides, Soo Hong Chew, Professor of the National University of Singapore and Special-term Professor of Shanghai University of Finance and Economics, leads a research team of behavioral and biological economics in collaboration with the School of Life Sciences of Fudan University and the research team of experimental economics of SUFE to study the relation between genetics and economic behavior. Such study is very meaningful. Once the relation between the two is figured out, it will lay down the foundation for economics to become a discipline of science just as natural science.

Surely, we must also admit that quite a few economic theories, like general equilibrium theory for comprehensive analysis, cannot or cannot easily be tested by social experiments in case of policy mistakes that will bring about huge risks to economy and society. This is the greatest difference from natural sciences, as in natural sciences, natural phenomena and objects can be studied through experiments and theories can be tested and further developed in the laboratory. Astronomy might be the only exception, but it involves no individual behavior. Once that is involved, things will become more complicated. Moreover, extreme accuracy can be attained in the application of theories in natural sciences. For instance, in the construction of buildings or bridges and manufacturing of missiles or nuclear weapons, accuracy of any extent is attainable for all the parameters are controllable and the interrelationship between variables can be experimented. However, in economics, many factors affecting economic phenomena are uncontrollable.

Economists are often criticized for inaccurate economic forecasts. We can explain this from two perspectives. From the subjective perspective, it

is due to the quality of some economists who have not been through systematic and rigorous training of modern economic theory. As a result, they cannot figure out main causes of problems or make correct logical analyses and inferences when they discuss and try to solve economic problems and thus will prescribe wrong medicines (if such medicine exists) to economic problems. From the objective perspective, some economic factors that influence economic results may suddenly and uncontrollably change, thus making the prediction inaccurate, even though they are made by well-trained economists with good economic intuition and insight. An economic issue involves not only human behavior which might complicate the matter but also many other uncontrollable factors. Although an economist might be wise and insightful, his predictions are liable to deviation because these uncontrollable factors that will influence economic results may change. For instance, a nation's leader, respected as he is, may manage affairs within his nation quite well but fail to do so in other countries. Likewise, even for a good economist with sound judgment, his economic forecasts might become quite inaccurate once sudden changes take place in the economic, political or social environment. Some people may question that no matter how economics develops and what the reason is, inaccurate prediction is the norm while accurate prediction is merely luck. It is true in principle, as economic fluctuation is a random variable, making it impossible to make accurate prediction. However, since the probability for an event to happen and the quality of economists vary, excellent economists can better judge the probability of an event and be more likely to be accurate in their prediction, influenced by the subjective factor of inaccurate prediction as mentioned above.

How can the problem that economic theory cannot experiment on society under many circumstances be compensated? The answer is logically inherent analysis, based on which inherent conclusions and inferences can be drawn and comparisons and empirical data test can be done through horizontal and vertical perspectives of history. In this way, when conducting economic analysis or giving policy suggestions, according to the three elements of economic analysis as discussed before, firstly, there should be theoretical analysis of inherent logic to define applicable boundary conditions and scope. Meanwhile, tools of statistics, econometrics or experimental economics shall be utilized for empirical analysis or test, and the great perspective of history should be taken for vertical and horizontal comparative analysis. So, when conducting economic analysis or giving policy suggestions, there should be not only theoretical analysis of inherent logic, historical comparative analysis from the great perspective, but also empirical quantitative analysis based on data and statistics, the three of which are all indispensable. The three-in-one research methodology compensates to a large extent the problem that many economic theories cannot or cannot easily experiment on society.

The method of logically inherent analysis of economics is to fully understand and characterize first relevant circumstances (economic environment, situation, and status quo) for a problem to be solved to ascertain what the problem is and its causes, apply proper economic theories accordingly, draw scientific conclusions, and make accurate forecasts and inferences. As long as the status quo accords with causes (economic environments and behavioral assumptions) presupposed in the economic model, logically inherent conclusions can be drawn according to economic theories, and thus solutions (certain institutional arrangement) can be obtained for different circumstances (varying with time, place, people, and case). The method of logically inherent analysis can help make scientific inferences on possible results under the circumstance where real economic and social environment, behavior pattern of economic agents, and economic institutional arrangement are given and thus provide guidance for solving real economic problems. In other words, once we make clear the problem and its causes and apply proper economic theory accordingly (like the prescription), if such theory exists, we can then use the right remedy of comprehensive treatment and draw logically inherent conclusions so as to make accurate predictions and inferences. Otherwise, severe consequences might arise.

It's true that the result of an economic theory cannot be tested by experiments on society under many circumstances, and data does not provide the sole basis for analysis. Practice is only the sole criterion for testing truth but not for predicting truth. So, it is the logically inherent analysis that should be relied upon. Like a doctor prescribing for his patient or a mechanic repairing a car, the hardest part is diagnosis of disease or cause of failure. The criterion for a good doctor lies in whether he can accurately find the true cause of disease. Once the cause is identified and there is a remedy, it is relatively easier to prescribe for the patient, unless he is a complete charlatan. For economic problems, the prescription is economic theories. Once we truly understand the characteristics of economic environment, thoroughly investigate the situation, and accurately assume individual's behavior, we will get twice the result with half the effort.

1.10 Basic Analytical Framework of Modern Economics

There are basic laws to follow for doing anything. The way that modern economics studies and solves problems is similar to how people deal with personal, family, economic, political and social affairs. As is known to all, in order to do something well and associate with people, the first thing is to learn about national conditions and customs, i.e., to know the real environment and the conduct and personality of the person to interact with; on such basis, one determines the strategy of dealing with people and affairs

1.10. BASIC ANALYTICAL FRAMEWORK OF MODERN ECONOMICS 63

accordingly and makes incentive response after trading off advantages and disadvantages to obtain the optimal outcome; last, one evaluates and compares the strategy and the outcome generated. The basic analytical framework and research methods of modern economics completely follow this mode to study economic phenomena, human behavior, and how people weigh trade-offs and make decisions. Of course, a major difference between these two is the rigorous reasoning of modern economics which uses formal models to strictly identify the logically inherent relationship between presupposed assumptions and conclusions. Such analytical framework is of great generality and consistency.

A standard academic article needs to spell out first of all the problems to be studied and resolved or the economic phenomena to be explained. That is to say, economists need to identify first research objectives and their significance, provide readers with information on the overview and progress of the issues under study through literature review, and illustrate the paper's innovation on technical analyses or theoretical conclusions. Then, they discuss how to address the issues raised and draw conclusions.

Economic issues under study may be quite different, but the basic analytical framework used can be the same. The analytical framework for a standard economic theory in modern economics consists of the following five parts or steps: (1) Specifying economic environment; (2) Making behavioral assumptions; (3) Setting institutional arrangements; (4) Determining equilibrium; and (5) Making evaluations. Any economics paper written with clarity and logical consistency is basically made up of these five parts,⁸ especially the former four parts, no matter what the conclusion is and whether the author realizes it or not. So to speak, writing an economics paper is innovative writing with logically inherent structure and analysis in such steps. Once you understand these components, you will grasp the basic writing pattern of modern economics papers and find it easier to study modern economics. These five steps are also helpful for understanding economic theory and its proof, selecting research topics and writing standard economics papers.

Before discussing the five components one by one, we should define first the term "institution". Institution is usually defined as a set of rules related with social, political and economic activities that dominate and restrict the behavior of various social classes (Schultz, 1968; Ruttan, 1980; North, 1990). When people consider an issue, they always treat some factors as exogenously given variables or parameters while others as endogenous variables or dependent variables. These endogenous variables depend on the exogenous variables, and thus are functions of those exoge-

⁸Detailed discussion on this section see my paper entitled *Basic Analytical Framework and Research Methodology of Modern Economics*, in which many famous theories or models consisting of these five components are listed and analyzed. An abbreviated version is published on *Economic Research Journal*, Issue 2, 2005.

nous variables. In line with the classification method of Davis-North (1971, pp 6-7) and the issue to be studied, we can divide institution into two categories: institutional environment and institutional arrangement. Institutional environment is the set of a series of basic economic, political, social and legal rules that form the basis for formulating production, exchange and distribution rules. Of these rules, the basic rules and policies that govern economic activities and property and contract rights constitute the economic institutional environment. Institutional arrangement is the set of rules that dominate the potential cooperation and competition between economic participants. It can be interpreted as the generally known rules of the game, with different game rules leading to different incentive reactions of individuals. In the long run, institutional environment and institutional arrangement will affect each other and change. Yet in most cases, as Davis-North clearly pointed out, people usually regard economic institutional environment as an exogenously given variable, while consider economic institutional arrangement (such as market system) as being exogenous or endogenous, depending on the issues to be studied and discussed.

1.10.1 Specifying Economic Environment

The primary component of the analytical framework of modern economics is to specify the economic environment where the issue or object to be studied lies. An economic environment is usually composed of economic agents, their characteristics, the institutional environment of economic society, the informational structure and so on, which cannot change in the short term (though may evolve in the long run) and thus are given as exogenous variables and parameters. They are the embodiment of the basic idea on constraints.

How can we specify economic environment? It can be divided into two levels: (1) objective and realistic description of economic environment and (2) concise and acute characterization of the essential features. The former is science and the latter is art, the two of which should be combined and balanced. The more clear and accurate the description of economic environment is, the greater the likelihood of obtaining correct theoretical conclusions. Also, the more refined and acute the characterization of economic environment is, the easier it is to argue and understand the theoretical conclusions. Only by combining these two levels together can we capture the essence of issues under study, as specifically discussed below:

Description of economic environment: The first step in every economic theory of modern economics is to give approximately objective description of economic environment where the issue or object to be studied lies. A reasonable, useful economic theory should exactly and properly describe the specific economic environment. Though different countries and areas have different economic environments, which usually lead to different con-

clusions, the basic analytical framework and methodologies utilized are the same. A basic common point of studying economic issues is to describe the economic environment. The more clear and accurate the description of economic environment is, the greater the likelihood of obtaining correct theoretical conclusions.

Characterization of economic environment: When describing the economic environment, a question equally important as clear and accurate description of the economic environment is how to concisely and acutely characterize economic environment in order to capture the essence of a problem. Because most facts and phenomena in reality are not very important or irrelevant to the economic issue to be analyzed and solved, completely objective description of the economic environment is not at all helpful but may even confuse people with trivial details. If we exactly depict all aspects, it surely is very accurate and truthful description of economic environment, but this kind of simple listing only presents numerous confusing facts without capturing key points and the essence of the problem. In order to avoid trivial aspects and focus on the most critical and central issues, we need to characterize economic environment specifically according to the demands of the issue to be studied. For example, when discussing consumer behavior in Chapter 3, we simply describe consumers as a composition of preference relation, consumption space, and initial endowment regardless of gender, age, or wealth of consumers. When discussing the production theory in Chapter 4, a producer can be characterized as the production possibilities set. When studying transitional economy, such as issues concerning China's economic transition, we cannot simply mimic and apply conclusions derived under normal market economic environment but need to characterize basic features of transitional economies, though the basic analytical framework and methodologies of modern economics still can be used.

We often hear people criticize modern economics as useless because it uses a few simple assumptions to plainly summarize complex situations. In fact, this is also the basic research methodology of physics. In the study of the relationship between two physical variables, both theoretical research and experimental operation will fix other variables that will influence the object of study. In many cases, clarifying every aspect (especially unrelated aspect) is not necessary and may even lead to the loss of focus. It is like drawing maps for different goals and purposes as we mentioned before: people need a tourist map for traveling, a traffic map for driving, and a military map for war. These maps only describe certain characteristics of a region, so they are not the whole picture of the real world. Why do people need tourist map, traffic map and military map? The reason is that they are for different purposes. If people depict the entire real world into a map, although it completely describes the objective reality, how can it be a useful map?

Therefore, economics is not only a science but also the art of abstracting and depicting real economic environment. Economics uses concise and profound characterization of economic environment to describe causes of problems, conduct analysis of inherent logic, and thus obtain logical conclusions and inferences. A good economist should be able to accurately grasp the most essential characteristics of the current economic situation in his/her study. Only when we truly make clear the causes and current situation can we solve the specific problems with proper remedies (economic theory adopted). Of course, to do this, one needs to have the basic training in economic theory.

1.10.2 Making Behavioral Assumptions

The second basic component of the analytical framework of modern economics is to make assumptions on the behavior mode of economic agents. This is the key difference between economics and natural science. The assumption is of fatal importance and is the foundation of economics. Whether an economic theory is convincing and has practical value and whether an institutional arrangement or economic policy is conducive to sustainable and rapid economic development mainly depends on whether the individual behavior assumed truly reflects the behavior of most people and whether the institutional arrangement and people's behavior are incentive-compatible, i.e., whether people's reaction to incentive is also beneficial to others or society.

In general, under a given environment and game rules, individuals will make trade-offs according to their behavioral disposition. Thus, when deciding the rules of game, policies, regulations or institutional arrangements, we need to take into account the behavioral pattern of participants and make correct judgments. Like dealing with different people in daily life, we need to know whether they are selfless and honest or not. Different rules of game should be instituted when faced with different participants. When facing an honest person who tends to tell the truth, the way to deal with him or the game rules imposed on him may often be comparatively simple. When facing totally selfless people like Lei Feng (the Chinese national role model for being selfless), the rules to deal with him can be even simpler, for one does not need to take precautions or invest much energy (in designing the rules of game) to deal with him, and the rules may seem not so important. On the contrary, when facing a cunning and dishonest person, the way to deal with him will be quite different and takes much energy so that the rules will be much more complicated. As such, making right judgments on individuals' behavior is a very important step for the study of how people react to incentives and make trade-off choices. When studying economic problems such as economic choice, interaction between economic variables and how they change, it is also important to determine

1.10. BASIC ANALYTICAL FRAMEWORK OF MODERN ECONOMICS 67

the behavioral pattern of economic agents.

As mentioned above, under normal circumstances, a reasonable and realistic assumption about individuals' behavior often used by economists is self-interest assumption, or the stronger rationality assumption, i.e., economic agents pursue the maximization of benefits. Bounded rationality is to make the optimal choice according to the knowledge and information an agent has, which belongs to the category of rationality assumption. In the consumer theory to be discussed later, we assume that consumer pursues the maximization of utility/satisfaction; in the producer theory, we assume that producer pursues the maximization of profits; in game theory, there are various equilibrium solution concepts describing the behavior of economic agents, which are given based on different behavioral assumptions. Any economic agent, in his contact with others, implicitly assumes others' behavior.

The assumption of (bounded) rationality is largely reasonable in some sense. From a practical point of view, as mentioned before, there are three basic kinds of institutional arrangements: institutional arrangement of mandatory regulation (for situations with small operation cost and relatively easy information symmetry), incentive mechanism (for cases of information asymmetry), and social norms (composed of ideology, ideal, morals, customs, etc., that are norms of self-discipline). If people are all selfless and of very high ideological levels, there will be no need for the rigid "stick-style" mandatory regulation that "enlightens with reason" or the flexible market system that "guides with interest". If everyone has no desire, the realization of Communism can be expected soon, but this is extremely unrealistic.

The essential reason for China's implementation of market economy is that under normal circumstances, individuals are self-interested, and market economy is in conformity with the self-interest assumption. This is also the foundation for institutional arrangements that we will then discuss.

1.10.3 Setting Economic Institutional Arrangements

The third basic component of the analytical framework of modern economics is to set up economic institutional arrangements, which are usually referred to as institutions or rules of game. Different strategies or game rules should be taken for different situations, different environments, and individuals with different goals and behavioral patterns. When the situation or environment changes, the strategies or game rules will also change accordingly in most cases. When an economic environment is given, agents need to decide the economic rules of game, which is called economic institutional arrangement in economics. Determination of institutional arrangement is very important for doing anything. Modern economics studies and gives various economic institutional arrangements or economic mechanisms according to different economic environments and behavioral

assumptions. Depending on the issue under discussion, an economic institutional arrangement could be exogenously given (in which case it will degenerate into the institutional environment) or endogenously determined.

As discussed before, there are three basic kinds of institutional arrangements to guide people's behavior: mandatory regulatory governance or government intervention, institutional norm of incentive mechanism, and didactic social norm. The three means play different roles and also have respective applicable ranges and limitations. Didactic social norm relies on the improvement of humanity and lacks constraining force; mandatory regulatory governance or government intervention incurs high information costs, and too much intervention will hurt individual freedom; compared with the other two means, incentive mechanism is the most effective one. This is why economists pay so much attention to institutions.

Therefore, for the enactment of an institutional arrangement of no matter regulatory governance or incentive mechanism, the purpose is not to change the self-interested nature of people but to make use of such unchangeable self-interestedness so as to guide people to do things that will objectively benefit society. Mechanism design should follow the nature of people but not try to change such nature. There is no good or evil in the discussion of people's self-interestedness, and the key lies in how to guide it with institutions. Different institutional arrangements will induce different incentive reactions and trade-off choices and thus may lead to very different results.

Any theory of modern economics involves economic institutional arrangements. Standard modern economics mainly focuses on market system and studies how individuals make trade-off decisions in a market system (such as the consumer theory, producer theory, and general equilibrium theory) and under what economic environments will market equilibrium exist. It also makes value judgments on the result of resource allocation under different market structures (the criterion is based on whether resource allocation is optimal and fair). In these studies, market system is normally assumed to be exogenously given. By so doing we can simplify the problem so as to focus on the study of individuals' economic behavior and how people make trade-offs.

Of course, as the exogeneity assumption of institutional arrangements is not entirely reasonable in many cases, different economic institutional arrangements should be given depending on different economic environments and individuals' behavioral patterns. As shall be discussed in Parts 4 and 5 of the book, there will be market failures (i.e., inefficient allocation of resources and non-existence of market equilibrium) in many situations, so people will need to find an alternative mechanism or a better economic mechanism. In that case, we need to treat institutional arrangement as an endogenous variable which is determined by the economic environment and individual behavior. Thus, economists should give a variety of alter-

native economic institutional arrangements for different purposes.

When studying the economic behavior and choice issues of a specific economic organization, economic institutional arrangements should especially be endogenously determined. New institutional economics, transition economics, modern theory of the firm, especially the economic mechanism design theory, information economics, optimal contract theory and auction theory that have developed in the last few decades, study and give various economic institutional arrangements for a wide range from the state to the family according to different economic environments and behavioral assumptions. Part 5 will give elaborate discussions on the issue of incentive design in economic institutional arrangement.

1.10.4 Determining Equilibrium

The fourth basic component of the analytical framework of modern economics is to make trade-off choices and determine the “best” outcome. Given the economic environment, institutional arrangement (rules of game) and other constraints that should be obeyed, individuals will react to incentives based on their own behavior, weigh and choose an outcome from many available or feasible outcomes. Such an outcome is called equilibrium. In fact, the concept of equilibrium is not hard to understand. It means that among various feasible and available choices, the one being finally chosen is called the equilibrium. Those who are self-interested will choose the best one for themselves; those who are altruistic may choose an outcome that is favorable to others. Thus, the so-called equilibrium, which refers to a state without deviation incentives for all economic agents, is a static concept.

The equilibrium defined above may be the most general definition in economics. It embraces the equilibria in textbooks that are reached by independent decisions under the drive of self-interested motivation and all kinds of technology or budget constraints. For instance, under market system, for the producer, a profit-maximizing production plan under the constraint of production technology is called equilibrium production plan; for the consumer, a utility-maximizing consumption set under the budget constraint is called consumption equilibrium. When producers, consumers, and their interactions reach a state where there is no incentive for deviations, a competitive market equilibrium for each commodity is obtained.

It should be noted that equilibrium is a relative concept. The equilibrium outcome depends on economic environments, participants’ behavior patterns (whether in terms of rationality assumption, bounded rationality assumption, or other behavioral assumptions), and the rules of game by which individuals react to incentives; it is the “best” choice relative to these factors. Note that due to bounded rationality, it may not be the optimal choice in objective reality, but is the “best” one chosen by individuals

according to their preferences and information and knowledge in hand.

1.10.5 Making Evaluations

The fifth basic component of the analytical framework of modern economics is to make evaluations and value judgments on the equilibrium outcome. After making their choices, individuals usually hope to evaluate the equilibrium that then arises and compare it with the ideal “best” outcome (for instance, efficient resource allocation, fair resource allocation, incentive compatibility, informational efficiency, etc.), so as to make further assessments and value judgments on the economic institutional arrangement—whether the economic institutional arrangement adopted has led to certain “optimal” outcome; and test whether the theoretical result is consistent with the empirical reality and whether it can provide correct predictions or practical significance. Finally, they evaluate the economic institution and rules adopted to find out whether there is room for improvement. In short, in order to achieve better results for doing something, after finishing it, we should evaluate the effects, whether it is worth continuing efforts, and whether there is a possibility for improvement. Thus, we need to make evaluations and value judgments on the equilibrium outcome under some economic institutional arrangement and trade-off choice so as to find out the institutions that are best suited to the development of a country.

When making evaluations on an economic mechanism or institutional arrangement, one of the most important criteria adopted in modern economics is whether the institutional arrangement is in line with the principle of efficiency. Surely, as economic environment and individuals’ behavioral patterns in reality keep changing and science and technology are also continuously developing, the exact Pareto optimality may never be truly realized. Just like Newton’s three laws of motion, free fall, and fluid flow without friction in physics, it only is an ideal state and provides the direction of improvement on economic efficiency. As long as the improvement of economic efficiency is desired, individuals will constantly pursue to approach this goal as much as possible. With such an ideal standard as Pareto optimality, we then have the benchmark to compare, measure, and evaluate various economic institutional arrangements in the real world and see how far they are from this ideal goal so as to learn the room of improvement on economic efficiency and make resource allocation approach the Pareto optimality as close as possible.

Nonetheless, Pareto optimality is not the sole criterion, and there is another value judgment called equality or fairness. Market system achieves efficient allocation of resources, but it also faces many problems, such as social injustice caused by huge wealth gap. There is a variety of definitions of equality and fairness. The fair allocation to be introduced in Chapter 12 takes into account both objective equality and subjective factors, and more

importantly, it can achieve fairness and efficiency at the same time. This is the basic conclusion of the Fairness Theorem to be introduced in Chapter 12. Another important criterion for evaluating an economic institutional arrangement is incentive compatibility.

In summary, the five components discussed above constitute the analytical framework underlying almost all standard economic theories, no matter how much mathematics is used, or whether the institutional arrangement is exogenously given or endogenously determined. In the study of economic issues, we should define first the economic environment, and then examine how the self-interested behavior of individuals affects each other under exogenously given or endogenously determined mechanisms. Economists usually take “equilibrium”, “efficiency”, “information” and “incentive compatibility” as key aspects of consideration to observe the effects of different mechanisms on individual behavior and economic organizations, explain how individual behavior achieves equilibrium, and evaluate and compare the equilibrium. Using such a basic analytical framework in the analysis of economic issues is not only compatible in methodology but may also lead to surprising (but logically consistent) conclusions.

1.11 Basic Research Methodology in Modern Economics

We have discussed the five components of the basic analytical framework of modern economics: specifying economic environment, making behavioral assumptions, setting institutional arrangements, determining equilibrium, and making evaluations. Broadly speaking, any economic theory consists of these five aspects. Discussion on the five components naturally leads to the question of how to combine them appropriately according to scientific research methodology, gradually deepen the study of various economic phenomena, and develop new economic theories. This is what we will discuss in this section: the basic research methodology and key points, which include establishing benchmarks, setting reference systems, building studying platforms, developing analytical tools, constructing rigorous models, and conducting positive and normative analyses.

The research methodology of modern economics is to firstly provide basic studying platforms for all levels and aspects and then establish benchmarks and reference system so as to present the criterion to evaluate equilibrium outcome and institutional arrangement. Building studying platform and setting reference system are of great importance to the construction and development of any discipline, and economics is no exception.

1.11.1 Establishing Benchmarks

Evaluation or judgement of anything can only be relative but not absolute, so there should be a benchmark, which also applies to the discussion of economic issues. In economics, **benchmark refers to a relatively ideal state or relatively simple economic environment**. As discussed in the first section of this chapter, to study a realistic economic issue and develop a new theory, we usually need to consider first it under a friction-free ideal economic environment to develop a rather simple result or theory. Then, we discuss the result in a non-ideal economic environment with frictions, which is closer to reality, develop a more general theory, and compare it with that developed under the benchmark situation.

In this sense, benchmark is relative to non-ideal economic environments and new theories to be developed that are closer to reality. For instance, complete information assumption is the benchmark for the study of incomplete information. When studying economic issues under information asymmetry, we need to fully understand first the situation of complete information (highly unrealistic though it is). Only when we are thoroughly clear about the situation of complete information can we study well economic issues taking place under the circumstance of incomplete information. So is the case with theoretical research in economics. We start from the ideal state or simple scenarios before considering more realistic or general scenarios; we learn from others' research results before innovating on the existing theories. New theories are always developed on the basis of prior research findings and results. It's like that Newton's mechanics makes possible Einstein's theory of relativity, while the theory of relativity makes possible the non-conservation of parity put forward by Chen-Ning Franklin Yang and Tsung-Dao Lee.

1.11.2 Setting Reference Systems

Reference system refers to economic models and theories generated in an ideal situation, such as the general equilibrium theory, that is, a perfectly competitive market will lead to efficient resource allocation. Setting reference system is of great importance to the construction and development of any discipline, including economics. Although economic theories set as the reference system may include many assumptions that do not accord with reality, at least they can help in: (1) simplifying the issue and capturing its characteristics directly; (2) establishing the measurement criterion for evaluating theoretical models, understanding the reality, and making improvements; and (3) theoretical innovations and further analyses on such basis.

Although the economic theories as the reference system might have many unrealistic assumptions, they are very useful and can serve as the

1.11. BASIC RESEARCH METHODOLOGY IN MODERN ECONOMICS 73

reference for further analysis. This is similar to our practice of setting role models in life. The importance of the reference system does not lie in whether it accurately describes the reality or not, but in establishing the measurement for a better understanding of the reality. Like a mirror, it helps reveal the gap between theoretical models or realistic economic institutions and the ideal state. It is essentially important in the sense that it points out the direction of efforts and adjustments and the extent of adjustments. If a person has no target and is unaware of the gap and rough direction of making efforts, how can he make improvements or be motivated to do anything? Not to mention achieving his goal.

General equilibrium theory is such a reference system. As we know that a perfectly competitive market will lead to efficient allocation of resources, although there is no such market in reality, if we make efforts toward that direction, efficiency will be enhanced. That is why we have institutional arrangements such as the anti-trust law to protect market competition. By virtue of the reference system with perfect competition and complete information as the benchmark, we can study what results can be obtained from economic institutional arrangements that are closer to the reality (such as some monopolistic or transitional economic institutional arrangements) where assumptions in the general equilibrium theory are not valid (incomplete information, imperfect competition, externalities), and compare them with the results obtained from the general equilibrium theory in the ideal state. In so doing, we will know whether an economic institutional arrangement (be it theoretical or realistic) is efficient in allocating resources and utilizing information, and how far away the economic institutional arrangement adopted in reality is from the ideal situation. Based on this, we can make policy suggestions accordingly. In this sense, general equilibrium theory also serves as a criterion for evaluating institutional arrangements and the corresponding economic policies in reality.

Just like a knife, however sharp, will not work if its holder does not cut in the right direction; a person, however smart, might go nowhere if he is unclear of his target and direction. Say we set Lei Feng (the Chinese national role model for being selfless) as the reference system and the ideal example for people's conduct. Though there is no living Lei Feng today, people still should be encouraged to learn from him, because even if we may only do one percent as he did, it will be much better than doing nothing at all. Therefore, we should have lofty ideals, which let us know the direction and goals of making efforts. Even if we cannot get there eventually, we can still be inspired to continuously approach the ideal.

1.11.3 Building Studying Platforms

A studying platform in modern economics consists of some basic economic theories and methods and also provides the basis for deeper analysis. The

methodology of modern economics is very similar to that of physics, i.e., simplifying the issue first to capture the core essence of the issue. In case that many factors breed an economic phenomenon, we need to make clear the impact of every factor. This can be done by studying the effect of one factor at a time while assuming other factors remain unchanged. The theoretical foundation of modern economics is modern microeconomics, and the most fundamental theory in microeconomics is individual choice theory to be discussed later – consumer theory and producer theory, which are the basic studying platform and cornerstone of modern economics. This is why almost all textbooks of modern economics start from the discussion of consumer theory and firm theory. They provide the fundamental theories explaining how agents make choices as consumers and firms and establish the studying platform for further study of individual choice.

Generally speaking, the equilibrium choice of an individual depends not only on one's own choice, but also on others' choices. In order to study individual choice, we need to clearly understand how an individual make his choice in the absence of influence by other agents. The consumer theory and producer theory are developed through this approach. It is assumed that economic agents are in the institutional arrangement of a perfectly competitive market. Therefore, every agent will take price as a given parameter, and individual choice will not be affected by others' choices. The optimal choice is determined by subjective factors (such as the pursuit of maximum utility or profit) and objective factors (such as budget line or production constraints).

Many people find this research method bewildering. Thinking that this kind of simple situation is too far away from reality and the assumption in the theory seems too unpractical, they doubt whether the theory is of any use. Actually, this sort of criticism indicates that they have not truly understood this scientific research methodology. This methodology that involves simplification and idealization of the question sets a basic platform for deepening research. It is like the approach in physics: in order to study a problem, we go to the essence first, start with the simplest situation that excludes frictions from consideration, and then gradually deepen the research and consider the more general and complicated cases. In microeconomics, theories of market structures like monopoly, oligopoly and monopolistic competition are generated from more general cases, where producers can influence each other. To study the choice issue under the more general situation where economic agents can influence each other's decision-making, economists have developed a very powerful analytical tool – game theory.

General equilibrium theory is a more sophisticated studying platform based on consumer theory and producer theory. Consumer theory and producer theory provide a fundamental platform for studying individual choice problems, while general equilibrium theory provides a fundamental

platform for analyzing how to reach market equilibrium through the interaction of all goods in the market. The mechanism design theory, which has developed for the past 40 years, provides an even higher-level platform for studying, designing and comparing various institutional arrangements and economic mechanisms (whether it is public ownership, private ownership, or mixed ownership). It can be used to study and prove the optimality and uniqueness of perfectly competitive market mechanism in resource allocation and information utilization, and more importantly, in case of market failures, it offers methods of how to design alternative mechanisms. Under some regularity conditions, the institutional arrangement of perfectly competitive market free of externalities will not only lead to efficient allocation of resources, but also prove the most efficient in terms of information utilization (mechanism operation cost, transaction cost) as it requires the least amount of information. Under other circumstances of market failure, we need to design a variety of alternative mechanisms for different economic environments. Furthermore, the studying platform also creates conditions for providing reference systems for evaluating various kinds of economic institutional arrangements. In other words, it provides a criterion for measuring the gap between reality and the ideal state.

1.11.4 Developing Analytical Tools

For the research of economic phenomena and economic behavior, we also need various analytical tools besides the analytical framework, benchmark, reference system, and studying platform. Modern economics requires not only qualitative analysis but also quantitative analysis to define the boundary condition for each theory to be true so that the theory will not be misused. Thus, a series of powerful analytical tools should be provided, which are usually given as mathematical models or graphics. The power of these tools lies in their ability to help us deeply analyze the intricate economic phenomena and economic behavior through a simple and clear diagram or mathematical structure. Examples include the demand-supply curve, game theory, principal-agent theory for studying information asymmetry, Paul A. Samuelson's (1915-2009, see 3.11.2 for his biography) overlapping generations model, dynamic optimality theory, etc. Of course, there are exceptions which are not expressed with analytical tools. For instance, the Coase theorem is established and demonstrated through words and basic logical deduction only.

1.11.5 Constructing Rigorous Analytical Models

Logical and rigorous theoretical analysis is needed when we explain economic phenomena or economic behavior and make conclusions or economic inferences. As mentioned above, each theory holds true under certain

conditions. Modern economics requires not only qualitative analysis but also quantitative analysis to define the boundary condition for each theory to be true so that the theory would not be misused, just like medication and pharmacology where we have to be clear about the application and functions of medicines. We need to establish rigorous analytical models identifying clearly the conditions under which a theory holds true. Lack of related mathematical knowledge will make it difficult to have an accurate understanding of the connotation of a definition, or to have discussions on related issues, not to mention defining boundary conditions or constraints for the research. Therefore, it's not surprising that mathematics and mathematical statistics are used as basic analytical tools, and they are also among the most important research methods in modern economics.

1.11.6 Positive Analysis and Normative Analysis

As per the research method, economic analysis can be divided into two types, one being positive or descriptive analysis, the other being normative or value analysis. Another major difference between economics and natural science is that the latter only makes positive analysis while the former involves both positive and normative analysis.

Positive analysis only explains how economy operates. It only gives facts and provides explanations (thus verifiable), but does not make value assessment of economic phenomena or offers means of revision. For example, an important task of modern economics is to describe, compare and analyze such phenomena as production, consumption, unemployment and price, and to predict possible outcomes of different policies. Consumer theory, producer theory, and game theory are typical examples of positive analysis.

Normative analysis makes judgments on economic phenomena. It not only explains how economy operates, but also attempts to find out means of revision. Therefore, it always involves value judgments and preferences of the economists and is thus not verifiable through facts. For instance, some economists lay more emphasis on economic benefits while others focus on equality or social justice. With the differences between the two methods in mind, we can avoid many disputes while discussing economic issues. Economic mechanism design theory is a typical example of normative analysis.

Positive analysis is the foundation for normative analysis, while normative analysis is the extension of positive analysis. In this sense, the foremost task of economics is to make positive analysis, and then normative analysis follows. General equilibrium theory includes both the former (such as the existence, stability and uniqueness of competitive equilibrium) and the latter (the first and second theorems of welfare economics).

1.12 Practical Role of the Analytic Framework and Methodologies

The most basic analytic framework and methodologies of modern economics that have been discussed are of practical uses. Even though these analytical framework and methodologies seem simple, it is actually difficult to comprehend and use them in your life, study, and research. However, once the knowledge is mastered, you will get lifelong benefits. They can make you smart, wise, profound, and able to think scientifically. They can help you to study the highbrow pure economic theories and also provide advices to solve practical issues in your life and work.

First of all, from the aspect of studying modern economics, if analytic framework and methodologies are mastered, you will not be confused by abstract models and abstruse mathematics. Because no matter how profound mathematics, how many formulae, and how complicated economic models are used in an economic theory, it still uses the above analytical framework and methodologies. If the basic analytical framework and methodologies are grasped and born in your mind as the main thread, you will not lose the direction or the focus and will basically understand its general idea. Therefore, you can temporarily put aside incomprehensible technical details and get the framework and conclusion straightened out first; only then we manage to comprehend the details. That is to say, you should grasp first the main point and general idea, know its objectives and conclusions, and then come to the details. Moreover, once you master the basic analytic framework and methodologies, you can have a correct idea of modern economics and may not be misled in the study of modern economics. The reason why people often criticize modern economics and its methodologies is that their judgements are mostly not based on methods of scientific analysis and may even only rely on their subjective assumptions. If you do not understand the basic analytic framework and methodologies, it's possible that their judgments will mislead you, make you lose the right direction of studying modern economics, and even overlook and resist the study of modern economics.

Second, in the aspect of modern economic research, once you understand and comprehend the basic analytical framework and methodologies, you will be more qualified to do the research. For many people who want to do economic research, even though they have understood the modern economics quite well and have read quite a number of related papers, they still find it hard to do research. They do not know how to do research, or their research findings are not significant or widely recognized. Actually, it wo not be so difficult to do research on economics as long as they understand the basic analytic framework and methodologies and have basic mathematic knowledge and the ability of logical analysis. Doing research,

to some extent, is step-by-step writing with inherent logic according to the five components of the basic analytical framework. The basic analytic framework and methodologies can help you improve your research and innovative abilities.

For example, if you are interested in a specific economic issue or phenomenon, or you want to put forward a new theory with stronger explanatory power to guide the resolution of practical economic issues, you will need to reasonably and precisely describe and depict the economic environment and economic man's behavior, employ existing analytical tools or develop new ones to build a model as simple as possible, and then do the deduction and proof. If you only want to extend and improve the original theory, you should analyze whether original assumptions about economic environment and behavior and models fit the reality and whether the assumptions can be relaxed to derive new or more general conclusions. For junior researchers, it can be easier to do the work of extensions and improvements, and may also be easier for your results to be accepted and published. Of course, you can also modify the definition of economic environment or other components so as to get a different or even more important conclusion. It is how the school of macroeconomics and many theories under the condition of information asymmetry were developed. If you are going to criticize a modern economic theory, you should criticize which parts of its analytic framework are unreasonable, illogical, or unrealistic and in what way rather than criticizing modern economics and its methodologies as a whole. Therefore, for those who criticize modern economics, totally deny it, want to abandon it and establish their own theories of economics, here is a suggestion that they may try first to understand the basic analytic framework and methodologies of modern economics, and, on such basis, then consider how to criticize certain theories of modern economics so as to be cautious enough in their words without misleading the public.

Finally, understanding modern economics, its methodologies, and analytical framework may help us do better thinking and deal with everyday concerns and people in a better way. It can make you more thoughtful, insightful, and capable in work. It is often heard that modern economics is highbrow and metaphysical since it involves so much mathematics and so much difficulty to learn but distance the reality hundreds of thousands of miles away; what will it be used for? Actually, the basic approach of dealing with people and things in daily life is similar to the basic framework in economic analysis. For example, when you are in a new place preparing to do something or teaming up with people, the first thing you need to do is acquainting yourself with the local environment and condition, which is similar to 'specifying economic environment' in the framework; then, you need to know the local culture and custom, the behavioral pattern and personality of your counterpart, etc., similar to 'making behavioral assumptions'; taken all the information together, you can decide your rules

of dealing with people, which is similar to 'setting economic institutional arrangements'; the next step is to choose the optimal scheme by trading off among feasible options, which is similar to the 'determining equilibrium'; the last step is to summarize and reflect your decisions, actions, and your ways of dealing with things to see whether they are the most effective approach, whether they achieve the best outcome, whether they are fair and reasonable, whether people's enthusiasm is aroused, whether people have reactions to the incentive, and whether the intended goal of incentive compatibility is achieved, which is similar to 'making evaluations'. Besides, when the environment and condition are changed, or the subject you are working with is changed, the rule should also be changed accordingly. If you can do according to the five aspects and adjust the rules with changes of the condition, better results can surely come about. This may be one of the best ways to deal with daily life and work. Moreover, some conclusions of economic theories can also help you to think and solve problems.

1.13 Basic Requirements for Mastering Modern Economics

There are three basic requirements for learning and mastering modern economic theory:

1. The basic requirement is to master the basic concepts and definitions, which is a reflection of logical thinking and clear mind. This is the prerequisite not only for discussion and analysis of questions and logically inherent analysis, but also for a good command of economics. Otherwise, different definitions of terms may cause great confusions and thus lead to unnecessary disputes.

2. One should be able to clearly state all theorems or propositions and also be clear about the basic conclusions and their conditions. Otherwise, a small error in application of economic theories to the analysis of issues will lead to serious mistakes. Just like there is a scope of application for medicines, any theory or institution also has its applicable boundaries. If we go beyond that boundary, problems are liable to arise, bringing about tremendous negative externalities. In many cases, social or economic problems occur just because economists misuse some theories without a good understanding of their boundaries and applicable conditions. In this sense, an eligible economist is like a qualified doctor who needs to know well the properties and efficacy of different kinds of medicines when prescribing for his patients.

3. One has to grasp how the basic theorems or propositions are proved (ideas and processes). An excellent economist, like a good doctor, should know what the matter is and why it is so, and also have medicine properties and pathology both in mind. Then, he can gain a deeper understanding

and a better command of the theories he has learned.

If you meet these requirements, it would be pretty easy to refresh your memory even if you forget the proof of some conclusions. Economics cannot be experimented for most cases and relies mainly upon the analysis of inherent logic, which is also the power of modern economic theory. That is why it is important to accurately grasp the theories and their application scopes.

1.14 Distinguishing Sufficient and Necessary Conditions

When discussing economic issues, it is very important to distinguish between sufficient conditions and necessary conditions, which can help people think clearly and avoid unnecessary debates. A necessary condition is a condition that is indispensable in order for a proposition to be true. A sufficient condition is a condition that guarantees the proposition to be true. For instance, some people often negate market economy based on the example of India, which adopts market economy but remains poor, so they argue that China should not embark on the path of market economy. These people do not realize the difference between sufficient and necessary conditions: the adoption of market economy is a necessary condition rather than a sufficient condition for a country to become prosperous and strong. In other words, if a country wants to be prosperous, it must adopt market economy. This is because one cannot find any wealthy country in the world that is not a market economy. However, market mechanism is just a necessary condition but not a sufficient condition for prosperity. We must also admit that market mechanism does not necessarily lead to national prosperity.

As discussed before, there is a distinction between good market economy and bad market economy, the reason of which is that although (based on observation of reality) market mechanism is indispensable for national prosperity, there are many other factors that also affect the prosperity of a nation, such as the degree of government intervention, political system, law, religion, culture, and social structure, thus making market mechanism be labeled as good or bad.

1.15 The Role of Mathematics and Statistics in Modern Economics

Mathematics and statistics are of extreme significance for people to have a good knowledge of the nature and to manage daily affairs. As the well-known statistician, C. Radhakrishna Rao pointed out that mathematics is

1.15. THE ROLE OF MATHEMATICS AND STATISTICS IN MODERN ECONOMICS 81

a kind of logic to deduce results on a given premise, while statistics is a rational method acquired through experience and a kind of logic to verify the premise with a given result. Rao believes that "All knowledge is, in final analysis, history. All sciences are, in the abstract sense, mathematics. All judgements are, in their rationale, statistics."⁹ His saying profoundly depicts the significance of mathematics and statistics and their respective connotations.

So mathematics and statistics are also important in modern economics. Almost every field in modern economics uses a lot of mathematics, statistics and econometrics. The width and depth of mathematics involved even exceed those in physical science. The reasons for this include: the fact that modern economics is increasingly becoming a science, the use of mathematical analytical tools, and the more complex and influential social systems. Thus, when considering and studying economic issues, we need to make logically inherent analyses by rigorous theoretical analytical models and conduct empirical test by quantitative analysis methods, and meanwhile clarify and determine the boundary condition for a theoretical conclusion to be true. Hence, it is not surprising that mathematics and mathematical statistics are used as the basic analytical tools and also become the most important analytical tools in modern economics. People who study and conduct research on modern economics must have a good knowledge of mathematics and mathematical statistics.

Modern Economics mainly adopts mathematical language to make assumptions on economic environments and individual behavior patterns, uses mathematical expressions to illustrate logical relations between economic variables and economic rules, builds mathematical models to study economic issues, and finally follows the logic of mathematical language to deduce conclusions. Without the related mathematical knowledge, it is hard to grasp the essence of concepts and discuss related issues, let alone conduct research and figure out the necessary boundary or constraints when giving conclusions. Therefore, it is of great necessity to master sufficient mathematical knowledge if you want to learn modern economics well, engage in research on modern economics, and become a good economist.

Many people with little knowledge of mathematics are unable to master the basic theories and analytical tools of modern economics or understand advanced economic textbooks or papers. Thus, they deny the function of mathematics in economic studies with excuses such as it is important to produce economic thoughts, or mathematics is far away from practical economic issues. No one could deny the importance of economic thoughts for they are the output of research. But, could we pursue academics with deep thoughts and also for deep thoughts? Just as Thomas A. Edison said,

⁹C. R. Rao, *Statistics and Truth: Putting Chance to Work*, Beijing: Science Press, 2004.

“Genius is the output of one percent inspiration and ninety-nine percent perspiration.” Without the tool of mathematics, how could we figure out boundary conditions and the applicable scope of economic thoughts or conclusions? Without knowledge of the conditions and scope, how could we defend against abuse or misuse of economic thoughts or conclusions? How many people could develop such profound economic thoughts without using mathematical models as Adam Smith and Ronald H. Coase did? Even so, economists have never stopped studying what conditions are required for their conclusions to be true. Besides, as the time in which we are living is different now, modern economics has become a quite rigorous discipline in social sciences. Without strict arguments, the thoughts or conclusions could not be widely recognized. It is indeed so. As mentioned above, the economic thoughts presented by philosophers in ancient China like JIANG Shang, LAO Tzu, SUN Tzu, GUAN Zhong and SIMA Qian are extremely profound, and some ideas of Adam Smith had already been mentioned by these philosophers long ago. However, their thoughts have never been known to the outside world because they were just conclusions of experience that did not form a scientific system nor involved logically inherent analyses using scientific methods.

There is another misunderstanding that research of economic issues with mathematics is remote from reality. Actually, most of the mathematical knowledge is developed on the basis of practical demands. People who have basic knowledge of physics, physical science history or history of mathematical thought will know that both primary and advanced mathematics are originated from the demands of scientific development and reality. As such, why can not mathematics be used to study practical economic issues? As a philosopher and mathematician, Karl Marx had used the most advanced mathematics of his time and written the book *Mathematical Manuscripts*. In his *Capital*, he had also used many statistical and mathematical methods. Though a lot of mathematics will be involved in the book, issues to be discussed are all from the real world, so they are still very practical and instructive. So, the foundation of mathematics and modern economics is very important for one to be a good economist. If you know mathematics well and master the fundamental analytical framework and research methodology of modern economics, you could learn modern economics more easily and greatly enhance study efficiency.

The functions of mathematics in the theoretical analysis of modern economics are as follows: (1) It makes the language more precise and concise and the statement of assumptions more clear, which can reduce many unnecessary debates resulting from ambiguous definitions. (2) It makes the analytical logic more rigorous and makes it possible to clearly state the boundary, application scope and conditions for a conclusion to hold true. Otherwise, the abuse of a theory may occur. For example, when discussing the issue of property right, many people would quote Coase

theorem, thinking that as long as the transaction cost is zero, there will be efficient allocation of resources. Up to now, there are still many people (including Coase himself when giving his argument) who do not know that this conclusion is normally not true if the utility (payment) function is not quasi-linear. (3) Mathematics can help obtain the results that cannot be easily attained through intuition. For example, from intuition, according to the law of supply and demand, competitive markets will achieve market equilibrium through the adjustment of market prices by the "invisible hand" as long as the supply and the demand are not equal. Yet this conclusion is not always true. Scarf (1960) gave a counterexample of market instability to prove that this result may not be true in some cases. (4) It helps improve and extend the existing economic theory. Examples in this respect are plenty in the study of economic theory. For instance, economic mechanism design theory is an improvement and extension of general equilibrium theory.

Qualitative theoretical analysis and quantitative empirical analysis are both needed for studying economic issues. Economic statistics and econometrics play an important role in these analyses. Economic statistics focuses more on data collection, description, sorting and providing statistical methods, whereas econometrics focuses more on testing economic theory, evaluating economic policy, making economic forecasts and identifying the causal relationship between economic variables. In order to better evaluate economic models and make more accurate predictions, theoretical econometricians have been continuously developing increasingly powerful econometric tools.

It is, however, noteworthy that economics is not mathematics. Mathematics in economics is only used as a tool to consider or study an economic behavior or phenomenon. Economists just employ mathematics to express their opinions and theories more rigorously and concisely and to analyze the interdependent relationship among economic variables. With the metrization of economics and the precision of various presupposed assumptions, economics has become a social science with a rigorous system.

However, a good knowledge of mathematics does not make a good economist. It also requires a deep understanding of the analytical framework and research methodologies of modern economics and a good intuition and insight of real economic environments and economic issues. The study of economics not only calls for the understanding of some terms, concepts and results from the perspective of mathematics (including geometry), but more importantly, even when those are given by mathematical language or geometric figures, we should try the best to get clear their economic meanings and the underlying economic thoughts. Thus we should avoid being confused by the mathematical formulas or symbols in the study of economics. So we say that, to become a good economist, one needs to pursue academics with deep thoughts and for deep thoughts.

1.16 Conversion between Economic and Mathematical Languages

The product of economic research is economic inferences and conclusions. A standard economics paper usually consists of three parts: (1) it raises questions, states the significance, and identifies the research objective; (2) it establishes economic models and rigorously expresses and proves the inferences; (3) it uses non-technical language to explain the conclusions and provides policy suggestions. That is to say, an economic conclusion is usually obtained through the following three stages: non-mathematical language stage - mathematical language stage - non-mathematical language stage. The first stage proposes economic ideas, concepts or conjectures, which may stem from economic intuition or historical and foreign experience. As they have not been proved by theories yet, they can be regarded as primary products of general production. The first stage is very important because it is the origin of theoretical research and innovation.

The second stage verifies whether the proposed economic ideas or conjectures hold true or not. The verification requires economists to give formal and rigorous proofs through economic models and analytical tools, and if possible, to test them with real empirical data. The conclusions and inferences obtained are usually expressed in mathematical language or technical terms, which may not be understandable to non-experts. Therefore, they may not be adopted by the public, government officials or policy makers. Thus, these conclusions expressed by technical language can be regarded as intermediate products of general production.

Economic studies should serve the real economic society. Therefore, the third stage is to express the conclusions and inferences by common language rather than technical language, making them more understandable to the general public. Policy implications and profound meanings of conclusions and insightful inferences conveyed through non-technical language will be final products of economics. It is notable that in both the first and the third stages, economic ideas and conclusions are presented by common, non-technical and non-mathematical language, but the third phase is a kind of enhancement of the first phase. As a matter of fact, the three-stage form of common language - technical language - common language is a normal research method widely adopted by many disciplines.

1.17 Biographies

1.17.1 Adam Smith

Adam Smith (1723-1790), the primary founder of economics, is well-known as the father of modern economics. Adam Smith finished his study of Latin,

Greek, mathematics, and ethics in the University of Glasgow in Britain. After that, he worked in the University of Glasgow as a professor in Logic and Moral Philosophy, and even once as the honorary position of Lord Rector. *The Wealth of Nations*, published in 1776, is Smith's most influential work and also a great contribution to the establishment of economics as an independent discipline. In western countries, this book even is the most influential work among all publications in the field of economics. His main academic thought was affected by Bernard de Mandeville (1670-1731), Francis Hutcheson (1694-1764), David Hume (1711-1776), J. Vanderlint (year of birth unknown, died in 1740), George Berkeley (1685-1753), and so on.

Smith suggested that the economic development of human society was the outcome of spontaneous action of tens of millions of individuals whose behavior followed the power of instinct and was driven by their self-interested nature. Smith regarded this power as the 'invisible hand', which is also his idea of allowing the rule of market to work in the organization of economic society. *The Wealth of Nations* denied the attention to land in physiocracy but valued labor as the most important and believed that the division of labor could increase the efficiency of production. Thomas Robert Malthus and David Ricardo focused on summarizing Smith's theories into a theory known as the classical economics in the twentieth century (where modern economics thus originated). Malthus further extended Smith's theories to the problem of surplus of population. Ricardo, on the other hand, put forward the iron law of wages, suggesting that the surplus of population may lead to the consequence that even workers' livelihood cannot be guaranteed. Smith assumed that the increase in wages would accompany the increase in production, which seems more correct from today's perspective. Theories involved in his book not only set up the division of labor theory, but also pioneered in areas including monetary theory, theory of value, theory of distribution, capital accumulation theory, theory of taxation, and so on. In addition, Marx's labor theory of value built upon the basis of Ricardo's political economy also received indirect influence from Smith's theory.

Before the foundational work *The Wealth of Nations*, Smith also wrote *The Theory of Moral Sentiments* (first published in 1759). In this book, he mainly argued that people should have sympathy and sense of justice, which may happen under unusual circumstances (for example, when people are in trouble or when the country is invaded). He tried hard to prove how self-interested individuals controlled their emotions and behavior, especially their selfish sentiment and behavior, so that incentive compatibility between social interest and self-interest could be achieved. The system of economic theory built by Smith in *The Wealth of Nations* is based on his arguments in *The Theory of Moral Sentiments*. Smith worked on the two works at the same time and modified them repeatedly until death. They became two organic and complementary components in his academic think-

ing system. *The Theory of Moral Sentiments* states the problem of morality, while *The Wealth of Nations* states the problem of economic development. Smith regarded *The Wealth of Nations* as the continuation of his thought in *The Theory of Moral Sentiments*. The two books differ in mood, scope of discussion, structure arrangement, and emphasis; for example, the control for self-interested behavior in *The Theory of Moral Sentiments* relied on the sympathy and sense of justice, while in *The Wealth of Nations*, it relied on competitive mechanism. Nonetheless, their discussions about motivation of self-interested behavior were the same in essence. It can be suggested that individuals' sympathy, sense of justice, and pursuit of self-interest are just different reactions to different circumstances (unusual or usual). In *The Theory of Moral Sentiments*, Smith saw "sympathy" as the core of judgment, but when it is regarded as the motivation of individual's behavior, it will turn out totally different result.

1.17.2 David Ricardo

David Ricardo (1772-1823), the representative of the Classical Political Economy, integrated Smith's theory into classical economics together with Thomas Robert Malthus (1766-1834). Ricardo was born to a Jewish family and a father who was a stock broker. He went to a business school in the Netherlands at 12 and worked on stock exchange with his father at 14. He was engaged in stock exchange independently in 1793, and by the time he was 25, he had already owned a wealth of two million pounds. After that, he began to study mathematics and physics. When he first read Adam Smith's *The Wealth of Nations* in 1799, he began to study economic issues. At the age of 37, he published his first paper in economics and then went very smooth in this field. During the 14 years of his short academic life, he left numerous of works, papers, notes, letters, and speeches. Among those, the most famous one is the *Principles of Political Economy and Taxation* published in 1817. Ricardo was rather self-conceited. He said that his point of view was different from Smith and Malthus who enjoyed great prestige then, and, in Britain, there would be less than 25 people who could understand his book. In 1819, Ricardo was elected as a member of parliament.

Starting with Bentham's version of utilitarianism, Ricardo established a theoretical system based on the labor theory of value and centered on the theory of distribution. He inherited scientific elements in Smith's theories and insisted in the principle stating that the value of a commodity was decided by the labor cost in it. He also criticized the mistake in Smith's theory of value by putting forward that labor that could decide value was socially necessary labor. In addition, labor that could decide the value of a commodity were not only direct living labor but also labor in factors of production. Ricardo suggested that all value was produced by labor and was distributed among three classes: wages that were decided by the value

of essential means of subsistence of workers; profit was the surplus when wages were deducted; and land rent was the surplus when wages and profit were deducted.

Based on the the labor theory of value, Ricardo established the theory of comparative advantage. In *On the Principles of Political Economy and Taxation*, he clearly stated that “The value of a commodity, or the quantity of any other commodity for which it will exchange, depends on the relative quantity of labour which is necessary for its production”. He further stated that “the exchangeable value of these commodities, or the rule which determines how much of one shall be given in exchange for another, depends almost exclusively on the comparative quantity of labour expended on each”. The profit of each party in international trade is also totally related to the exchangeable value of all commodities in the international market, namely, the relative price level. Ricardo regarded the free flow of factors of production, such as capital and labor, among regions and industries within a country as the fundamental reason of equalized rate of profit. The flow of factors between countries, however, would inevitably be interrupted by force or even totally stopped due to various reasons. Ricardo concluded that, it was the immobility of factors between countries that decided “the same rule which regulates the relative value of commodities in one country, does not regulate the relative value of the commodities exchanged between two or more countries”. Since there are numerous reasons for different relative prices of one commodity in different countries, there is room of profit for all the participating countries in the international trade. The premise of it, however, is that each country knows its advantage compared with others; that is to say, they are sure about their own comparative advantages.

1.18 Exercises

Exercise 1.1 (*Economics and three elements of scientific economic analysis*) Answer the following questions:

1. What is the definition of Economics?
2. What are the two great objective realities facing the study of economic problems?
3. What is modern economics?
4. Why should scientific economic analysis consist of the three elements of economic theories, statistics, and history?

Exercise 1.2 (*Differences between Economics and natural science*) Answer the following questions:

1. What are the main differences between Economics and natural science?

2. Why these differences make economic research more complex and difficult?

Exercise 1.3 (Two basic categories of economic theory) Answer following questions:

1. Which two categories of economic theories can be classified according to their functions?
2. Please describe the connotation and function of each category, as well as the relationship between these two categories.
3. How should we correctly regard and deal with the interaction between these two kinds of economic theory?

Exercise 1.4 (The fundamental functions of economic theory) Answer the following questions:

1. Please write down the three main functions of economic theory.
2. Why is there not a best kind of economic theory that is always right and fits every development stage but a kind that fits certain institutional environment the best?
3. What are two common misunderstandings of economic theories?

Exercise 1.5 (Market and market mechanism) Answer the following questions:

1. From the perspectives of information and incentive, what are the advantages of the market economic system compared with the planned economic system?
2. Under the condition of market economy, what are three basic functions of price?
3. What is the superiority of market system?
4. What are the three development stages an economy will go through? How can the efficiency-driven and even innovation-motivated development be realized? What is the basic economic institution behind this?

Exercise 1.6 (The dialectical relationship between competition and monopoly) Answer the following questions:

1. Why do people want to introduce competitive mechanism in the view of social resource allocation, but enterprises want monopoly? Please state the dialectical relationship between competition and monopoly.
2. What does the Innovation Theory of Schumpeter tell us? Please state the importance of innovation-driven development.

Exercise 1.7 (*The boundaries between government and market and society*) Answer the following questions:

1. Why is it necessary to reasonably define the boundaries between government and market and between government and society?
2. How should the boundaries between government and market and between government and society be roughly defined?
3. Why does a well-governed country need to reasonably define not only the boundaries between government and market but also those between government and society?

Exercise 1.8 (*The three basic institutional arrangements for state governance*) Answer the following questions:

1. What are three basic institutional arrangements for state governance?
2. Please state the range of application and limitation of the three basic institutional arrangements. Which is the most basic and important one?

Exercise 1.9 (*The logic of development and governance*) Answer the following questions:

1. How shall we correctly understand the logic of development and governance and dialectical relationship between the two?
2. Please explain the achievements and limitations of the economic reform in China according to this framework.

Exercise 1.10 (*Ancient Chinese Economic Thought*) Answer the following questions:

1. Please give five examples to state the thought of market economy in ancient China.
2. Why could not those numerous deep economic thoughts in ancient China form a scientific economic system?

Exercise 1.11 (*The cornerstone assumptions of modern economics*) Answer the following questions:

1. Please state the relationship and distinction between self-love, selfishness, and self-interestedness?
2. Why does economics use the self-interest assumption as the most basic, important, and central assumption?
3. How shall we regard self-interestedness and altruism?

Exercise 1.12 (*Key points in modern economics*) Answer the following questions:

1. What are the key points of modern economics? Please state each of them generally.
2. Please state the meanings, advantages, and limitations of centralized and decentralized decision-making.
3. Why are economic freedom and competition extremely important to economic development?
4. Why doing things under constraints is one of the most fundamental principles in economics?
5. What is the relationship between incentive and information?
6. Why are clearly defined property rights important to efficient allocation of resources?
7. Please discuss the differences between outcome fairness and equity in opportunity. Which one does not conflict with efficiency?

Exercise 1.13 (*Proper understanding of modern economics*) Answer the following questions:

1. How shall we regard the scientific nature of modern economics?
2. How shall we regard the mathematical nature of modern economics?
3. How shall we regard the economic theory correctly?
4. How shall we regard the critics that economics cannot be tested through experiment?

Exercise 1.14 (*Basic analytical framework of modern economics*) Answer the following questions:

1. What are the components that constitute the basic analytical framework of a standard modern economic theory?
2. Why do different economic environments need different economic theories?
3. Why are different economic theories needed even for the same economic reality or environment under many circumstances?
4. Why should evaluation be included in the analytical framework?
5. Taking Coase Theorem as an example, please expound on its economic analytical framework.

6. *What are practical usages of the basic analytical framework and research methodologies of modern economics?*

Exercise 1.15 (*Benchmarks and reference system*) Answer the following questions:

1. *What are definitions of benchmark and reference system?*
2. *Why are establishing benchmarks and setting reference system the premise of discussing economic problems?*
3. *What are the typical examples of reference systems?*

Exercise 1.16 (*Methodologies*) Answer the following questions:

1. *When considering the problem of economic reform, why is it important to distinguish necessary condition from sufficient condition?*
2. *Why does it need both empirical and normative analysis when discussing economic problems?*
3. *How shall we regard the role of mathematics and statistics in modern economics?*
4. *How shall we complete the conversion between economic and mathematical languages?*

1.19 References

Books and Monographs:

- Shleifer, A. and Vishny, R.W. (2004). *The Grabbing Hand: Government Pathologies and their Cures*. Chinese version, translated by Zhao, Hongjun. Beijing: Citic Press.
- Hayek, F. (2000). *Law, Legislation and Liberty (Volume II and III)*. Chinese version, translated by Deng, Zhenglai. et al. Beijing: Encyclopedia of China Publishing House.
- Kornai, J. (2007). *The Socialist System: The Political Economy of Communism*. Chinese version, translated by Zhang, An. Beijing: Central Compilation and Translation Press.

- Tian, G. (2010). *From the Wealth of Nations to the Wealth of People: A Transition from Development-oriented Government to Public-service-oriented Government*. Preface to *The Wealth of People* by Wang, Yijiang. Beijing: Citic Press.
- Tian, G. (2014). *The Reform of China: History, Logic, and Future*. Beijing: Citic Press.
- Tian, G. & Zhang, F. (1993). *Public Market Economics*. Shanghai: Shanghai People's Publishing House.
- Wang, Y. (2010). *The Wealth of the People*. Beijing: Citic Press.
- Debreu, G. (1959). *Theory of Value*, Wiley.
- Friedman, M. and R. Friedman (1980). *Free to Choose*, HBJ.
- Kreps, D. M. (2013). *Microeconomic Foundation I: Choice and Competitive Markets*, Princeton University Press.
- Luenberger, D. (1995). *Microeconomic Theory*, McGraw-Hill.
- Mas-Colell, A., M. D. Whinston, and J. Green (1995). *Microeconomic Theory*, Oxford University Press.
- Jehle, G. A., and P. Reny (2011). *Advanced Microeconomic Theory*, Addison-Wesley.
- Olson, M. (1965). *The Logic of Collective Action*, Cambridge: Harvard University Press. (Chinese version: Olson, M. *The Logic of Collective Action*. Translated by Chen, Y. et al. Shanghai: SDX Joint Publishing.)
- Olson, M. (1982). *The Rise and Decline of Nations*, Yale University Press. (Chinese version: Olson, M. *The Rise and Decline of Nations*. Translated by Lv, Y. Beijing: The Commercial Press.)
- Olson, M. (2000). *The Power and Prosperity, Outgrowing Communist and Capitalist Dictatorships*, Basic Books.
- Rawls, J. (1971). *A Theory of Justice*, Cambridge: Harvard University Press. (Chinese version: Rawls, J. *A Theory of Justice*. Translated by He, H., et al. Beijing: China Social Sciences Press.)
- Rubinstein, Ariel (2005). *Lecture Notes in Microeconomics (modeling the economic agent)*, Princeton University Press.
- Smith, Adam (1759). *The Theory of Moral Sentiments*. (Chinese version: Smith, Adam. *The Theory of Moral Sentiments*. Translated by Xie, Z. Beijing: Central Compilation and Translation Press.)

Smith, Adam (1776). (*An Inquiry into the Nature and Causes of the Wealth of Nations*) *The Wealth of Nations*. W. Strahan and T. Cadell. Reprinted, Oxford: Clarendon Press. (Chinese version: Smith, Adam. (*An Inquiry into the Nature and Causes of the Wealth of Nations*) *The Wealth of Nations*. Translated by Guo, D. & Wang, Y. Beijing: The Commercial Press.)

Varian, H. R. (1992). *Microeconomic Analysis (Third Edition)*, W. W. Norton and Company.

Papers:

Hellman, Joel S. (2009). "Strategies to Combat State Capture and Administrative Corruption in Transition Economies" Translated by Ye, Q. & Bin, J. *Comparative Economic and Social Systems*, No. 2.

Roland, G. (2005). "Understanding Institutional Change: Fast-Moving and Slow-Moving Institutions". *Nanjing Business Review*. No. 5.

Qian, Y. (2002). "Understanding Modern Economics". *Comparative Economic and Social Systems*, No. 2.

Qian, Y. (2000). "Market and the Rule of Law". *Comparative Economic and Social Systems*, No. 3.

Tian, G. (1996). "The Theory of Endogenous Property Rights and the Smooth Transition of Economic System". *Economic Research Journal*, No. 11.

Tian, G. (2005). "The Basic Analytical Framework and Methodologies in Modern Economics". *Economic Research Journal*, No. 2.

Tian, G. (2007). "The Construction of a Harmonious Society and Improvement of Modern Market System—Efficiency, Justice, and Rule of Law". *Economic Research Journal*, No. 3.

Tian, G. (2008). "From Bringing Order out of Chaos, Market Economy, to the Construction of a Harmonious Society—The Key to Efficiency, Justice, and Harmonious Development is the Reasonable Definition of the Boundaries between Government and Market". Excellent Paper in Call for Papers themed "30 Years of China's Reform and Opening up and Development" by Wenhui Paper, Jiefang Daily, and Shanghai Institution for Management Science Research, July.

Tian, G. (2009). "The Thought and Methodology of Economics". Manuscript, October.

Tian, G. (2011). "Deep-Seated Problems in the Economic Development of China". *Academic Monthly*, No. 3.

- Tian, G. (2012). "China's Reform and Transition of the Government Role in the Next Step". *People's Forum: Academic Frontiers*, No. 3.
- Tian, G. (2012). "An Interpretation of the Connotative Features and Realistic Bottlenecks of China's Economic Transition". *People's Forum*, No. 35.
- Tian, G. (2012). "China's Reform and Transition of the Government Role under the Global Context of Change". *Academic Monthly*, No. 6.
- Tian, G. (2013). "The Future Path and Point to Break in for China's Reform". *Comparative Studies*, No. 1.
- Tian, G. (2013). "The 'Middle-Income Trap' and Reconstruction of the State Governance Mode". *People's Forum*, No. 8.
- Tian, G. (2013). "The Rule of Law: the Most Important Cornerstone of Modern Governance System". *People's Forum: Academic Frontiers*, No. 23.
- Tian, G. (2014). "The Reform of China's Political System from the Perspective of Modern State Governance". *Academic monthly*, No. 3.
- Tian, G. (2014). "The Fourth Great Social and Economic Reform in Modern China: Lessons and Reflections on the Reform of State-owned Enterprises". *Exploration and Free View*, No. 6.
- Tian, G. (2015). "Discussion on the Reasonable Range of China's Economic Growth Rate: How to Consider the Logic of both Development and Governance". *People's Forum: Academic Frontiers*, No. 6.
- Tian, G. (2015). "The Handle for Reconstructing the Relationship between Politics and Business in the New Era". *People's Forum: Academic Frontiers*, No. 5.
- Tian, G., Xia, J., & Chen, X. (2010). "Breaking the Myth of China's Mode and Insisting on Market-Oriented Reform". *Comparative Studies*, Vol. 50.
- Tian, G. & Yang, L. (2006). "A Solution to the Happiness-Income Puzzle: Theory and Evidence". *Economic Research Journal*, No. 11.
- Wang, J. "State and Economy". *Comparative Studies*, Vol. 18.
- Arrow, K. and G. Debreu (1954). "Existence of Equilibrium for a Competitive Economy", *Econometrica*, Vol. 22, No. 3, 265-290.
- Coase, R. (1960). "The Problem of Social Cost", *Journal of Law and Economics*, No. 3, 1-44.

- Modigliani, F. and M. Miller (1958). "The Cost of Capital, Corporation Finance and the Theory of Investment", *American Economic Review*, Vol. 48, No. 3, 261-297.
- Rothschild, M. and J. E. Stiglitz (1971). "Increasing Risk II: Its Economic Consequences", *Journal of Economic Theory*, Vol. 3, No. 1, 66-84.
- Scarf, H. (1960). "Some Examples of Global Instability of the Competitive Equilibrium", *International Economic Review*, Vol. 1, No. 3, 157-172.

Chapter 2

Knowledge and Methods of Mathematics

This chapter briefly introduces basic mathematical knowledge and results required by advanced microeconomics, including basic knowledge and common results of topology, linear algebra, mathematical analysis, fixed point theory, static optimization, dynamic optimization and probability theory. In the later discussion, we will use relevant conclusions.

2.1 Basic Set Theory

This section introduces some basic concepts and results of set theory.

2.1.1 Set

A set is a collection of some elements. According to the number of elements, a set can be a finite **set**, for example, $S = \{1, 3, 5, 7, 9\}$; be an infinite countable set, for example, $S = \mathcal{N}$, where \mathcal{N} is the set of all natural numbers; or be an infinite uncountable set, for example, $S = \mathcal{R}$, where \mathcal{R} is the set of all real numbers. A countable set can be finite or infinite. A set can also be described with some properties, for example, $S = \{1, 3, 5, 7, 9\} = \{x : x < 10, x \in \mathcal{N}, \frac{x}{2} \notin \mathcal{N}\}$. The empty set \emptyset is a set consisting of no elements.

A subset T of a set S is also a set, and any element in T belongs to S , denoted by $T \subseteq S$. If T is a subset of S and S has at least one element that does not belong to T , then T is a **proper subset** of S . If T and S are subsets of each other, then these two sets are same, that is, $T = S$.

The union of two sets T and S is denoted by $T \cup S = \{x : x \in T \text{ or } x \in S\}$; the intersection of two sets T and S is denoted by $T \cap S = \{x : x \in T \text{ and } x \in S\}$.

The complement set of S in the universal set U is denoted by $S^c = \{x : x \in U, x \notin S\}$. The complement set of the universal set U is the empty set, and the complement of the empty set is the universal set. The difference of sets S and T is denoted by $S \setminus T$ or $S - T$, which is defined as $S \setminus T = S - T = \{x : x \in S, x \notin T\}$.

The complement of the union or the intersection of any number of sets satisfies De Morgan's law:

$$\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c;$$

$$\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c.$$

The product of sets T and S is denoted by $S \times T = \{(s, t) | s \in S, t \in T\}$. The n -dimensional real space is $\mathcal{R}^n = \underbrace{R \times R \times R \times \cdots \times R}_{\text{product of } n \text{ spaces}}$.

2.1.2 Mapping

In order to discuss the number of elements of a collection, we first introduce the concept of functions or correspondences.

Definition 2.1.1 (*Mapping*) Given sets A and B , if for each element x in A , there always have an element y in B related to it, then this relation is called **mapping** or **function**, denoted by $f : A \rightarrow B$. The set A is called **domain** of f , and the set B is called **codomain**.

The following definition gives the types of mapping.

Definition 2.1.2 Given sets A, B , and a mapping $f : A \rightarrow B$, we call f a **surjection** if $\{y \in B : f(x) = y, x \in A\} \equiv f(A) = B$; f is called an **injection** if $f(x) \neq f(x')$ holds for all $x \neq x'$; it is called a **bijection** if f is both a surjection and an injection.

If there is a bijection f between sets A and B , we call sets A and B are **equivalent**, denoted by $A \sim B$. Next We will discuss the number of elements of a set.

Definition 2.1.3 $J_n = \{1, 2, \dots, n\}$ is the set of first n positive integers, and J is the set of all positive integers.

- (1) A set A is finite, if there exists a certain n such that $A \sim J_n$;
- (2) A set A is countable, if $A \sim J$;
- (3) A set A is at most countable, if it is either a finite set or a countable set;

- (4) A set A is uncountable, if it is neither a finite set nor a countable set.

Both sets of natural and rational numbers are countable sets, but the real number set is not. The following conclusion shows that the set of all binary numbers is not countable.

Theorem 2.1.1 A is a set consisting of all sequences made up of 0 and 1, then it is uncountable.

See the proof of Theorem 2.14 in Rudin (1976) *Principles of Mathematical Analysis*.

Since real numbers can be represented in binary, the set of real numbers is equivalent to the set A above, which is uncountable. Any an interval

(a, b) is equivalent to the real space (since $f = \frac{y}{1 - |y|}, y \equiv \frac{x - \frac{(a+b)}{2}}{(b-a)}$ is a

bijection of them), thus any real interval is uncountable.

2.2 Basic Linear Algebra

2.2.1 Matrix and Vector

We use \mathcal{R}^n to represent a set of all n -tuple real numbers. The elements of it

are called points or vectors. $\mathbf{x} = \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}$ represents a column vector, and

x_i is the i th component of the vector \mathbf{x} . $\mathbf{x}' = (x_1, \cdots, x_n)$ is defined as the transpose or the row vector of \mathbf{x} . If not specially specified, vectors refer to column vectors in general.

The inequality signs \geq, \geq and $>$ about vectors are defined as follows. Let $\mathbf{a}, \mathbf{b} \in \mathcal{R}^n$, then $\mathbf{a} \geq \mathbf{b}$ represents that $a_s \geq b_s$ for all $s = 1, \cdots, n$; $\mathbf{a} \geq \mathbf{b}$ represents that $\mathbf{a} \geq \mathbf{b}$ but $\mathbf{a} \neq \mathbf{b}$; $\mathbf{a} > \mathbf{b}$ represents that for all $s = 1, \cdots, n$, $a_s > b_s$.

In economics, it is usually required to solve the system of linear equations, which now can be easily expressed and solved by linear algebra.

We consider a system of m linear equations with n variables (x_1, x_2, \cdots, x_n) :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\ \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= d_m \end{aligned}$$

where the letter with double subscript a_{ij} denotes the coefficient that the i th equation gives to the j th variable x_j , and d_j denotes the constant term on the right side of j th equation.

It can be expressed more succinctly by the following matrix form:

$$A\mathbf{x} = \mathbf{d},$$

where A , \mathbf{x} , \mathbf{d} are respectively:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$$

$$\mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ \cdots \\ d_m \end{pmatrix}$$

A is called the **coefficient matrix** of a $m \times n$ system of equations, which is arranged in m rows and n columns; \mathbf{x} is called a **variable vector**, and \mathbf{d} is a **constant vector**. An n -dimensional vector can be viewed as a special $n \times 1$ matrix.

2.2.2 Matrix Operations

Here we give a brief introduction to some common matrix operations.

Two equivalent matrices: $A = B$ if and only if $a_{ij} = b_{ij}$ holds for all $i = 1, 2, \dots, m, j = 1, 2, \dots, n$.

Addition and subtraction of two matrices: $A \pm B = [a_{ij}] \pm [b_{ij}] = [a_{ij} \pm b_{ij}]$. It is remarkable that addition and subtraction make sense only if the dimensions of matrices are same.

Scalar multiplication of matrices: $\lambda A = \lambda[a_{ij}] = [\lambda a_{ij}]$.

Matrix multiplication: Given two matrices $A_{m \times n}$ and $B_{p \times q}$, matrix multiplication requires a compatibility condition: the number of columns in matrix A is the same as the number of rows in matrix B , that is $n = p$. If the compatibility condition is satisfied, the dimension of the product of AB is $m \times q$. AB is defined as:

$$AB = C$$

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{l=1}^n a_{il}b_{lj}$. Obviously, the matrix product of AB is not equal to BA in general.

Identity Matrix

The identity matrices discussed below are the square matrix with the same number of rows and columns, which is assumed to be n . An identity matrix of order n , denoted by I_n , is a square matrix with ones on the main diagonal and zeros elsewhere.

It has the following properties:

Property 1:

$$I_m A_{m \times n} = A_{m \times n} I_n = A_{m \times n}$$

Property 2:

$$A_{m \times n} I_n B_{n \times p} = (A_{m \times n} I_n) B_{n \times p} = A_{m \times n} B_{n \times p}$$

Property 3:

$$(I_n)^k = I_n$$

This matrix is analogous to 1 in real space, and the zero matrix discussed below (not necessarily square matrix) is analogous to the 0 in real space.

Zero Matrix

A $m \times n$ zero matrix is a matrix all of whose entries are zero.

It satisfies some operations below:

$$A_{m \times n} + 0_{m \times n} = A_{m \times n};$$

$$A_{m \times n} 0_{n \times p} = 0_{m \times p};$$

$$0_{q \times m} A_{m \times n} = 0_{q \times n}.$$

Remark:

(1) $CD = CE, C \neq 0$ does not imply that $D = E$. e.g.

$$C = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad E = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}.$$

(2) Even if A and $B \neq 0$, we still have $AB = 0$. e.g.

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}.$$

2.2.3 Linear Dependence of Vectors

One of the most important properties among vectors is the linear dependence.

Definition 2.2.1(Linear Dependence) A set of vectors $\mathbf{v}^1, \dots, \mathbf{v}^n$ is **linearly dependent**, if and only if there exist a vector \mathbf{v}^i which is a linear combination of the others, namely, $\mathbf{v}^i = \sum_{j \neq i} \alpha_j \mathbf{v}^j$.

Example 2.2.1 The following three vectors

$$\mathbf{v}^1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \mathbf{v}^2 = \begin{bmatrix} 1 \\ 8 \end{bmatrix}, \mathbf{v}^3 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

are linearly dependent, since

$$3\mathbf{v}^1 - 2\mathbf{v}^2 = \begin{bmatrix} 6 \\ 21 \end{bmatrix} - \begin{bmatrix} 1 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \mathbf{v}^3.$$

or

$$3\mathbf{v}^1 - 2\mathbf{v}^2 - \mathbf{v}^3 = \mathbf{0}$$

where $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a zero vector.

2.2.4 Transpose and Inverse of a Matrix

We first discuss the transpose of a matrix.

Definition 2.2.2(Transpose of A Matrix) The transpose of a $m \times n$ matrix A is an operator which interchanges rows and columns of the matrix. If $a_{ji} = b_{ij}$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$, then $B = [b_{ij}]_{n \times m}$ is called the **transpose** of the matrix $A = [a_{ij}]_{m \times n}$, denoted by A' or A^T .

The transpose of matrices have following properties.

If $A' = A$, the matrix A is **symmetric**.

If $A' = -A$, the matrix A is **antisymmetric**.

If $A'A = I$, the matrix A is **orthogonal**.

The followings are some operations about transpose:

- $(A')' = A$;
- $(A + B)' = A' + B'$;
- $(\alpha A)' = \alpha A'$, where α is a real number;
- $(AB)' = B' A'$.

Next we discuss the inverse of a square matrix. The inverse of matrix A , denoted by A^{-1} , should satisfy

$$AA^{-1} = A^{-1}A = I.$$

The transpose of matrix always exists, but the inverse does not necessarily exist.

Remark:

- (1) Not all square matrices have inverse. A square matrix that is not invertible is called **singular**.
- (2) If A is nonsingular, A and A^{-1} are inverses of each other, that is,

$$(A^{-1})^{-1} = A.$$

- (3) If A is an $n \times n$ matrix, then A^{-1} is also $n \times n$.
- (4) The inverse of A is unique.
- (5) Suppose both A and B are $n \times n$ invertible matrices, then

$$(a) (AB)^{-1} = B^{-1}A^{-1};$$

$$(b) (A')^{-1} = (A^{-1})'.$$

2.2.5 Solving a Linear System

We shall now discuss the inverse matrix and how to solve the systems of linear equations. Consider a system of n equations with n unknowns:

$$Ax = \mathbf{d}.$$

If A is nonsingular, then multiplying both sides by A^{-1} gives:

$$A^{-1}Ax = A^{-1}\mathbf{d}.$$

Therefore, $\mathbf{x} = A^{-1}\mathbf{d}$ is the unique solution of the linear system $Ax = \mathbf{d}$, where A^{-1} is unique.

Before applying the method of inverse matrix to solve linear systems, we first need to determine what kind of matrix A is nonsingular. Secondly, we shall solve it by an efficient way, that is, Cramer's rule.

There are two ways to judge the singularity of a square matrix A . One is to see whether the row or column vectors of a matrix are linearly dependent or not; the other is to see whether the determinant of a square matrix is equal to zero.

An $n \times n$ square matrix A could be written as a set of vectors in terms of rows.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} \mathbf{v}^{1'} \\ \mathbf{v}^{2'} \\ \cdots \\ \mathbf{v}^{n'} \end{bmatrix}$$

where $\mathbf{v}^{i'} = [a_{i1}, a_{i2}, \dots, a_{in}]$, $i = 1, 2, \dots, n$. Whether a square matrix A is singular is determined by whether the vectors $\mathbf{v}^{i'}$, $i = 1, 2, \dots, n$ are linearly independent.

Determinant of a Matrix

The determinant of an n th order square matrix $A = (a_{ij})$, denoted by $|A|$ or $\det(A)$, is defined by:

$$|A| = \sum_{(\alpha_1, \dots, \alpha_n)} (-1)^{I(\alpha_1, \dots, \alpha_n)} a_{1\alpha_1} \cdot a_{n\alpha_n},$$

where $(\alpha_1, \dots, \alpha_n)$ is the permutation of $(1, \dots, n)$, and $I(\alpha_1, \dots, \alpha_n)$ is the number of inversion times when reordering $(1, \dots, n)$.

For example, $(2, 1, 3)$ is reordered once by $(1, 2, 3)$, and $(2, 3, 1)$ is reordered twice by $(1, 2, 3)$.

There is a simple method to calculate the determinant of a matrix, that is, Laplace expansion:

$$|A| = \sum_{k=1}^n (-1)^{l+k} a_{lk} \times \det(M_{lk}), \text{ for any } l \in \{1, \dots, n\},$$

where M_{lk} is the $n - 1$ th order square matrix that results from A by removing the l -th row and the k -th column, called the cofactor of a_{lk} .

The followings show some operations of matrix:

$$\det(AB) = \det(A)\det(B);$$

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Hence, a necessary condition for the existence of A^{-1} is that $\det(A) \neq 0$.

Next is a formula for solving the inverse of a nonsingular square matrix. Let $A^{-1} = (d_{ij})$, then

$$d_{ij} = \frac{1}{\det(A)} (-1)^{i+j} \det(M_{ij}).$$

The Cramer's Rule given below summarizes how to solve a linear system. For a system of linear equations:

$$A\mathbf{x} = \mathbf{d},$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

$$\mathbf{d}' = (d_1, \dots, d_n),$$

$$\mathbf{x} = (x_1, \dots, x_n).$$

the solution is:

$$x_j = \frac{\det(A_j)}{\det(A)},$$

where:

$$A_j = \begin{bmatrix} a_{11} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nj-1} & b_n & a_{nj+1} & \cdots & a_{nn} \end{bmatrix}$$

2.2.6 Quadratic Form and Matrix

A function q with n variables is called a **Quadratic Form** if it has the following expression:

$$\begin{aligned} q(x_1, x_2, \dots, x_n) &= a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + 2a_{1n}x_1x_n \\ &\quad + a_{22}x_2^2 + 2a_{23}x_2x_3 + \cdots + 2a_{2n}x_2x_n \\ &\quad \cdots \\ &\quad + a_{nn}x_n^2. \end{aligned}$$

Let $a_{ji} = a_{ij}$, $i < j$, and then $q(x_1, x_2, \dots, x_n)$ could be written as

$$\begin{aligned} q(x_1, x_2, \dots, x_n) &= a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n \\ &\quad + a_{12}x_2x_1 + a_{22}x_2^2 + \cdots + a_{2n}x_2x_n \\ &\quad \cdots \\ &\quad + a_{n1}x_nx_1 + a_{n2}x_nx_2 + \cdots + a_{nn}x_n^2 \\ &= \sum_{i=1}^n \sum_{j=1}^n a_{ij}x_i x_j \\ &= \mathbf{x}' \mathbf{A} \mathbf{x}. \end{aligned}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

is called a **matrix of quadratic form**. Since $a_{ij} = a_{ji}$, A is an n th-order symmetric square matrix.

Definition 2.2.3 For a matrix of quadratic form A ,

if for any n -dimensional vectors \mathbf{x} , we have $\mathbf{x}'A\mathbf{x} \geq 0$, then the function $q(x_1, x_2, \dots, x_n) = \mathbf{x}'A\mathbf{x}$ or A is defined as **positive semi-definite**;

if for any n -dimensional vectors $\mathbf{x} \neq 0$, we have $\mathbf{x}'A\mathbf{x} > 0$, then the function $q(x_1, x_2, \dots, x_n) = \mathbf{x}'A\mathbf{x}$ or A is **positive definite**;

if for any n -dimensional vectors \mathbf{x} , we have $\mathbf{x}'A\mathbf{x} \leq 0$, then the function $q(x_1, x_2, \dots, x_n) = \mathbf{x}'A\mathbf{x}$ or A is **negative semi-definite**;

if for any n -dimensional vectors $\mathbf{x} \neq 0$, we have $\mathbf{x}'A\mathbf{x} < 0$, then the function $q(x_1, x_2, \dots, x_n) = \mathbf{x}'A\mathbf{x}$ or A is **negative definite**.

The following theorem characterizes the relation between the eigenvalues and positive definiteness (negative definiteness).

Theorem 2.2.1 *A Matrix of quadratic form A is*

positive definite, if and only if eigenvalues $\lambda_i > 0$ for all $i = 1, 2, \dots, n$;

negative definite, if and only if eigenvalues $\lambda_i < 0$ for all $i = 1, 2, \dots, n$;

positive semi-definite, if and only if eigenvalues $\lambda_i \geq 0$ for all $i = 1, 2, \dots, n$;

negative semi-definite, if and only if eigenvalues $\lambda_i \leq 0$ for all $i = 1, 2, \dots, n$;

indefinite, if at least one eigenvalue is positive and at least one eigenvalue is negative.

A necessary and sufficient condition for a matrix of quadratic form A to be positive definite is that all its minors are positive. That is,

$$\begin{aligned} |A_1| &= A_{11} > 0; \\ |A_2| &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0; \\ &\dots \\ |A_n| &= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} > 0. \end{aligned}$$

A necessary and sufficient condition for a quadratic form A to be negative definite is that its minors are negative and positive alternately, that is,

$$|A_1| < 0, |A_2| > 0, |A_3| < 0, \dots, (-1)^n |A_n| > 0.$$

2.2.7 Eigenvalues, Eigenvectors and Traces

If a square matrix A and a real number λ satisfies the equation $A\mathbf{x} = \lambda\mathbf{x}$, then λ is called the eigenvalue of A , and the vector \mathbf{x} is called the eigenvector of A belonging to the eigenvalue λ .

Eigenvalues and some properties of matrix, such as positive or negative definite, have close connections.

For a symmetric matrix A , there is a convenient decomposition method.

If $U'U = I_n$ holds, or $U' = U^{-1}$, the n th order square matrix U is called a **normed orthogonal matrix**, where U' is the transpose of U . If A is symmetric, and $\lambda_i, i = 1, \dots, n$ are its eigenvalues, then there exists a normed orthogonal matrix U such that:

$$U'AU = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix},$$

or

$$A = U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U'.$$

The power operation of symmetric matrix has a convenient form:

$$A^k = U \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} U'.$$

If the eigenvalues of A are nonzero real numbers, then the inverse of A can be reformulated as follows:

$$A^{-1} = U \begin{bmatrix} \lambda_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{-1} \end{bmatrix} U'.$$

Another common concept about square matrix is the trace. The trace of an n th-order A is $tr(A) = \sum_i^n a_{ii}$. It also has following properties:

- (1) $tr(A) = \lambda_1 + \dots + \lambda_n$;
- (2) If A and B have the same dimension, then $tr(A + B) = tr(A) + tr(B)$;
- (3) If a is a real number, $tr(aA) = atr(A)$;

(4) $tr(AB) = tr(BA)$, if AB is a square matrix;

(5) $tr(A') = tr(A)$;

(6) $tr(A'A) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}^2$.

2.3 Basic Topology

Topology, a branch of mathematics, studies the basic properties of topological spaces and all kinds of mathematical structures defined on them. This branch originated from the detailed study of point sets on real axis, manifolds, metric spaces, and early functional analysis.

There are two branches of topology. One focusing on using analysis method is called point set topology, or analytic topology. If further subdivided, point set topology also has a branch: differential topology. The other branch emphasizes on the use of algebraic method, called algebraic topology. However, these branches tend to be unified. Topology is widely applied in functional analysis, Lie Group, differential geometry, differential equations and many other branches of mathematics.

Here we give a brief introduction to the basic knowledge of point set topology, and apply them to establish some important conclusions about the properties of sets and continuous mapping between sets.

2.3.1 Topological Space

Definition 2.3.1 Suppose X is a nonempty set, and then τ is a family of subsets of X if

- (1) both X and the empty set belong to τ ;
- (2) the union of any number of members in τ is still in τ ;
- (3) the intersect of a finite number of members in τ is still in τ .

We call τ a topology of X , and the set X together with its topology τ is called a topological space, denoted by (X, τ) ; members in τ are called the open sets of this topological space.

Example: Examples of topological spaces:

- (1) (Discrete Topology) Suppose X is a nonempty set, topology $\tau = 2^X$.
- (2) (Trivial Topology) Suppose X is a nonempty set, topology $\tau = \{X, \emptyset\}$.
- (3) (Euclidean Topology) Suppose R is the set of all real numbers, topology τ is a collection of open sets as usually known. (See the following definition).

- (4) (Quotient Topology) Suppose X is a nonempty set, according to a given equivalence relation R , we partition X into disjoint subsets, all of which make up a new collection, denoted by X/R . We specify that the subsets of X/R are open sets if and only if the union of any elements of U is an open set, and call X/R a quotient topology.

Although the study object of topology can be an arbitrary type of sets, considering the convenience of understanding and application, the followings mainly involve the introduction of some commonly used topological spaces: especially the metric spaces in the finite dimensional real space.

2.3.2 Metric Space

We first illustrate the definition of metric and metric space. Metric is a measure of distance. A metric space (X, d) is composed of a set X and the metric d defined on the elements of set. The metric space may be finite or infinite dimension, depending on the topology structure defined on X . Meanwhile, metric should satisfy three basic assumptions. For any $\mathbf{p}, \mathbf{q}, \mathbf{r} \in X$, we have

- (1) $d(\mathbf{p}, \mathbf{q}) > 0$ if and only if $\mathbf{p} \neq \mathbf{q}$;
- (2) $d(\mathbf{p}, \mathbf{q}) = d(\mathbf{q}, \mathbf{p})$;
- (3) $d(\mathbf{p}, \mathbf{q}) \leq d(\mathbf{p}, \mathbf{r}) + d(\mathbf{r}, \mathbf{q})$.

Remark: If the metrics on the same set are different, then the metric spaces are different. For example,

- (1) Metric Space 1: $(X = \mathcal{R}^n, d_1), \forall \mathbf{x}^1, \mathbf{x}^2 \in X, d_1(\mathbf{x}^1, \mathbf{x}^2) = \sqrt{\sum_i (x_i^1 - x_i^2)^2}$, Metric Space 1 is called n -dimensional Euclidean space.
- (2) Metric Space 2: $(X = \mathcal{R}^n, d_2), \forall \mathbf{x}^1, \mathbf{x}^2 \in X, d_2(\mathbf{x}^1, \mathbf{x}^2) = \sum_i |x_i^1 - x_i^2|$.
- (3) Metric Space 3: $(X = \mathcal{R}^n, d_3), \forall \mathbf{x}^1, \mathbf{x}^2 \in X, d_3(\mathbf{x}^1, \mathbf{x}^2) = \max\{|x_1^1 - x_1^2|, \dots, |x_n^1 - x_n^2|\}$.

Although the remaining discussions in this section are also true for general metric spaces, we mainly focus on Euclidean spaces for convenience of statement.

2.3.3 Open Sets, Closed Sets and Compact Sets

With the concept of metric, we can clearly define the proximity between points. In an n -dimensional Euclidean space, given $\mathbf{x}^0 \in \mathcal{R}^n$, the set of

all points of distance less than ϵ from \mathbf{x}^0 is called an open ball with radius ϵ and center \mathbf{x}^0 , denoted by $B_\epsilon(\mathbf{x}^0)$. A related concept is closed ball, the metric of which changes to less than or equal to instead of less than above, denoted by $B_\epsilon^*(\mathbf{x}^0)$.

Next, we give the definition of closed sets and compact sets.

Definition 2.3.2 The set $S \subseteq \mathcal{R}^n$ is an **open set**, and if for any $\mathbf{x} \in S$, there always exists an $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subseteq S$.

Based on the definition of open sets, the following theorem gives some basic properties of open sets:

Theorem 2.3.1(Open Sets in \mathcal{R}^n) *In term of open sets, there are following conclusions:*

- (1) *The empty set \emptyset is an open set.*
- (2) *The universal space \mathcal{R}^n is an open set.*
- (3) *The union of open sets is an open set.*
- (4) *The intersection of a finite number of open sets is an open set.*

PROOF. (1) If \emptyset has no elements, then the proposition "for each points in \emptyset , there is an ϵ, \dots ", satisfies the definition of an empty set.

(2) For any point in \mathcal{R}^n and any $\epsilon > 0$, according to the definition of an open ball, the set $B_\epsilon(\mathbf{x})$ consist of points in \mathcal{R}^n . Hence, $B_\epsilon(\mathbf{x}) \subseteq \mathcal{R}^n$, and then \mathcal{R}^n is open.

(3) For all $i \in I$, suppose S_i is an open set, we need to show $\cup_{i \in I} S_i$ is an open set. Suppose $\mathbf{x} \in \cup_{i \in I} S_i$, then for some $i' \in I$, we have $\mathbf{x} \in S_{i'}$. Since $S_{i'}$ is open, for an $\epsilon > 0$, we have $B_\epsilon(\mathbf{x}) \subseteq S_{i'}$. It follows that $B_\epsilon(\mathbf{x}) \subseteq \cup_{i \in I} S_i$, and then $\cup_{i \in I} S_i$ is open.

(4) Suppose $B = \bigcap_{k=1}^n B_k$. If $B = \emptyset$, it is clear that B is an open set. If $B \neq \emptyset$, for any $\mathbf{x} \in B$, obviously, we have: for any $k \in \{1, \dots, n\}$, $\mathbf{x} \in B_k$. Since B_k is an open set, there must exist an $\epsilon_k > 0$ such that $B_{\epsilon_k}(\mathbf{x}) \subseteq B_k$. Let $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\}$, then for any $k \in \{1, \dots, n\}$, $B_\epsilon(\mathbf{x}) \subseteq B_k$, so $B_\epsilon(\mathbf{x}) \subseteq B$. Hence, B is a open set. \square

The following theorem discusses the relationship between open sets and open balls.

Theorem 2.3.2(Each open set is a union of open balls.) *Suppose S is an open set, then for each $\mathbf{x} \in S$, there exists an $\epsilon_{\mathbf{x}} > 0$ such that $B_{\epsilon_{\mathbf{x}}}(\mathbf{x}) \subseteq S$, then:*

$$S = \bigcup_{\mathbf{x} \in S} B_{\epsilon_{\mathbf{x}}}(\mathbf{x}).$$

PROOF. Suppose $S \subseteq \mathcal{R}^n$ is an open set, then it follows from the definition of open sets that for any $\mathbf{x} \in S$, there exists an $\epsilon_{\mathbf{x}} > 0$ such that

$B_{\epsilon_x}(\mathbf{x}) \subseteq S$. We now need to show that $\mathbf{x}' \in S$ implies $\mathbf{x}' \in \cup_{\mathbf{x} \in S} B_{\epsilon_x}(\mathbf{x})$, and $\mathbf{x}' \in \cup_{\mathbf{x} \in S} B_{\epsilon_x}(\mathbf{x})$ implies $\mathbf{x}' \in S$.

If $\mathbf{x}' \in S$, then it follows from the definition of open balls with centre \mathbf{x}' that $\mathbf{x}' \in B_{\epsilon_{\mathbf{x}'}}(\mathbf{x})$. But \mathbf{x}' belongs to any union containing this open ball. Hence, we have $\mathbf{x}' \in \cup_{\mathbf{x} \in S} B_{\epsilon_x}(\mathbf{x})$.

If $\mathbf{x}' \in \cup_{\mathbf{x} \in S} B_{\epsilon_x}(\mathbf{x})$, then $\mathbf{x}' \in B_{\epsilon_{\mathbf{x}'}}(s)$. Since $B_{\epsilon_{\mathbf{x}'}}(s) \subseteq S$, it follows that $\mathbf{x}' \in S$. \square

Here we discuss closed sets and the definition of closed set based on the definition of open sets.

Definition 2.3.3(Closed Sets in \mathcal{R}^n) If the complement of S , that is S^c , is an open set, then S is a **closed set**.

We also have some conclusions about the basic properties of closed sets.

Theorem 2.3.3(Closed Sets in \mathcal{R}^n) In term of closed sets, there are following conclusions:

- (1) The empty set \emptyset is a closed set.
- (2) The universal space \mathcal{R}^n is a closed set.
- (3) The intersection of any closed sets is a closed set.
- (4) The union of a finite number of closed sets is a closed set.

PROOF. (1) Since $\emptyset = \{\mathcal{R}^n\}^c$, and \mathcal{R}^n is an open set, it follows from the definition of closed sets that \emptyset is a closed set.

(2) Since $\{\mathcal{R}^n\}^c = \emptyset$, and \emptyset is an open set, it follows from the definition of closed sets that \mathcal{R}^n is a closed set.

(3) Suppose for all $i \in I$, S_i is a closed set in \mathcal{R}^n , then we need to show that $\cup_{i \in I} S_i$ is closed. Since S_i is closed, its complement S_i^c is an open set. The intersection $\cap_{i \in I} S_i^c$ is also open since the intersection of a finite number of open sets is open. It follows from the De Morgan's laws that $i \in I$, $(\cap_{i \in I} S_i^c)^c = \cup_{i \in I} S_i$ holds for a finite number of sets. Since $(\cap_{i \in I} S_i^c)$ is open, and then its complement $\cup_{i \in I} S_i$ is closed.

(4) Let C_1, C_2 be closed sets and denote $C = C_1 \cap C_2$. Since C_1, C_2 are closed, $C_k^c = B_k, k = 1, 2$ are open. It follows from the properties of open sets above that $B_1 \cup B_2$ is an open set, and hence $C = (B_1 \cup B_2)^c$ is a closed set. \square

Next we discuss the concept of point sets related to open and closed sets.

Definition 2.3.4 For set S , a point $\mathbf{x} \in S$ is called **limit point** if for any $\epsilon > 0$, $B_\epsilon(\mathbf{x}) \cap S \neq \emptyset, B_\epsilon(\mathbf{x}) \cap S^c \neq \emptyset$, where $S^c = X \setminus S$, i.e., the complement of S . The collection of all limit points of set S is denoted by ∂S ; for set S , a point $\mathbf{x} \in S$ is called **interior point**, if there is an $\epsilon > 0$ such that $B_\epsilon^*(\mathbf{x}) \subseteq S$.

Now, we can redefine the open set as follows: a set is open if every element in the set is a interior point. Similarly, closed sets can also be defined: a set is called a closed set if all limit points of a set belong to itself. In addition, for any set S in a metric space, the smallest closed set containing S is called the **closure** of the set, denoted by $\bar{S} = S \cup \partial S$ or $\text{cl}S$. Obviously, if S is closed, $\bar{S} = S$.

Next, we discuss a class of special but widely used closed sets, that is compact sets. We first introduce the concept of bounded sets.

Definition 2.3.5(Bounded Sets) If a set S in \mathcal{R}^n is totally contained in a ball (open or closed ball) with radius ϵ , then S is called **bounded**. In other words, if for $\mathbf{x} \in \mathcal{R}^n$, there is an $\epsilon > 0$ such that $S \subseteq B_\epsilon(\mathbf{x})$, then S is **bounded**.

Definition 2.3.6(Compact Sets) If a set $S \subseteq \mathcal{R}^n$ is closed and bounded, then it is **compact**.

Compact set is a crucial concept in mathematical analysis. Next, we discuss an important feature of compact set. We first introduce the concept of open covering.

Definition 2.3.7(Open Covering) For a set S and a collection of open sets $\{G_\alpha\}$ in metric space X , if $S \subseteq \bigcup_\alpha G_\alpha$, then $\{G_\alpha\}$ is called a **open covering** of S ; if the index set $\{\alpha\}$ is finite, it is called a finite open covering.

The definition of compact set above is only for the finite dimensional spaces. The definition of compact set for the infinite dimensional spaces is based on the concept of open covering. Whether a set is in finite or infinite dimension, there is another way to define compact sets, that is, each open covering of a set has a finite subcover. The following Heine-Borel theorem, is also known as the finite covering theorem, proved that the above two ways of definition are consistent for the compact sets in finite dimensional spaces.

Theorem 2.3.4(Heine-Borel Theorem or Finite Covering Theorem) For a set $S \subseteq \mathcal{R}^n$, the following two arguments are consistent:

- (1) S is a bounded closed set;
- (2) Any open covering of S has a finite subcover $\{G_\alpha\}$. That is, for $\{G_\alpha\}$, there is a finite set $\{1, \dots, n\} \subseteq \{\alpha\}$ such that $S \subseteq \bigcup_{i=1}^n G_i$.

See the proof of Theorem 2.41 in Rudin's *Principles of Mathematical*.

2.3.4 Connectedness of Set

Now we introduce the concept and properties of connected sets .

Definition 2.3.8(Connected Sets) For a set S in metric spaces, if there do not exist two sets A and B such that $A \cap B = B \cap A = \emptyset$, and $S \subseteq A \cup B$, then S is called a **connected set**.

The following theorem illustrates the characteristic of connected sets.

Theorem 2.3.5 *The set $S \subseteq \mathcal{R}^1$ is connected, if and only if it satisfies the following property: for any $x, y \in S$, if $x < z < y$, then $z \in S$.*

See the proof of the theorem 2.47 in textbook *Principles of Mathematical Analysis*. Obviously, the whole real space is connected, and intervals in real space, such as (a, b) , $[a, b]$, are all connected sets.

2.3.5 Sequence and Convergence

We first give some concepts of the sequences and convergence.

Definition 2.3.9(Sequences in \mathcal{R}^n) The **sequences** in \mathcal{R}^n is a function which maps the finite subset I of the positive integers into \mathcal{R}^n , represented by $\{\mathbf{x}^k\}_{k \in I}$, and for each $k \in I$, $\mathbf{x}^k \in \mathcal{R}^n$.

For all sufficiently large k , if each element of sequence $\{\mathbf{x}^k\}$ can arbitrarily approach a point in \mathcal{R}^n , then we call that the sequence converges to this point. Formally, we have the following definition:

Definition 2.3.10(Convergent Sequence) If for each $\epsilon > 0$, there is a \bar{k} such that for all $k \in I$ larger than \bar{k} , $\mathbf{x}^k \in B_\epsilon(\mathbf{x})$ holds, then we call that the sequence $\{\mathbf{x}^k\}_{k \in I}$ **converges to** $\mathbf{x} \in \mathcal{R}^n$.

Like subsets of a set, we have the concept of subsequences of a sequence.

Definition 2.3.11(Subsequences) If J is an infinite subset of I , then $\{\mathbf{x}^k\}_{k \in J}$ is called a **subsequence** of $\{\mathbf{x}^k\}_{k \in I}$ in \mathcal{R}^n .

Definition 2.3.12(Bounded Sequences) If for $M \in \mathcal{R}$ and any $k \in I$, $\|\mathbf{x}^k\| \leq M$ holds, then the sequence $\{\mathbf{x}^k\}_{k \in I}$ in \mathcal{R}^n is **bounded**.

The following is a property of the subsequence of a bounded sequence.

Theorem 2.3.6(Bounded Sequences) *Each bounded sequence in \mathcal{R}^n has a convergent subsequence.*

2.3.6 Convex Set and Convexity

Convex sets are a very important kind of sets, widely used in microeconomics. For example, sets of budget constraints are generally convex sets and have a strong economic meaning. We first define the convex sets.

Definition 2.3.13 If for any two elements $\mathbf{x}^1, \mathbf{x}^2 \in S$ and any $t \in [0, 1]$, we have $t\mathbf{x}^1 + (1-t)\mathbf{x}^2 \in S$, then the set $S \subseteq \mathcal{R}^n$ is a **convex set**.

If $\mathbf{z} = t\mathbf{x}^1 + (1-t)\mathbf{x}^2, t \in (0, 1)$, the point \mathbf{z} is the weighted mean or convex combination of $\mathbf{x}^1, \mathbf{x}^2$. If $\mathbf{z} = \sum_{l=1}^k \alpha^l \mathbf{x}^l, \mathbf{x}^l \in S, \alpha^l \in [0, 1], l \in \{1, \dots, k\}, \sum_l \alpha^l = 1$, then \mathbf{z} is also a convex combination of $\{\mathbf{x}^l\}$.

We have following conclusions about convex sets:

Theorem 2.3.7 If both sets S and T are convex, then their intersection $T \cap S$ is also convex.

Any set can be convexified, i.e., convex hull, denoted by $co S$.

Definition 2.3.14 The **convex hull** of a set $S \subseteq \mathcal{R}^n$ is the smallest convex set containing S , denoted by $co S$.

The following theorem illustrates how to convexify a set.

Theorem 2.3.8 For a set $S \subseteq \mathcal{R}^n$, its convex hull is

$$co S = \left\{ \mathbf{y} \in \mathcal{R}^n \mid \mathbf{y} = \sum_{l=1}^k \alpha^l \mathbf{x}^l, \mathbf{x}^l \in S, \alpha^l \in [0, 1], \forall l \in \{1, \dots, k\}, \sum_l \alpha^l = 1 \right\},$$

that is, the convex hull of S is formed by convex combination of all finite points in S .

The points of convex hull are made up of convex combinations of finite points. The following Caratheodory theorem simplifies the way of convexification in finite dimensional real space.

Theorem 2.3.9(Caratheodory Theorem) If the set is in finite dimensional real space, $S \subseteq \mathcal{R}^n$, the points of its convex hull $co S$ can be written as the convex combination of at most $n + 1$ points in S .

The following theorem characterizes for compact sets that convex hull will keep the properties of compact sets.

Theorem 2.3.10 If $S \subseteq \mathcal{R}^n$ is a compact set, then its convex hull $co S$ is also a compact set.

See A3.1 of Kreps (2013) for the proof of the above three theorems.

Every point in a convex hull is a convex combination of finite points in a set, but it does not mean that it must be a convex combination formed by other points. If a point is not a convex combination formed by other points, we define such a point as the extreme point. For compact sets, the structure of convex hull will be more simplified. The following Krein-Milman theorem characterizes the convex hull of compact sets.

Theorem 2.3.11(Krein-Milman Theorem) *If a set S is a compact set of the finite dimensional real space, and $EX(S)$ is the set of the extreme points of set S , then $coS = co EX(S)$, which means the convex hull of the compact set is composed of a finite convex combinations of all the extreme points.*

2.4 Single-Valued Function and Its properties

2.4.1 The continuity of function

The continuity of functions can be defined in any topological space. However, in order to state conveniently and introduce the commonly used results, here, without loss of generality, suppose $X \subseteq \mathcal{R}^n$.

Definition 2.4.1(continuity) For a function $f : X \rightarrow \mathcal{R}$ and $\mathbf{x}_0 \in X$, if

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0),$$

we call that f is **continuous** at x_0 ; or equivalently, given any $\epsilon > 0$, there is $\delta > 0$ such that for any $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{x}_0| < \delta$, we have

$$|f(\mathbf{x}) - f(\mathbf{x}_0)| < \epsilon,$$

or equivalently, **the upper contour set** of f at \mathbf{x}_0

$$U(\mathbf{x}_0) \equiv \{\mathbf{x}' \in X : f(\mathbf{x}') \geq f(\mathbf{x}_0)\}$$

and its **lower contour set**

$$L(\mathbf{x}_0) \equiv \{\mathbf{x}' \in X : f(\mathbf{x}') \leq f(\mathbf{x}_0)\}$$

are the closed subsets of X .

If f is continuous at any $\mathbf{x} \in X$, then the function $f : X \rightarrow \mathcal{R}$ is defined to be continuous on X .

Although the three definitions of continuity are all equivalent, the third definition is easier to verify. The idea of continuity is very intuitive. If we draw the function, the curve has no break point.

The function is continuous, and then the change of $f(x)$ is also extremely small when x changes slightly.

The following theorem illustrates the relationship between the continuity of functions and the open sets.

Theorem 2.4.1(Continuity and Inverse Image) *Let D be a subset of \mathcal{R}^m , and the following conditions are equivalent.*

- (1) $f : D \rightarrow \mathcal{R}^n$ is continuous.
- (2) For each open ball B in \mathcal{R}^n , $f^{-1}(B)$ is also open in D .
- (3) For each open set S in \mathcal{R}^n , $f^{-1}(S)$ is also open in D .

PROOF. We will show that (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1).

(1) \Rightarrow (2). Suppose that (1) holds and B is an open ball in \mathcal{R}^n . Picking any $\mathbf{x} \in f^{-1}(B)$, we have $f(\mathbf{x}) \in B$. Since B is open in \mathcal{R}^n , then there is an $\varepsilon > 0$ such that $B_\varepsilon(f(\mathbf{x})) \subseteq B$, and it follows from the continuity of f that there is a δ such that $f(B_\delta(\mathbf{x}) \cap D) \subseteq B_\varepsilon(f(\mathbf{x})) \subseteq B$. Hence, $B_\delta(\mathbf{x}) \cap D \subseteq f^{-1}(B)$. Since $\mathbf{x} \in f^{-1}(B)$ is arbitrary, it can be seen that $f^{-1}(B)$ is open in D , thus (2) is established.

(2) \Rightarrow (3). Suppose that (2) holds and S is open in \mathcal{R}^n , then S can be written as a union of open balls $B_i (i \in I)$ such that $S = \cup_{i \in I} B_i$. Hence, $f^{-1}(S) = f^{-1}(\cup_{i \in I} B_i) = \cup_{i \in I} f^{-1}(B_i)$. It follows from (2) that each set $f^{-1}(B_i)$ is open in D , and then $f^{-1}(S)$ is the union of open sets in D . Therefore $f^{-1}(S)$ is also open in D . Since S is an arbitrary open set in \mathcal{R}^n , (3) is established.

(3) \Rightarrow (1). Suppose that (3) holds and pick $\mathbf{x} \in D$ and $\varepsilon > 0$. Then, since $B_\varepsilon(f(\mathbf{x}))$ is open in \mathcal{R}^n , it follows from (3) that $f^{-1}(B_\varepsilon(f(\mathbf{x})))$ is open in D . Since $\mathbf{x} \in f^{-1}(B_\varepsilon(f(\mathbf{x})))$, there is a $\delta > 0$ such that $B_\delta(\mathbf{x}) \cap D \subseteq f^{-1}(B_\varepsilon(f(\mathbf{x})))$. It means that $f(B_\delta(\mathbf{x}) \cap D) \subseteq B_\varepsilon(f(\mathbf{x}))$. Therefore, f is continuous at \mathbf{x} . Since \mathbf{x} is arbitrary, (1) is established. \square \square

We have the following conclusion for a continuous function whose domain is a compact set.

Theorem 2.4.2(The continuous image of a compact set is a compact set) *Suppose $f : D \subseteq \mathcal{R}^m \rightarrow \mathcal{R}^n$ is a continuous function. If $S \subseteq D$ is a compact set in D (for example, S is closed and bounded in D), then its image $f(S) \subseteq \mathcal{R}^n$ is compact in \mathcal{R}^n .*

2.4.2 Upper Semi-continuity and Lower Semi-continuity

Upper semi-continuity and **lower semi-continuity** are weaker than continuity. Suppose that X is an arbitrary topological space.

Definition 2.4.2 A function $f : X \rightarrow \mathbb{R}$ is said to be *upper semi-continuous* if at point $x_0 \in X$, we have

$$\limsup_{x \rightarrow x_0} f(x) \leq f(x_0),$$

or equivalently, for any $\epsilon > 0$, there is a $\delta > 0$ such that for any $x \in X$ satisfying $|x - x_0| < \delta$, we have

$$f(x) < f(x_0) + \epsilon.$$

or equivalently, the upper contour set $U(x_0)$ of f is a closed set of X .

A function $f : X \rightarrow \mathbb{R}$ is said to be *upper semi-continuous* on X if f is upper semi-continuous at every point $x \in X$.

Definition 2.4.3 A function $f : X \rightarrow \mathbb{R}$ is said to be *lower semi-continuous* on X if $-f$ is upper semi-continuous.

It is clear that a function $f : X \rightarrow \mathbb{R}$ is continuous on X if and only if it is both upper and lower semi-continuous.

2.4.3 Transfer Upper and Lower Continuity

A weaker concept of continuity is **transfer continuity**. It is used to completely characterize the extremum problems of functions or preferences (see a series of papers by authors: Tian (1992, 1993, 1994), Tian&Zhou (1995) and Zhou& Tian (1992)).

Definition 2.4.4 The function $f : X \rightarrow \mathcal{R}$ is defined as **transfer (weakly) continuous** on X , if for any point $\mathbf{x}, \mathbf{y} \in X$, $f(\mathbf{y}) < f(\mathbf{x})$ means that there exist a point $\mathbf{x}' \in X$ and a neighbourhood $\mathcal{N}(\mathbf{y})$ of \mathbf{y} such that $f(\mathbf{z}) < f(\mathbf{x}')$ ($f(\mathbf{z}) \leq f(\mathbf{x}')$) for any $\mathbf{z} \in \mathcal{N}(\mathbf{y})$.

Definition 2.4.5 The function $f : X \rightarrow \mathcal{R}$ is defined as **transfer (weakly) lower continuous** on X , if $-f$ is transfer (weakly) upper continuous functions on X .

Remark: It is clear that the upper (lower) semi-continuity implies the transfer upper (lower) continuity (let $\mathbf{x}' = \mathbf{x}$); while the transfer upper (lower) continuity implies the transfer weakly upper (lower) continuity, and it may not be true for the converse. We will then prove that a function f has the largest (smallest) value on the compact set X if and only if f is transfer weakly upper continuous on X , and the set of maximum (minimum) of f is compact if and only if f is transfer upper (lower) continuous on X .

2.4.4 Differentiation and Partial Differentiation of Function

The differential in one-dimensional real space measures the sensitivity to change of the function value with respect to a change in the independent variable. Let X be a subset of \mathcal{R} .

Definition 2.4.6(Derivative) The **derivative** of $f : X \rightarrow \mathcal{R}$ at point $x_0 \in X$ is defined as

$$f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x},$$

where $\Delta x = x - x_0$.

Obviously, if a function has derivative at a certain point, then it must be continuous, but it may not be true for the converse.

We can use the derivatives to solve the limit value of the continuous functions of which the numerator and denominator tends to zero (infinity), that is, the following L'Hopital rule:

Theorem 2.4.3(L'Hopital Rule) Suppose that $f(x)$ and $g(x)$ are continuous functions, and $f(0) = g(0) = 0$, then

$$(1) \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)};$$

$$(2) \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}.$$

Higher order derivatives and partial derivatives are widely used in economics. Let's define the derivatives of order n .

Definition 2.4.7(Multiorder Derivative) The n th order derivative of $f : X \rightarrow \mathcal{R}$ at $x_0 \in X$ is defined as

$$f^{[n]}(x_0) = \lim_{\Delta x \rightarrow 0} \frac{f^{[n-1]}(x_0 + \Delta x) - f^{[n-1]}(x_0)}{\Delta x}.$$

In a multidimensional real space $X \subseteq \mathcal{R}^n$, we introduce the concept of partial differentiation of the function $f : X \rightarrow \mathcal{R}$, $f(x_1, \dots, x_n)$, to measure the degree of change of a function value with respect to one of those variables, with the others held constant.

Definition 2.4.8(Partial Derivative) The **partial derivative** of $f : X \rightarrow \mathcal{R}$, $X \subseteq \mathcal{R}^n$ with respect to x_i at $\mathbf{x}^0 = (x_1^0, \dots, x_n^0) \in X$ is defined as

$$\frac{\partial f(\mathbf{x}^0)}{\partial x_i} = \lim_{\Delta x_i \rightarrow 0} \frac{f(x_1^0, \dots, x_i^0 + \Delta x_i, \dots, x_n^0) - f(\mathbf{x}^0)}{\Delta x_i}.$$

We illustrate the degree of change of a multidimensional function in different directions in the way of matrix, which is called gradient vector.

Definition 2.4.9(Gradient Vector) Let f be defined as a function on \mathcal{R}^n which has partial derivatives. We define the **gradient** of f as a vector

$$Df(\mathbf{x}) = \left[\frac{\partial f(\mathbf{x})}{\partial x_1}, \frac{\partial f(\mathbf{x})}{\partial x_2}, \dots, \frac{\partial f(\mathbf{x})}{\partial x_n} \right].$$

Suppose f has second-order partial derivative. We define the Hessian matrix of f at \mathbf{x} is a $n \times n$ matrix $D^2f(\mathbf{x})$, where

$$D^2f(\mathbf{x}) = \left[\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} \right],$$

and if all the second-order partial derivatives are continuous, then

$$\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} = \frac{\partial^2 f(\mathbf{x})}{\partial x_j \partial x_i},$$

thus the above matrix is a symmetric matrix.

2.4.5 Mean Value Theorem and Taylor Expansion

Theorem 2.4.4(Férmat lemma) *Let X be a subset of \mathcal{R} . If*

- (i) *a function $f : X \rightarrow \mathcal{R}$ has definition in a neighborhood $N(x_0)$ at x_0 , and we always have $f(x) \leq f(x_0)$ or $f(x) \geq f(x_0)$ in this neighborhood;*
- (ii) *the function $f(x)$ is derivable at point x_0 ,*

then we have

$$f'(x_0) = 0.$$

The following mean value theorem, or called Lagrange formula depicts the relationship between function values and derivatives in an interval.

Theorem 2.4.5(The Mean Value Theorem or the Lagrange Formula) *If $f : [a, b] \rightarrow \mathcal{R}$ is: (i) continuous on $[a, b]$; (ii) differentiable on (a, b) , then there exist $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{(b - a)}.$$

The above mean value theorem is also true for multivariate x . If function $f : \mathcal{R}^n \rightarrow \mathcal{R}$ is differentiable, there is $z \in \mathcal{R}^n$ such that

$$f(y) = f(x) + Df(z)(\mathbf{y} - \mathbf{x}).$$

One of the transformations of the mean value theorem is the mean value theorem for integrals:

Theorem 2.4.6(The Mean Value Theorem for Integrals) *If $f : [a, b] \rightarrow \mathcal{R}$ is continuous on $[a, b]$, then there exists $c \in (a, b)$ such that*

$$\int_a^b f(x)dx = f(c)(b - a).$$

Another transformation of the mean value theorem is the extended mean value theorem, or called Cauchy's mean value theorem:

Theorem 2.4.7(Extended Mean Value Theorem or Cauchy's Mean Value Theorem) *If $f : [a, b] \rightarrow \mathcal{R}$ and $g : [a, b] \rightarrow \mathcal{R}$ satisfy that: (i) are continuous on $[a, b]$; (ii) are differentiable on (a, b) , then there must exist $c \in (a, b)$ such that*

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

Taylor's expansion is a very useful method for solving approximation.

Consider a continuously differentiable function $f : \mathcal{R}^n \rightarrow \mathcal{R}$, $x, y \in \mathcal{R}^n$, and it follows from the mean value theorem that there exists $z, w \in (\mathbf{x}, \mathbf{y})$ such that the following two equations hold:

$$f(y) = f(x) + Df(z)(\mathbf{y} - \mathbf{x}),$$

$$f(y) = f(x) + Df(x)(\mathbf{y} - \mathbf{x}) + \frac{1}{2}(\mathbf{y} - \mathbf{x})' D^2 f(w)(\mathbf{y} - \mathbf{x}),$$

where $(\mathbf{y} - \mathbf{x})'$ is the transpose of the vector $(\mathbf{y} - \mathbf{x})$.

Generally, we have the following results:

Proposition 2.4.1(Taylor's Theorem) *Given any function $f(x)$, if the derivative of order $(n + 1)$ exists at x_0 , then the function can be expanded at x_0 :*

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n + R_n \\ &\equiv P_n + R_n, \end{aligned}$$

where P_n represents the n -th order polynomial, and R_n is the Lagrange's remainder:

$$R_n = \frac{f^{(n+1)}(P)}{(n+1)!}(x - x_0)^{n+1},$$

where P is a point between x and x_0 , and $n!$ is the factorial of n

$$n! \equiv n(n-1)(n-2) \cdots (3)(2)(1).$$

We have the following approximation of function by Taylor's expansion, if y approximates x ,

$$f(y) \approx f(x) + Df(x)(y - x),$$

$$f(y) \approx f(x) + Df(x)(\mathbf{y} - \mathbf{x}) + \frac{1}{2}(\mathbf{y} - \mathbf{x})' D^2 f(x)(\mathbf{y} - \mathbf{x}).$$

2.4.6 Homogeneous Functions and Euler's Theorem

Definition 2.4.10 A function $f : X \rightarrow \mathbb{R}$ is said to be *homogeneous of degree k* if for any t , $f(tx) = t^k f(x)$.

An important result concerning homogeneous function is the Euler's theorem.

Theorem 2.4.8(Euler's Theorem) If a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is homogeneous of degree k if and only if

$$kf(x) = \sum_{i=1}^n \frac{\partial f(x)}{\partial x_i} x_i.$$

2.4.7 Implicit Function Theorem

If the variable y is clearly expressed as a function in terms of \mathbf{x} , we call $y = f(x_1, x_2, \dots, x_n)$ an **explicit function**. In many cases, y is not an explicit function, and the relationship between y and x_1, \dots, x_n is expressed by the following equations:

$$F(y, x_1, x_2, \dots, x_n) = 0.$$

If for each vector \mathbf{x} in a certain range, there is a unique definite values of y satisfying the above equation, then y is an **implicit function** in terms of \mathbf{x} , denoted by $y = f(x_1, x_2, \dots, x_n)$. Thus, the question is how to determine whether there is a unique value y satisfying this equation for every \mathbf{x} in a certain range. The following implicit function theorem indicates that under certain conditions the implicit function $y = f(x_1, x_2, \dots, x_n)$ determined by $F(y, x_1, x_2, \dots, x_n) = 0$ not only exists but is derivable.

Theorem 2.4.9(Implicit Function Theorem) Let $X = \mathcal{R}^n$. Suppose that a function $F(y, x_1, x_2, \dots, x_n) = 0$ satisfies the four conditions below:

- (a) $F_y, F_{x_1}, F_{x_2}, \dots, F_{x_n}$ are continuous in the domain X containing $(y^0, x_1^0, x_2^0, \dots, x_n^0)$;
- (b) $F(y, x_1, x_2, \dots, x_n)$ has continuous partial derivatives with respect to \mathbf{x} and y in the domain X ;
- (c) $F(y^0, x_1^0, x_2^0, \dots, x_n^0) = 0$;
- (d) The partial derivative F_y of $F(y, x_1, x_2, \dots, x_n)$ with respect to y at $(y^0, x_1^0, x_2^0, \dots, x_n^0)$ is not equal to zero.

Then:

- (1) In a neighbourhood $N(\mathbf{x}^0)$ of a point $(x_1^0, x_2^0, \dots, x_n^0)$, the function $y = f(x_1, x_2, \dots, x_n)$ of (x_1, x_2, \dots, x_n) can be defined

implicitly, which satisfies $F(y(x_1, \dots, x_n), x_1, x_2, \dots, x_n) = 0$
and $y^0 = f(x_1^0, x_2^0, \dots, x_n^0)$.

(2) $y = f(x_1, x_2, \dots, x_n)$ is continuous in $N(\mathbf{x}^0)$.

(3) $y = f(x_1, x_2, \dots, x_n)$ has continuous partial derivative in $N(\mathbf{x}^0)$,
which is given by:

$$\frac{\partial y}{\partial x_i} = -\frac{F_i}{F_y}, \quad i = 1, \dots, n.$$

2.4.8 Concave and Convex Function

Next we discuss the concavity and convexity of functions. Concave functions, convex functions and quasi-concave functions are common functions in microeconomics and have strong economic significance. They hold a special position in optimization problems.

Definition 2.4.11 Suppose a convex set X and a function $f : X \rightarrow \mathcal{R}$, if for any $\mathbf{x}, \mathbf{x}' \in X$ and any $t \in [0, 1]$, we have

$$f(t\mathbf{x} + (1-t)\mathbf{x}') \geq tf(\mathbf{x}) + (1-t)f(\mathbf{x}'),$$

then f is defined to be **concave** on X .

If for all $\mathbf{x} \neq \mathbf{x}' \in X$ and $0 < t < 1$, we have

$$f(t\mathbf{x} + (1-t)\mathbf{x}') > tf(\mathbf{x}) + (1-t)f(\mathbf{x}'),$$

then f is **strictly concave** on X .

Definition 2.4.12 If $-f$ is (strictly) concave on X , then $f : X \rightarrow \mathcal{R}$ is called **(strictly) convex** function on X .

Remark: (1) A linear function is both a convex and a concave function.
(2) The sum of two concave (convex) functions is still concave (convex).
(3) The sum of a concave (convex) function and a strictly concave (convex) function is strictly concave (convex).

Remark: The statement that the function $f : X \rightarrow \mathcal{R}$ is concave on X is equivalent to the statement that for any $\mathbf{x}_1, \dots, \mathbf{x}_m \in X$ and any $t_i \in [0, 1]$, we have

$$f(t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \dots + t_m\mathbf{x}_m) \geq t_1f(\mathbf{x}_1) + \dots + t_mf(\mathbf{x}_m),$$

This formula is also called **Jensen's inequality**. If t_i is regarded as the probability of \mathbf{x}_i , when $f : X \rightarrow \mathcal{R}$ is concave on X , Jensen's inequality implies that the expectation of function value with respect to a random

variable is not less than the function value with respect to the expectation of the random variable, denoted by

$$f(E(X)) \geq E(f(X)).$$

In terms of differentiable functions, it can be determined by whether the second derivative or the second partial derivative matrix is positive (negative) definite.

Remark: A function f defined on X has a continuous second partial derivative, and then it is a concave (convex) function if and only if its **Hessian matrix** $D^2 f(\mathbf{x})$ is negative (positive) semi-definite on X . It is strictly concave (convex) if and only if its Hessian matrix $D^2 f(\mathbf{x})$ is negative (positive) definite on X .

Remark: The strict concavity of the function $f(\mathbf{x})$ can be determined by testing whether the principal minors of the Hessian matrix change signs alternately, namely,

$$\begin{aligned} \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} &> 0, \\ \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} &< 0, \end{aligned}$$

and so on, where $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. This algebraic condition is very useful for testing second order conditions of optimality.

2.4.9 Quasi-concave and Quasi-convex Function

In economic theory, quasi-concave functions are frequently used, especially in the representation of utility functions. The concept of quasi-concavity is relatively weaker than that of the concavity.

Definition 2.4.13 Suppose a convex set X and a function $f : X \rightarrow \mathcal{R}$, and if the set

$$\{x \in X : f(\mathbf{x}) \geq c\}$$

is a convex set for all real numbers c , then f is called **quasi-concave** on X . If the set

$$\{\mathbf{x} \in X : f(\mathbf{x}) > c\}$$

is a convex set for all c , then f is **strictly quasi-concave** on X .

If $-f$ is (strictly) quasi-concave on X , then the function $f : X \rightarrow \mathcal{R}$ is (strictly) **quasi-convex** on X .

Remark: The following facts are clear:

- (1) If a function f is (strictly) concave (convex), then it is (strictly) quasi-concave (convex);
- (2) The function f is (strictly) quasi-concave if and only if $-f$ is (strictly) quasi-convex;
- (3) An arbitrary (strictly) monotone function defined on the subset of one-dimensional real number space is both (strictly) quasi-concave and (strictly) quasi-convex;
- (4) The sum of two quasi-concave (convex) functions is generally not a quasi-concave (convex) function.

The following theorem correlates the quasi-concavity of a function to the convexity of upper contour set.

Theorem 2.4.10(Quasi-concavity and Upper Contour Sets) $f : X \rightarrow \mathcal{R}$ is a quasi-concave function if and only if for any $y \in \mathcal{R}$, $S(y) \equiv \{\mathbf{x} \in X : f(\mathbf{x}) \geq y\}$ is a convex set.

PROOF. Sufficiency: we first show that: if f is quasi-concave, then for all $y \in \mathcal{R}$, $S(y)$ is a convex set. Suppose that \mathbf{x}^1 and \mathbf{x}^2 are two arbitrary points of $S(y)$ (if $S(y)$ is the empty set, then we can complete the proof immediately since the empty set is convex). We need to show: if f is quasi-concave, all points in the form of $\mathbf{x}^t \equiv t\mathbf{x}^1 + (1-t)\mathbf{x}^2$, $t \in [0, 1]$ also belong to $S(y)$.

Since $\mathbf{x}^1 \in S(y)$ and $\mathbf{x}^2 \in S(y)$, it follows from the definition of upper contour set that both \mathbf{x}^1 and \mathbf{x}^2 belong to X and satisfy

$$f(\mathbf{x}^1) \geq y, f(\mathbf{x}^2) \geq y.$$

Now, we consider any \mathbf{x}^t . Since we suppose that X is a convex set, then $\mathbf{x}^t \in X$. If f is quasi-concave, then:

$$f(\mathbf{x}^t) \geq \min[f(\mathbf{x}^1), f(\mathbf{x}^2)] \geq y.$$

Thus, $\mathbf{x}^t \in X$ and $f(\mathbf{x}^t) \geq y$, and then \mathbf{x}^t meets the requirement that it is in $S(y)$. Therefore, $S(y)$ must be a convex set. Sufficiency is proved.

Necessity: we need to show: if for all $y \in \mathcal{R}$, $S(y)$ is a convex set, then $f(\mathbf{x})$ is a quasi-concave function. Let \mathbf{x}^1 and \mathbf{x}^2 be two arbitrary points in X . Without loss of generality, suppose $f(\mathbf{x}^1) \geq f(\mathbf{x}^2)$. Since for all $y \in \mathcal{R}$, $S(y)$ is a convex set, then $S(f(\mathbf{x}^2))$ must be convex obviously. It is also clear that $\mathbf{x}^2 \in S(f(\mathbf{x}^2))$, and since $f(\mathbf{x}^1) \geq f(\mathbf{x}^2)$, we have $\mathbf{x}^1 \in S(f(\mathbf{x}^2))$. Hence, for any convex combination of \mathbf{x}^1 and \mathbf{x}^2 , we must have $\mathbf{x}^t \in S(f(\mathbf{x}^2))$. It follows from the definition of $S(f(\mathbf{x}^2))$ that $f(\mathbf{x}^t) \geq f(\mathbf{x}^2)$. As a result, it can be obtained:

$$f(\mathbf{x}^t) \geq \min[f(\mathbf{x}^1), f(\mathbf{x}^2)].$$

Therefore, $f(\mathbf{x})$ is quasi-concave. Necessity is proved. \square \square

The following theorem characterizes the properties of quasi-concave functions, that is, quasi-concavity holds under monotonic functions, and note that concave functions do not have such properties.

Theorem 2.4.11 *Suppose the function $f : X \rightarrow \mathcal{R}$ is quasi-concave on X , and $h : \mathcal{R} \rightarrow \mathcal{R}$ is a monotonically non-decreasing function, then the composite function $h(f(\mathbf{x}))$ is also quasi-concave. If f is strictly quasi-concave and h is strictly increasing, the composite function is strictly quasi-concave.*

When a function f defined on a convex set X has continuous second order partial derivatives, the bordered Hessian determinant is defined as follows:

$$|B| = \begin{vmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ f_2 & f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_n & f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}$$

The principal minors of the bordered Hessian determinant B are as follows:

$$|B_1| = \begin{vmatrix} 0 & f_1 \\ f_1 & f_{11} \end{vmatrix}, \quad |B_2| = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix}, \quad \cdots, \quad |B_n| = |B|.$$

then the necessary condition for $f : X \rightarrow \mathcal{R}$ to be a quasi-concave function is

$$|B_1| \leq 0, \quad |B_2| \geq 0, \quad |B_3| \leq 0, \quad \cdots, \quad (-1)^n |B_n| \geq 0.$$

The sufficient condition for $f : X \rightarrow \mathcal{R}$ to be a strictly quasi-concave function is :

$$|B_1| < 0, \quad |B_2| > 0, \quad |B_3| < 0, \quad \cdots, \quad (-1)^n |B_n| > 0.$$

The necessary condition for $f : X \rightarrow \mathcal{R}$ to be a quasi-convex function is :

$$|B_1| \leq 0, \quad |B_2| \leq 0, \quad \cdots, \quad |B_n| \leq 0.$$

The sufficient condition for $f : X \rightarrow \mathcal{R}$ to be a strictly quasi-convex function is:

$$|B_1| < 0, \quad |B_2| < 0, \quad \cdots, \quad |B_n| < 0.$$

2.4.10 Separating Hyperplane Theorem

Separating hyperplane theorem also has crucial application in economics. First recall that if $X \subseteq \mathcal{R}^n$ is a **compact set**, then it is bounded and closed. X is **convex set**, if for any $\mathbf{x}, \mathbf{x}' \in X$ and any $0 \leq t \leq 1$, $t\mathbf{x} + (1-t)\mathbf{x}' \in X$. The convex set implies that the connections between any two points in the set belong to this set.

Theorem 2.4.12(Separating Hyperplane Theorem) *Suppose that $A, B \subseteq \mathcal{R}^m$ are convex and $A \cap B = \emptyset$. Then, there is a vector $\mathbf{p} \in \mathcal{R}^m, \mathbf{p} \neq 0$ and $c \in \mathcal{R}$ such that*

$$\mathbf{p}\mathbf{x} \leq c \leq \mathbf{p}\mathbf{y} \quad \forall \mathbf{x} \in A, \forall \mathbf{y} \in B.$$

Furthermore, suppose that $B \subseteq \mathcal{R}^m$ is convex and closed, $A \subseteq \mathcal{R}^m$ is convex and compact, and $A \cap B = \emptyset$. Then, there is a vector $\mathbf{p} \in \mathcal{R}^m, \mathbf{p} \neq 0$ and $c \in \mathcal{R}$ such that A, B are strictly separated, namely

$$\mathbf{p}\mathbf{x} < c < \mathbf{p}\mathbf{y} \quad \forall \mathbf{x} \in A, \forall \mathbf{y} \in B.$$

Definition 2.4.14 Let $C \subseteq \mathcal{R}^m$. If for any $\mathbf{x} \in C$ and $\lambda \in \mathcal{R}$, we have $\lambda\mathbf{x} \in C$, then C is called a **cone**.

Proposition 2.4.2 *A cone C is convex if and only if $\mathbf{x}, \mathbf{y} \in C$ implies $\mathbf{x} + \mathbf{y} \in C$.*

Proposition 2.4.3 *Let $C \subseteq \mathcal{R}^m$ be a closed and convex cone, and $K \subseteq \mathcal{R}^m$ be a compact and convex cone, then $C \cap K \neq \emptyset$ if and only if for any $\mathbf{p} \in C$, there is $\mathbf{z} \in K$ such that*

$$\mathbf{p} \cdot \mathbf{z} \leq 0.$$

2.5 Multi-Valued Function and Its properties

Set-valued mapping refers to the situation where the image of a mapping is not a point but a set. The followings give some basic concepts.

2.5.1 Point-to-Set Mappings

Suppose that X and Y are the two subsets of a topological vector space (such as Euclidean space).

Point-to-set mapping, is also called **correspondence** or **multi-valued function**. A correspondence F maps points \mathbf{x} in a domain X into sets in Y (for example, maps the point \mathbf{x} in $X \subseteq \mathcal{R}^n$ into the range $Y \subseteq \mathcal{R}^m$), denoted by $F : X \rightarrow 2^Y$. We also use $F : X \rightrightarrows Y$ or $F : X \rightarrow\rightarrow Y$ to denote the multi-valued mapping $F : X \rightarrow 2^Y$.

Definition 2.5.1 Let $F : X \rightarrow 2^Y$ be a correspondence.

- (1) If $F(\mathbf{x})$ is non-empty for all $\mathbf{x} \in X$, then the correspondence F is **non-empty valued**;
- (2) If $F(\mathbf{x})$ is a convex set for all $\mathbf{x} \in X$, then the correspondence F is **convex valued**;
- (3) If $F(\mathbf{x})$ is a closed set for all $\mathbf{x} \in X$, then the correspondence F is **closed valued**;
- (4) If $F(\mathbf{x})$ is compact for all $\mathbf{x} \in X$, then the correspondence F is **compact valued**;
- (5) If $F(\mathbf{x})$ is open for all $\mathbf{x} \in X$, then the correspondence F has **upper open sections**;
- (6) The preimage $F^{-1}(y) = \{\mathbf{x} \in X : y \in F(\mathbf{x})\}$ is open, then the correspondence F has **upper open sections**.

Definition 2.5.2 Let $F : X \rightarrow 2^X$ be a correspondence from X to X itself.

- (1) If for any $\mathbf{x}_1, \dots, \mathbf{x}_m \in X$ and its convex combination $\mathbf{x}_\lambda = \sum_{i=1}^m \lambda_i \mathbf{x}_i$, we have

$$\mathbf{x}_\lambda \in \bigcup_{i=1}^m F(\mathbf{x}_i),$$

then F is **FS-convex**.¹

- (2) If for any $\mathbf{x} \in X$, $\mathbf{x} \notin co F(\mathbf{x})$, then the correspondence F is **SS-convex**.²

Remark: It is easy to verify that the corresponding $P : X \rightarrow 2^X$ is SS-Convex if and only if the corresponding $G : X \rightarrow 2^X$ defined by $G^{-1}(\mathbf{x}) = X \setminus P(\mathbf{x})$ is FS-convex.

Specially, for the function $f : X \rightarrow \mathcal{R}$, it can be defined that the upper contour set is

$$U(\mathbf{x}) = \{\mathbf{y} \in X : f(\mathbf{y}) \geq f(\mathbf{x})\}, \forall \mathbf{x} \in X,$$

strict upper contour set is

$$U_s(\mathbf{x}) = \{\mathbf{y} \in X : f(\mathbf{y}) > f(\mathbf{x})\}, \forall \mathbf{x} \in X,$$

lower contour set is

$$L(\mathbf{x}) = \{\mathbf{y} \in X : f(\mathbf{y}) \leq f(\mathbf{x})\}, \forall \mathbf{x} \in X,$$

¹The concept of FS-convex is introduced by Fan (1984)& Sonnenschein (1971), so it is called FS-convex.

²The concept of SS-convex is introduced by Shafer & Sonnenschein (1975), so it is called SS-convex.

and strict lower contour set is

$$L_s(\mathbf{x}) = \{\mathbf{y} \in X : f(\mathbf{y}) < f(\mathbf{x})\}, \forall \mathbf{x} \in X.$$

We should pay attention to the following equivalence results which are used later in this book.

Proposition 2.5.1 *The following arguments are equivalent:*

- (1) *The function $f : X \rightarrow \mathcal{R}$ is quasi-concave;*
- (2) *$U : X \rightarrow 2^X$ is a convex-valued correspondence;*
- (3) *$U_s : X \rightarrow 2^X$ is a convex-valued correspondence;*
- (4) *$U_s : X \rightarrow 2^X$ is SS-convex;*
- (5) *$U : X \rightarrow 2^X$ is FS-convex.*

PROOF. It is clear that (1) implies (2), (2) implies (3), (3) implies (4), and (5) implies (1). We just need to show that (4) implies (5). Suppose not, there is a finite set $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\} \subset X$ and its a certain convex combination $\mathbf{x}_\lambda = \sum_{j=1}^m \lambda_j \mathbf{x}_j$ such that $\mathbf{x}_\lambda \notin \cup_{j=1}^m U(\mathbf{x}_j)$. Thus, for all j , we have $\mathbf{x}_\lambda \in L_s(\mathbf{x}_j)$, that is, $\mathbf{x}_j \in U_s(\mathbf{x}_\lambda)$ and hence $\mathbf{x}_\lambda \in co U_s(\mathbf{x}_\lambda)$, contradiction. \square

2.5.2 Upper Hemi-continuous and Lower Hemi-continuous Correspondence

Intuitively, a correspondence is continuous if small changes in x produce small changes in the set $F(x)$. Unfortunately, giving a formal definition of continuity for correspondences is not so simple. Figure 2.1 shows a continuous correspondence.

The notions of **hemi-continuity** are usually defined in terms of sequences (see Debreu (1959) and Mask-Collell et al. (1995)). Although they are relatively easy to verify, it seems that they depend on the assumption that a correspondence is compacted-valued. The following definitions are more formal (see Border, 1985).

Definition 2.5.3 For a correspondence $F : X \rightarrow 2^Y$ and a point \mathbf{x} , if for each open set U containing $F(\mathbf{x})$, there is an open set $N(\mathbf{x})$ containing \mathbf{x} such that $F(\mathbf{x}') \subseteq U$ for all $\mathbf{x}' \in N(\mathbf{x})$, then F is **upper hemi-continuous** at \mathbf{x} .

If the correspondence F is *upper hemi-continuous* at every $\mathbf{x} \in X$, then F is defined as **upper hemi-continuous** on X ; or equivalently, for every open subset V of Y , $\{\mathbf{x} \in X : F(\mathbf{x}) \subset V\}$ is always a open set of X .

Remark: Upper hemi-continuity captures the idea that $F(\mathbf{x})$ will not “suddenly contain new points” when passing through a point \mathbf{x} , in other words, $F(\mathbf{x})$ does not jump if \mathbf{x} changes slightly. That is, if one starts at a point \mathbf{x} and moves a little way to \mathbf{x}' , upper hemi-continuity at \mathbf{x} implies that there will be no point in $F(\mathbf{x}')$ that is not close to some point in $F(\mathbf{x})$.

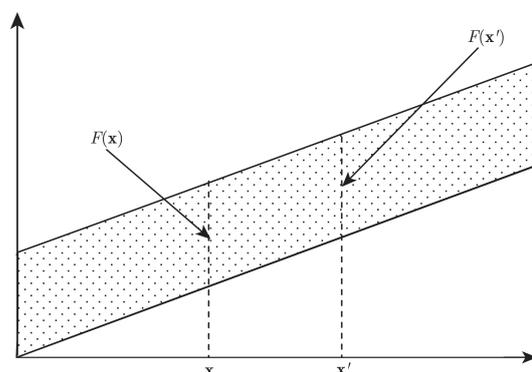


Figure 2.1: Continuous Correspondence

Definition 2.5.4 For a correspondence $F : X \rightarrow 2^Y$ and a point x , if for every open set V , $F(x) \cap V \neq \emptyset$, there exists a neighborhood $N(x)$ of x such that $F(x') \cap V \neq \emptyset$ for all $x' \in N(x)$, A correspondence F is **lower hemi-continuous** at x .

If F is lower hemi-continuous at every x , or equivalently, the set $\{x \in X : F(x) \cap V \neq \emptyset\}$ is open in X for every open set V of Y , then F is **lower hemi-continuous** on X .

Remark: Lower hemi-continuity captures the idea that any element in $F(x)$ can be “approached” from all directions, in other words, $F(x)$ does not suddenly becomes much smaller if one changes the argument x slightly. That is, if one starts at some point x and some point $y \in F(x)$, lower hemi-continuity at x implies that if one moves a little way from x to x' , there will be some $y' \in F(x')$ that is close to y .

Combining the concepts of upper and lower hemi-continuity, we can define the continuity of a correspondence.

Definition 2.5.5 A correspondence $F : X \rightarrow 2^Y$ is said to be continuous at $x \in X$ if it is both upper hemi-continuous and lower hemi-continuous at $x \in X$. The correspondence $F : X \rightarrow 2^Y$ is **continuous** on X if it is both upper hemi-continuous and lower hemi-continuous.

Figure 2.2 shows the correspondence is upper hemi-continuous, but not lower hemi-continuous. To see why it is upper hemi-continuous, imagine an open interval U that encompasses $F(x)$. Now consider moving a little to the left of x to a point x' . Clearly $F(x') = \{\hat{y}\}$ is in the interval. Similarly, if we move to a point x' a little to the right of x , then $F(x)$ will inside the interval so long as x' is sufficiently close to x . So it is upper hemi-continuous. On the other hand, the correspondence it not lower hemi-continuous. To see this, consider the point $y \in F(x)$, and let U be a very small interval

around y that does not include \hat{y} . If we take any open set $N(x)$ containing x , then it will contain some point x' to the left of x . But then $F(x') = \{\hat{y}\}$ will contain no points near y , i.e., it will not intersect U . Thus, the correspondence is not lower hemi-continuous.

Figure 2.3 shows the correspondence is lower hemi-continuous, but not upper hemi-continuous. To see why it is lower hemi-continuous: For any $0 \leq x' \leq x$, note that $F(x') = \{\hat{y}\}$. Let $x_n = x' - 1/ne$, $y_n = \hat{y}$. Then $x_n > 0$ for sufficiently large n , $x_n \rightarrow x'$, $y_n \rightarrow \hat{y}$, and $y_n \in F(x_n) = \{\hat{y}\}$. So it is lower hemi-continuous. It is clearly lower hemi-continuous for $x_i > x$. Thus, it is lower hemi-continuous on X . On the other hand, the correspondence is not upper hemi-continuous. If we start at x by noting that $F(x) = \{\hat{y}\}$, and make a small move to the right to a point x' , then $F(x')$ suddenly contains many points that are not close to \hat{y} . So this correspondence fails to be upper hemi-continuous.

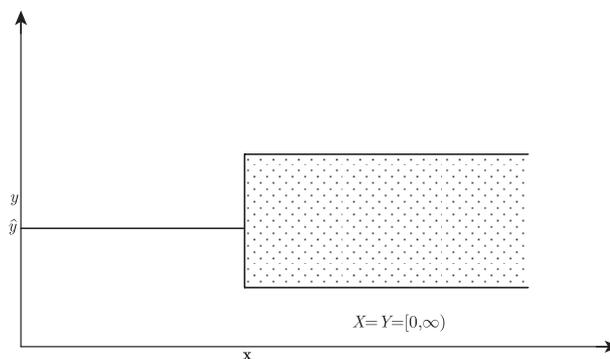


Figure 2.2: The correspondence is upper hemi-continuous, but not lower hemi-continuous

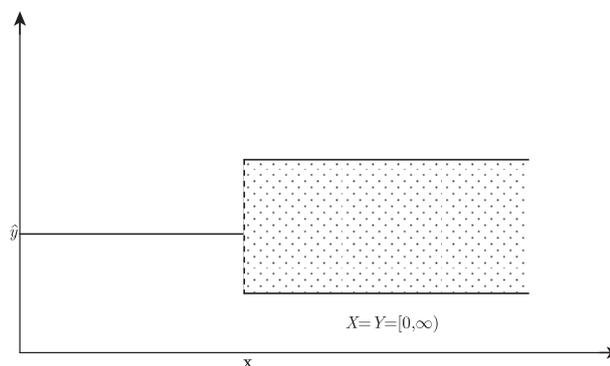


Figure 2.3: The correspondence is lower hemi-continuous, but not upper hemi-continuous

Remark: As it turns out, the notions of upper and hemi-continuous correspondence both reduce to the standard notion of continuity for a function if $F(\cdot)$ is a single-valued correspondence, i.e., a function. That is, $F(\cdot)$ is a single-valued upper (or lower) hemi-continuous correspondence if and only if it is a continuous function.

Remark: Based on the following two facts, both notions of hemi-continuity can be characterized by sequences.

- (a) If a correspondence $F : X \rightarrow 2^Y$ is compacted-valued, then it is upper hemi-continuous if and only if for any $\{x_k\}$ and $\{y_k\}$, where $x_k \rightarrow x$, $y_k \in F(x_k)$, there exists a converging subsequence $\{y_{k_m}\}$, such that $y_{k_m} \rightarrow y$ and $y \in F(x)$.
- (b) A correspondence $F : X \rightarrow 2^Y$ is said to be lower hemi-continuous at x if and only if for any $\{x_k\}$ and $y \in F(x)$, where $x_k \rightarrow x$, there is a sequence $\{y_k\}$ such that $y_k \rightarrow y$ and $y_k \in F(x_k)$.

2.5.3 The Open and Closed Graphs of Correspondence

Definition 2.5.6 A correspondence $F : X \rightarrow 2^Y$ is said to be **closed** at x if for any $\{x_k\}$ and $\{y_k\}$, where $x_k \rightarrow x$ and $y_k \rightarrow y$, $y_k \in F(x_k)$, we have $y \in F(x)$. F is said to be **closed** if F is closed for all $x \in X$, or equivalently

$$Gr(F) = \{(x, y) \in X \times Y : y \in F(x)\} \text{ is closed.}$$

Regarding the relationship between upper hemi-continuity and closed graph, the following facts can be proved.

Proposition 2.5.2 Let $F : X \rightarrow 2^Y$ is a correspondence.

- (i) Suppose Y is compact and $F : X \rightarrow 2^Y$ is closed-valued, If F has closed graph, it is upper hemi-continuous.
- (ii) Suppose X and Y are closed and $F : X \rightarrow 2^Y$ is closed-valued, if F is upper hemi-continuous, then it has closed graph.

Because of fact (i), a correspondence with closed graph is sometimes used to define a hemi-continuous correspondence in the literature. But one should keep in mind that they are not the same in general. For example, let $F : \mathcal{R}_+ \rightarrow 2^{\mathcal{R}}$ be defined by

$$F(x) = \begin{cases} \left\{ \frac{1}{x} \right\}, & \text{if } x > 0, \\ \{0\}, & \text{if } x = 0. \end{cases}$$

The correspondence is closed but not upper hemi-continuous. Also, define $F : \mathcal{R}_+ \rightarrow 2^{\mathcal{R}}$ by $F(x) = (0, 1)$. Then F is upper hemi-continuous but not closed.

Definition 2.5.7 Correspondence $F : X \rightarrow 2^Y$ is said to be *open* if its graph

$$Gr(F) = \{(\mathbf{x}, y) \in X \times Y : y \in F(\mathbf{x})\} \text{ is open.}$$

Proposition 2.5.3 Let $F : X \rightarrow 2^Y$ be a correspondence. Then,

- (1) if a correspondence $F : X \rightarrow 2^Y$ has an open graph, then it has upper and lower open sections.
- (2) If a correspondence $F : X \rightarrow 2^Y$ has lower open sections, then it must be lower hemi-continuous.

2.5.4 Transfer Closed-valued Correspondence

Tian Guoqiang and collaborators introduced the concepts of transfer closed, transfer open, transfer convex and others for the multivalued mapping (correspondence) in Tian (1992, 1993) and Zhou and Tian (1992), which weakens the conditions for the establishment of some basic mathematical theorems in nonlinear analysis and the existence of equilibrium solution of the optimization problem, and gets many characterization results, such as the existence of the maximum element of preference relations and the existence of Nash equilibrium in the game. These conclusions are given in the corresponding chapters in this book.

Denote $\text{int } D$ and $\text{cl } D$ as the interior points and closure of set D respectively.

Definition 2.5.8 If for any $\mathbf{x} \in X$, $y \notin G(\mathbf{x})$ implies that there is a $\mathbf{x}' \in X$ such that $y \notin \text{cl } G(\mathbf{x}')$, then the correspondence $G : X \rightarrow 2^Y$ is **transfer closed-valued** on X .

Definition 2.5.9 If for any $\mathbf{x} \in X$ and $y \in Y$, $\mathbf{x} \in P(y)$ implies that there is a point $\mathbf{x}' \in X$ such that $y \in \text{int } P(\mathbf{x}')$, then the correspondence $P : X \rightarrow 2^Y$ has **transfer upper open sections** on X .

Remark: If a correspondence is closed-valued, then it is a transfer closed-valued(it is obtained by $\mathbf{x}' = \mathbf{x}$); if a corresponding has upper open sections, then it has the transfer upper open sections (let $\mathbf{x}' = \mathbf{x}$). Meanwhile, the correspondence $P : X \rightarrow 2^Y$ has upper open sections in X if and only if $G : X \rightarrow 2^Y$ defined by $G(\mathbf{x}) = Y \setminus P(\mathbf{x})$ is transfer closed-valued in X .

Remark: For any function $f : X \rightarrow \mathcal{R}$, the correspondence $G : X \rightarrow 2^Y$ defined by

$$G(\mathbf{x}) = \{y \in X : f(y) \geq f(\mathbf{x})\}, \forall \mathbf{x} \in X$$

is transfer closed-valued if and only if f is continuous on X .

The following proposition can effectively weaken the continuity conditions when proving many optimization problems.

Proposition 2.5.4(Tian (1992)) *Let X and Y be two topological spaces, $G : X \rightarrow 2^Y$ is a correspondence from point to set. Then*

$$\bigcap_{x \in X} cl G(x) = \bigcap_{x \in X} G(x)$$

if and only if G is transfer closed-valued on X .

PROOF. Sufficiency: we need to show

$$\bigcap_{x \in X} cl G(x) = \bigcap_{x \in X} G(x).$$

It is clear that

$$\bigcap_{x \in X} G(x) \subseteq \bigcap_{x \in X} cl G(x),$$

thus we just need to show that

$$\bigcap_{x \in X} cl G(x) \subseteq \bigcap_{x \in X} G(x).$$

Suppose not, then there is a y such that $y \in \bigcap_{x \in X} cl G(x)$, but $y \notin \bigcap_{x \in X} G(x)$. Hence, for a $z \in X$, we have $y \notin G(z)$. Now that G is transfer closed-valued on X , then there exists a $z' \in X$ such that $y \notin cl G(z')$, thus $y \notin cl G(z')$, contradiction.

Necessity: suppose that

$$\bigcap_{x \in X} cl G(x) = \bigcap_{x \in X} G(x).$$

If $y \notin G(x)$, then

$$y \notin \bigcap_{x \in X} cl G(x) = \bigcap_{x \in X} G(x),$$

and thus $y \notin cl G(x')$ for $x' \in X$. Hence, G is a transfer closed-valued correspondence on X \square

Similarly, we can define the transfer convexity.

Definition 2.5.10(Transfer FS-convex) *Let X be a topological space and Z be a convex subset of the topological space. For the correspondence $G : X \rightarrow 2^Z$ and any a finite set $\{x_1, x_2, \dots, x_n\} \subseteq X$, if there is a corresponding finite set $\{y_1, y_2, \dots, y_n\} \subseteq Z$ such that for any subset $\{y_{i_1}, y_{i_2}, \dots, y_{i_s}\} (1 \leq s \leq n)$, we have*

$$co \{y_{i_1}, y_{i_2}, \dots, y_{i_s}\} \subseteq \bigcup_{r=1}^s G(y_{ir}),$$

then G is **transfer FS-convex** on X .

Definition 2.5.11(Transfer SS-convex) Let X be a topological space, and Z be a convex subset of the topological space. For the correspondence $P : Z \rightarrow 2^X$ and any a finite set $\{y_1, y_2, \dots, y_n\} \subseteq X$, if there always exists a finite set $\{y_1, y_2, \dots, y_n\} \subseteq Z$ such that for any subset $\{y_{i_1}, y_{i_2}, \dots, y_{i_s}\} (1 \leq s \leq n)$ and any $y_{i_0} \in co \{y_{i_1}, y_{i_2}, \dots, y_{i_s}\}$, we have $x_{i_0} \notin P(y_{i_0})$, then P is **transfer SS-convex** on X .

Remark: Unlike defining FS-convex and SS-Convex, when defining the transfer FS-convex and transfer SS-convex, we do not assume that correspondences are mapping from itself to itself. It is clear that when $X = Z$ and picking $y_i = x_i$, FS-convex implies transfer FS-convex and SS-convex implies transfer SS-convex. Similarly, it is not difficult to verify that the correspondence $P : X \rightarrow 2^Z$ is transfer SS-convex if and only if $G : X \rightarrow 2^X$ defined by $G^{-1}(x) = Z \setminus P(x)$ is transfer FS-convex.

2.6 Static Optimization

The optimization problem is the core issue in economics. Rationality is the most basic assumption about individual decision-makers in economics. Individuals pursuit maximizing the interests of themselves, and the foundation of their analysis is solving optimization problems. This section introduces the problem of solving static optimization. The basic results will be used in the whole book. Finally, we discuss the problem of dynamic optimization

2.6.1 Unconstrained Optimization

The basic optimization problem is that of maximizing or minimizing a function on some set. Let X be an arbitrary topological space. First we give the following concepts:

Definition 2.6.1(Local Optimum) If $f(x^*) \geq f(x)$ for all x in some neighbourhoods of x^* , then the function has **local maximum** at point x^* .

If $f(\tilde{x}) \leq f(x) (f(\tilde{x}) < f(x))$ for all $x \neq \tilde{x}$ in some neighbourhoods of \tilde{x} , then the function has **local minimum (unique local minimum)** at \tilde{x} .

Definition 2.6.2(Global Optimum) If $f(x^*) \geq f(x) (f(x^*) > f(x))$ for all x in the domain of the function, then the function has **global (unique) maximum** at x^* ; if $f(x^*) \leq f(x) (f(x^*) < f(x))$ for all x in the domain of the function, then the function has **global (unique) minimum** at x^* .

A classical conclusion about global optimization is the so-called **Weierstrass theorem**.

Theorem 2.6.1(Weierstrass Theorem) *Any upper (lower) semi-continuous function must attain a maximum and a minimum on a compact set, and the set of maximum points is compact.*

Tian Guoqiang and collaborators introduce some very weak continuity—transfer continuity, then generalize the Weierstrass theorem and give a sufficient and necessary condition for a function f to have a global maximum (minimum) value on a compact set X , sufficient and necessary conditions for the global maximum (minimum) set to be compact and the characteristics of a function that has a global maximum (minimum) value in Tian (1992, 1993, 1994), Tian & Zhou (1995) and Zhou & Tian (1992).

Theorem 2.6.2(Tian-Zhou Theorem I) *Suppose that X is a compact set in an arbitrary topological space. The function $f : X \rightarrow \mathcal{R}$ has a maximum (minimum) on X if and only if f is transfer weakly upper continuous on X .*

PROOF. Since f is transfer weakly upper continuous on X if and only if it is transfer weakly lower continuous, we just need to show the case that the function has a maximum value.

Sufficiency: proof by contradiction, suppose that f cannot reach a maximum value on X , then for each $\mathbf{y} \in X$, there is $\mathbf{x} \in X$ such that $f(\mathbf{x}) > f(\mathbf{y})$. It follows from the transfer weakly continuity of f that there is a $\mathbf{x}' \in X$ and a neighbourhood $\mathcal{N}(\mathbf{y})$ of \mathbf{y} such that $f(\mathbf{x}') \geq f(\mathbf{y}')$ for all $\mathbf{y}' \in \mathcal{N}(\mathbf{y})$. Hence, we have $X = \cup_{\mathbf{y} \in X} \mathcal{N}(\mathbf{y})$. Since X is compact, there is a finite number of points $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ such that $X = \cup_{i=1}^n \mathcal{N}(\mathbf{y}_i)$. Let \mathbf{x}'_i be all the points such that $f(\mathbf{x}'_i) \geq f(\mathbf{y}')$ for all $\mathbf{y}' \in \mathcal{N}(\mathbf{y}_i)$. f must have the maximum in the finite subset $\{\mathbf{x}'_1, \mathbf{x}'_2, \dots, \mathbf{x}'_n\}$. Without loss of generality, suppose \mathbf{x}'_1 satisfying $f(\mathbf{x}'_1) \geq f(\mathbf{x}'_i)$ for $i = 1, 2, \dots, n$. It follows that f has no maximum on X , that is, \mathbf{x}'_1 is not the maximum point of f on X , Thus there exist $\mathbf{x} \in X$ such that $f(\mathbf{x}) > f(\mathbf{x}'_1)$. However, since $X = \cup_{i=1}^n \mathcal{N}(\mathbf{y}_i)$, there is j such that $\mathbf{x} \in \mathcal{N}(\mathbf{y}_j)$ and then $f(\mathbf{x}'_j) \geq f(\mathbf{x})$. Thus $f(\mathbf{x}) > f(\mathbf{x}'_1) \geq f(\mathbf{x}'_j) \geq f(\mathbf{x})$, contradiction. Therefore, f is sure to reach the maximum on X .

Necessity: Clearly. Let \mathbf{x}' be any maximum point of f , then $f(\mathbf{x}') \geq f(\mathbf{y}')$ holds for all $\mathbf{y}' \in X$.

□

In many cases, when proving the existence of competitive equilibrium and game equilibrium, we need not only prove the existence of an optimal result, but also prove that the set of the optimal results is compact.

Theorem 2.6.3(Tian-Zhou Theorem) II *Suppose X is a compact set in an arbitrary topological space, and $f : X \rightarrow \mathcal{R}$ is a function. The set of maximum (minimum) points of f on X is nonempty and compact if and only if f is transfer upper (lower) continuous on X .*

PROOF. We only need to prove the case of a set of maximum points.

Necessity: Suppose the set of maximum points of f on X is nonempty and closed. If $f(\mathbf{y}) < f(\mathbf{x})$ for any $\mathbf{x}, \mathbf{y} \in X$, then \mathbf{y} cannot be a maximum point of f on X . It follows from the compactness of the set of maximum points that there is a neighbourhood $\mathcal{N}(\mathbf{y})$ of \mathbf{y} that does not contain any maximum points of f on X . Let \mathbf{x}' be a maximum point of f on X , then $f(\mathbf{z}) < f(\mathbf{x}')$ for all $\mathbf{z} \in \mathcal{N}(\mathbf{y})$. Thus, f is transfer upper continuous on X .

Sufficiency: First note that $G : X \rightarrow 2^X$ defined by

$$G(\mathbf{x}) = \{\mathbf{y} \in X : f(\mathbf{y}) \geq f(\mathbf{x})\}, \forall \mathbf{x} \in X$$

is a transfer closed-valued correspondence if and only if f is transfer upper continuous on X . Since f is transfer continuous on X , according to Proposition 2.5.4, we have $\bigcap_{\mathbf{x} \in X} cl G(\mathbf{x}) = \bigcap_{\mathbf{x} \in X} G(\mathbf{x})$, and hence the set of maximum points is closed.

Since f has a maximum point on any finite subset $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\} \subseteq X$, and suppose \mathbf{x}_1 , that is, $f(\mathbf{x}_1) \geq f(\mathbf{x}_i)$ holds for all $i = 1, \dots, m$. Then we have $\mathbf{x}_1 \in G(\mathbf{x}_i)$ for $i = 1, \dots, m$, and thus

$$\emptyset \neq \bigcap_{i=1}^m G(\mathbf{x}_i) \subseteq \bigcap_{i=1}^m cl G(\mathbf{x}_i),$$

namely, the class of sets $\{cl G(\mathbf{x}) : \mathbf{x} \in X\}$ has the property of finite intersection on X . Since $\{cl G(\mathbf{x}) : \mathbf{x} \in X\}$ is a collection of closed sets in compact set X , $\emptyset \neq \bigcap_{\mathbf{x} \in X} cl G(\mathbf{x}) = \bigcap_{\mathbf{x} \in X} G(\mathbf{x})$. It implies that there is $\mathbf{x}^* \in X$ such that $f(\mathbf{x}^*) \geq f(\mathbf{x})$ for all $\mathbf{x} \in X$. Since the set of maximum points $\bigcap_{\mathbf{x} \in X} cl G(\mathbf{x})$ is a closed subset of the compact set X , it is also compact. q.e.d \square

\square

In order to easily judge whether a function has an extreme value, the following gives the conclusion of finding extreme values by differential method. We first show the necessary conditions for interior extreme points without constraints, and then give the sufficient conditions.

Generally, there are two necessary conditions for the interior extreme point, that is, the first and second order necessary conditions.

Theorem 2.6.4(The first-order necessary condition for interior extreme points) Suppose $X \subseteq \mathcal{R}^n$. If a differentiable function $f(\mathbf{x})$ reaches a local maximum or minimum at an interior point $\mathbf{x}^* \in X$, then \mathbf{x}^* is the solution to the following system of simultaneous equations:

$$\frac{\partial f(\mathbf{x}^*)}{\partial x_1} = 0$$

$$\frac{\partial f(\mathbf{x}^*)}{\partial x_2} = 0$$

$$\begin{aligned} & \vdots \\ & \frac{\partial f(\mathbf{x}^*)}{\partial x_n} = 0 \end{aligned}$$

PROOF. Suppose that $f(\mathbf{x})$ reaches the local extreme value at an interior point \mathbf{x}^* , and we need to prove that $Df(\mathbf{x}^*) = 0$. Although this proof is not simple, it will be very useful when considering the second order condition.

Choose any vector $\mathbf{z} \in \mathcal{R}^n$, and then construct a familiar univariate function of any scalar t :

$$g(t) = f(\mathbf{x}^* + t\mathbf{z})$$

First, for $t \neq 0$, $\mathbf{x}^* + t\mathbf{z}$ gives a vector that is different from \mathbf{x}^* . For $t = 0$, $\mathbf{x}^* + t\mathbf{z}$ is equal to \mathbf{x}^* , thus $g(0)$ is exactly the value of f at \mathbf{x}^* . According to the assumption that f attains an extremum at \mathbf{x}^* , $g(t)$ must reach a local extreme at $t = 0$. It follows from Fermat Theorem given by the proposition 2.4.4 that $g'(0) = 0$. Take the derivative of $g(t)$ by the Chain Rule:

$$g'(t) = \sum_{i=1}^n \frac{\partial f(\mathbf{x}^* + t\mathbf{z})}{\partial x_i} z_i.$$

When $t = 0$ and using $g'(0) = 0$, we have

$$g'(0) = \sum_{i=1}^n \frac{\partial f(\mathbf{x}^*)}{\partial x_i} z_i = Df(\mathbf{x}^*)\mathbf{z} = 0.$$

Since the equation above holds for any vector \mathbf{z} , including for those n unit vectors, it means that each partial derivative of f must equal to zero, namely

$$Df(\mathbf{x}^*) = 0. \quad \square$$

\square

Theorem 2.6.5(The second-order necessary conditions for interior extreme points) *Suppose that $f(\mathbf{x})$ is twice continuously differentiable on $X \subseteq \mathcal{R}^n$.*

(1) *If $f(\mathbf{x})$ reaches a local maximum at the interior point \mathbf{x}^* , then $\mathbf{H}(\mathbf{x}^*)$ is negative semi-definite.*

(2) *If $f(\mathbf{x})$ reaches a local minimum at the interior point $\tilde{\mathbf{x}}$, then $\mathbf{H}(\tilde{\mathbf{x}})$ is positive semi-definite.*

PROOF. Let $g(t) = f(\mathbf{x} + t\mathbf{z})$, $\mathbf{z} \in \mathcal{R}^n$ and \mathbf{x} be a stationary point of f . If f attains a stationary point at \mathbf{x} , then g gets a stationary point at $t = 0$. For any t , we have

$$g'(t) = \sum_{i=1}^n \frac{\partial f(\mathbf{x} + t\mathbf{z})}{\partial x_i} z_i.$$

We have the second order derivatives:

$$g''(t) = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f(\mathbf{x} + t\mathbf{z})}{\partial x_i \partial x_j} z_i z_j.$$

Now suppose that f reaches maximum at $\mathbf{x} = \mathbf{x}^*$. Since $g''(0) \leq 0$, then the value of $g''(t)$ at \mathbf{x}^* and $t = 0$ is:

$$g''(0) = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f(\mathbf{x}^*)}{\partial x_i \partial x_j} z_i z_j \leq 0$$

or $\mathbf{z}^T \mathbf{H}(\mathbf{x}) \mathbf{z} \leq 0$. Since \mathbf{z} is arbitrary, it implies that $\mathbf{H}(\mathbf{x}^*)$ is negative semi-definite. Similarly, if f is minimized at $\mathbf{x} = \tilde{\mathbf{x}}$, then $g''(0) \geq 0$ and $\mathbf{H}(\tilde{\mathbf{x}})$ is positive semi-definite. \square \square

Then the sufficient conditions for optimization are discussed.

Theorem 2.6.6 (The sufficient conditions for interior extreme points) *Suppose that $f(\mathbf{x})$ is twice continuously differentiable on $X \subseteq \mathcal{R}^n$, and we have*

- (1) if $f_i(\mathbf{x}^*) = 0$, and $(-1)^i D_i(\mathbf{x}^*) > 0$, $i = 1, \dots, n$, then $f(\mathbf{x})$ has a local maximum at \mathbf{x}^* .
- (2) if $f_i(\tilde{\mathbf{x}}) = 0$, and $D_i(\tilde{\mathbf{x}}) > 0$, $i = 1, \dots, n$, then $f(\mathbf{x})$ has a local minimum at $\tilde{\mathbf{x}}$.

Generally, the local optimum is not equal to the global optimum, but under certain conditions, these two are consistent.

Theorem 2.6.7 (Local and Global Optimum) *Suppose that f is a concave and twice continuously differentiable real function on $X \subseteq \mathcal{R}^n$, and here the point \mathbf{x}^* is a interior point of X , then the following three propositions are equivalent:*

- (1) $Df(\mathbf{x}^*) = 0$.
- (2) f has a local maximum at \mathbf{x}^* .
- (3) f has a global maximum at \mathbf{x}^* .

PROOF. It is clear that (3) \Rightarrow (2), and it follows from the previous theorem that (2) \Rightarrow (1). Thus, we just need to prove that (1) \Rightarrow (3).

Suppose that $Df(\mathbf{x}^*) = 0$, then f is concave implies that for all \mathbf{x} in the domain, we have:

$$f(\mathbf{x}) \leq f(\mathbf{x}^*) + Df(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*).$$

These two formulas mean that for all \mathbf{x} , we must have

$$f(\mathbf{x}) \leq f(\mathbf{x}^*).$$

Therefore, f reaches a global maximum at \mathbf{x}^* . \square \square

Theorem 2.6.8(Strict Concavity/Convexity and Uniqueness of Global Optimization) *Let X be a topological vector space.*

- (1) *If a strictly concave function f defined on X reaches a local maximum value at \mathbf{x}^* , then \mathbf{x}^* is the unique global maximum point.*
- (2) *If a strictly convex function f reaches a local minimum value at $\tilde{\mathbf{x}}$, then $\tilde{\mathbf{x}}$ is the unique global minimum point.*

PROOF. Proof by contradiction. If \mathbf{x}^* is a global maximum point of function f but not unique, then there is a point $\mathbf{x}' \neq \mathbf{x}^*$ such that $f(\mathbf{x}') = f(\mathbf{x}^*)$. Suppose $\mathbf{x}^t = t\mathbf{x}' + (1-t)\mathbf{x}^*$, then strict concavity requires that for all $t \in (0, 1)$,

$$f(\mathbf{x}^t) > tf(\mathbf{x}') + (1-t)f(\mathbf{x}^*).$$

Since $f(\mathbf{x}') = f(\mathbf{x}^*)$,

$$f(\mathbf{x}^t) > tf(\mathbf{x}') + (1-t)f(\mathbf{x}') = f(\mathbf{x}').$$

Contradict with the assumption that \mathbf{x}' is a global maximum point of f . Therefore, the global maximum point of a strictly concave function is unique. Similar to the proof of Proposition (2). \square

Theorem 2.6.9(The sufficient condition for the uniqueness of global optimum) *Suppose that $f(\mathbf{x})$ is twice continuously differentiable on $X \subseteq \mathcal{R}^n$. We have:*

- (1) *if $f(\mathbf{x})$ is strictly concave and $f_i(\mathbf{x}^*) = 0, i = 1, \dots, n$, then \mathbf{x}^* is a unique global maximum point of $f(\mathbf{x})$.*
- (2) *if $f(\mathbf{x})$ is strictly convex and $f_i(\tilde{\mathbf{x}}) = 0, i = 1, \dots, n$, then $\tilde{\mathbf{x}}$ is a unique global minimum point of $f(\mathbf{x})$.*

2.6.2 Optimization with Equality Constraints

Equality Constrained Optimization

An optimization problem with equality constraints has the following form: suppose a function n variable defined on $X \subseteq \mathcal{R}^n$ with m constraints, where $m < n$. The optimization problem is:

$$\begin{aligned} & \max_{x_1, \dots, x_n} f(x_1, \dots, x_n) \\ & \text{s.t. } g^1(x_1, \dots, x_n) = 0, \\ & \quad g^2(x_1, \dots, x_n) = 0, \\ & \quad \vdots \end{aligned}$$

$$g^n(x_1, \dots, x_n) = 0.$$

The most important conclusion of the equality constrained optimization problem is the **Lagrange theorem**, which gives a necessary condition for a point to be the solution of the optimization problem.

The Lagrange function of above equality constrained problem is defined as:

$$\mathcal{L}(\mathbf{x}, \lambda) = f(\mathbf{x}) + \sum_{j=1}^m \lambda_j g^j(\mathbf{x}). \quad (2.6.1)$$

where $\lambda_1, \dots, \lambda_m$ are called **Lagrange multipliers**.

The following Lagrange theorem presents how to solve optimization problems under equality constraints.

Theorem 2.6.10(The First-Order Necessary Condition for the interior extremum points with equality constraint) Suppose $f(\mathbf{x})$ and $g^j(\mathbf{x}), j = 1, \dots, m$, are continuously differentiable real functions defined on $X \subseteq \mathcal{R}^n$, \mathbf{x}^* is an interior point of X and \mathbf{x}^* is a extreme point (maximum or minimum) of f —here f is subjected to the constaint of $g^j(\mathbf{x}^*) = 0$, where $j = 1, \dots, m$. If the gradient $Dg^j(\mathbf{x}^*) = 0, j = 1, \dots, m$ are linearly independent, then there is a unique $\lambda_j^*, j = 1, \dots, m$ such that:

$$\frac{\partial \mathcal{L}(\mathbf{x}^*, \Lambda^*)}{\partial x_i} = \frac{\partial f(\mathbf{x}^*)}{\partial x_i} + \sum_{j=1}^m \lambda_j^* \frac{\partial g^j(\mathbf{x}^*)}{\partial x_i} = 0, \quad i = 1, \dots, n.$$

The following propositions give the sufficient conditions for the extreme value at interior points with equality constraints.

Proposition 2.6.1(The Second-Order Necessary Condition for the interior extremum points with equality constraint) Suppose f and g^1, \dots, g^m are twice continuously differentiable functions, and \mathbf{x}^* satisfy the necessary conditions of Theorem(2.6.10). Let bordered Hessian determinant

$$|\bar{H}_r| = \det \begin{pmatrix} 0 & \cdots & 0 & \frac{\partial g^1}{\partial x_1} & \cdots & \frac{\partial g^1}{\partial x_r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{\partial g^m}{\partial x_1} & \cdots & \frac{\partial g^m}{\partial x_r} \\ \frac{\partial g^1}{\partial x_1} & \cdots & \frac{\partial g^m}{\partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g^1}{\partial x_r} & \cdots & \frac{\partial g^m}{\partial x_r} & \frac{\partial^2 \mathcal{L}}{\partial x_r \partial x_1} & \cdots & \frac{\partial^2 \mathcal{L}}{\partial x_r \partial x_r} \end{pmatrix}, r = m+1, 2, \dots, n$$

take value at \mathbf{x}^* . Thus

- (1) if $(-1)^{r-m+1}|\bar{H}_r(\mathbf{x}^*)| > 0, r = m + 1, \dots, n$, then \mathbf{x}^* is the local maximum of the optimization problem.
- (2) if $|\bar{H}_r(\mathbf{x}^*)| < 0, r = m + 1, \dots, n$, then \mathbf{x}^* is the local minimum of the optimization problem.

Specially, when there is only one equality constraint, that is, $m = 1$, the bordered Hessian determinant $|\bar{H}|$ becomes:

$$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & \mathcal{L}_{11} & \mathcal{L}_{12} & \cdots & \mathcal{L}_{1n} \\ g_2 & \mathcal{L}_{21} & \mathcal{L}_{22} & \cdots & \mathcal{L}_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_n & \mathcal{L}_{n1} & \mathcal{L}_{n2} & \cdots & \mathcal{L}_{nn} \end{vmatrix}.$$

where $\mathcal{L}_{ij} = f_{ij} - \lambda g_{ij}$. Note that the first order condition is

$$\lambda = \frac{f_1}{g_1} = \frac{f_2}{g_2} = \cdots = \frac{f_n}{g_n}.$$

The principal minors of the bordered Hessian are

$$|\bar{H}_2| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & \mathcal{L}_{11} & \mathcal{L}_{12} \\ g_2 & \mathcal{L}_{21} & \mathcal{L}_{22} \end{vmatrix}, \quad |\bar{H}_3| = \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ g_2 & \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ g_3 & \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{vmatrix}, \quad \cdots.$$

It leads to the following two conclusions.

Conditions for Minimum with Equality Constraints

- (1) $\mathcal{L}_\lambda = \mathcal{L}_1 = \mathcal{L}_2 = \cdots = \mathcal{L}_n = 0$ [First Order Necessary Condition];
- (2) $|\bar{H}_2| > 0, |\bar{H}_3| < 0, |\bar{H}_4| > 0, \dots, (-1)^n |\bar{H}_n| > 0$.

Conditions for Maximum with Equality Constraints

- (1) $\mathcal{L}_\lambda = \mathcal{L}_1 = \mathcal{L}_2 = \cdots = \mathcal{L}_n = 0$ [First Order Necessary Condition];
- (2) $|\bar{H}_2| < 0, |\bar{H}_3| < 0, |\bar{H}_4| < 0, \dots, |\bar{H}_n| < 0$.

Note that when the constraint function g is linear, $g(\mathbf{x}) = a_1x_1 + \cdots + a_nx_n = c$, all the twice partial derivatives of g are equal to zero, so the bordered determinant $|B|$ and the bordered Hessian determinant have the following relations:

$$|B| = \lambda^2 |\bar{H}|$$

Thus, the sequential principal minors of the bordered determinant have the same signs. So, as long as the objective function is strictly quasi-concave, the first order necessary condition is also a sufficient condition to get the maximum value.

2.6.3 Optimization with Inequality Constraints

Consider an optimization problem with inequality constraints:

$$\begin{aligned} & \max f(\mathbf{x}) \\ \text{s.t.} & \quad g_i(\mathbf{x}) \leq d_i, \quad i = 1, 2, \dots, k. \end{aligned}$$

If for all points \mathbf{x} making all constraints held with equality, $Dg_1(\mathbf{x}), Dg_2(\mathbf{x}), \dots, Dg_k(\mathbf{x})$ are linearly independent,

then \mathbf{x} is said to satisfy the **constrained qualification**, where the symbol D represents the partial differential operator.

Theorem 2.6.11(Kuhn-Tucker Theorem) *Suppose x solves the inequality constrained optimization problem and satisfies the constrained qualification condition. Then, there is a set of Kuhn-Tucker multipliers $(\lambda_i \geq 0, i = 1, \dots, k)$ such that*

$$Df(\mathbf{x}) = \sum_{i=1}^k \lambda_i Dg_i(\mathbf{x}).$$

Furthermore, we have the complementary slackness conditions:

$$\begin{aligned} \lambda_i & \geq 0, \quad \text{for all } i = 1, 2, \dots, k. \\ \lambda_i & = 0, \quad \text{if } g_i(\mathbf{x}) < d_i. \end{aligned}$$

Comparing the Kuhn-Tucker theorem to the Lagrange multipliers in the equality constrained optimization problem, we see that the major difference is that the signs of the Kuhn-Tucker multipliers are nonnegative while the signs of the Lagrange multipliers can be anything. This additional information can occasionally be very useful.

The Kuhn-Tucker theorem only provides a necessary condition for a maximum. The following theorem states conditions that guarantee the above first-order conditions are sufficient.

Theorem 2.6.12(Kuhn-Tucker Sufficiency)

Suppose f is concave and $g_i, i = 1, \dots, k$ is convex. If \mathbf{x} satisfies the Kuhn-Tucker first-order conditions specified in the above theorem, then \mathbf{x} is a global solution to the constrained optimization problem.

We can weaken the conditions in the above theorem when there is only one constraint. Let $C = \{\mathbf{x} \in \mathcal{R}^n : g(\mathbf{x}) \leq d\}$. We have the following propositions.

Proposition 2.6.2 Suppose f is quasi-concave and the set C is convex (this is true if g is quasi-convex). If \mathbf{x} satisfies the Kuhn-Tucker first-order conditions, then x is a global solution to the constrained optimization problem.

Sometimes we require \mathbf{x} to be nonnegative. Suppose we had optimization problem:

$$\begin{aligned} \max \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq d_i, \quad i = 1, 2, \dots, k, \\ & \mathbf{x} \geq 0. \end{aligned}$$

Then the Lagrange function in this case is given by

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \sum_{l=1}^k \lambda_l [d_l - g_l(\mathbf{x})] + \sum_{j=1}^n \mu_j x_j,$$

where μ_1, \dots, μ_k are the multipliers associated with constraints $x_j \geq 0$. The first-order conditions are

$$\begin{aligned} \frac{L(\mathbf{x}, \lambda)}{\partial x_i} &= \frac{\partial f(\mathbf{x})}{\partial x_i} - \sum_{l=1}^k \lambda_l \frac{\partial g_l(\mathbf{x})}{\partial x_i} + \mu_i = 0, \quad i = 1, 2, \dots, n. \\ \lambda_l &\geq 0, \quad l = 1, 2, \dots, k. \\ \lambda_l &= 0, \quad \text{if } g_l(\mathbf{x}) < d_l. \\ \mu_i &\geq 0, \quad i = 1, 2, \dots, n. \\ \mu_i &= 0, \quad \text{if } x_i > 0. \end{aligned}$$

Eliminating μ_i , we can equivalently write the above first-order conditions with nonnegative choice variables as

$$\frac{L(\mathbf{x}, \lambda)}{\partial x_i} = \frac{\partial f(\mathbf{x})}{\partial x_i} - \sum_{l=1}^k \lambda_l \frac{\partial g_l(\mathbf{x})}{\partial x_i} \leq 0, \quad \text{with equality if } x_i > 0, \quad i = 1, 2, \dots, n,$$

or in matrix notation,

$$\begin{aligned} Df - \lambda Dg &\leq 0, \\ \mathbf{x}[Df - \lambda Dg] &= 0, \end{aligned}$$

where we have written the product of two vector \mathbf{x} and \mathbf{y} as the inner product, i.e., $\mathbf{xy} = \sum_{i=1}^n x_i y_i$. Thus, if we are at an interior optimum, we have

$$Df(\mathbf{x}) = \lambda Dg.$$

2.6.4 The Envelope Theorem

Consider a following maximization problem:

$$M(\mathbf{a}) = \max_{\mathbf{x}} f(\mathbf{x}, \mathbf{a}).$$

The function $M(\mathbf{a})$ gives the maximized value of the objective function as a function of the parameter \mathbf{a} .

Let $\mathbf{x}(\mathbf{a})$ be the value of \mathbf{x} that solves the maximization problem. Then we can also write $M(\mathbf{a}) = f(\mathbf{x}(\mathbf{a}), \mathbf{a})$. It is often of interest to know how $M(\mathbf{a})$ changes as \mathbf{a} changes. The envelope theorem tells us the answer:

$$\frac{dM(\mathbf{a})}{d\mathbf{a}} = \left. \frac{\partial f(\mathbf{x}, \mathbf{a})}{\partial \mathbf{a}} \right|_{\mathbf{x}=\mathbf{x}(\mathbf{a})}.$$

The conclusion is particularly useful. This expression says that the derivative of M with respect to \mathbf{a} is given by the partial derivative of f with respect to \mathbf{a} , holding \mathbf{x} fixed at the optimal choice. This is the meaning of the vertical bar to the right of the derivative. The proof of the envelope theorem is a relatively straightforward calculation.

Now consider a more general parameterized constrained maximization problem of the form

$$\begin{aligned} M(\mathbf{a}) &= \max_{x_1, x_2} g(x_1, x_2, \mathbf{a}) \\ \text{s.t.} \quad &h(x_1, x_2, \mathbf{a}) = 0. \end{aligned}$$

The Lagrangian for this problem is

$$\mathcal{L} = g(x_1, x_2, \mathbf{a}) - \lambda h(x_1, x_2, \mathbf{a}),$$

and the first-order conditions are

$$\begin{aligned} \frac{\partial g}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} &= 0, \\ \frac{\partial g}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} &= 0, \\ h(x_1, x_2, \mathbf{a}) &= 0. \end{aligned} \tag{2.6.2}$$

These conditions determine the optimal choice functions $(x_1(\mathbf{a}), x_2(\mathbf{a}), \lambda(\mathbf{a}))$, which in turn determine the maximum value function

$$M(\mathbf{a}) \equiv g(x_1(\mathbf{a}), x_2(\mathbf{a}), \lambda(\mathbf{a})). \tag{2.6.3}$$

The envelope theorem gives us a formula for the derivative of the value function with respect to a parameter in the maximization problem. Specifically, the formula is

$$\begin{aligned} \frac{dM(\mathbf{a})}{d\mathbf{a}} &= \left. \frac{\partial \mathcal{L}(\mathbf{x}, \mathbf{a})}{\partial \mathbf{a}} \right|_{\mathbf{x}=\mathbf{x}(\mathbf{a})} \\ &= \left. \frac{\partial g(x_1, x_2, \mathbf{a})}{\partial \mathbf{a}} \right|_{x_i=x_i(\mathbf{a})} - \lambda \left. \frac{\partial h(x_1, x_2, \mathbf{a})}{\partial \mathbf{a}} \right|_{x_i=x_i(\mathbf{a})} \end{aligned}$$

As before, the interpretation of the partial derivatives needs special care: they are the derivatives of g and h with respect to \mathbf{a} holding x_1 and x_2 fixed at their optimal values.

2.6.5 Maximum Theorems

In many optimization problems, we need to check if an optimal solution is continuous in parameters, say, to check the continuity of the demand function. We can apply the so-called **maximum theorem**

Berge's Maximum Theorem

Theorem 2.6.13(Berge's Maximum Theorem) *Let A and X be two topological spaces. Suppose $f(\mathbf{x}, \mathbf{a}) : A \times X \rightarrow \mathcal{R}$ is a continuous function, and the constraint set $F : A \rightarrow 2^X$ is a continuous correspondence with non-empty compact values. Then, the optimal valued function (also called marginal function):*

$$M(\mathbf{a}) = \max_{\mathbf{x} \in F(\mathbf{a})} f(\mathbf{x}, \mathbf{a})$$

is a continuous function, and the optimal solution:

$$\mu(\mathbf{a}) = \arg \max_{\mathbf{x} \in F(\mathbf{a})} f(\mathbf{x}, \mathbf{a})$$

is a upper hemi-continuous correspondence.

Walker's Maximum Theorem

In many cases of optimal problems, the preference of an economic man may not be represented by a utility function. Walker (1979) generalize the Berge's maximum theorem to the case of maximum element under the open preference relation. The Walker's maximum theorem allows the preference relations and constraint sets to vary with parameters.

Theorem 2.6.14(Walker's Maximum Theorem) *Let A and Y be two topological spaces. Suppose $U : Y \times A \rightarrow 2^Y$ is a correspondence with an open graph.*

The constraints set $F : A \rightarrow 2^Y$ is a continuous and non-empty compact-valued correspondence. Define an optimal-valued correspondence $\mu : A \rightarrow 2^Y$ as:

$$\mu(\mathbf{a}) := \{y \in F(\mathbf{a}) : U(y, \mathbf{a}) \cap F(\mathbf{a}) = \emptyset\},$$

μ is a compact-valued upper semi-continuous correspondence.

Tian-Zhou Maximum Theorem

Both Berge's and Walker's maximum theorems depend on the continuity(with open graph) of the constraint correspondence and the objective function (preference correspondence).

Tian Guoqiang and Zhou Jianxin relax these assumptions, generalize and characterize the Berge's and Walker's maximum theorems in Tian & Zhou(1995). Here only gives a description of the generalized Berge's maximum theorem. We first give the following definition of the transfer continuity.

Definition 2.6.3 Let A and Y be two topological spaces, and $F : A \rightarrow 2^Y$ be a correspondence. If $u(\mathbf{a}, z) > u(\mathbf{a}, y)$ for every $(\mathbf{a}, y) \in A \times Y$ and $z \in F(\mathbf{a})$ satisfying $y \in F(\mathbf{a})$, which implies that there is a neighbourhood $\mathcal{N}(\mathbf{a}, y)$ of (\mathbf{a}, y) such that for any $(\mathbf{a}', y') \in \mathcal{N}(\mathbf{a}, y)$, $y' \in F(\mathbf{a}')$, and there is a $z' \in F(\mathbf{a}')$, such that

$$u(\mathbf{a}', z') > u(\mathbf{a}', y'),$$

then the function $u : A \times Y \rightarrow \mathcal{R} \cup \{\infty\}$ is **quasi-transfer upper continuous with respect to F** .

The following definition is a natural extension of transfer upper continuity:

Definition 2.6.4 Let A and Y be two topological spaces, and $F : A \rightarrow 2^Y$ be a correspondence. If for every $(\mathbf{a}, y) \in A \times Y$ and $z \in F(\mathbf{a})$ that satisfying $y \in F(\mathbf{a})$, $u(\mathbf{a}, z) > u(\mathbf{a}, y)$ implies that there is a point $z' \in Y$ and a neighbourhood $\mathcal{N}(y)$ of y , such that for any $y' \in \mathcal{N}(y)$ satisfying $y' \in F(\mathbf{a})$, we have $u(\mathbf{a}, z') > u(\mathbf{a}, y')$ and $z' \in F(\mathbf{a})$. Then the function $u : A \times Y \rightarrow \mathcal{R} \cup \{\infty\}$ is **transfer upper continuous on F** .

Theorem 2.6.15(Tian-Zhou Maximum Theorem) Let A and Y be two topological spaces, and $u : A \times Y \rightarrow \mathcal{R} \cup \{\infty\}$ be a real function defined on A and Y . Suppose that $F : A \rightarrow 2^Y$ is a compact and closed valued correspondence. Then the correspondence of extreme value $\mu : A \rightarrow 2^Y$ is a nonempty, compact and close valued correspondence if and only if u is transfer upper continuous on F , and quasi-transfer upper continuous with respect to F . Further, if F is upper hemi-continuous, then correspondence of extreme value μ is also upper hemi-continuous.

This theorem relaxes the upper semi-continuity of the objective function and the correspondence of constraints in the Berge's maximum theorem.

2.6.6 Continuous Selection Theorems

The continuous selection theorem is a powerful tool to prove the existence of equilibrium, and it is closely related to the fixed point theorem to be introduced below. The basic conclusion of a continuous selection theorem is that if a correspondence is lower hemi-continuous with non-empty convex values, there is a continuous function, so that for all points at the region, the function value is a corresponding subset.

Definition 2.6.5 Let $X \subseteq \mathcal{R}^n$, $Y \subseteq \mathcal{R}^m$ and $F : X \rightarrow 2^Y$ be a correspondence from X to Y . If for any $\mathbf{x} \in X$, we have $f(\mathbf{x}) \in F(\mathbf{x})$, then the single valued function $f : X \rightarrow Y$ is a **selection** corresponding to F .

Theorem 2.6.16(Browder, (1968)) Let $X \subseteq \mathcal{R}^n$. Suppose $F : X \rightarrow 2^{\mathcal{R}^m}$ is a correspondence with lower open sections and convex values, then F has a continuous selection, that is, a single-valued function $f : X \rightarrow Y$ such that $f(\mathbf{x}) \in F(\mathbf{x})$ for all $\mathbf{x} \in X$.

Theorem 2.6.17(Michael(1956)) Let $X \subseteq \mathcal{R}^n$ be compact. Suppose $F : X \rightarrow 2^{\mathcal{R}^m}$ is a lower hemi-continuous correspondence with close and convex values, then F has a continuous selection, that is, a single-valued function $f : X \rightarrow \mathcal{R}^m$ such that $f(\mathbf{x}) \in F(\mathbf{x})$ for all $\mathbf{x} \in X$.

2.6.7 Fixed Point Theorems

The fixed point theorem plays a crucial role in proving the existence of equilibrium. It is the most commonly used method for determining whether there is a solution of equilibrium equations. John von Neumann (1903-1957, See section 5.8.1 for his biography) first proposed the concept of fixed points in the two papers published in 1928 and 1937 respectively. We first gave the fixed point theorem which can be used to prove the existence of solutions of nonlinear equations.

Definition 2.6.6 Let X be a topological space and $f : X \rightarrow X$ is a single-valued function from X to itself. If there is a point $\mathbf{x}^* \in X$ such that $f(\mathbf{x}^*) = \mathbf{x}^*$, then \mathbf{x}^* is called a **fixed point** of function f .

Definition 2.6.7 Let X be a topological space and $F : X \rightarrow 2^X$ is a correspondence from X to itself. If there is a point $\mathbf{x}^* \in X$ such that $\mathbf{x}^* \in F(\mathbf{x}^*)$, then \mathbf{x}^* is called a **fixed point** of correspondence f .

There are some important fixed point theorems which are widely used in economics.

Brouwer's Fixed Theorem

The Brouwer's fixed point theorem is one of the most fundamental and important fixed point theorems.

Theorem 2.6.18(Brouwer's Fixed Theorem) *Let X be a non-empty, compact, and convex subset of \mathcal{R}^m . If a function $f : X \rightarrow X$ is continuous on X , then f has a fixed point, i.e., there is a point $\mathbf{x}^* \in X$ such that $f(\mathbf{x}^*) = \mathbf{x}^*$ (See Figure 2.4).*

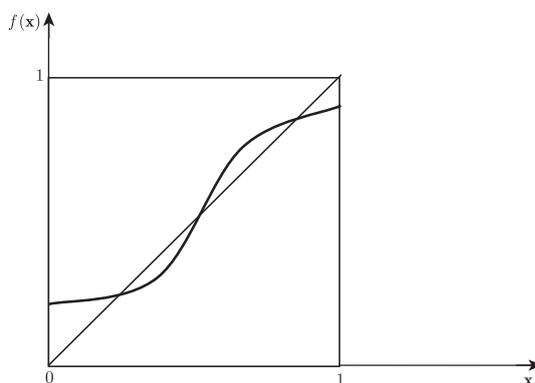


Figure 2.4: The intersection point of 45° line and the curve of a function is a fixed point. There are three fixed points in this example

Example 2.6.1 $f : [0, 1] \rightarrow [0, 1]$ is continuous, then f has a fixed point \mathbf{x} . To see this, let $g(\mathbf{x}) = f(\mathbf{x}) - \mathbf{x}$. Then, we have

$$g(0) = f(0) \geq 0$$

$$g(1) = f(1) - 1 \leq 0.$$

From the mean-value theorem, there is a point $\mathbf{x}^* \in [0, 1]$ such that $g(\mathbf{x}^*) = f(\mathbf{x}^*) - \mathbf{x}^* = 0$.

Kakutani's Fixed Point Theorem

In applications, mapping is often a correspondence, so Brouwer's fixed point theorem can not be used directly, so then the Kakutani's fixed point theorem is commonly used.

Theorem 2.6.19(Kakutani's Fixed Point Theorem (1941)) *Let $X \subseteq \mathcal{R}^m$ be a non-empty, compact, and convex subset. If a correspondence $F : X \rightarrow 2^X$ is an upper hemi-continuous correspondence with non-empty compact and convex values on X , then F has a fixed point, i.e., there is a point $\mathbf{x}^* \in X$ such that $\mathbf{x}^* \in F(\mathbf{x}^*)$.*

Browder's Fixed Point Theorem

It follows from Theorem 2.6.16 that we have the following Browder's Fixed Point Theorem.

Theorem 2.6.20(Browder (1968)) *Let $X \subseteq \mathcal{R}^n$ be a compact and convex subset. Suppose a correspondence $F : X \rightarrow 2^{\mathcal{R}^m}$ is convex valued with lower open sections, then F has a fixed point, i.e. there is a point $\mathbf{x}^* \in X$ such that $\mathbf{x}^* \in F(\mathbf{x}^*)$.*

Michael's Fixed Point Theorem

It follows from Theorem 2.6.17 that the Michael's Fixed Point Theorem is given.

Theorem 2.6.21(Michael (1956)) *Let $X \subseteq \mathcal{R}^n$ be a compact and convex subset. Suppose that $F : X \rightarrow 2^{\mathcal{R}^m}$ is a lower hemi-continuous correspondence with closed and convex values, then F has fixed point, i.e. there is a point $\mathbf{x}^* \in X$ such that $\mathbf{x}^* \in F(\mathbf{x}^*)$.*

Tarsky's Fixed Point Theorem

The fixed point theorem of Tarsky is a distinct fixed point theorem. It does not require a function to have any type of continuity, but only requires that the function is monotonous and non-decreasing, and it is defined on the domain composed of intervals. It is becoming more and more important in the application of economics, especially in the game with a monotonous payment function.

Theorem 2.6.22(Tarsky's (1955)Fixed Point Theorem) *Denote $[0, 1]^n$ as the n times product of interval $[0, 1]$. If $f : [0, 1]^n \rightarrow [0, 1]^n$ is a non-decreasing function, then f has a fixed point, i.e., there is a point $\mathbf{x}^* \in X$ such that $f(\mathbf{x}^*) = \mathbf{x}^*$.*

Contraction Mapping Theorem

In many economic dynamic models, we not only need to prove the existence of equilibrium, but also prove the uniqueness of equilibrium. The principle of contraction mapping is an important tool to solve this problem. It is also the most basic and simplest theorems of existence in functional analysis. Many of the existence theorems in mathematical analysis are its special cases. Its basic conclusion is that a contraction mapping from complete metric space to itself has a unique fixed point.

Definition 2.6.8 Let (X, d) be a complete metric space, and $f : X \rightarrow X$ be a single-valued function form X to itself. If for any point $\mathbf{x}, \mathbf{x}' \in X$, there

is $\alpha \in (0, 1)$ such that $d(f(\mathbf{x}), f(\mathbf{x}')) < \alpha d(\mathbf{x}, \mathbf{x}')$, then the function f is a **contraction mapping**.

Theorem 2.6.23(Banach Contraction Mapping Theorem) *Suppose $f : X \rightarrow X$ is a contraction mapping from a complete metric space X to itself, then f has a unique fixed point on X .*

The characterization of the existence of a fixed point

All the above fixed point theorems are only sufficient conditions for the existence of fixed points. Recently, Tian Guoqiang (Tian (2016)) introduces a series of concepts of recursive transfer continuity, and gives a sufficient and necessary condition for the existence of fixed points.

We first introduce the concept of diagonal transfer continuity introduced by Baye-Tian-Zhou (1993).

Definition 2.6.9 If $\varphi(\mathbf{x}, \mathbf{y}) > \varphi(\mathbf{y}, \mathbf{y})$ holds for all $\mathbf{x}, \mathbf{y} \in X$, there is a point $\mathbf{z}^0 \in X$ and a neighbourhood $\mathcal{V}_{\mathbf{y}}$ of \mathbf{y} such that $\varphi(\mathbf{z}, \mathcal{V}_{\mathbf{y}}) > \varphi(\mathcal{V}_{\mathbf{y}}, \mathcal{V}_{\mathbf{y}})$ holds for all $\mathbf{y}' \in \mathcal{V}_{\mathbf{y}}$, then $\varphi : X \times X \rightarrow \mathcal{R}$ is **diagonally transfer continuous with respect to \mathbf{y}** .

Now we define the concept of recursively diagonally transfer continuity.

Definition 2.6.10 If $\varphi(\mathbf{x}, \mathbf{y}) > \varphi(\mathbf{y}, \mathbf{y})$ holds for all $\mathbf{x}, \mathbf{y} \in X$, there is a point $\mathbf{z}^0 \in X$ (may be $\mathbf{z}^0 = \mathbf{y}$) and a neighbourhood $\mathcal{V}_{\mathbf{y}}$ of \mathbf{y} such that $\varphi(\mathbf{z}, \mathcal{V}_{\mathbf{y}}) > \varphi(\mathcal{V}_{\mathbf{y}}, \mathcal{V}_{\mathbf{y}})$ holds for any finite set $\{\mathbf{z}^0, \mathbf{z}^1, \dots, \mathbf{z}^m\} \subseteq X$, where $\mathbf{z}^m = \mathbf{z}$ and $\varphi(\mathbf{z}, \mathbf{z}^{m-1}) > \gamma$, $\varphi(\mathbf{z}^{m-1}, \mathbf{z}^{m-2}) > \gamma, \dots, \varphi(\mathbf{z}^1, \mathbf{z}^0) > \gamma$, $m \geq 1$, then $\varphi : X \times X \rightarrow \mathcal{R}$ is **recursively diagonally transfer continuous with respect to \mathbf{y}** , where $\varphi(\mathbf{z}, \mathcal{V}_{\mathbf{y}}) > 0$ represents $\varphi(\mathbf{z}, \mathbf{y}') > 0$ holds for all $\mathbf{y}' \in \mathcal{V}_{\mathbf{y}}$.

Theorem 2.6.24(Tian's Fixed Point Theorem(2016)) *Let X be a nonempty and compact set in a metric space (E, d) , and $f : X \rightarrow X$ be a function. Define $\varphi : X \times X \rightarrow \mathcal{R} \cup \{\pm\infty\}$:*

$$\varphi(\mathbf{x}, \mathbf{y}) = -d(\mathbf{x}, f(\mathbf{y})),$$

then the sufficient and necessary condition for f to have a fixed point is that φ is recursively diagonally transfer continuous with respect to \mathbf{y} .

2.6.8 Variation Inequality

Ky-Fan minimax inequality is one of the most important results in nonlinear analysis. It is equivalent to many important mathematical theorems in certain sense, such as KKM lemma, Sperner lemma, Brouwer's fixed point theorem and Kakutani's fixed point theorem (can be derived from each other). In many disciplines, such as variation inequalities, mathematical programming, partial differential equations and economical models, it can be used to prove the existence of equilibrium solutions.

Theorem 2.6.25(Fan-Ky minimax inequality) *Let $X \subseteq \mathcal{R}^m$ be a nonempty, convex and compact set, and $\phi: X \times X \rightarrow \mathcal{R}$ satisfy the following conditions:*

- (1) *for all $\mathbf{x} \in X$, $\phi(\mathbf{x}, \mathbf{x}) \leq 0$;*
- (2) *ϕ is lower semi-continuous with respect to \mathbf{y} ;*
- (3) *ϕ is quasi-concave with respect to \mathbf{x} .*

Then there exists a point $\mathbf{y}^ \in X$ such that $\phi(\mathbf{x}, \mathbf{y}^*) \leq 0$ holds for all $\mathbf{x} \in X$.*

Ky-Fan inequality is generalized in various forms in mathematical literature. Tian Guoqiang fully characterizes the existence of solutions to Ky-Fan inequality, and gives the sufficient and necessary condition for the existence of Ky-Fan inequality in Tian (2014).

Definition 2.6.11 Let X be a topological space, $\gamma \in \mathcal{R}$. If for $\mathbf{x}, \mathbf{y} \in X$, $\phi(\mathbf{x}, \mathbf{y}) > \gamma$, then there is a point $\mathbf{z}^0 \in X$ (maybe $\mathbf{z}^0 = \mathbf{y}$) and a neighbourhood $\mathcal{V}_{\mathbf{y}}$ of \mathbf{y} such that $\phi(\mathbf{z}, \mathcal{V}_{\mathbf{y}}) > \gamma$ for any finite number of points $\{\mathbf{z}^0, \mathbf{z}^1, \dots, \mathbf{z}^{m-1}, \mathbf{z}\}$ and $\phi(\mathbf{z}, \mathbf{z}^{m-1}) > \gamma$, $\phi(\mathbf{z}^{m-1}, \mathbf{z}^{m-2}) > \gamma, \dots, \phi(\mathbf{z}^1, \mathbf{z}^0) > \gamma$, $m \geq 1$. Then $\phi: X \times X \rightarrow \mathcal{R} \cup \{\pm\infty\}$ is **γ -recursively diagonally transfer lower hemi-continuous with respect to \mathbf{y}**

Theorem 2.6.26(Tian (2016)) *Let X be a compact subset in a topological space, $\gamma \in \mathcal{R}$, and $\phi: X \times X \rightarrow \mathcal{R} \cup \{\pm\infty\}$ be a function satisfying $\phi(\mathbf{x}, \mathbf{x}) \leq \gamma, \forall \mathbf{x} \in X$, then there is a point $\mathbf{y}^* \in X$ such that $\phi(\mathbf{x}, \mathbf{y}^*) \leq \gamma$ for all $\mathbf{x} \in X$ if and only if ϕ is γ -recursively diagonally transfer lower hemi-continuous with respect to \mathbf{y} .*

2.6.9 FKKM Theorems

The Knaster-Kuratowski-Mazurkiewicz (KKM) lemma is quite basic and in some ways more useful than Brouwer's fixed point theorem.

Theorem 2.6.27(KKM Theorem) *Let $X \subseteq \mathcal{R}^m$ be a convex set .Suppose $F: X \rightarrow 2^X$ is a correspondence such that*

- (1) *$F(\mathbf{x})$ is closed for all $\mathbf{x} \in X$;*
- (2) *F is FS-convex, , i.e, for any $\mathbf{x}_1, \dots, \mathbf{x}_m \in X$ and its convex combination $\mathbf{x}_\lambda = \sum_{i=1}^m \lambda_i \mathbf{x}_i$, we have*

$$\mathbf{x}_\lambda \in \bigcup_{i=1}^m F(\mathbf{x}_i).$$

then

$$\bigcap_{\mathbf{x} \in X} F(\mathbf{x}) \neq \emptyset.$$

The following is a generalized version of KKM lemma due to Ky Fan (1984).

Theorem 2.6.28(FKKM Theorem) *Suppose $X \subseteq \mathcal{R}^m$ is a convex set, and $\emptyset \neq X \subseteq Y$. And $F : X \rightarrow 2^Y$ is a correspondence such that*

- (1) $F(\mathbf{x})$ is closed for all $\mathbf{x} \in X$;
- (2) $F(\mathbf{x}_0)$ is compact for some $\mathbf{x}_0 \in X$;
- (3) F is FS-convex, i.e, for any $\mathbf{x}_1, \dots, \mathbf{x}_m \in X$ and its convex combination $\mathbf{x}_\lambda = \sum_{i=1}^m \lambda_i \mathbf{x}_i$, we have

$$\mathbf{x}_\lambda \in \bigcup_{i=1}^m F(\mathbf{x}_i).$$

then

$$\bigcap_{\mathbf{x} \in X} F(\mathbf{x}) \neq \emptyset.$$

This theorem has many generalizations in Tian (2016), and the author also gives the sufficient and necessary conditions for the establishment of the FKKM theorem:

Theorem 2.6.29(Tian (2016)) *Let X be a nonempty compact set in a topological space T , and $F : X \rightarrow 2^X$ be a correspondence satisfying $\mathbf{x} \in F(\mathbf{x}), \mathbf{x} \in X$. Then $\bigcap_{\mathbf{x} \in X} F(\mathbf{x}) \neq \emptyset$ if and only if the correspondence $\phi : X \times X \rightarrow R \cup \{\pm\infty\}$ defined by y*

$$\phi(\mathbf{x}, \mathbf{y}) = \begin{cases} \gamma, & \text{if } (\mathbf{x}, \mathbf{y}) \in G, \\ +\infty, & \text{others} \end{cases}$$

is γ -recursively transfer semi-continuous with respect to y , where $\gamma \in R$ and $G = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : \mathbf{y} \in F(\mathbf{x})\}$.

2.7 Dynamic Optimization

We generally encounter various constraints when making optimization decisions, and the constrained optimization problems mentioned in the last section are all among different variables in the same period. However, in reality, people often need to make decisions in the dynamic environment, and the early decision variables will affect the variables in the later period. Dynamic optimization, dynamic programming, or optimal control provides an analytical framework and tool for solving the optimal problems in dynamic environments. In this section, we will show the variational method,

Hamilton equation and the basic results of dynamic programming. We focus mainly on continuous cases of dynamic optimization problems defined on $X \subseteq \mathcal{R}$.

2.7.1 Variational Method

The general dynamic optimization problems have the following forms:

$$\max \int_{t_0}^{t_1} F[t, \mathbf{x}(t), \mathbf{x}'(t)] dt \quad (2.7.4)$$

$$\text{s.t. } \mathbf{x}(t_0) = \mathbf{x}_0, \mathbf{x}(t_1) = \mathbf{x}_1. \quad (2.7.5)$$

The above optimization problem is to choose a function $\mathbf{x}(T)$ under the constraint condition (2.7.5) to maximize the objective function (2.7.4). Variational method is a common method to solve such problems. Let $\mathbf{x}^*(T)$ be the solution to the above optimization problem, and the necessary condition is that the solution must satisfy the Euler equation:

$$F_{\mathbf{x}}[t, \mathbf{x}^*(t), \mathbf{x}'^*(t)] = \frac{dF_{\mathbf{x}'}[t, \mathbf{x}^*(t), \mathbf{x}'^*(t)]}{dt}, \quad t \in [t_0, t_1]. \quad (2.7.6)$$

Next, we will derive the Euler equation of dynamic optimization.

We say the function satisfying the constraint (2.7.5) is admissible. Assuming that $\mathbf{x}(T)$ is admissible, $\mathbf{h}(T) = \mathbf{x}(T) - \mathbf{x}^*(T)$ represents the difference between $\mathbf{x}(T)$ and the optimal selection. We have $\mathbf{h}(t_0) = \mathbf{h}(t_1) = 0$.

For any constant a , $\mathbf{y}(T) = \mathbf{x}^*(T) + a\mathbf{h}(T)$ is also admissible. In this way, the dynamic optimization problems can be transformed into solving under what conditions $a = 0$ is the optimal choice under dynamic optimization.

$$\begin{aligned} g(a) &= \int_{t_0}^{t_1} F[t, \mathbf{y}(t), \mathbf{y}'(t)] dt \\ &= \int_{t_0}^{t_1} F[t, \mathbf{x}^*(t) + a\mathbf{h}(t), \mathbf{x}'^*(t) + a\mathbf{h}'(t)] dt. \end{aligned} \quad (2.7.7)$$

The first order condition of optimization is the derivative of the equation (2.7.7) with respect to a :

$$\begin{aligned} g'(0) &= \int_{t_0}^{t_1} F_{\mathbf{x}}[t, \mathbf{x}^*, \mathbf{x}'^*(t)]\mathbf{h}(t) + F_{\mathbf{x}'}[t, \mathbf{x}^*, \mathbf{x}'^*(t)]\mathbf{h}'(t) dt \\ &= 0. \end{aligned} \quad (2.7.8)$$

Using integration by parts on the second part of the right side of the equation (2.7.8) gets:

$$\int_{t_0}^{t_1} \left\{ F_{\mathbf{x}}[t, \mathbf{x}^*, \mathbf{x}'^*(t)] - \frac{dF_{\mathbf{x}'}[t, \mathbf{x}^*(t), \mathbf{x}'^*(t)]}{dt} \right\} \mathbf{h}(t) dt = 0. \quad (2.7.9)$$

If equation (2.7.9) holds for any continuous function $h(t)$ that satisfies the constraint $\mathbf{h}(t_0) = \mathbf{h}(t_1) = 0$, it is proved (see Kamien & Schwartz (1991)) that the Euler equation (2.7.6) also holds.

Example 2.7.1(Kamien & Schwartz (1991)) Suppose an enterprise receives an order, requiring B units products delivered at time T . Assume that the production capacity of the enterprise is limited, and the unit cost of production is proportional to the output. In addition, finished products need to put into stock and the inventory cost per unit is a constant. Business managers need to consider production problems from now (time 0) to delivery date (time T). Suppose that at time $t \in [0, T]$, the inventory of the enterprise is $x(t)$, and the changes of inventory depends on the production of the enterprise, that is, $\dot{x}(t) \equiv x'(t) = y(t)$, where $y(t)$ is the productivity at time t . At t , the cost of the enterprise is $c_1 x'(t)x'(t) + c_2 x(t)$ or $c_1 u(t)u(t) + c_2 x(t)$, where $c_1 u(t)$ is the unit cost of production when yield is $u(t)$, and c_2 is the unit cost of inventory. The goal of the enterprise is to minimize costs (including both production costs and inventory costs), therefore the dynamic optimization problem is

$$\begin{aligned} \min \int_0^T [c_1 x'^2(t) + c_2 x(t)] dt & \quad (2.7.10) \\ \text{s.t. } x(0) = 0, x(T) = B, x'(t) \geq 0. & \end{aligned}$$

In expression (2.7.10), $u(t)$ is called a **control variable**, and $x(t)$ is called a **state variable**. Using the variational method to solve the optimization problem, we have

$$F[t, x(t), x'(t)] = c_1 x'^2(t) + c_2 x(t).$$

The Euler equation is:

$$c_2 = 2c_1 x''^*(t).$$

According to the constraint conditions: $x^*(0) = 0, x^*(T) = B$, solving the above Euler equation gets:

$$x^*(t) = \frac{c_2}{4c_1} t(t - T) + Bt/T, \quad t \in [0, T].$$

Integrating Euler equation (2.7.6) gives:

$$F_{\mathbf{x}} = F_{\mathbf{x}'t} + F_{\mathbf{x}'\mathbf{x}}\mathbf{x}' + F_{\mathbf{x}'\mathbf{x}'}\mathbf{x}'' \quad (2.7.11)$$

Now introduce the Hamilton equations to avoid taking second order differential. Let $\mathbf{p}(t) = F_{\mathbf{x}'}[t, \mathbf{x}(t), \mathbf{x}'(t)]$, and Hamilton equation is:

$$H(t, \mathbf{x}, \mathbf{p}) = -F(t, \mathbf{x}, \mathbf{x}') + \mathbf{p}\mathbf{x}'. \quad (2.7.12)$$

In above equations (2.7.12), $\mathbf{p}(t)$ can be regarded as shadow price. The total differential of equation (2.7.12) is:

$$dH = -F_t dt - F_x dx - F_{x'} dx' + \mathbf{p} d\mathbf{x}' + \mathbf{x}' d\mathbf{p} = -F_t dt - F_x dx + \mathbf{x}' d\mathbf{p}$$

The partial derivatives of equation (2.7.12) with respect to \mathbf{x} , \mathbf{p} respectively are:

$$\begin{aligned} \partial H / \partial \mathbf{x} &= -F_x; \\ \partial H / \partial \mathbf{p} &= \mathbf{x}'. \end{aligned}$$

Since $-F_x = -(dF_{x'}/dt) = -\mathbf{p}'$, we have two Euler equations under first order conditions:

$$\begin{aligned} \partial H / \partial \mathbf{x} &= -\mathbf{p}'; \\ \partial H / \partial \mathbf{p} &= \mathbf{x}'. \end{aligned}$$

Thus, the Euler equations above are only the necessary conditions for solving dynamic optimization, and the sufficient conditions involves the second order conditions. After deriving the first order conditions by the variational method, it is clear that the second order condition is:

$$g''(0) = \int_{t_0}^{t_1} [F_{xx} h^2 + 2F_{xx'} \mathbf{h}\mathbf{h}' + F_{x'x'} (\mathbf{h}')^2] dt \leq 0.$$

It is easy to verify that if the objective function F is concave in \mathbf{x} and \mathbf{x}' , then the second order condition is met automatically.

Denote $F = F(t, \mathbf{x}, \mathbf{x}')$, $F^* = F(t, \mathbf{x}^*, \mathbf{x}'^*)$, and let $\mathbf{h}(t) = \mathbf{x}(t) - \mathbf{x}^*(t)$, then we have $\mathbf{h}'(t) = \mathbf{x}'(t) - \mathbf{x}'^*(t)$, hence

$$\begin{aligned} \int_{t_0}^{t_1} (F - F^*) dt &\leq \int_{t_0}^{t_1} [(\mathbf{x} - \mathbf{x}^*) F_x^* + (\mathbf{x}' - \mathbf{x}'^*) F_{x'}^*] dt \\ &= \int_{t_0}^{t_1} (\mathbf{h} F_x^* + \mathbf{h}' F_{x'}^*) dt \\ &= \int_{t_0}^{t_1} h (F_x^* - dF_{x'}^*/dt) dt = 0. \end{aligned}$$

It can be proved (see Kamiem & Schwartz (1991, p.43)) that the first order condition, namely, the Euler equation will be met as long as $F_{x'x'} \leq 0$, and thus the dynamic maximization problem is solved. As for the dynamic minimization, the first order condition is also a sufficient condition if the second order condition satisfies $F_{x'x'} \geq 0$.

2.7.2 Optimum Control

We have two kinds of variables in the previous examples: state variables and control variables, and we can also discuss the dynamic optimization problem using the analytical framework of optimal control.

The optimal control problem can be generally expressed as follows:

$$\max \int_{t_0}^{t_1} f[t, \mathbf{x}(t), \mathbf{u}(t)] dt \quad (2.7.13)$$

$$\text{s.t. } \mathbf{x}'(t) = g(t, \mathbf{x}(t), \mathbf{u}(t)), \quad (2.7.14)$$

$$\mathbf{x}(t_0) = \mathbf{x}_0. \quad (2.7.15)$$

In the above statement, $\mathbf{x}(t)$ is a state variable, and $\mathbf{u}(t)$ is a control variable that affects the change of the state variable, and the objective (2.7.13) is a function of the state variable and the control variable.

The necessary and sufficient conditions for optimal control are given below. Similar to the optimization problem under static constraints, the dynamic Lagrange equation is established:

$$L = \int_{t_0}^{t_1} \{f[t, \mathbf{x}(t), \mathbf{u}(t)] + \lambda_t [g(t, \mathbf{x}(t), \mathbf{u}(t)) - \mathbf{x}'(t)]\} dt, \quad (2.7.16)$$

Here, λ_t is the multiplier of constraint on state changes at time t , commonly known as **costate variable**. Integrating by parts gives:

$$L = \int_{t_0}^{t_1} \{f[t, \mathbf{x}(t), \mathbf{u}(t)] + \lambda_t g(t, \mathbf{x}(t), \mathbf{u}(t)) + \mathbf{x}(t) \lambda_t'\} dt - \lambda_{t_1} \mathbf{x}(t_1) + \lambda_{t_0} \mathbf{x}(t_0).$$

The necessary conditions for the optimal control method can be derived by using similar process of deducing the variational method. Assuming that $\mathbf{u}^*(t)$ is the optimal control (function), we introduce another control function $\mathbf{u}^*(t) + a\mathbf{h}(t)$ which is the optimal control function when $a = 0$. The optimal state function $\mathbf{x}^*(t)$ can be determined by giving the optimal control function $\mathbf{u}^*(t)$ and the initial state $\mathbf{x}(t_0) = \mathbf{x}_0$. Denote the state variable generated by control function $\mathbf{u}^*(t) + a\mathbf{h}(t)$ and initial state \mathbf{x}_0 as $\mathbf{y}(t, a)$, which meets: $\mathbf{y}(t, a) = \mathbf{x}^*(t)$, $\mathbf{y}(t, 0) = \mathbf{x}_0$, and $d\mathbf{y}(t, a)/dt = g(t, \mathbf{y}(t, a), \mathbf{u}^*(t) + a\mathbf{h}(t))$. Establish the function:

$$\begin{aligned} J(a) &= \int_{t_0}^{t_1} f[t, \mathbf{y}(t, a), \mathbf{u}^*(t) + a\mathbf{h}(t)] dt \\ &= \int_{t_0}^{t_1} \{f[t, \mathbf{y}(t, a), \mathbf{u}^*(t) + a\mathbf{h}(t)] \\ &\quad + \lambda_t [g(t, \mathbf{y}(t, a), \mathbf{u}^*(t) + a\mathbf{h}(t)) + \mathbf{y}'(t, a) \lambda_t']\} dt \\ &\quad - \lambda_{t_1} \mathbf{y}(t_1, a) + \lambda_{t_0} \mathbf{y}(t_0, a). \end{aligned} \quad (2.7.17)$$

The derivative of the function (2.7.17) at $a = 0$ is:

$$J'(a) = \int_{t_0}^{t_1} [(f_{\mathbf{x}} + \lambda g_{\mathbf{x}} + \lambda') \mathbf{y}_a + (f_{\mathbf{u}} + \lambda g_{\mathbf{u}}) h] dt - \lambda_{t_1} \mathbf{y}'(t_1, 0).$$

Thus, $\lambda(t)$ is required to be differentiable, and the optimization needs to satisfy the following three conditions:

First is the first order condition with respect to the control variable:

$$f_{\mathbf{u}}[t, \mathbf{x}(t), \mathbf{u}(t)] + \lambda g_{\mathbf{u}}(t, \mathbf{x}(t), \mathbf{u}(t)) = 0. \quad (2.7.18)$$

Second is the first order condition with respect to the costate variable:

$$\lambda'(t) = -[f_{\mathbf{x}}[t, \mathbf{x}(t), \mathbf{u}(t)] + \lambda(t)g_{\mathbf{x}}([t, \mathbf{x}(t), \mathbf{u}(t)])], \lambda(t_1) = 0. \quad (2.7.19)$$

The state function is:

$$\mathbf{x}'(t) = g(t, \mathbf{x}(t), \mathbf{u}(t)), \mathbf{x}(t_0) = \mathbf{x}_0. \quad (2.7.20)$$

The Hamilton equation for optimal control, is similar with the Lagrange equations for constrained optimization, is defined as:

$$H(t, \mathbf{x}(t), \mathbf{u}(t)) \equiv f(t, \mathbf{x}(t), \mathbf{u}(t)) + \lambda(t)g(t, \mathbf{x}(t), \mathbf{u}(t)). \quad (2.7.21)$$

Third is the first order condition of dynamic optimization, which is obtained by taking derivative of Hamilton equation with respect to $t, \mathbf{x}(t), \mathbf{u}(t)$:

$$\partial H / \partial \mathbf{u} = 0 : \quad \partial H / \partial \mathbf{u} = f_{\mathbf{u}} + \lambda g_{\mathbf{u}} = 0, \quad (2.7.22)$$

i.e., the equation (2.7.18);

$$-\partial H / \partial \mathbf{x} = \lambda' : \quad \lambda'(t) = -\partial H / \partial \mathbf{x} = -(f_{\mathbf{x}} + \lambda g_{\mathbf{x}}), \quad (2.7.23)$$

i.e., the equation (2.7.19);

$$\partial H / \partial \lambda = \mathbf{x}' : \quad \mathbf{x}'(t) = \partial H / \partial \lambda = g, \quad (2.7.24)$$

i.e., the equation (2.7.20).

Example 2.7.2 Recall Example 2.7.1, the problem is

$$\begin{aligned} & \min \int_0^T [c_1 u^2(t) + c_2 x(t)] dt \\ \text{s.t.} \quad & x'(t) = u(t), x(0) = 0, x(T) = B, x'(t) \geq 0. \end{aligned}$$

It follows from the three conditions of optimization above that:

$$2c_1 u(t) = -\lambda(t); \lambda'(t) = -c_2; x'(t) = u(t), x(0) = 0, x(T) = B.$$

and hence we have:

$$x^{*''}(t) = \frac{c_2}{2c_1}, \quad t \in [0, T],$$

$$x^*(t) = \frac{c_2}{4c_1}t(t - T) + Bt/T, \quad t \in [0, T],$$

$$u^*(t) = \frac{c_2}{2c_1}t + k, \quad t \in [0, T]; \quad k = \frac{-c_2}{4c_1}T + B/T.$$

Similarly, the second order conditions of optimal control can be derived: in the case of maximization problems, if the objective function and the state change function f, g are concave with respect to \mathbf{x} and \mathbf{u} , then the first order necessary conditions are also the sufficient conditions, and the proof can refer to Kamien & Schwartz (1991).

2.7.3 Dynamic Programming

The third method of dealing with the dynamic optimization is dynamic programming method proposed by Richard Bellman, and its basic logic can be summed up as the principle of optimality. An optimal path satisfies the property that whatever the states and the control variables are before a certain time, the selection of decision function must constitute an optimal policy from now to the end with regard to the current state .

The general form of dynamic programming problems is:

$$\max \int_0^T f(t, \mathbf{x}(t), \mathbf{u}(t))dt + \phi(\mathbf{x}(T), T) \quad (2.7.25)$$

$$\text{s.t.} \quad \mathbf{x}'(t) = g(t, \mathbf{x}(t), \mathbf{u}(t)), \mathbf{x}(0) = a, \quad t \in [0, T]. \quad (2.7.26)$$

Define the optimal value function $J(t_0, \mathbf{x}_0)$ as the best value starting at time t_0 in state \mathbf{x}_0 :

$$J(t_0, \mathbf{x}_0) = \max_{\mathbf{u}} \int_{t_0}^T f(t, \mathbf{x}(t), \mathbf{u}(t))dt + \phi(\mathbf{x}(T), T) \quad (2.7.27)$$

$$\text{s.t.} \quad \mathbf{x}'(t) = g(t, \mathbf{x}(t), \mathbf{u}(t)), \mathbf{x}(t_0) = \mathbf{x}_0, \quad t \in [t_0, T], \quad \forall t_0 \in [0, T].$$

The value function is: $J(T, \mathbf{x}(T)) = \phi(\mathbf{x}(T), T)$ when $t = T$.

We break up the equation (2.7.27) and gets:

$$J(t_0, \mathbf{x}_0) = \max_{\mathbf{u}} \left\{ \int_{t_0}^{t_0+\Delta t} f dt + \int_{t_0+\Delta t}^T f dt + \phi(\mathbf{x}(T), T) \right\}. \quad (2.7.28)$$

At time $t_0 + \Delta t$, the state changes to $\mathbf{x}_0 + \Delta \mathbf{x}$, and it follows the principle

of optimality that the equation (2.7.28) is equivalent to:

$$\begin{aligned} J(t_0, \mathbf{x}_0) &= \max_{\mathbf{u}} \int_{t_0}^{t_0+\Delta t} f dt + \max_{\mathbf{u}} \left(\int_{t_0+\Delta t}^T f dt + \phi(\mathbf{x}(T), T) \right) \\ &= \max_{\mathbf{u}} \int_{t_0}^{t_0+\Delta t} f dt + J(t_0 + \Delta t, \mathbf{x}_0 + \Delta \mathbf{x}), \\ &\quad \mathbf{x}' = g, \mathbf{x}(t_0 + \Delta t) = \mathbf{x}_0 + \Delta \mathbf{x}. \end{aligned} \quad (2.7.29)$$

The equation (2.7.29) depicts the principle of optimality. Expand the right side of (2.7.29) by Taylor's theorem and gives:

$$\begin{aligned} J(t_0, \mathbf{x}_0) &= \max_{\mathbf{u}} [f(t_0, \mathbf{x}_0, \mathbf{u})\Delta t + J(t_0, \mathbf{x}_0) + J_t(t_0, \mathbf{x}_0)\Delta t \\ &\quad + J_{\mathbf{x}}(t_0, \mathbf{x}_0)\Delta \mathbf{x} + h.o.t.]. \end{aligned} \quad (2.7.30)$$

Let $\Delta t \rightarrow 0$, the equation (2.7.30) becomes:

$$0 = \max_{\mathbf{u}} [f(t, \mathbf{x}, \mathbf{u}) + J_t(t, \mathbf{x}) + J_{\mathbf{x}}(t, \mathbf{x})\mathbf{x}'],$$

and hence we have

$$-J_t(t, \mathbf{x}) = \max_{\mathbf{u}} [f(t, \mathbf{x}, \mathbf{u}) + J_{\mathbf{x}}(t, \mathbf{x})g(t, \mathbf{x}, \mathbf{u})]. \quad (2.7.31)$$

Compared to the analytical framework of optimal control, $J_{\mathbf{x}}(t, \mathbf{x})$ on the right side of 2.7.31 plays the role of the costate variable λ . We just define $\lambda(t) = J_{\mathbf{x}}(t, \mathbf{x})$, and hence the economic meaning behind the costate variables is the marginal contribution of states to the value function.

The derivative of (2.7.31) with respect to \mathbf{x} gives:

$$-J_{t\mathbf{x}}(t, \mathbf{x}^*) = f_{\mathbf{x}}(t, \mathbf{x}^*, \mathbf{u}^*) + J_{\mathbf{x}}(t, \mathbf{x}^*)g_{\mathbf{x}}. \quad (2.7.32)$$

Since

$$\lambda'(t) = \frac{dJ_{\mathbf{x}}(t, \mathbf{x})}{dt} = -J_{t\mathbf{x}} + J_{\mathbf{x}\mathbf{x}}g,$$

together with (2.7.32), we get:

$$-\lambda'(t) = f_{\mathbf{x}} + \lambda g_{\mathbf{x}}. \quad (2.7.33)$$

The equation (2.7.33) is just the first order condition of optimal control framework with respect to the state:

$$-\partial H / \partial \mathbf{x} = \lambda'.$$

The derivative of the right side of (2.7.31) with respect to \mathbf{u} gives:

$$f_{\mathbf{u}} + J_{\mathbf{x}}g_{\mathbf{u}} = 0,$$

and this is the first order condition of optimal control framework with respect to control variables:

$$\frac{\partial H}{\partial \mathbf{u}} = f_{\mathbf{u}} + \lambda g_{\mathbf{u}} = 0.$$

Therefore, the optimal control and dynamic programming are essentially consistent.

In the discrete case, the method of dynamic programming may be more convenient. The following results are given only for unbounded situations.

Suppose the state set $S \subseteq \mathbb{R}^n$ is a nonempty and compact set, $U : S \times S \rightarrow \mathbb{R}$ is a continuous function, which generally represents the utility function in a period. Given the initial state $\mathbf{s}_0 = \mathbf{z}$, the general dynamic optimization problem is:

$$\max_{\{\mathbf{s}_t\}} \sum_{t=0}^{\infty} \delta^t U(\mathbf{s}_t, \mathbf{s}_{t+1}) \quad (2.7.34)$$

$$\text{s.t.} \quad \mathbf{s}_t \in S, \quad \forall t, \quad (2.7.35)$$

$$\mathbf{s}_0 = \mathbf{z}.$$

It is proved by using the contraction mapping theorem that there is a sequence of maximum points in the problem (2.7.34) and hence there exist a maximum value of $V(\mathbf{z})$. Function $V : S \rightarrow \mathcal{R}$ is called the value function of problem (2.7.34). Like function $U(\cdot, \cdot)$, the value function is also continuous. In addition, if S is a convex set and $U(\cdot, \cdot)$ is concave, then $V(\cdot)$ is also concave, and is equivalent to the Bellman's principle of optimality, that is, the solution to the following Bellman equation:

$$V(\mathbf{s}) = \max_{\hat{\mathbf{s}} \in S} U(\mathbf{s}, \hat{\mathbf{s}}) + \delta V(\hat{\mathbf{s}}).$$

The equivalence results provide the basis for solving the dynamic optimization problem by Bellman method. The following theorem reveals that the value function is the only function satisfying the Bellman equation.

Theorem 2.7.1

$$f(\mathbf{s}) = \max_{\hat{\mathbf{s}} \in S} U(\mathbf{s}, \hat{\mathbf{s}}) + \delta V(\hat{\mathbf{s}}) \quad (2.7.36)$$

that is, $f(\cdot) = V(\cdot)$.

PROOF. Using (2.7.36) repeatedly gives: for each T ,

$$f(\mathbf{z}) = \max_{\{\mathbf{s}_t\}_{t=0}^T} \sum_{t=0}^{T-1} \delta^t U(\mathbf{s}_t, \mathbf{s}_{t+1}) + \delta^T f(\mathbf{x}_T)$$

$$\text{s.t.} \quad \mathbf{s}_t \in S, \quad \forall t,$$

$$\mathbf{s}_0 = \mathbf{z}.$$

When $T \rightarrow \infty$, the contribution of $\delta^T f(\mathbf{x}_T)$ to the above summation is increasingly negligible, and thus $f(\cdot) = V(\cdot)$. \square

The above theorem provides a way to calculate the value function that starting from any continuous function $f_0(\cdot) : S \rightarrow \mathcal{R}$, imagine $f_0(\hat{s})$ as a trial "valuation" function which gives the estimated value from time 0. Then let

$$f_1(\mathbf{s}) = \max_{\hat{\mathbf{s}} \in S} U(\mathbf{s}, \hat{\mathbf{s}}) + \delta f_0(\hat{\mathbf{s}})$$

holds for any a $\mathbf{s} \in S$, and thus we obtain a new valuation function $f_1(\hat{\mathbf{s}})$.

Value function $v(\cdot)$ can be also found by the iterative method. If $f_1(t) = f_0(t)$, then $f_0(t)$ satisfies the Bellman equation. It follows from the theorem above that $f_0(t) = V(t)$. If $f_1(t) \neq f_0(t)$, we get a new valuation function from $f_1(t)$, and get the whole sequence of functions $\{f_r(\cdot)\}_{r=0}^{\infty}$. The theory of dynamic programming proves that for each $\mathbf{s} \in S$, we have

$$\lim_{r \rightarrow \infty} f_r(\mathbf{s}) = V(\mathbf{s}),$$

that is, it will converge to value function with r increasing.

If the function is differentiable, there is a similar first order condition that called the Euler equation of dynamic optimization:

$$0 = F_y(s_t^*, s_{t+1}^*) + \beta F_x(s_{t+1}^*, s_{t+2}^*), \quad t = 0, 1, 2, \dots \quad (2.7.37)$$

The first order condition of optimal decision gives:

$$0 = F_y[\mathbf{x}, g(\mathbf{x})] + \beta v'[g(\mathbf{x})], \quad (2.7.38)$$

where $g(\mathbf{x})$ is the state of next periods determined by \mathbf{x} following principle of optimality. It follows from the envelope theorem that

$$v'(\mathbf{x}) = F_x[\mathbf{x}, g(\mathbf{x})]. \quad (2.7.39)$$

The Euler equation is derived from these two equations above.

2.8 Differential Equations

We first introduce the general concept of the ordinary differential equations defined on Euclidean spaces.

Definition 2.8.1 The equation containing the independent variable x , unknown function $y = y(x)$ of this independent variable, and functions from its first derivative $y' = y'(x)$ to the derivative of n order $y^{(n)} = y^{(n)}(x)$,

$$F(x, y, y', \dots, y^{(n)}) = 0, \quad (2.8.40)$$

is called **ordinary differential equation**. If the highest order of derivative in the equation is n , the equation is also called n th order ordinary differential equation

If for all $x \in I$, the function $y = \psi(x)$ satisfies

$$F(x, \psi(x), \psi'(x), \dots, \psi^{(n)}(x)) = 0,$$

then $y = \psi(x)$ is called a **solution to the ordinary differential equation (2.8.40)**.

Sometimes the solutions for the ordinary differential equations are not unique, and there may even exist infinite solutions. For example, $y = \frac{C}{x} + \frac{1}{5}x^4$ is the solution to the ordinary differential equation $\frac{dy}{dx} + \frac{y}{x} = x^3$, where C is an arbitrary constant. Next we introduce the concept of general solutions and particular solutions to ordinary differential equations.

Definition 2.8.2

The solution to the n th order ordinary differential equation (2.8.40)

$$y = \psi(x, C_1, \dots, C_n) \quad (2.8.41)$$

containing n independent arbitrary constant C_1, \dots, C_n , is called the **general solution** to an ordinary differential equation (2.8.40). Here, independence means that the Jacobi determinant

$$\frac{D[\psi, \psi^{(1)}, \dots, \psi^{(n-1)}]}{D[C_1, \dots, C_n]} \stackrel{\text{def}}{=} \begin{vmatrix} \frac{\partial \psi}{\partial C_1} & \frac{\partial \psi}{\partial C_2} & \dots & \frac{\partial \psi}{\partial C_n} \\ \frac{\partial \psi^{(1)}}{\partial C_1} & \frac{\partial \psi^{(1)}}{\partial C_2} & \dots & \frac{\partial \psi^{(1)}}{\partial C_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \psi^{(n-1)}}{\partial C_1} & \frac{\partial \psi^{(n-1)}}{\partial C_2} & \dots & \frac{\partial \psi^{(n-1)}}{\partial C_n} \end{vmatrix}$$

is not equal to 0.

If the solution to an ordinary differential equation $y = \psi(x)$ does not contain any constant, it is called the particular solution. Obviously, a general solution becomes a particular solution when the arbitrary constants are determined. In general, the restrictions of some initial conditions determine the value of any constants. For example, for an ordinary differential equation (2.8.40), if there are some given initial conditions:

$$y(x_0) = y_0, y^{(1)}(x_0) = y_0^{(1)}, \dots, y^{(n-1)}(x_0) = y_0^{(n-1)}, \quad (2.8.42)$$

an ordinary differential equation (2.8.40) and the initial value conditions (2.8.42) are said to be the Cauchy or initial value problems for n th order ordinary differential equations. Then the question is what conditions the function F should satisfy so that the above ordinary differential equations are uniquely solvable. This problem is the existence and uniqueness about solutions for ordinary differential equations.

2.8.1 Existence and Uniqueness Theorem of Solutions for Ordinary Differential Equations

First, we consider that an ordinary differential equation of first order $y' = f(x, y)$ satisfies initial condition (x_0, y_0) , that is, $y(x_0) = y_0$. Let $y(x)$ is a solution to the differential equation.

First, we introduce the concept of Lipschitz conditions.

Definition 2.8.3 Considering a function $f(x, y)$ defined on $D \subseteq \mathcal{R}^2$, if there exist a neighborhood $U \subseteq D$ of (x_0, y_0) , and a positive number L such that

$$f(x, y) - f(x, \hat{y}) \leq L|y - z|, \forall (x, y), (x, z) \in U,$$

then we say f satisfies the **local Lipschitz condition** with respect to y at the point $(x_0, y_0) \in D$.

If there is a positive number L , such that

$$f(x, y) - f(x, \hat{y}) \leq L|y - z|, \forall (x, y), (x, z) \in D,$$

we call $f(x, y)$ satisfies **global Lipschitz condition** in $D \subseteq \mathcal{R}^2$.

The following lemma characterizes the properties of the function satisfying Lipschitz condition.

Lemma 2.8.1 $f(x, y)$ defined on $y \in D \subseteq \mathcal{R}^2$ is continuously differentiable. If there is an $\epsilon > 0$ such that $f_y(x, y)$ is bounded on $U = \{(x, y) : |x - x_0| < \epsilon, |y - y_0| < \epsilon\}$, then $f(x, y)$ satisfies the local Lipschitz condition. If D is bounded on $f_y(x, y)$, $f(x, y)$ satisfies the global Lipschitz condition.

Theorem 2.8.1 If f is continuous on an open set D , then for any $(x_0, y_0) \in D$, there always exists a solution $y(x)$ to the differential equation, which satisfies the $y' = f(x, y)$ and $y(x_0) = y_0$.

The following is the theorem of the uniqueness of the solution for differential equations.

Theorem 2.8.2 if f is continuous on an open set D , and f satisfies the Lipschitz condition with respect to y , then for any $(x_0, y_0) \in D$, there always exists a unique solution $y(x)$ satisfying $y' = f(x, y)$ and $y(x_0) = y_0$.

For n th order ordinary differential equations $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$, if the Lipschitz condition is changed to for $y, y', \dots, y^{(n-1)}$ instead of for y , we have similar conclusions about existence and uniqueness. See Ahmad and Ambrosetti (2014) for the specific proof of existence and uniqueness.

2.8.2 Some Common Ordinary Differential Equations with Explicit Solutions

Generally, we hope to obtain the concrete form of solutions for differential equations, that is, explicit solutions. However, in many cases, there is no explicit solution. Here we give some common cases that differential equations can be solved explicitly.

Case of Separable Equations

Consider a separable differential equation $y' = f(x)g(y)$, and $y(x_0) = y_0$. It can be rewrite as:

$$\frac{dy}{g(y)} = f(x)dx.$$

integrating both sides, then we get the solution to the differential equation.

For example, for $(x^2 + 1)y' + 2xy^2 = 0$, $y(0) = 1$, using above solving procedure, we get the solution

$$y(x) = \frac{1}{\ln(x^2 + 1) + 1}.$$

In addition, the differential equation with the form $y' = f(y)$ is called autonomous system, since y' is only depend on y .

Homogeneous Type of Differential Equation

Some differential equations with constant coefficients have explicit solutions. We first give a definition of homogeneous functions.

Definition 2.8.4 If for any λ , there is $f(\lambda x, \lambda y) = \lambda^n f(x, y)$, we call the function $f(x, y)$ **homogeneous function of degree n** .

Differential equations has the form of homogeneous functions if $M(x, y)dx + N(x, y)dy = 0$, where $M(x, y)$ and $N(x, y)$ are homogeneous functions with same orders.

By conversion $z = \frac{y}{x}$, the above differential equations can be transformed into the separable form. Suppose $M(x, y), N(x, y)$ are homogeneous functions of degree n , $M(x, y)dx + N(x, y)dy = 0$ is transformed

to $z + x \frac{dz}{dx} = -\frac{M(1, z)}{N(1, z)}$, and then the final form is $\frac{dz}{dx} = -\frac{z + \frac{M(1, z)}{N(1, z)}}{x}$,

where $z + \frac{M(1, z)}{N(1, z)}$ is a function respect to z .

Exact Differential Equation

If the form of differential equation is $M(x, y)dx + N(x, y)dy = 0$, satisfying $\frac{\partial M(x, y)}{\partial y} \equiv \frac{\partial N(x, y)}{\partial x}$, then the solution is $F(x, y) = C$, where the constant

C is determined by the initial value, and $F(x, y)$ satisfies $\frac{\partial F}{\partial x} = M(x, y)$ or $\frac{\partial F}{\partial y} = N(x, y)$.

It is clear that a separable differential equation is a special case of an exact differential equation $y' = f(x)g(y)$ or $\frac{1}{g(y)}dy - f(x)dx = 0$, and then we have $M(x, y) = -f(x)$, $N(x, y) = \frac{1}{g(y)}$, $\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x} = 0$.

For example, $2xy^3dx + 3x^2y^2dy = 0$ is the exact function of which the general solution is $x^2y^3 = C$, and C is a constant.

When solving differential equations with explicit solutions, we usually convert differential equations into the form of exact equations.

First Order Linear Equation

Consider the first order linear equation of the following form:

$$\frac{dy}{dx} + p(x)y = q(x). \quad (2.8.43)$$

When $q(x) = 0$, the above differential equation (2.8.43) is a separable differential equation, and its solution is assumed to be $y = \psi(x)$.

Suppose that $\psi_1(x)$ is a particular solution to the differential equation (2.8.43), then $y = \psi(x) + \psi_1(x)$ is clearly the solution to the equations (2.8.43).

It is easy to be obtained that the solution to $\frac{dy}{dx} + p(x)y = 0$ is $y = Ce^{-\int p(x)dx}$. Next we find a particular solution to the differential equation (2.8.43).

Suppose

$$y = c(x)e^{-\int p(x)dx},$$

and differentiating gives

$$y' = c'(x)e^{-\int p(x)dx} + c(x)p(x)e^{-\int p(x)dx},$$

then substituting back into the original differential equation, we have

$$c'(x)e^{-\int p(x)dx} + c(x)p(x)e^{-\int p(x)dx} = p(x)c(x)e^{-\int p(x)dx} + q(x),$$

and get

$$c'(x) = q(x)e^{\int p(x)dx}.$$

We have

$$c(x) = \int q(x)e^{\int p(x)dx} dx + C.$$

Thus, the solution is

$$y(x) = e^{-\int p(x)dx} \left(\int q(x)e^{\int p(x)dx} dx + C \right).$$

Bernoulli Equation

The following differential equation is called the **Bernoulli equation**:

$$\frac{dy}{dx} + p(x)y = q(x)y^n \quad (2.8.44)$$

where $n \neq 0, 1$ is a natural number.

Multiplying both sides by $(1 - n)y^{(-n)}$ gives:

$$(1 - n)y^{(-n)}\frac{dy}{dx} + (1 - n)y^{(1-n)}p(x) = (1 - n)q(x).$$

Let $z = y^{(1-n)}$, and get:

$$\frac{dz}{dx} + (1 - n)zp(x) = (1 - n)q(x),$$

which becomes a first order linear equation specified above which the explicit solution can be obtained.

The differential equations with explicit solutions have other forms, such as some special forms of Riccati equations, and the equations similar to $M(x, y)dx + N(x, y)dy = 0$, but not satisfying

$$\frac{\partial M(x, y)}{\partial y} \equiv \frac{\partial N(x, y)}{\partial x}$$

. See Ding Tongren and Li Chengzhi for a detailed discussion of these issues (2004).

2.8.3 Higher Order Linear Equations with Constant Coefficients

Considering a differential equation of degree n with constant coefficients,

$$y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = f(x). \quad (2.8.45)$$

If $f(x) \equiv 0$, then the differential equation (2.8.45) is called **homogeneous differential equation of degree n** , otherwise it is called **nonhomogeneous differential equation**.

There is a method for finding the general solution $y_g(x)$ of a homogeneous differential equation of degree n . The general solution is the sum of n bases of solutions y_1, \dots, y_n , that is,

$y_g(x) = C_1 y_1(x) + \dots + C_n y_n(x)$, where C_1, \dots, C_n are arbitrary constants.

These arbitrary constants are determined only by initial value conditions. Find a function $y(x)$ to meet:

$$y(x) = y_{00}, y'(x) = y_{01}, \dots, y^{(n-1)}(x) = y_{0_{n-1}}, \text{ when } x = x_0,$$

where, $x_0, y_0, y'_0, \dots, y_0^{(n-1)}$ are the known initial values.

The procedures of solving the basis of solution for homogeneous differential equations are given below:

(1) Solve the characteristic equation with respect to λ :

$$\lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n = 0.$$

Suppose the roots of the characteristic equation are: $\lambda_1, \dots, \lambda_n$ respectively. Some roots may be complex and some are multiple.

(2) If λ_i is the non-multiple real characteristic root, then the basis of solution corresponding to this root is $y_i(x) = e^{\lambda_i x}$.

(3) If λ_i is the real characteristic root of multiplicity k , then there are k bases of solution:

$$y_{i_1}(x) = e^{\lambda_i x}, y_{i_2}(x) = x e^{\lambda_i x}, \dots, y_{i_k}(x) = x^{k-1} e^{\lambda_i x}.$$

(4) If λ_j is the non-multiple complex characteristic root $\lambda_j = \alpha_j + i\beta_j$, $i = \sqrt{-1}$, its complex conjugate denoted by $\lambda_{j+1} = \alpha_j - i\beta_j$ is also the characteristic root, thus there are two bases of solution generating by these complex conjugate roots λ_j, λ_{j+1} :

$$y_{j_1} = e^{\alpha_j x} \cos \beta_j x, \quad y_{j_2} = e^{\alpha_j x} \sin \beta_j x.$$

(5) If λ_j is the complex characteristic root of multiplicity l , $\lambda_j = \alpha_j + i\beta_j$, its complex conjugate is also the complex characteristic root of multiplicity l , thus these $2l$ complex roots generates $2l$ bases of solution:

$$y_{j_1} = e^{\alpha_j x} \cos \beta_j x, y_{j_2} = x e^{\alpha_j x} \cos \beta_j x, \dots, y_{j_l} = x^{l-1} e^{\alpha_j x} \cos \beta_j x;$$

$$y_{j_{l+1}} = e^{\alpha_j x} \sin \beta_j x, y_{j_{l+2}} = x e^{\alpha_j x} \sin \beta_j x, \dots, y_{j_{2l}} = x^{l-1} e^{\alpha_j x} \sin \beta_j x.$$

The following is a general method for solving nonhomogeneous differential equations.

The general form of solution to nonhomogeneous differential equations is $y_{nh}(x) = y_g(x) + y_p(x)$, where $y_g(x)$ is the corresponding general solution

to the homogeneous equation, and $y_p(x)$ is the particular solution to the nonhomogeneous equations.

Next are some procedures for solving particular solutions to nonhomogeneous equations.

(1) If $f(x) = P_k(x)e^{bx}$, $P_k(x)$ is the polynomial of degree k , then a form of particular solutions is:

$$y_p(x) = x^s Q_k(x)e^{bx},$$

where $Q_k(x)$ is also a polynomial of degree k . If b is not a characteristic root corresponding to the characteristic equation, then $s = 0$; if b is a characteristic root of multiplicity m , then $s = m$.

(2) If $f(x) = P_k(x)e^{px} \cos qx + Q_k(x)e^{px} \sin qx$, $P_k(x), Q_k(x)$ are all polynomials of degree k , then a form of particular solutions is:

$$y_p(x) = x^s R_k(x)e^{px} \cos qx + x^s T_k(x)e^{px} \sin qx,$$

where $R_k(x), T_k(x)$ are also polynomials of degree k . If $p + iq$ is not a root of the characteristic equation, then $s = 0$; if $p + iq$ is a characteristic root of multiplicity m , then $s = m$.

(3) **The variation of parameters, or the method of undetermined coefficients** is a general method for solving nonhomogeneous differential equations.

Suppose the known general solution to a homogeneous equation is as follows:

$$y_g = C_1 y_1(x) + \cdots + C_n y_n(x),$$

where $y_i(x)$ is the basis of solution. Regard constants C_1, \cdots, C_n as the functions with respect to x , such as $u_1(x), \cdots, u_n(x)$, so the form of particular solutions to a nonhomogeneous equation can be expressed as

$$y_p(x) = u_1(x)y_1(x) + \cdots + u_n(x)y_n(x),$$

where $u_1(x), \cdots, u_n(x)$ are the solution following equations

$$u_1'(x)y_1(x) + \cdots + u_n'(x)y_n(x) = 0,$$

$$u_1'(x)y_1'(x) + \cdots + u_n'(x)y_n'(x) = 0,$$

$$\vdots$$

$$u_1'(x)y_1^{(n-2)}(x) + \cdots + u_n'(x)y_n^{(n-2)}(x) = 0,$$

$$u_1'(x)y_1^{(n-1)}(x) + \cdots + u_n'(x)y_n^{(n-1)}(x) = f(x).$$

(4) If $f(x) = f_1(x) + f_2(x) + \cdots + f_r(x)$, and $y_{p1}(x), \cdots, y_{pr}(x)$ are the particular solutions corresponding to $f_1(x), \cdots, f_r(x)$, then

$$y_p(x) = y_{p1}(x) + \cdots + y_{pr}(x).$$

Here is an example to familiarize the application of this method.

Example 2.8.1 Solve $y'' - 5y' + 6y = t^2 + e^t - 5$.

The characteristic roots now are $\lambda_1 = 2$ and $\lambda_2 = 3$. Thus, the general solution to the homogeneous equation is:

$$y(t) = C_1 e^{2t} + C_2 e^{3t}.$$

Next find a particular solution to the nonhomogeneous equation, and its form is written as:

$$y_p(t) = at^2 + bt + c + de^t.$$

We first substitute this particular solution in the initial equation to determine the coefficients a, b, c, d :

$$2a + de^t - 5(2at + b + de^t) + 6(at^2 + bt + c + de^t) = t^2 - 5 + e^t.$$

The coefficients of both sides should be consistent, thus we get:

$$6a = 1, \quad -5 \times 2a + 6b = 0, \quad 2a - 5b + 6c = -5, \quad d - 5d + 6d = 1,$$

Therefore, $d = 1/2$, $a = 1/6$, $b = 5/18$, $c = -71/108$.

Finally, the general solution to the nonhomogeneous differential equation is:

$$y(t) = C_1 e^{2t} + C_2 e^{3t} + \frac{t^2}{6} + \frac{5t}{18} - \frac{71}{108} + \frac{e^t}{2}.$$

2.8.4 System of Ordinary Differential Equations

The general form is:

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t), \quad \mathbf{x}(0) = \mathbf{x}_0,$$

where t is an independent variable (time), $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))'$ is a vector of dependent variables, $A(t) = (a_{ij}(t))_{[n \times n]}$ is an $n \times n$ matrix of real variable coefficients, and $\mathbf{b}(t) = (b_1(t), \dots, b_n(t))'$ is an n -dimensional variable vector.

Consider the case that A is a constant coefficient matrix and \mathbf{b} is a constant vector, also called the **system of differential equations with constant coefficients**:

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + \mathbf{b}, \quad \mathbf{x}(0) = \mathbf{x}_0. \quad (2.8.46)$$

where suppose A is nonsingular.

The system of differential equations (2.8.46) can be solved by the following two steps.

First step, we consider the system of homogeneous equations (i.e. $\mathbf{b} = 0$):

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0. \quad (2.8.47)$$

and its solution is denoted by $\mathbf{x}_c(t)$.

Second step, find a particular solution \mathbf{x}_p to the nonhomogeneous equation (2.8.46). The constant vector \mathbf{x}_p is a particular solution: $A\mathbf{x}_p = -\mathbf{b}$, namely $\mathbf{x}_p = -A^{-1}\mathbf{b}$.

Given the general solution to the homogeneous equation and the particular solution to the nonhomogeneous equation, the general solution to the system of differential equations (2.8.47) is:

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p.$$

There are two methods for solving the system of homogeneous differential equations (2.8.47).

The first one is that we can eliminate $n - 1$ dependent variables so that the system of differential equations becomes the differential equation of order n , such as the following example.

Example 2.8.2 The system of differential equation is:

$$\begin{cases} \dot{x} = 2x + y, \\ \dot{y} = 3x + 4y. \end{cases}$$

Differentiate the first equation so that eliminate y and \dot{y} . Since $\dot{y} = 3x + 4y = 3x + 4\dot{x} - 4 \cdot 2x$, we obtain a corresponding quadratic homogeneous differential equation:

$$\ddot{x} - 6\dot{x} + 5x = 0,$$

thus the general solution is $x(t) = C_1 e^t + C_2 e^{5t}$. Since $y(t) = \dot{x} - 2x$, $y(t) = -C_1 e^t + 3C_2 e^{5t}$.

The second method is that rewrite the homogeneous differential equation (2.8.47) as:

$$\mathbf{x}(t) = e^{At} \mathbf{x}_0,$$

where

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \dots$$

Now we solve e^{At} in three different cases.

Case 1: A has different real eigenvalues

Matrix A has different real eigenvalues, which means that its eigenvectors are linearly independent. Thus A can be diagonalization, that is,

$$A = P\Lambda P^{-1},$$

where $P = [v_1, v_2, \dots, v_n]$ consist of the eigenvectors of A , and moreover Λ is a diagonal matrix whose diagonal elements are the eigenvalues of A , thus we have

$$e^A = Pe^\Lambda P^{-1}.$$

Therefore, the solution to the system of differential equation (2.8.47) is:

$$\begin{aligned} \mathbf{x}(t) &= Pe^{\Lambda t} P^{-1} \mathbf{x}_0 \\ &= Pe^{\Lambda t} \mathbf{c} \\ &= c_1 v_1 e^{\lambda_1 t} + \dots + c_n v_n e^{\lambda_n t}, \end{aligned}$$

where $\mathbf{c} = (c_1, c_2, \dots, c_n)$ is a vector of arbitrary constants, and it is determined by initial value, namely ($\mathbf{c} = P^{-1} \mathbf{x}_0$).

Case 2: A has multiple real eigenvalues, but no complex eigenvalues

First, consider a simple case, that is, A has only one eigenvalue of multiplicity m . In this case, in general, there are at most m linearly independent eigenvectors, which means that the matrix P can not be constructed as a matrix consisting of linearly independent eigenvectors, so A can not be diagonalized.

Thus, the solution has the following form:

$$\mathbf{x}(t) = \sum_{i=1}^m c_i h_i(t),$$

Where $h_i(t)$ are quasi-polynomials, and c_i are determined by initial conditions. For example, when $m = 3$, we have:

$$\begin{aligned} h_1(t) &= e^{\lambda t} v_1, \\ h_2(t) &= e^{\lambda t} (t v_1 + v_2), \\ h_3(t) &= e^{\lambda t} (t^2 v_1 + 2t v_2 + 3v_3), \end{aligned}$$

where v_1, v_2, v_3 are determined by following conditions:

$$(A - \lambda I)v_i = v_{i-1}, v_0 = 0.$$

If A has more than one multiple real eigenvalues, then the solution to the differential equation (2.8.47) can be obtained by summing up the solutions corresponding to each eigenvalue.

Case 3: A has complex eigenvalues

Since A is a real matrix, complex eigenvalues will be generated in the form of conjugate pairs.

If an eigenvalues of A is $\alpha + \beta i$, then its conjugate complex $\alpha - \beta i$ is also an eigenvalue.

Now consider a simple case: A has only one pair of complex eigenvalues, $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$.

Let \mathbf{v}_1 and \mathbf{v}_2 be the eigenvectors belong to λ_1 and λ_2 ; then we have $\mathbf{v}_2 = \bar{\mathbf{v}}_1$, where $\bar{\mathbf{v}}_1$ refers to the conjugation of \mathbf{v}_1 . The solution to the differential equation (2.8.47) can be expressed as:

$$\begin{aligned}\mathbf{x}(t) &= e^{At} \mathbf{x}_0 \\ &= P e^{At} P^{-1} \mathbf{x}_0 \\ &= P e^{At} \mathbf{c} \\ &= c_1 v_1 e^{(\alpha + \beta i)t} + c_2 v_2 e^{(\alpha - \beta i)t} \\ &= c_1 v_1 e^{\alpha t} (\cos \beta t + i \sin \beta t) + c_2 v_2 e^{\alpha t} (\cos \beta t - i \sin \beta t) \\ &= (c_1 v_1 + c_2 v_2) e^{\alpha t} \cos \beta t + i(c_1 v_1 - c_2 v_2) e^{\alpha t} \sin \beta t \\ &= h_1 e^{\alpha t} \cos \beta t + h_2 e^{\alpha t} \sin \beta t,\end{aligned}$$

where $h_1 = c_1 v_1 + c_2 v_2$, $h_2 = i(c_1 v_1 - c_2 v_2)$ is a real vector.

If A has many pairs of conjugate complex eigenvalues, then the solution to the differential equation (2.8.47) is summing up the solutions corresponding to all eigenvalues.

2.8.5 Simultaneous Differential Equations and Types of Equilibrium Stability

Consider a case of simultaneous differential equations with respect to two variables $x = x(t)$, $y = y(t)$, where t is a (time) independent variable.

The two dimensional autonomous differential equations has the following form:

$$\begin{cases} \frac{dx}{dt} = f(x, y), \\ \frac{dy}{dt} = g(x, y). \end{cases}$$

Such simultaneous differential equations are called **planar dynamical systems**, and multidimensional dynamical systems can be similarly defined.

If $f(x^*, y^*) = g(x^*, y^*) = 0$ is established, the point (x^*, y^*) is called the **equilibrium** of this dynamical system.

Let \mathcal{J} be the value of the Jacobian determinant

$$\mathcal{J} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

at (x^*, y^*) , let λ_1 and λ_2 be the eigenvalues of this Jacobian determinant.

Then the equilibrium stability :

(1) is a **stable (or unstable) node**, if λ_1 and λ_2 are different real numbers and both are negative (or positive);

(2) is **saddle point type of stable**, if eigenvalues are real numbers but with opposite signs, namely $\lambda_1 \lambda_2 < 0$;

(3) is a **stable (or unstable) focus**, if λ_1 and λ_2 are complex numbers, and $Re(\lambda_1) < 0$ (or $Re(\lambda_1) > 0$);

(4) is a **center**, if λ_1 and λ_2 are complex, and $Re(\lambda_1) = 0$;

(5) is a **stable (or unstable) improper node**, if λ_1 and λ_2 are real, $\lambda_1 = \lambda_2 < 0$ (or $\lambda_1 = \lambda_2 > 0$), and the Jacobian determinant is not a diagonal matrix;

(6) is a **stable (or unstable) star node**, if λ_1 and λ_2 are real, $\lambda_1 = \lambda_2 < 0$ (or $\lambda_1 = \lambda_2 > 0$), and the Jacobian determinant is a diagonal matrix

Figure 2.5 below depicts the types of the 6 equilibrium points above.

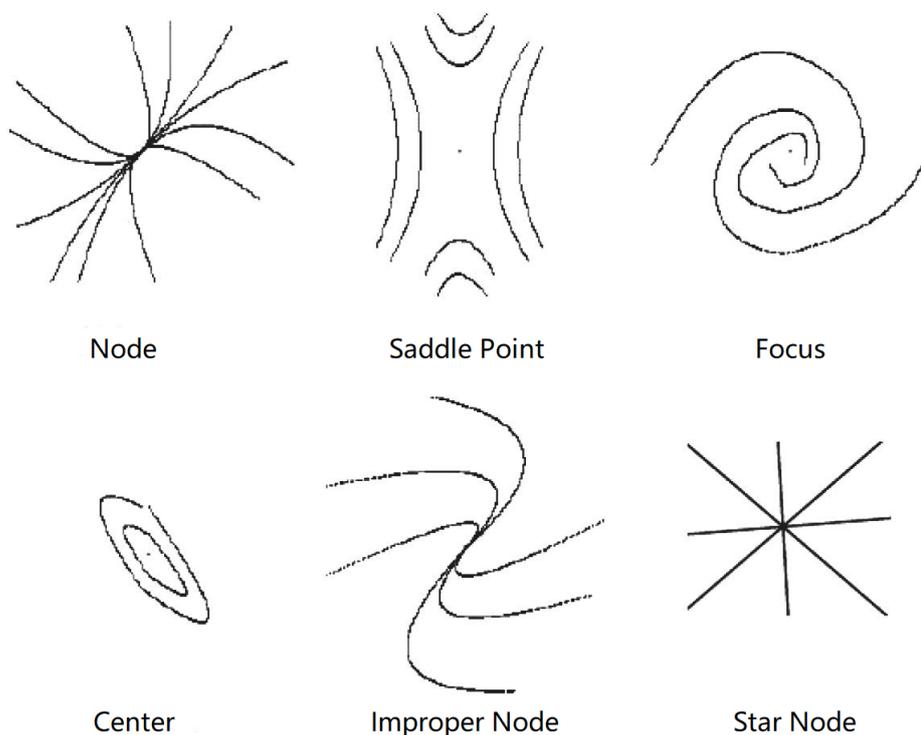


Figure 2.5: The Types of Equilibrium Points

2.8.6 The Stability of Dynamical System

In a dynamical system, Lyapunov method studies the global stability of equilibrium points.

The following dynamical system:

$$\dot{\mathbf{x}} = f(t, \mathbf{x}), \quad (2.8.48)$$

where $\mathbf{x} = (x_1, \dots, x_n)$, $f(t, \mathbf{x})$ is continuously differentiable with respect to $\mathbf{x} \in \mathcal{R}^n$ and meanwhile satisfies the initial condition $\mathbf{x}(0) = \mathbf{x}_0$. \mathbf{x}^* is the equilibrium point of the dynamical system, namely $f(t, \mathbf{x}^*) = 0$.

Let $\bar{\mathbf{x}}(t, \mathbf{x}_0)$ be the unique solution to the dynamical system (2.8.48) and initial conditions. $B_r(\mathbf{x}) = \{\mathbf{x}' \in D : |\mathbf{x}' - \mathbf{x}| < r\}$ is an open ball of radius r centered at \mathbf{x} .

The following is the definition of stability of equilibrium point.

Definition 2.8.5 *The equilibrium point \mathbf{x}^* of the dynamical system (2.8.48)*

- (1) is **globally stable**, if for any $r > 0$, there is a neighbourhood U of \mathbf{x}^* such that

$$\bar{\mathbf{x}}(t, \mathbf{x}_0) \in B_r(\mathbf{x}^*), \forall \mathbf{x}_0 \in U.$$

- (2) is **globally asymptotically stable**, if for any $r > 0$, there is a neighbourhood U' of \mathbf{x}^* such that

$$\lim_{t \rightarrow \infty} \bar{\mathbf{x}}(t, \mathbf{x}_0) = \mathbf{x}^*, \forall \mathbf{x}_0 \in U'.$$

- (3) is **globally unstable**, if it is neither globally stable nor asymptotically globally stable.

Definition 2.8.6 *Let \mathbf{x}^* be the equilibrium point of the dynamical system (2.8.48), $Q \subseteq \mathcal{R}^n$ be an open set containing \mathbf{x}^* , and $V(\mathbf{x}) : Q \rightarrow \mathcal{R}$ be a continuously differentiable function. If it satisfies:*

- (1) $V(\mathbf{x}) > V(\mathbf{x}^*)$, $\forall \mathbf{x} \in Q$, $\mathbf{x} \neq \mathbf{x}^*$;
 (2) $\dot{V}(\mathbf{x})$ is defined as:

$$\dot{V}(\mathbf{x}) \stackrel{\text{def}}{=} \nabla V(\mathbf{x})f(t, \mathbf{x}) \leq 0, \forall \mathbf{x} \in Q, \quad (2.8.49)$$

where $\nabla V(\mathbf{x})$ is the gradient of V with respect to \mathbf{x} ,

thus it is called **Lyapunov function**.

The following is the Lyapunov theorem about equilibrium points of dynamical systems.

Theorem 2.8.3 *If the dynamical system (2.8.48) exists a Lyapunov function V , then the equilibrium point \mathbf{x}^* is globally stable.*

If the Lyapunov function (2.8.49) of the dynamical system satisfies $\dot{V}(\mathbf{x}) < 0$, $\forall \mathbf{x} \in Q$, $\mathbf{x} \neq \mathbf{x}^$, then the equilibrium point \mathbf{x}^* is asymptotically globally stable.*

2.9 Difference Equations

Difference equations can be regarded as discretized differential equations, and many of their properties are similar to that of differential equations.

Let y be a real-valued function defined on natural numbers. y_t means $y(t)$, that is, the value of y at t , where $t = 0, 1, 2, \dots$, which can be regarded as time points.

Definition 2.9.1 The first order difference of y at t is:

$$\Delta y(t) = y(t+1) - y(t).$$

The second order difference of y at t is:

$$\Delta^2 y(t) = \Delta(\Delta y(t)) = y(t+2) - 2y(t+1) + y(t).$$

Generally, the n th-order difference of y at t is:

$$\Delta^n y(t) = \Delta(\Delta^{n-1} y(t)), \quad n > 1.$$

Definition 2.9.2 The difference equation contains y and its difference $\Delta y, \Delta^2 y, \dots, \Delta^{n-1} y$,

$$F(y, \Delta y, \Delta^2 y, \dots, \Delta^n y, t) = 0, \quad t = 0, 1, 2, \dots \quad (2.9.50)$$

if n is the highest order difference of nonzero coefficient in the formula (2.9.50), the above equation is called **n th order difference equation**.

If $F(\psi(t), \Delta\psi(t), \Delta^2\psi(t), \dots, \Delta^n\psi(t), t) = 0$, holds for $\forall t$ then we call $y = \psi(k)$ **is a solution to the difference equation**. Similar to differential equations, the solutions of difference equations also have general solutions and particular solutions. The general solutions usually contains some arbitrary constants that can be determined by initial conditions.

The difference equations can also be expressed in the following form by variables conversion:

$$F(y(t), y(t+1), \dots, y(t+n), t) = 0, \quad t = 0, 1, 2, \dots \quad (2.9.51)$$

If the coefficients of $y_0(k), y_n(k)$ are not zero, and the highest corresponding order is n , then it is called **n th order difference equations**.

The followings are mainly focused on the difference equations with constant coefficients. A common expression is written as:

$$f_0 y(t+n) + f_1 y(t+n-1) + \dots + f_{n-1} y(t+1) + f_n y(t) = g(t), \quad t = 0, 1, 2, \dots, \quad (2.9.52)$$

where f_0, f_1, \dots, f_n are real numbers, and $f_0 \neq 0, f_n \neq 0$.

Dividing both sides of the equation by f_0 , and making $a_i = \frac{f_i}{f_0}$ for $i = 0, \dots, n$, $r(t) = \frac{g(t)}{f_0}$, the n th order difference equation can be written as a simpler form:

$$y(t+n) + a_1y(t+n-1) + \dots + a_{n-1}y(t+1) + a_ny(t) = r(t), t = 0, 1, 2, \dots \quad (2.9.53)$$

Here are three procedures that are usually used to solve n th order linear difference equations:

The first step:

find the general solution to the homogeneous difference equation

$$y(t+n) + a_1y(t+n-1) + \dots + a_{n-1}y(t+1) + a_ny(t) = 0,$$

and let the general solution be $y(t)$.

The second step:

find a particular solution y^* to the difference equation (2.9.53).

The third step:

The solution to the difference equation is (2.9.53)

$$y(t) = Y + y^*.$$

The followings give the solutions to first, second and n th difference equations respectively.

2.9.1 First Difference Equations

The first order difference equation is defined as:

$$y(t+1) + ay(t) = r(t), t = 0, 1, 2, \dots \quad (2.9.54)$$

The corresponding homogeneous difference equation is:

$$y(t+1) + ay(t) = 0,$$

and the general solution is $y(t) = c(-a)^t$, where c is an arbitrary constant.

Next, we discuss how to get a particular solution of a nonhomogeneous difference equation (2.9.54):

First, consider $r(t) = r$, that is, the case of not changing with time.

Obviously, a particular solution is as follows:

$$y^* = \frac{r}{1+a}, a \neq -1,$$

$$y^* = rt, a = -1.$$

Hence, the solution to the nonhomogeneous difference equation(2.9.54) is:

$$y(t) = \begin{cases} c(-a)^t + \frac{r}{1+a}, & \text{if } a \neq -1, \\ c + rt, & \text{if } a = -1. \end{cases} \quad (2.9.55)$$

If the initial condition is $y(0) = y_0$ is known, the solution to the difference equation (2.9.54) is:

$$y(t) = \begin{cases} \left(y_0 - \frac{r}{1+a}\right) \times (-a)^t + \frac{r}{1+a}, & \text{if } a \neq -1, \\ y_0 + rt, & \text{if } a = -1. \end{cases} \quad (2.9.56)$$

If r is dependent to k , a particular solution is:

$$y^* = \sum_{i=0}^{t-1} (-a)^{t-1-i} r(i),$$

thus the solution to the difference equation (2.9.54) is:

$$y(t) = (-a)^t y_0 + \sum_{i=0}^{t-1} (-a)^{t-1-i} r(i), \quad t = 1, 2, \dots .$$

For a general function $r(t) = f(t)$, the coefficients of A_0, \dots, A_m can be determined by using method of undetermined coefficients $y^* = f(A_0, A_1, \dots, A_m; t)$. The following is to solve a particular solution in a case that $r(t)$ is a polynomial.

Example 2.9.1 Solve the following difference equation:

$$y(t+1) - 3y(t) = t^2 + t + 2.$$

The homogeneous equation is:

$$y(t+1) - 3y(t) = 0,$$

The general explanation is:

$$Y = C \cdot 3^t.$$

Using the undetermined coefficients method to solve the particular solution of the nonhomogeneous equation, suppose the particular solution has the form:

$$y^* = At^2 + Bt + D.$$

Substitute y^* into the nonhomogeneous difference equation, and get:

$$A(t+1)^2 + B(t+1) + D - 3At^2 - 3Bt - 3D = t^2 + k + 2,$$

or

$$-2At^2 + 2(A - B)t + A + B - 2D = t^2 + t + 2.$$

Since equality holds for each k , we must have:

$$\begin{cases} -2A = 1 \\ 2(A - B) = 1 \\ A + B - 2D = 2, \end{cases}$$

which gives: $A = -\frac{1}{2}, B = -1, D = -\frac{3}{4}$, thus we have a particular solution: $y^* = -\frac{1}{2}t^2 - t - \frac{3}{4}$. Therefore, a particular solution to the nonhomogeneous equation is $y(t) = Y + y^* = C3^t - \frac{1}{2}t^2 - t - \frac{3}{4}$.

We can also solve exponential function by using undetermined coefficient method.

Example 2.9.2 Consider the first order difference equation:

$$y(t+1) - 3y(t) = 4e^k.$$

Suppose the form of particular solution is $y^* = Ae^t$, then substituting it to the nonhomogeneous difference equation gives: $A = \frac{4}{e-3}$. Therefore, the general solution to the first order difference equation above is: $y(t) = Y + y^* = C3^t + \frac{4e^t}{e-3}$.

Here are some of the common forms of finding particular solutions:

- (1) when $r(t) = r$, a usual form of particular solution is: $y^* = A$;
- (2) when $r(t) = r + ct$, a usual form of particular solution is: $y^* = A_1t + A_2$;
- (3) when $r(t) = t^n$, a usual form of particular solution is: $y^* = A_0 + A_1t + \dots + A_nt^n$;
- (4) when $r(t) = c^t$, a usual form of particular solution is: $y^* = Ac^t$;
- (5) when $r(t) = \alpha \sin(ct) + \beta \cos(ct)$, a usual form of particular solution is: $y^* = A_1 \sin(ct) + A_2 \cos(ct)$.

2.9.2 Second Order Difference Equation

The second order difference equation is defined as:

$$y(t+2) + a_1y(t+1) + a_2y(t) = r(t).$$

The corresponding homogeneous differential equation is:

$$y(t+2) + a_1y(t+1) + a_2y(t) = 0.$$

Then, its general solution depends on the roots of the following linear equation in two unknowns:

$$m^2 + a_1m + a_2 = 0,$$

which is called the **auxiliary equation or characteristic equation** of second order difference equations. Let m_1, m_2 be the roots of this equation. Since $a_2 \neq 0$, both m_1 and m_2 are not 0.

Case 1: m_1 and m_2 are different real roots.

Now the general solution of the homogeneous equation is $Y = C_1m_1^t + C_2m_2^t$, where C_1, C_2 are arbitrary constants.

Case 2: m_1, m_2 are same real roots.

Now the general solution of the homogeneous equation is $Y = (C_1 + C_2t)m_1^k$.

Case 3: m_1, m_2 are two complex roots, whose the form are $r(\cos \theta \pm i \sin \theta), r > 0, \theta \in (-\pi, \pi]$. Now the general solution of the homogeneous equation is $Y = C_1r^t \cos(t\theta + C_2)$.

For a general function $r(t)$, it can also be obtained by undetermined coefficient method.

2.9.3 Difference Equation of Order n

The general n th order difference equation is defined as:

$$y(t+n) + a_1y(t+n-1) + \cdots + a_{n-1}y(t+1) + a_ny(t) = r(t), t=0, 1, 2, \cdots . \quad (2.9.57)$$

The corresponding homogeneous equation is:

$$y(t+n) + a_1y(t+n-1) + \cdots + a_{n-1}y(t+1) + a_ny(t) = 0,$$

and its characteristic equation is:

$$m^n + a_1m^{n-1} + \cdots + a_{n-1}m + a_n = 0.$$

Let its n characteristic roots be m_1, \cdots, m_n .

The general solutions of the homogeneous equations are the sum of the bases generated by these eigenvalues, and its concrete forms are as follows:

Case 1: The formula generating by a single real root m is $C_1 m^k$.

Case 2: The formula generating by the real root m of multiplicity p is:

$$(C_1 + C_2 t + C_3 t^2 + \cdots + C_p t^{p-1}) m^t.$$

Case 3: The formula generating by a pair of nonrepeated conjugate complex roots $r(\cos \theta \pm i \sin \theta)$:

$$C_1 r^t \cos(t\theta + C_2).$$

Case 4: The formula generating by a pair of conjugate complex roots $r(\cos \theta \pm i \sin \theta)$ of multiplicity p :

$$r^t [C_{1,1} \cos(t\theta + C_{1,2}) + C_{2,1} t \cos(t\theta + C_{2,2}) + \cdots + C_{p,1} t^{p-1} \cos(t\theta + C_{p,2})].$$

The general solution of the homogeneous difference equation is obtained by summing up all formula generating by eigenvalues.

A particular solution y^* of a nonhomogeneous difference equation can be generated by similar undetermined coefficient method.

Or a particular solution is:

$$y^* = \sum_{s=1}^n \theta_s \sum_{i=0}^{\infty} m_s^i r(t-i),$$

where

$$\theta_s = \frac{m_s}{\prod_{j \neq s} (m_s - m_j)}.$$

2.9.4 The Stability of n th Order Difference Equations

Consider an n th order difference equation

$$y(t+n) + a_1 y(t+n-1) + \cdots + a_{n-1} y(t+1) + a_n y(t) = r(t), \quad t = 0, 1, 2, \dots \quad (2.9.58)$$

The corresponding homogeneous equation is:

$$y(t+n) + a_1 y(t+n-1) + \cdots + a_{n-1} y(t+1) + a_n y(t) = 0, \quad t = 0, 1, 2, \dots \quad (2.9.59)$$

Definition 2.9.3 If an arbitrary solution $Y(k)$ of the homogeneous equation (2.9.59) satisfies $Y(k)|_{k \rightarrow \infty} = 0$, then the difference equation (2.9.53) is **asymptotically stable**.

Let m_1, \dots, m_n be the solution of their characteristic equation (2.9.60):

$$m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0. \quad (2.9.60)$$

Theorem 2.9.1 *If the modulus of all eigenvalues of the characteristic equation are less than 1, the difference equation (2.9.53) is asymptotically stable.*

When the following inequality conditions are satisfied, the modulus of all eigenvalues of the characteristic equation are less than 1.

$$\begin{aligned} & \begin{vmatrix} 1 & a_n \\ a_n & 1 \end{vmatrix} > 0, \\ & \begin{vmatrix} 1 & 0 & a_n & a_{n-1} \\ a_1 & 1 & 0 & a_n \\ a_n & 0 & 1 & a_1 \\ a_{n-1} & a_n & 0 & 1 \end{vmatrix} > 0, \\ & \begin{vmatrix} 1 & 0 & \cdots & 0 & a_n & a_{n-1} & \cdots & a_1 \\ a_1 & 1 & \cdots & 0 & 0 & a_n & a_{n-1} & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & 1 & 0 & 0 & \cdots & a_n & \\ a_n & 0 & \cdots & 0 & 1 & a_1 & \cdots & a_{n-1} & \\ a_{n-1} & a_n & \cdots & 0 & 0 & 1 & \cdots & a_{n-2} & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & a_2 & \cdots & a_n & 0 & 0 & \cdots & 1 & \end{vmatrix} > 0. \end{aligned}$$

2.9.5 Difference Equations with Constant Coefficients

The difference equation with constant coefficients is defined as:

$$\mathbf{x}(t) = A\mathbf{x}(t - 1) + \mathbf{b}, \tag{2.9.61}$$

where $\mathbf{x} = (x_1, \dots, x_n)'$, $\mathbf{b} = (b_1, \dots, b_n)'$. Suppose that the matrix A can be diagonalizable, there are corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ and the matrix P formed by linear independent eigenvectors such that

$$A = P^{-1} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} P.$$

A necessary and sufficient condition for the differential equation (2.9.61) to be (asymptotically) stable is that the modulus of all eigenvalues λ_i are less than 1. When the modulus of all eigenvalues λ_i are less than 1, the equilibrium point $\mathbf{x}^* = \lim_{t \rightarrow \infty} \mathbf{x}(t) = (I - A)^{-1}\mathbf{b}$.

2.10 Basic Probability

Risk and uncertainty, as well as some of their basic operations, are widely used in economics. This section briefly introduces some concepts involved in textbooks.

2.10.1 Probability and Conditional Probability

Compared with other fields of mathematics, the development of probability theory is rather late. However, after axiomatic, probability theory has developed profoundly and rapidly and become a very important field in mathematics. All of it are attributed to Andrey Nikolaevich Kolmogorov (1903-1987), the greatest Russian probability scientist in twentieth Century. In 1933, Kolmogorov published a book less than 100 pages, *Foundations of the Theory of Probability*, laying the foundations of probability theory as the name says. In this book, he believed that the probability theory, as a mathematical discipline, could and should begin to develop from axioms, just like geometry and algebra.

When dealing with probability problems, we must clearly define the situation, that is, giving the probability space explicitly. Classical probability (i.e. explaining probability as the same possibility) is often associated with permutation and combination.

Statistics requires getting data from sampling, so that the characteristics of randomness in probability theory are revealed.

The probability of random variable \tilde{R}_a being \tilde{R}_{as} is $\pi_s, s \in S$.

S can be divided into two different situations, discrete and continuous. When $S = \{1, \dots, n\}$, where n can be finite or infinite, this situation is about a discrete random variable. If S is an interval of real space, then it is called a continuous random variable.

If there is a correlation between two random variables, then the value of a random variable will provides information for the value of another random variable, which is the concept of conditional probability.

When the two random variables \tilde{R}_a, \tilde{R}_b share a same probability distribution $\pi_{ss'}$, with the known information that $\tilde{R}_a = \tilde{R}_{as}$, the probability of $\tilde{R}_b = \tilde{R}_{bs'}$ is called conditional probability:

$$P(\tilde{R}_b = \tilde{R}_{bs'} | \tilde{R}_a = \tilde{R}_{as}) = \frac{\pi_{ss'}}{\sum_{t' \in S'} \pi_{st'}}$$

This formula is also called the Bayes' rule.

2.10.2 Mathematical Expectation and Variance

The **expectation** of random variable \tilde{R}_a is the weighted average of all possible values, is defined as $E(\tilde{R}_a) \equiv \bar{R}_a = \sum_{s \in S} \pi_s R_{as}$, which in a continuous

case is to solve an integral.

The operation rule of expected utility is that if \tilde{R}_a, \tilde{R}_b are two random variables, then we have

$$E(a\tilde{R}_a + b\tilde{R}_b) = a\bar{R}_a + b\bar{R}_b.$$

The variance of a random variable \tilde{R}_a measuring the degree of variation of its value is defined as

$$\text{Var}(\tilde{R}_a) \equiv \sigma_{\tilde{R}_a}^2 = \sum_{s \in S} \pi_s (\tilde{R}_{as} - \bar{R}_a)^2.$$

thus, the larger the variance, the greater the variation degree is.

There may be some correlation between the two random variables \tilde{R}_a, \tilde{R}_b .

Suppose the value space of \tilde{R}_a is $\{\tilde{R}_{as}\}_{s \in S}$, and that of \tilde{R}_b is $\{\tilde{R}_{bs'}\}_{s' \in S'}$, then their covariance measures the correlations between their values.

Let $\pi_{ss'}$ be the probability of $\tilde{R}_a = \tilde{R}_{as}, \tilde{R}_b = \tilde{R}_{bs'}$. **Covariance**, denoted by $\text{Cov}(\tilde{R}_a, \tilde{R}_b)$ is defined as:

$$\text{Cov}(\tilde{R}_a, \tilde{R}_b) = \sum_{s \in S, s' \in S'} \pi_{ss'} (\tilde{R}_{as} - \bar{R}_a) (\tilde{R}_{bs'} - \bar{R}_b)$$

or

$$\text{Cov}(\tilde{R}_a, \tilde{R}_b) = E(\tilde{R}_a - \bar{R}_a)(\tilde{R}_b - \bar{R}_b) = E(\tilde{R}_a \tilde{R}_b) - E(\tilde{R}_a)E(\tilde{R}_b).$$

If the random variables \tilde{R}_a, \tilde{R}_b are independent, then $\pi_{ss'} = \pi_s \pi_{s'}$, thus we have $\text{Cov}(\tilde{R}_a, \tilde{R}_b) = 0$.

There is a following operation for linear combinations of variance:

$$\text{Var} \left(\sum_{a \in A} \alpha_a \tilde{R}_a \right) = \sum_{a \in A, b \in A} \alpha_a \alpha_b \text{Cov}(\tilde{R}_a, \tilde{R}_b).$$

2.10.3 Continuous Distributions

Given a random variable X , which takes on values in $[0, \omega]$, we define its cumulative distribution function $F : [0, \omega] \rightarrow [0, 1]$ by

$$F(x) = \text{Prob}[X \leq x]$$

the probability that X takes on a value not exceeding x . By definition, the function F is nondecreasing and satisfies $F(0) = 0$ and $F(\omega) = 1$ (if $\omega = \infty$, then $\lim_{x \rightarrow \infty} F(x) = 1$). In this course, we always suppose that F is increasing and continuously differentiable.

The derivative of F is called the associated probability density function and is usually denoted by the corresponding lowercase letter $f \equiv F'$. By

assumption, f is continuous and we will suppose that for all $x \in (0, \omega)$, $f(x)$ is positive. The interval $[0, \omega]$ is called the support of the distribution.

If X is distributed according to F , then the expectation of X is

$$E(X) = \int_0^{\omega} x f(x) dx$$

and if $\gamma : [0, \omega] \rightarrow \mathbb{R}$ is some arbitrary function, then the expectation of $\gamma(X)$ is analogously defined as

$$E[\gamma(X)] = \int_0^{\omega} \gamma(x) f(x) dx.$$

Sometimes the expectation of $\gamma(X)$ is also written as

$$E[\gamma(X)] = \int_0^{\omega} \gamma(x) dF(x).$$

The *conditional expectation* of X given that $X < x$ is

$$E[X|X < x] = \frac{1}{F(x)} \int_0^x t f(t) dt$$

and so

$$F(x)E[X|X < x] = \int_0^x t f(t) dt = xF(x) - \int_0^x F(t) dt$$

which is obtained by integrating the right-hand side of the first equality by parts.

2.10.4 Common Probability Distributions

Next we review some common distributions, and their expectations and variances.

Binomial Distribution

In a box there are many balls with two colors. The proportion of red balls is p and that of black balls is $1 - p$. The value of the random variable \tilde{X} is 1 if the red ball is drawn, otherwise it is 0. If it is only taken once, the probability distribution of the random variable is $p(\tilde{X} = 1) = p$, $p(\tilde{X} = 0) = 1 - p$.

The expectation and variance are:

$$E(\tilde{X}) = p; \quad \text{Var}(\tilde{X}) = p(1 - p).$$

If we draw n times (the ball is put back to the box at each time), the random variable is defined as the number of times that the red ball is get.

The probability distribution of random variables is

$$p(\tilde{X} = k) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k},$$

Its expectation and variance are:

$$E(\tilde{X}) = np; \quad \text{Var}(\tilde{X}) = np(1-p).$$

Poisson Distribution

If the probability of a random variable \tilde{X} is

$$P(\tilde{X} = k) = e^{-\lambda} \frac{\lambda^k}{k!},$$

then it is said that \tilde{X} have a **Poisson distribution** with parameter (λ), and its expectation and variance are:

$$E(\tilde{X}) = \lambda; \quad \text{Var}(\tilde{X}) = \lambda.$$

Uniform Distribution

If the probability density function of a random variable \tilde{X} is

$$f(x) = \frac{1}{b-a}, \quad x \in [a, b],$$

, then it is said that \tilde{X} have an **uniform distribution** with a range of $[a, b]$. Its expectation and variance are:

$$E(\tilde{X}) = \frac{b+a}{2}; \quad \text{Var}(\tilde{X}) = \frac{(b-a)^2}{12}.$$

Normal Distribution

If the probability density function of a random variable \tilde{X} is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in (-\infty, \infty),$$

then it is said that \tilde{X} have a **normal distribution** with parameters (μ, σ^2). The expectation and variance are:

$$E(\tilde{X}) = \mu; \quad \text{Var}(\tilde{X}) = \sigma^2.$$

Exponential Distribution

If the probability density function of a random variable \tilde{X} is

$$f(x) = \lambda e^{-\lambda x}, x \in [0, \infty),$$

then it is said that \tilde{X} have an **exponential distribution** with a parameter (λ), and its expectation and variance are

$$E(\tilde{X}) = \frac{1}{\lambda}; \quad \text{Var}(\tilde{X}) = \frac{1}{\lambda^2}.$$

2.11 Stochastic Dominance and Affiliation

2.11.1 Hazard Rates

Let F be a distribution function with support $[0, \omega]$. The hazard rate of F is the function $\lambda : [0, \omega) \rightarrow \mathcal{R}_+$ defined by

$$\lambda(x) \equiv \frac{f(x)}{1 - F(x)}.$$

If F represents the probability that some event will happen before time x , then the hazard rate at x represents the instantaneous probability that the event will happen at x , given that it has not happened until time x . The event may be the failure of some component-a lightbulb, for instance-and hence it is sometimes also known as the "failure rate".

Solving for F , we have

$$F(x) = 1 - \exp\left(-\int_0^x \lambda(t)dt\right).$$

This shows that any arbitrary function $\lambda : [0, \omega) \rightarrow \mathcal{R}_+$ such that for all $x < \omega$,

$$\int_0^x \lambda(t)dt < \infty, \quad \lim_{x \rightarrow \omega} \int_0^x \lambda(t)dt = \infty,$$

is the hazard rate of *some* distribution.

Closely related to the hazard rate is the function $\sigma : (0, \omega] \rightarrow \mathcal{R}_+$ defined by

$$\sigma(x) \equiv \frac{f(x)}{F(x)},$$

sometimes known as the *reverse hazard rate* or is referred to as the *inverse of the Mills' ratio*. Similarly, Solving for F , we have

$$F(x) = \exp\left(-\int_x^\omega \sigma(t)dt\right),$$

This shows that any arbitrary function $\sigma : (0, \omega] \rightarrow \mathcal{R}_+$ such that for all $x > 0$,

$$\int_x^\omega \sigma(t)dt < \infty \text{ meanwhile } \lim_{x \rightarrow 0} \int_x^\omega \sigma(t)dt = \infty.$$

is the "reverse hazard rate" of some distribution.

2.11.2 Stochastic Dominance

First-Order Stochastic Dominance

Definition 2.11.1 (First-Order Stochastic Dominance) Given two distribution functions F and G , we say that F *first-order stochastically dominates* G if for all $z \in [0, \omega]$, $F(z) \leq G(z)$.

The first-Order stochastic dominance means that for any outcome x , the probability of obtaining at least x under $F(\cdot)$ is at least as high as that under $G(\cdot)$, i.e., $F(\cdot) \leq G(\cdot)$ implies that the probability in lower part under $F(\cdot)$ is smaller than under $G(\cdot)$, or that the probability in higher part under $F(\cdot)$ is larger than under $G(\cdot)$. This is analogous to the monotonicity concept under certainty.

There is another test criterion for F to first-order stochastically dominate G : Does every expected utility maximizer with an increasing utility function prefer $F(\cdot)$ over $G(\cdot)$?

The following theorem shows that these two criteria are equivalent.

Theorem 2.11.1 $F(\cdot)$ *first-order stochastically dominates* $G(\cdot)$ if and only if for any function $u : [0, \omega] \rightarrow \mathcal{R}$ that is (weakly) increasing and differentiable, we have

$$\int u(z)dF(z) \geq \int u(z)dG(z).$$

PROOF. Define $H(z) = F(z) - G(z)$. We need to prove that $H(z) \leq 0$ if and only if $\int u(z)dH(z) \geq 0$ for any increasing and differentiable function $u(\cdot)$.

Sufficiency: We prove this by way of contradiction. Suppose there is a \hat{z} such that $H(\hat{z}) > 0$. We choose a weakly increasing and differentiable function $u(z)$ as

$$u(z) = \begin{cases} 0, & z \leq \hat{z}, \\ 1, & z > \hat{z}, \end{cases}$$

then immediately $\int u(z)dH(z) = -H(\hat{z}) < 0$, a contradiction.

Necessity:

$$\int u(z)dH(z) = [u(z)H(z)]_{-\infty}^{\infty} - \int u'(z)H(z)dz = 0 - \int u'(z)H(z)dz \geq 0,$$

in which the first equality is based on the formula of integration by parts, the second equality is based on

$$F(-\infty) = G(-\infty) = 0, \quad F(\infty) = G(\infty) = 1,$$

while the inequality is based on the assumptions that $u(\cdot)$ is weakly increasing ($u'(\cdot) \geq 0$) and $H(z) \leq 0$. \square

Since a monotone function can be arbitrarily approximated by a sequence of monotone and differentiable functions, the differentiability requirement imposed on u is not necessary. For any two gambles F and G , as long as an agent's utility is (weakly) increasing in outcomes, he prefers the one that first-order stochastically dominates the other one.

Second-Order Stochastic Dominance

Definition 2.11.2 (Second-Order Stochastic Dominance) Given two distribution functions F and G with the same expectation, we say that $F(\cdot)$ second-order stochastically dominates $G(\cdot)$ if

$$\int_{-\infty}^z F(r)dr \leq \int_{-\infty}^z G(r)dr$$

for all z .

It is clear that first-order stochastic dominance implies second-order stochastic dominance. The second-order stochastic dominance implies not only monotonicity but also lower risk. To do so, we introduce the notion of "Mean-Preserving Spreads".

Suppose X is a random variable with distribution function F . Let Z be a random variable whose distribution conditional on $X = x$, $H(\cdot|X = x)$, is such that for all x , $E[Z|X = x] = 0$. Suppose $Y = X + Z$ is the random variable obtained from first drawing X from F and then for each realization $X = x$, drawing a Z from the conditional distribution $H(\cdot|X = x)$ and adding it to X . Let G be the distribution of Y so defined. We will then say that G is a *mean-preserving spread* of F .

While random variables X and Y have the same mean, namely $E[X] = E[Y]$, variable Y is "more spread-out" than X since it is obtained by adding a "noise" variable Z to X . Now suppose $u : [0, \omega] \rightarrow \mathcal{R}$ is a concave function. Using Jensen's inequality

$$E(u(X)) \leq E(u(Y)),$$

we obtain

$$\begin{aligned} E_Y[u(Y)] &= E_X[E_Z[u(X + Z)|X = x]] \\ &\leq E_X[u(E_Z[X + Z|X = x])] \\ &= E_X[u(X)]. \end{aligned}$$

As such, similar to Theorem 2.11.1, we have the following conclusion for the second-order stochastic dominance.

Theorem 2.11.2 *If distributions $F(\cdot)$ and $G(\cdot)$ have the same mean, then the following statements are equivalent.*

- (1) $F(\cdot)$ second-order stochastically dominates $G(\cdot)$;
- (2) for any nondecreasing concave function $u : \mathcal{R} \rightarrow \mathcal{R}$, we have $\int u(z)dF(z) \geq \int u(z)dG(z)$;
- (3) $G(\cdot)$ is a mean-preserving spread of $F(\cdot)$.

PROOF. (3) \Rightarrow (2): It is obtained by using

$$\begin{aligned} \int u(z)dF(z) &= \int u \left(\int (x+z)dH_z(x) \right) dF(z) \\ &\geq \int \left(\int u(x+z)dH_z(x) \right) dF(z) \\ &= \int u(z)dG(z), \end{aligned}$$

in which the inequality follows from the concavity of $u(\cdot)$.

(1) \Rightarrow (2): For expositional convenience, we set $\omega = 1$. We have

$$\begin{aligned} &\int u(z)dF(z) - \int u(z)dG(z) \\ &= -u'(1) \int_0^1 (F(z) - G(z))dz + \int \left(\int_0^z (F(x) - G(x))dx \right) u''(z)dz \\ &= \int \left(\int_0^z (F(x) - G(x))dx \right) u''(z)dz \\ &\geq 0, \end{aligned}$$

in which the inequality follows from the definition of second-order stochastic dominance, namely

$$\int_{-\infty}^z F(r)dr \leq \int_{-\infty}^z G(r)dr$$

, and also $u''(\cdot) \leq 0$ for any z . We thus have

$$\int u(z)dF(z) - \int u(z)dG(z) \geq 0.$$

(1) \Rightarrow (3): We just show the case with discrete distributions.

Define

$$\begin{aligned} S(z) &= G(z) - F(z), \\ T(x) &= \int_0^x S(z)dz. \end{aligned}$$

By the definition of second-order stochastic dominance, we have $T(x) \geq 0$ and $T(1) \geq 0$, which imply that there exists some \hat{z} such that $S(z) \geq 0$ for $z \leq \hat{z}$ and $S(z) \leq 0$ for $z \geq \hat{z}$.

Since the random variable follows a discrete distribution, $S(z)$ must be a step function. Let $I_1 = (a_1, a_2)$ be the first interval over which $S(z)$ is positive, and $I_2 = (a_3, a_4)$ be the first interval over which $S(z)$ is negative. If no such $I_1 = (a_1, a_2)$ exists, then $S(z) \equiv 0$ and hence statement (3) is immediate. If $I_1 = (a_1, a_2)$ does exist, then $I_2 = (a_3, a_4)$ must exist as well.

So, $S(z) \equiv \gamma_1 > 0$ for $z \in I_1$, and $S(z) \equiv -\gamma_2 < 0$ for $z \in I_2$. By $T(x) \geq 0$, we must have $a_2 < a_3$. If $\gamma_1(a_2 - a_1) \geq \gamma_2(a_4 - a_3)$, then there exist $a_1 < \hat{a}_2 \leq a_2$ and $\hat{a}_4 = a_4$ such that $\gamma_1(\hat{a}_2 - a_1) = \gamma_2(\hat{a}_4 - a_3)$. If $\gamma_1(a_2 - a_1) < \gamma_2(a_4 - a_3)$, then there exists $a_3 < \hat{a}_4 \leq a_4$ such that $\gamma_1(\hat{a}_2 - a_1) = \gamma_2(\hat{a}_4 - a_3)$.

Letting

$$S_1(z) = \begin{cases} \gamma_1, & \text{if } a_1 < z < \hat{a}_2, \\ -\gamma_2, & \text{if } a_3 < z < \hat{a}_4, \\ 0, & \text{otherwise.} \end{cases}$$

If $F_1 = F + S_1$, then F_1 is a mean-preserving spread of F . Letting $S^1 = G - F_1$, then we can similarly construct $S_2(z)$ and F_2 . Since $S(z)$ is a step function, then there exists an n such that $F_0 = F$, $F_n = G$, and F_{i+1} is a mean-preserving spread of F_i . Also, a finite summation of mean-preserving spreads is still a mean-preserving spread. \square

Though a continuous function can be arbitrarily approximated by step functions, the formal proof is complicated, and Rothschild and Stiglitz (1971) provide a complete proof for the case with continuous distributions.

2.11.3 Hazard Rate Dominance

Definition 2.11.3 (Hazard Rate Dominance) For any two distributions F and G with hazard rates λ_F and λ_G , respectively. We say that F **dominates** G **in terms of the hazard rate** if $\lambda_F(x) \leq \lambda_G(x)$ for all x . This order is also referred in short as **hazard rate dominance**.

If F dominates G in terms of the hazard rate, then

$$F(x) = 1 - \exp\left(-\int_0^x \lambda_F(t) dt\right) \leq 1 - \exp\left(-\int_0^x \lambda_G(t) dt\right) = G(x),$$

and hence F stochastically dominates G . Thus, hazard rate dominance implies first-order stochastic dominance.

2.11.4 Reverse Hazard Rate Dominance

Definition 2.11.4 (Reverse Hazard Rate Dominance) For any two distributions F and G with reverse hazard rates σ_F and σ_G , respectively. We say

that F **dominates G in terms of the reverse hazard rate** if $\sigma_F(x) \geq \sigma_G(x)$ for all x . This order is also referred in short as **reverse hazard rate dominance**.

If F dominates G in terms of the reverse hazard rate, then

$$F(x) = \exp\left(-\int_x^\omega \sigma_F(t)dt\right) \leq \exp\left(-\int_x^\omega \sigma_G(t)dt\right) = G(x),$$

and hence, again, F stochastically dominates G . Thus, reverse hazard rate dominance also implies first-order stochastic dominance.

2.11.5 Order Statistic

Let X_1, X_2, \dots, X_n be n independently draws from a distribution F with associated density f . Let $Y_1^{(n)}, Y_2^{(n)}, \dots, Y_n^{(n)}$ be a rearrangement of these so that

$$Y_1^{(n)} \geq Y_2^{(n)} \geq \dots \geq Y_n^{(n)}.$$

These random variables $Y_k^{(n)}$, $k = 1, 2, \dots, n$ are referred to as **order statistics**.

Let $F_k^{(n)}$ denote the distribution of $Y_k^{(n)}$, with corresponding probability density function $f_k^{(n)}$. When the "sample size" n is fixed and there is no ambiguity, we simply write Y_k instead of $Y_k^{(n)}$, F_k instead of $F_k^{(n)}$ and f_k instead of $f_k^{(n)}$. In auction theory, we will typically be interested in properties of the highest and second highest order statistics, namely Y_1 and Y_2 .

Highest Order Statistic

The distribution of the **highest order statistic** Y_1 is easy to derive. The event that $Y_1 \leq y$ is the same as the event: for all k , $X_k \leq y$. Since each X_k is an independent draw from the same distribution F , we have that

$$F_1(y) = F(y)^n.$$

The associated probability density function is

$$f_1(y) = nF(y)^{n-1}f(y).$$

Observe that if F stochastically dominates G , and F_1 and G_1 are the distributions of the highest order statistics of n draws from F and G , respectively, then F_1 stochastically dominates G_1 .

Second-Highest Order Statistic

The distribution of the **second-highest order statistic** Y_2 can also be easily derived. The event that $Y_2 \leq y$ is the union of the following disjoint events: (1) all X_k 's are less than or equal to y ; and (2) $n - 1$ of the X_k 's are less than or equal to y and one is greater than y . There are n different ways in which (2) can occur, so we have that

$$\begin{aligned} F_2(y) &= \underbrace{F(y)^n}_{(i)} + \underbrace{nF(y)^{n-1}(1 - F(y))}_{(ii)} \\ &= nF(y)^{n-1} - (n - 1)F(y)^n. \end{aligned}$$

The associated probability density function is

$$f_2(y) = n(n - 1)(1 - F(y))F(y)^{n-2}f(y).$$

Again, it can be verified that if F stochastically dominates G and also F_2 and G_2 are the distributions of the second-highest order statistics of n draws from F and G , respectively, then F_2 stochastically dominates G_2 .

2.11.6 Affiliation

Affiliation is a basic assumption used to study auction with interdependent values which are non-negative correlated.

Definition 2.11.5 Suppose the random variables X_1, X_2, \dots, X_n are distributed on some product of intervals $\mathcal{X} \subseteq \mathcal{R}^n$ according to the joint density function f . The variables $\mathbf{X} = (X_1, X_2, \dots, X_n)$ are said to be **affiliated** if for all $\mathbf{x}', \mathbf{x}'' \in \mathcal{X}$,

$$f(\mathbf{x}' \vee \mathbf{x}'')f(\mathbf{x}' \wedge \mathbf{x}'') \geq f(\mathbf{x}')f(\mathbf{x}''), \quad (2.11.62)$$

in which

$$\mathbf{x}' \vee \mathbf{x}'' = (\max(x'_1, x''_1), \dots, \max(x'_n, x''_n))$$

denotes the component-wise **maximum** of \mathbf{x}' and \mathbf{x}'' , and

$$\mathbf{x}' \wedge \mathbf{x}'' = (\min(x'_1, x''_1), \dots, \min(x'_n, x''_n))$$

denotes the component-wise **minimum** of \mathbf{x}' and \mathbf{x}'' . If (2.11.62) is satisfied, then we also say that f is affiliated.

Suppose that the density function $f : \mathcal{X} \rightarrow \mathcal{R}_+$ is strictly positive in the interior of \mathcal{X} and twice continuously differentiable. It is "easy" to verify that f is affiliated if and only if, for all $i \neq j$,

$$\frac{\partial^2}{\partial x_i \partial x_j} \ln f \geq 0.$$

In other words, the off-diagonal elements of the Hessian of $\ln f$ are nonnegative.

Proposition 2.11.1 Let X_1, X_2, \dots, X_n be random variables, and Y_1, Y_2, \dots, Y_{n-1} be the largest, second largest, ..., smallest order statistics from among X_2, X_3, \dots, X_n . If X_1, X_2, \dots, X_n are symmetrically distributed and affiliated, then we have

- (1) variables in any subset of X_1, X_2, \dots, X_n are also affiliated;
- (2) $X_1, Y_1, Y_2, \dots, Y_{n-1}$ are affiliated.

Monotone Likelihood Ratio Property

Suppose the two random variables X and Y have a joint density $f : [0, \omega]^2 \rightarrow \mathcal{R}$. If X and Y are affiliated, then for all $x' \geq x$ and $y' \geq y$, we have

$$f(x', y)f(x, y') \leq f(x, y)f(x', y') \Leftrightarrow \frac{f(x, y')}{f(x, y)} \leq \frac{f(x', y')}{f(x', y)} \quad (2.11.63)$$

and

$$\frac{f(y'|x)}{f(y|x)} \leq \frac{f(y'|x')}{f(y|x')}$$

so the **likelihood ratio**

$$\frac{f(\cdot|x')}{f(\cdot|x)}$$

is increasing and this is referred to as the **monotone likelihood ratio property**.

Likelihood Ratio Dominance

Definition 2.11.6 (Likelihood Ratio Dominance) The distribution function F dominates G in terms of the **likelihood ratio** if for all $x < y$,

$$\frac{f(x)}{f(y)} \leq \frac{g(x)}{g(y)}.$$

We thus have the following conclusion.

Proposition 2.11.2 If X and Y are affiliated, the following properties hold:

- (1) For all $x' \geq x$, $F(\cdot|x')$ dominates $F(\cdot|x)$ in terms of **hazard rate**; that is,

$$\lambda(y|x') \equiv \frac{f(y|x')}{1 - F(y|x')} \leq \frac{f(y|x)}{1 - F(y|x)} \equiv \lambda(y|x).$$

Or equivalently, for all y , $\lambda(y|\cdot)$ is nonincreasing.

- (2) For all $x' \geq x$, $F(\cdot|x')$ dominates $F(\cdot|x)$ in terms of the **reverse hazard rate**; that is,

$$\sigma(y|x') \equiv \frac{f(y|x')}{F(y|x')} \leq \frac{f(y|x)}{F(y|x)} \equiv \sigma(y|x),$$

or equivalently, for all y , $\sigma(y|\cdot)$ is nondecreasing.

(3) For all $x' \geq x$, $F(\cdot|x')$ **stochastically dominate** $F(\cdot|x)$; that is,

$$F(y|x') \leq F(y|x),$$

or equivalently, for all y , $F(y|\cdot)$ is nonincreasing.

All of these results extend in a straightforward manner to the case where the number of conditioning variables is more than one. Suppose Y, X_1, X_2, \dots, X_n are affiliated and let $F_Y(\cdot|\mathbf{x})$ denote the distribution of Y conditional on $X = \mathbf{x}$. Then, using the same arguments as above, it can be deduced that for all $\mathbf{x}' \geq \mathbf{x}$, $F_Y(\cdot|\mathbf{x}')$ dominates $F_Y(\cdot|\mathbf{x})$ in terms of the likelihood ratio. The other dominance relationships then follow as usual.

2.12 Biographies

2.12.1 Friedrich August Hayek

Friedrich August Hayek (1899-1992), the greatest economic thinker of the 20th century and a representative of the Austrian school, won the Nobel Prize in economics in 1974 for his contributions to the theory of monetary and economic cycles as well as to the thorough analysis of the interrelationship between economic, political and institutional phenomena. The Nobel Prize Committee believed that Hayek's in-depth analysis of the economic cycle made him one of the very few economists who had warned about possible great economic depression before 1929. In fact, in the 20th century, both academically and practically speaking, it was the competition of the market economy system and the planned economy system and the dispute between their advantages and disadvantages. Hayek's thorough analysis of different economic systems had led him to point out very early that the planned economy is not feasible from the standpoint of information efficiency, incentive compatibility or resource allocation efficiency. The results of practice proved Hayek's extraordinary judgment and insight, and finally it was ended with the disappearance of the planned economic system, which made him one of the most influential economists of the 20th century.

Hayek was born in an intellectual family in Vienna and received a doctorate from the University of Vienna (1921-1923). When Hayek was at the University of Vienna, he attended a class by Ludwig von Mises (1881-1973). It was Mises's thorough critique of socialism published in 1922 that eventually pulled Hayek out of the Fabian socialist ideological trend. The best way to understand Hayek's great contribution to economics and classical liberalism is to analyze it from the perspective of Mises's paradigm of social collaboration. Hayek taught mainly at the London School of Economics and Political Science (1931-1950), the University of Chicago (1950-1962),

and Freiburg University (1962-1968), etc. Hayek was a professor of social and ethical science at the University of Chicago, was affiliated with the "Social Thoughts Committee" and did not get a teaching post in the Department of Economics. Professor Friedman, a friend of his economics department, was also critical of Hayek's books on economics. When he first arrived at the University of Chicago, Hayek did political studies and did not engage in economics research, and he was hostile to some research methods at the University of Chicago's Department of Economics. Even so, Hayek often interacted frequently with some of the Chicago School of Economics and his political views were also compatible with many people of the Chicago school. Hayek made a remarkable contribution to the University of Chicago. He strongly supported Aaron Director, a Chicago school economist and the founder of law and economics, to carry out "Law and Society" project at the Law School of Chicago. While the latter persuaded the University of Chicago Press to publish Hayek's *The Road to Slavery*, which became popular around the world. Hayek collaborated with Friedman and others co-founded the International Forum of Liberal Economists and so on.

Hayek had two profound debates in his life: one was the great debate on socialism in the 1920s and 1930s. He criticized the drawbacks of the planned economy from the perspective of information and incentives. He thought that the planned economy was theoretically impracticable, and emphasized the importance of a spontaneous social order based on freedom, competition, and rules. His advanced internal logic judgment was verified before his death. The second was the theoretical debate with Keynes in the 1930s. He pointedly criticized Keynes's theoretical claims and academic viewpoints put forward in *Monetary Theory*, and thought that Keynes's economic proposition of achieving full employment by lowering interest rate and increasing money supply was fundamentally wrong. In 1947, Hayek advocated the establishment of the Pilgrimage Mountain Society, an important academic organization of liberals. He advocated thorough economic freedom and opposed any form of state intervention, which also required the "non-nationalization" of currency issuance.

Hayek's profound thought of revealing the importance of institution will undoubtedly continue to influence and guide the world, especially China's next reform.

2.12.2 Joseph Alois Schumpeter

Joseph Alois Schumpeter (1883-1950), an Austrian American political economist with profound influence, is known as the originator of the Innovation Theory. In connection with most of the concepts and knowledge about market economy and innovation, he is called one of the greatest economists in history. He proposed four most representative and well-

known economic terms, namely, innovation, entrepreneurship, corporate strategy, and creative destruction. He believes that "creative destruction" is a double-edged sword which can engender economic growth, but can also weaken some values that people have traditionally cherished. "What poverty brings is a tragic life, while It is difficult for prosperity to maintain the peace of mind."

In 1883, Schumpeter was born in the family of a weaving factory owner in Triesch, Habsburg Moravia (now in Czech, so someone also think Schumpeter as a Czech-American), Austria Hungary. He enrolled in a noble middle school in Vienna. He studied law and sociology at University of Vienna from 1901 to 1906 and received his doctor's degree in law in 1906. In 1908, he became an associate professor in University of Czernowitz through his instructor's recommendation just at the beginning of the road of economist. Czernowitz is a remote city but a good place for learning with tranquility outside modern industrial civilization. Here, Schumpeter wrote his first masterpiece, *the theory of economic development*, published in 1912, which put forward "innovation" and its role in the economic development, and made a stir in the circles of western economics. According to statistics, the concept of "creative destruction" proposed by Schumpeter was cited frequently, second only to "invisible hand" of Adam Smith. The theory of economic development has become one of the classical economic literatures in twentieth Century. Later he emigrated to the United States, and has been teaching at Harvard University.

In his famous book *History of Economic Analysis*, he argued that the difference between an economic scientist and an average economist is the using of the following three elements in the process of economic analysis. First is the economic theory with intrinsic logic analysis. Second is history with historical perspective analysis. Third is statistics with data and empirical analysis. In recent years, Schumpeter has been increasingly renowned in mainland China. Especially when it comes to innovation, Schumpeter's five innovation concepts are often quoted and mentioned by people. What's more, as the founder of the Innovation Theory and the research of business history, in the West, Schumpeter's influence is being rediscovered.

Innovation refers to an economic process that rearranges and integrates the original production factors into new production methods in order to increase efficiency and reduce costs. In Schumpeter's economic model, those who can successfully innovate can survive from the dilemmas of diminishing returns, while those who cannot successfully reconfigure production factors will be eliminated first. The creative destruction of capitalism is the time when the economy cycles to the bottom. Meanwhile, it's the time when some entrepreneurs have to consider exiting the market and others need to innovate in order to survive. As long as the excess competitors are excluded or some successful innovations are produced, the e-

conomy will be improved and the production efficiency will be increased. But when an industry becomes profitable again, it will attract the investment of new competitors. Then it becomes a process of diminishing returns again and returns to the previous state. Therefore, every depression is implied by the possibility of another technological innovation, namely, the result of technological innovation is the next expected depression. In Schumpeter's view, the creativity and destructiveness of capitalism are homologous. However, Schumpeter does not believe that the superiority of capitalism is due to its own impetus which can promote its own development continually. He believes that the capitalist economy will eventually collapse on its own scale because it cannot withstand the energy of its rapid expansion. The business cycle, also known as the economic cycle, is Schumpeter's economic claims, which is frequently cited by later generations. Schumpeter's concept of creative destruction has a great influence on the development of modern economics. The combination of the dynamic market mechanism and R&D economics provides economists with a perspective of endogenous technology. Schumpeter's technological innovation has become a core element of the endogenous growth theory in macroeconomics.

In *capitalism, socialism and democracy*, Schumpeter gave a modern definition of Democracy: the democratic method is the institutional arrangement for making political decisions, in which some people gain the power to make decisions by gaining the votes of the people. He believes that democracy is the process that political elite competes for power and people choose political leaders. The essence of democracy lies in a competitive election process. Political elites master political power and implement ruling, but their legitimacy comes from the choice of the people. Schumpeter also believes that, in the political market with democracy, politicians provide political programs and policies according to the preferences of voters, and compete freely in the general election to campaigning for votes. Schumpeter's definition of democracy symbolizes the great transformation of democratic theory from the classical democracy directly ruled by the people to the modern election democracy.

Although Schumpeter is a senior scholar, he is not a boring man. There were many setbacks and difficulties in his personal life. But he was always well-groomed, handsome and gentlemanly, with a pair of smart eyes. Therefore, countless women fell in love with him, and Schumpeter was always passionate about them. He had quite contentedly said: "I was born to be a ladies man." The economic development of any country needs to go through three stages, namely, factor-driven, efficiency-driven and innovation-driven, which is the same in China. China needs to move from elemental drive to efficiency driven smoothly, thereby turning to innovation drive. The ideas and theories of Hayek and Schumpeter play an vital role in theoretically guiding for the two driver stages.

2.13 Exercises

Exercise 2.1 Consider an economy with two sectors: the industrial sector and the monetary sector, characterized by the following equation:

$$Y = C + I + G,$$

$$C = a + b(1 - t)Y,$$

$$I = d - ei,$$

$$G = G_0,$$

where Y, C, I and i (i is the interest rate) are endogenous variables, G_0 is an exogenous variable, and a, b, d, e and t are all structure parameters.

In the newly introduced monetary market, we have:

the equilibrium conditions: $M_d = M_s,$

the money demand: $M_d = kY - li,$

and the money supply: $M_s = M_0,$

where M_0 is the exogenous variable of money stock, k and l are parameters. Given this economy, solve the following problems: (using Cramer's rule)

1. Equilibrium income Y^* ;
2. Money supply multiplier;
3. Government expenditure multiplier.

Exercise 2.2 Q represents the set of rational numbers, and as a metric space, its distance is defined by $d(p, q) = |p - q|$. The set $E = \{p \in Q : 2 < p < 40\}$ is defined in this space.

1. Show that E is closed and bounded in Q .
2. Show that E is not compact.
3. Is E is open in Q ? Why?

Exercise 2.3 Given a metric space X , consider a series of open sets $\{E_n\}_{n \in \mathbb{N}}$ in X .

1. Show that $\bigcup_{n \in \mathbb{N}} E_n$ is a open set.
2. Show that it may not be true that the intersection of a series of open sets is open (Give an example).

Exercise 2.4 Prove the following theorems:

1. The difference of a open set and a closed set is also open, while the difference of a closed set and a open set is also closed.

2. Each closed set is the intersection of a countable number of open sets; each open set is the union of a countable number of closed set.

Exercise 2.5 Let $S \subseteq \mathcal{R}^L$. Show that the following propositions are equivalent:

1. S is compact.
2. S is bounded and closed.
3. Every sequence in S has a convergent subsequence with the limit point in S .
4. Every infinite subset of S has a cluster point in S .
5. Each closed subset of the set S with finite intersection property (i.e., the intersection over any finite subcollection is nonempty) is nonempty.

Exercise 2.6 Prove the following propositions:

1. Every closed subset of a compact set is compact.
2. If $f : X \rightarrow Y$ is continuous and K is compact in X , then $f(K)$ is compact in Y .
3. S_i is compact, $i \in I$, if and only if $\prod_{i \in I} S_i$ is compact
4. S_i is compact, $i = 1, 2, \dots, m$, if and only if $\sum_{i=1}^m S_i$ is compact.

Exercise 2.7 (Shapley-Folkman Theorem) Prove the theorem: Let S_i be a nonempty subset of \mathcal{R}^l , $i = 1, 2, \dots, m$. For each $x \in \text{co}(\sum_{i=1}^m S_i)$, there is a $x_i \in \text{co}S_i$, $i = 1, 2, \dots, m$ such that $x = \sum_{i=1}^m x_i$ and $\#\{i : x_i \notin S_i\}$.

Exercise 2.8 Prove the following theorems:

1. If f is a differentiable function defined on \mathcal{R}^1 , then f is concave if and only if the first order condition $f'(x)$ is non-increasing.
2. If f is a twice differentiable function defined on \mathcal{R}^1 , then f is concave if and only if the second order condition $f''(x)$ is non-positive.
3. If f is a differentiable function defined on \mathcal{R}^1 , then f is concave if and only if $f(y) \leq f(x) + f'(x)(y - x)$ for any $x, y \in \mathcal{R}^1$.

Exercise 2.9 Suppose that $f(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T A \mathbf{x} + \mathbf{b}^T \mathbf{x} + c$, where $\mathbf{x} \in \mathcal{R}^n$, \mathbf{x}^T is the transpose of vector \mathbf{x} , A is an $n \times n$ symmetric matrix, \mathbf{b} is an n -dimensional vector, and c is a constant.

1. Show that if A is a positive semi-definite matrix, then $f(\mathbf{x})$ is a convex function.

2. Show that if A is a positive definite matrix, then $f(\mathbf{x})$ is a strictly convex function.

Exercise 2.10 Determine whether Kuhn-Tucker conditions is applicable for the following optimization problems and solve it.

$$\begin{aligned} & \max x_1 \\ \text{s.t. } & x_1^3 - x_2 \leq 0, \\ & x_2 \leq 0. \end{aligned}$$

Exercise 2.11 Solve the following optimization problems using Kuhn-Tucker conditions.

$$\begin{aligned} & \max xyz \\ \text{s.t. } & x^2 + y^2 + z^2 \leq 6, \\ & x \geq 0, y \geq 0, z \geq 0. \end{aligned}$$

Exercise 2.12 The maximization problem is as follows:

$$\begin{aligned} & \max f(\mathbf{x}) \\ \text{s.t. } & g^1(\mathbf{x}) = 0, \dots, g^m(\mathbf{x}) = 0, \end{aligned}$$

where $f : \mathcal{R}^n \rightarrow \mathcal{R}$ and $g : \mathcal{R}^n \rightarrow \mathcal{R}$ are increasing functions with respect to \mathbf{x} , and $m < n$. Try to prove that: If f is quasi-concave and all g^j are quasi-convex functions, then any local optimum is the global optimal solution.

Exercise 2.13 Let $u : \mathcal{R}^n \rightarrow \mathcal{R}$ be a function, $\mathbf{p}, \mathbf{x} \in \mathcal{R}^n$, and $y \in \mathcal{R}$. Considering the following optimization problem:

$$\begin{aligned} & \max_{\mathbf{x}} u(\mathbf{x}) \\ \text{s.t. } & \mathbf{p}\mathbf{x} = y. \end{aligned}$$

Suppose that there is an optimum solution $x^*(\mathbf{p}, y) > 0$ such that $v(\mathbf{x}, y) = u(\mathbf{x}^*(\mathbf{p}, y))$.

1. Show that $v(\mathbf{p}, y)$ is a zero order homogeneous function.
2. Show that $v(\mathbf{p}, y)$ is a quasi-concave function.

Exercise 2.14 Suppose that a Cobb-Douglas utility function $u : \mathcal{R}^2 \rightarrow \mathcal{R}$ is defined as:

$$u(x_1, x_2) = x_1^\alpha x_2^\beta, \quad \alpha, \beta > 0.$$

Prove:

1. If $\alpha + \beta \leq 1$, then u is a concave function.
2. If $\alpha + \beta \geq 1$, then u is a quasi-concave function, but not concave.

3. For any $\alpha > 0$ and $\beta > 0$, $h(x_1, x_2) = \ln(u(x_1, x_2))$ is a concave function.

Exercise 2.15 Suppose that \bar{X} is a nonempty, closed and convex set in \mathcal{R}^n , $x_0 \notin \bar{X}$. Prove that the following propositions are true.

1. There is a point $a \in \bar{X}$ such that $d(x_0, a) < d(x_0, x)$ for all $x \in \bar{X}$, and $d(x_0, a) > 0$.
2. There is a point $p \in \mathcal{R}^n$, $p \neq 0$, $\|p\| \equiv (\sum_{i=1}^n p_i^2)^{1/2} < \infty$ and $\alpha \in \mathcal{R}$ such that

$$p \cdot x \geq \alpha, \text{ for all } x \in \bar{X} \text{ and } p \cdot x_0 < \alpha.$$

Namely, \bar{X} and x_0 is separated by a hyperplane $H = \{x : p \cdot x = \alpha, x \in \mathcal{R}^n\}$.

Exercise 2.16 Consider following functions:

(1) $3x^5y + 2x^2y^4 - 3x^3y^3$.

(2) $3x^5y + 2x^2y^4 - 3x^3y^4$.

(3) $x^{3/4}y^{1/4} + 6x + 4$.

(4) $\frac{x^2 - y^2}{x^2 + y^2} + 3$.

(5) $x^{1/2}y^{-1/2} + 3xy^{-1} + 7$.

(6) $x^{3/4}y^{1/4} + 6x$.

1. Find out homogeneous functions among them and determine their orders.
2. Test whether the above functions satisfy the Euler theorem.

Exercise 2.17 There is a simple application of upper (lower) hemi-continuity. Suppose that $f : X \times Y \rightarrow \mathcal{R}$,

$$G(x) = \{y \in \Gamma(x) : f(x, y) = \max_{y \in \Gamma(x)} f(x, y)\}.$$

1. Suppose that $X = \mathcal{R}$, $\Gamma(x) = Y = [-1, 1]$. For all $x \in X$, $f(x, y) = xy^2$. Draw the graph of $G(x)$ and show that $G(x)$ is upper hemi-continuous at $x = 0$, but not lower hemi-continuous.
2. Suppose that $x = \mathcal{R}$ and $\Gamma(x) = Y = [0, 4]$ for all $x \in X$. Define that

$$f(x, y) = \max\{2 - (y - 1)^2, x + 1 - (y - 2)^2\}.$$

Draw the graph of $G(x)$ and show that: $G(x)$ is upper hemi-continuous but not lower hemi-continuous, and specify at which points it is not lower hemi-continuous.

3. Suppose that $X = \mathcal{R}_+$, $\Gamma(x) = Y = \{y \in \mathcal{R} : -x \leq y \leq x\}$. For all $x \in X$, define that $f(x, y) = \cos(y)$, then draw the graph of $G(x)$ and show that: $G(x)$ is upper hemi-continuous but not lower hemi-continuous, and specify at which points it is not lower hemi-continuous.

Exercise 2.18 Let $S = \{x \in \mathcal{R}^2 : \|x\| = 4\}$ be the boundary of a circle with a radius of 2. The mapping $\psi : \mathcal{R}^2 \rightarrow S$ is defined as:

$$\psi(x) = \arg \min_{x' \in S} d(x, x'),$$

namely, $\psi(x)$ contains the closest point in S to x . Discuss the upper and lower hemi-continuity of $\psi(x)$.

Exercise 2.19 Consider a correspondence $\Gamma : D \subseteq \mathcal{R}^l \rightarrow \mathcal{R}^k$ of which the graph is defined as

$$G(\Gamma) = \{(x, y) \in D \times \mathcal{R}^k : y \in \Gamma(x)\}.$$

If $G(\Gamma)$ is a closed set, then we call Γ has a closed graph; If $G(\Gamma)$ is a bounded and closed set, we call Γ is compact-valued. Suppose that Γ is compact-valued, and show that:

1. If Γ is upper hemi-continuous, then it has a closed graph.
2. If Γ is locally bounded and the graph of it is closed, then Γ is upper hemi-continuous. (Hint: The definition of locally bounded correspondence Γ : $G(\Gamma) = \{(x, y) \in D \times \mathcal{R}^k : y \in \Gamma(x)\}$ is locally bounded, if for each $x \in D$, there is an $\epsilon > 0$ and a bounded set $Y(x) \subseteq \mathcal{R}^k$ such that for all $x' \in N_\epsilon(x) \cap D$, $\Gamma(x') \subseteq Y(x)$.)

Exercise 2.20 Suppose $X \subseteq \mathcal{R}_+$ is a nonempty compact set. Show that:

1. If $f : X \rightarrow X$ is a continuous increasing function, then f has a fixed point.
2. Specially, suppose that $X = [0, 1]$. If $f : X \rightarrow X$ is a increasing function (not necessarily continuous), does f has another fixed point?

Exercise 2.21 Suppose that X is a complete metric space, T is the mapping from X to X . Denote

$$a_n = \sup_{x \neq x'} \frac{d(T^n x, T^n x')}{d(x, x')}, n = 1, 2, \dots$$

Show that : If $\sum_{n=1}^{\infty} a_n < \infty$, then the mapping T has a unique fixed point.

Exercise 2.22 Suppose $n \in \mathbb{N}$, and an n th order square matrix $A = (a_{ij})_{n \times n}$. For any $x \in \mathbb{R}^n$, we have

$$Ax = \left(\sum_{j=1}^n a_{1j}x_j, \sum_{j=1}^n a_{2j}x_j, \dots, \sum_{j=1}^n a_{nj}x_j \right)^T.$$

Suppose that f is a differentiable mapping from \mathbb{R} to \mathbb{R} such that

$$s = \sup\{|f'(t)| : t \in \mathbb{R}\} < \infty.$$

Define a mapping F from \mathbb{R}^n to \mathbb{R}^n , namely,

$$F(x) = (f(x_1), \dots, f(x_n))^T.$$

For a given n -dimensional vector w , we can solve the following system of nonlinear equations:

$$z = AF(z) + w. \quad (*)$$

1. Show that: if $\max\{\sum_{j=1}^n |a_{ij}| : i = 1, \dots, n\} < \frac{1}{s}$, then there is a unique $z \in \mathbb{R}^n$ satisfying the above system of equations (*).
2. Show that: if $\sum_{i=1}^n \sum_{j=1}^n |a_{ij}| < \frac{1}{s^2}$, then there is a unique $z \in \mathbb{R}^n$ satisfying the above system of equations. (*).

Exercise 2.23 Suppose that h is a mapping from \mathbb{R}_+ to \mathbb{R}_+ , and $H : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is a bounded function such that there is a $K \in (0, 1)$,

$$|H(x, y) - H(x, z)| < K|y - z|, \text{ for any } x \geq 0, y, z \in \mathbb{R}.$$

Show that: there is a unique bounded function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

$$f(x) = H(x, f(h(x))), \text{ for any } x \geq 0.$$

Exercise 2.24 Find the extremum curve of the following functional:

1. $V(y) = \int_0^1 (t^2 + y'^2) dt, y(0) = 0, y(1) = 2;$
2. $V(y) = \int_0^1 (y + yy' + y' + 0.5y'^2) dt, y(0) = 2, y(1) = 5;$
3. $V(y) = \int_0^T (1 + y'^2)^{0.5} dt, y(0) = A, y(T) = Z.$

Exercise 2.25 Solve the following optimal problem:

$$\begin{aligned} \max \int_0^3 (x-2)^2 (x'(t) - 1)^2 dt \\ \text{s.t. } x(0) = 0, x(3) = 2. \end{aligned}$$

Exercise 2.26 Consider the following optimal control problem, write the Hamilton equation, and solve the optimal function.

$$\begin{aligned} & \max \int_0^1 (x + u) dt \\ \text{s.t. } & x'(t) = 1 - u^2, x(0) = 1. \end{aligned}$$

Exercise 2.27 Consider the following optimization problem:

$$v(q) = \max_{x \in \mathcal{R}^+} \ln(2x + q) - 6x + 2q,$$

where $q \in (0, 2)$.

1. Solve $v(q)$ and its derivative $v'(q)$.
2. Verify the envelope theorem holds.

Exercise 2.28 Find the general solution to the extremum curve of the following functional:

$$V(y, z) = \int_a^b (y'^2 + z'^2 + 3y'z') dt.$$

Exercise 2.29 In the problem of functional $\int_0^T F(t, y, z, y', z') dt$, suppose that $y(0) = A$, $z(0) = B$, $y_T = C$, $z_T = D$, T are free, and A, B, C and D are constants.

1. How many transversal conditions is required for the problem? Why?
2. Write these transversal conditions.

Exercise 2.30 The integrand function of the target functional is $F(t, y, y') = 4y^2 + 4yy' + y'^2$.

1. Write the Euler equation.
2. Is the above Euler equation sufficient for the Maximization or minimization problems? Why?

Exercise 2.31 Solve the paths of $y(t)$ and $z(t)$ of extremum curves of $V(y, z) = \int_0^T (y'^2 + z'^2) dt$ subjected to $y - z' = 0$.

Exercise 2.32 Solve the optimal paths of control variables, state variables and costate variables as follows:

1. $\max \int_0^T -(t^2 + 2u^2) dt$ subjected to $y' = u$, $y(0) = 2$, and $y(T) = 3$, and T is free.
2. $\max \int_0^T -(u^2 + y^2 + 3uy) dt$ subjected to $y' = u$, $y(0) = y_0$, and $y(t)$ is free.
3. $\max \int_0^4 2y dt$ subjected to $y' = y + u$, $y(0) = 3$, $y(4) \geq 200$.

Exercise 2.33 Try to find the optimal consumption path of exhaustible resource problem is as follows:

$$\begin{aligned} & \max \int_0^T \ln q e^{-\delta t} dt \\ \text{s.t.} \quad & s' = -q, s(0) = s_0, s(t) \geq 0. \end{aligned}$$

Exercise 2.34 Using the revised transversal conditions expressed by the present value Hamilton function, solve the problems:

1. with the end curve $y_T = \phi(t)$.
2. with truncated vertical end line.
3. with truncated horizontal end line.

Exercise 2.35 In a maximization problem, there are two known state variables, (y_1, y_2) , two control variables (u_1, u_2) , an inequality constraint and an inequality integral constraint. The initial state is fixed, but the final state is free at fixed T .

1. State the maximization problem.
2. Define the Hamilton's equation and the Lagrange function.
3. Suppose that there is a internal solution, then write the conditions of the maximum principle.

Exercise 2.36 Consider the problem of "eating cake" as follows. The agent has a $A_0 > 0$ units commodity for consumption in the period 0 and can save the commodity to the next period without costs, and its utility function is $\sum_{t=0}^{\infty} \beta^t \ln c_t$.

1. Write the Bellman equations of the problem.
2. Define the state variable and control variable.
3. Find the value function.

Exercise 2.37 Consider the following problem of "tree cutting": the growth a tree can be represented by the function h , that is, $k_{t+1} = h(k_t)$, where k_t is the scale of the tree at time t .

There is no cost for cutting trees, and the timber price is $p = 1$. Interest rate r remains unchanged, $\beta = 1/(1+r)$.

1. Assuming that trees cannot be replanted, we write the maximization problem of present value as $v(k) = \max\{k, \beta v[h(k)]\}$. Under what assumptions about h is there a simple rule that can be used to describe when to cut trees?

2. Suppose that another tree can be planted where the original tree is cut down, and the replanting cost $c \geq 0$ remains unchanged for a long term, under what assumptions about h and c is there a simple rule that can be used to describe when to cut trees?

Exercise 2.38 Solving following dynamic programming problems by three methods: value function iteration, guessing value function and guessing policy function respectively:

$$\begin{aligned} & \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln c_t \\ \text{s.t.} \quad & c_t + k_{t+1} = Ak_t^\alpha, \end{aligned}$$

where k_0 is given.

Exercise 2.39 Solve the following differential equations:

1. $y' = t^2 y$.
2. $y'' - 4y' + 5y = 0$.
3. $y'' - 2y' - 3y = 9t^2$.

Exercise 2.40 Consider the following two dimensional autonomous differential equations:

$$\begin{aligned} \frac{dx}{dt} &= x(4 - x - y), \\ \frac{dy}{dt} &= y(6 - y - 3x). \end{aligned}$$

1. Solve the equilibrium of the power system.
2. Verify the stability of each equilibrium.

Exercise 2.41 Solve the following difference equations:

1. $y(t+1) - 2y(t) = 4^t$.
2. $y(t+2) + 3y(t+1) + 2y(t) = 0$.
3. $y(t+2) - y(t+1) - 6y(t) = t + 2$.

Exercise 2.42 Suppose X_1, X_2, \dots, X_n are n independent and identically distributed random variables. The distribution function is F , and probability density function is f . Let $Y_1^{(n)}, Y_2^{(n)}, \dots, Y_n^{(n)}$ be the corresponding order statistics satisfying $Y_1^{(n)} \geq Y_2^{(n)} \geq \dots \geq Y_n^{(n)}$.

1. Solve the distribution function and probability density function of $Y_n^{(n)}$.
2. Solve $E(Y_n^{(n)})$ and $\text{Var}(Y_n^{(n)})$.
3. Solve $\text{Cov}(Y_1^{(n)}, Y_n^{(n)})$.

Exercise 2.43 Suppose that X is a non-negative random variable, the distribution function and density function are F and f respectively. The risk rate of random variable X is defined as

$$\lambda_X : \mathcal{R}_+ \longrightarrow \mathcal{R}_+, \quad \lambda_X(t) = \frac{f(t)}{1 - F(t)}.$$

If $\lambda_X(\cdot) \leq \lambda_Y(\cdot)$, then it is said that the random variable X is larger than or equal to (or not less than) random variable Y in the sense of risk rate. Suppose that G and g are respectively the distribution function and density function of random variables Y . If $f(\cdot)/g(\cdot)$ is a non-decreasing function, then we will say that (in the sense of likelihood ratio), the random variable X is larger than or equal to (not less than) the random variable Y . Show the following statements:

1. $\lambda_X(\cdot) \leq \lambda_Y(\cdot)$ if and only if $1 - G(t)/[1 - F(t)]$ is a non-increasing function.
2. The likelihood ratio sequence is stronger than the risk rate sequence, that is, if the random variable X in the sense of likelihood ratio is larger than or equal to Y , then X in the sense of risk rate must be larger than or equal to (or not less) Y .

Exercise 2.44 Show that: if X_1, X_2, \dots, X_N are correlated, and $\gamma(\cdot)$ is an increasing function, then for $x'_1 > x_1$, we have

$$E[\gamma(Y_1)|X_1 = x'_1] \geq E[\gamma(Y_1)|X_1 = x_1].$$

2.14 Reference

Books and Monographs:

- Ding Tongren and Li Chengzhi (2004). *A Textbook on Ordinary Differential Equations*, Beijing: Higher Education Press.
- Jiang Zhongyi (1999). *Dynamic optimization Foundation*, Beijing: Commercial Press.
- Ahmad, Shair and Antonio Ambrosetti (2004). *A Textbook on Ordinary Differential Equations*, Springer.
- Bellman, R. (1957). *Dynamic Programming*, Princeton University Press.

- Border, K. C. (1985). *Fixed Point Theorems with Applications to Economics and Game Theory*, Cambridge University Press.
- Debreu, G. (1959). *Theory of Value*, Wiley.
- Hildenbrand, W. and A. P. Kirman (1988). *Equilibrium Analysis: Variations on Themes by Edgeworth and Walras*, North-Holland.
- Jehle, G. A. and P. Reny (1998). *Advanced Microeconomic Theory*, Addison-Wesley.
- Krishna, K. (2002). *Auction Theory*, Academic Press.
- Kamien, M. and N. L. Schwartz (1991). *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management*, North-Holland.
- Kreps, D. M. (2013). *Microeconomic Foundation I: Choice and Competitive Markets*, Princeton University Press.
- Luenberger, D. (1995). *Microeconomic Theory*, McGraw-Hill.
- Mas-Colell, A. , M. D. Whinston, and J. Green (1995). *Microeconomic Theory*, Oxford University Press.
- Royden, H. L. (1989). *Real Analysis*, Prentice Hall.
- Rubinstein, Ariel (2009). *Lecture Notes in Microeconomics (modeling the economic agent)*, Princeton Univeristy Press.
- Rudin, Walter (1976). *Principles of Mathematical Analysis*, McGraw-Hill.
- Stockey, N. , and R. Lucas (1989). *Recursive Methods in Economic Dynamics*, Harvard University Press.
- Sydsaeter, Knut, Arne Strom and Peter Berck (2010). *Economist's Mathematical Manual (4th Edition)*, Springer.
- Takayama, A. (1985). *Mathematical Economics(Second Edition)*, Cambridge University Press.
- Tian, G. (2015). *Mathematical Economics(Lecture Notes)*.
- Varian, H. R. (1992). *Microeconomic Analysis(Third Edition)*, W. W. Norton and Company.
- Vinnogradov, V. (1999). *A Cook-Book of Mathematics*, CERGE-EI Lecture Notes.

Papers:

- Arrow, K and G. Debreu (1954). "Existence of an Equilibrium for a Competitive Economy", *Econometrica*, Vol. 22, No. 3, 265-290.
- Browder, F. E. (1968). "The Fixed Point Theory of Multi-valued Mappings in Topological Vector Spaces" , *Mathematische Annale*, Vol. 177, 283-301.
- Fan, K. (1984). "Some Properties of Convex Sets Related to Fixed Point Theorem", *Mathematics Annals*, Vol. 266, No. 4, 519-537.
- Kakutani, S. (1941). "A Generalization of Brouwer's Fixed Point Theorem", *Duke Mathematical Journal*, No. 8, 457-459.
- Michael, E. (1956). "Continuous Selections I", *Annals of Mathematics*, Vol. 63, No. 2, 361-382.
- Modigliani, F. and M. Miller (1958). "The Cost of Capital, Corporation Finance and the Theory of Investment", *American Economic Review*, Vol. 48, No. 3, 261-297.
- Nessah, R. and G. Tian (2013). "Existence of Solution of Minimax Inequalities, Equilibria in Games and Fixed Points without Convexity and Compactness Assumptions", *Journal of Optimization Theory and Applications*, Vol. 157, No. 1, 75-95.
- Samuelson, P. (1958). "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money", *Journal of Political Economy*, Vol. 66, No. 6, 467-482.
- Sonnenschein, H. (1971). "Demand Theory Without Transitive Preferences, with Application to the Theory of Competitive Equilibrium", in *Preferences, Utility, and Demand*, (ed.) by J. S. Chipman, L. Hurwicz, M. K. Richter, and H. Sonnenschein, New York: Harcourt Brace Jovanovich.
- Shafer, W. and H. Sonnenschein (1975). "Equilibrium in Abstract Economies without Ordered Preferences", *Journal of Mathematical Economics*, Vol. 2, Iss. 3, 345-348.
- Tarski, A. (1955). "A Lattice-theoretical Fixpoint Theorem and Its Applications", *Pacific Journal of Mathematics*, Vol. 5, No. 2, 285-309.
- Tian, G. (1991). "Fixed Points Theorems for Mappings with Non-Compact and Non-Convex Domains", *Journal of Mathematical Analysis and Applications*, Vol. 158, No. 1, 161-167.

- Tian, G. (1992). "Generalizations of the FKKM Theorem and Ky-Fan Minimax Inequality, with Applications to Maximal Elements, Price Equilibrium, and Complementarity", *Journal of Mathematical Analysis and Applications*, Vol. 170, No. 2, 457-471.
- Tian, G. (1993). "Necessary and Sufficient Conditions for Maximization of a Class of Preference Relations", *Review of Economic Studies*, Vol. 60, No. 4, 949-958
- Tian, G. (1994). "Generalized KKM Theorem and Minimax Inequalities and Their Applications", *Journal of Optimization Theory and Applications*, Vol. 83, 375-389.
- Tian, G. (2016). "Characterizations of Minimax-Inequality, Fixed-Point Theorem, Saddle Point, and KKM Theorem in-Arbitrary Topological Spaces", *Journal of Fixed Point Theory and Applications*, forthcoming.
- Tian, G. and R. Nessah (2013). "Existence of Solution of Minimax Inequalities, Equilibria in Games and Fixed Points without Convexity and Compactness Assumptions", *Journal of Optimization Theory and Applications*, Vol. 157, No. 1, 75-95.
- Tian, G. and R. Nessah (2014). "On the Existence of Strong Nash Equilibria", *Journal of Mathematical Analysis and Applications*, Vol. 414, 871-885.
- Tian, G. and J. Zhou (1992). "The Maximum Theorem and the Existence of Nash Equilibrium of (Generalized) Games without Lower Semicontinuity", *Journal of Mathematical Analysis and Applications*, Vol. 166, No. 2, 351-364.
- Tian, G. and J. Zhou (1995). "Transfer Continuities, Generalizations of the Weierstrass Theorem and Maximum Theorem—A Full Characterization", *Journal of Mathematical Economics*, Vol. 24, No. 3, 281-303.
- Zhou and Tian (1992). "Transfer Method for Characterizing the Existence of Maximal Elements of Binary Relations on Compact or Noncompact Sets", *SIAM Journal on Optimization*, Vol. 2, No. 3, 360-375.

Part V

**Information, Incentives, and
Mechanism Design**

Information economics, incentive theory, mechanism design theory, principal-agent theory, contract theory, and auction theory have been very important and active research areas and had wide applications in various fields in economics, finance, management, and corporate law, political sciences in last five decades. Because of this, more than twenty economists of founding contributors of mechanism design and the associated files of game theory so far have been rewarded with the Nobel prize in economics, including Friedrich A. Hayek, Kenneth Arrow, George J. Stigler, Gerard Debreu, Ronald Coase, Herbert Simon, John Nash, Reinhard Selten, William Vickrey, James Mirrlees, George Akerlof, Joseph Stiglitz, Michael Spence, Robert Auman, Leo Hurwicz, Eric Maskin, Roger Myerson, Peter Diamond, Oliver Williamson, Alvin E. Roth, Lloyd S. Shapley, Jean Tirole, Oliver Hart, and Bengt Holmstrom.

The notion of incentives is a basic and key concept in modern economics. To many economists, economics is to a large extent a matter of incentives: incentives to work hard, to produce good quality products, to study, to invest, to save, etc.

Until about 40 year ago, economics was mostly concerned with understanding the theory of value in large economies. A central question asked in general equilibrium theory was whether a certain mechanism (especially the competitive mechanism) generated Pareto-efficient allocations, and if so – for what categories of economic environments. In a perfectly competitive market, the pressure of competitive markets solves the problem of incentives for consumers and producers. The major project of understanding how prices are formed in competitive markets can proceed without worrying about incentives.

The question was then reversed in the economics literature: instead of regarding mechanisms as given and seeking the class of environments for which they work, one seeks mechanisms which will implement some desirable outcomes (especially those which result in Pareto-efficient and individually rational allocations) for a given class of environments without destroying participants' incentives, and have a low cost of operation and other desirable properties. In a sense, the theorists went back to basics.

The reverse question was stimulated by two major lines in the history of economics. Within the capitalist/private-ownership economics literature, a stimulus arose from studies focusing upon the failure of the competitive market to function as a mechanism for implementing efficient allocations in many nonclassical economic environments such as the presence of externalities, public goods, incomplete information, imperfect competition, increasing return to scale, etc. At the beginning of the seventies, works by Akerlof (1970), Hurwicz (1972), Spence (1974), and Rothschild and Stiglitz (1976) showed in various ways that asymmetric information was posing a much greater challenge and could not be satisfactorily imbedded in a proper generalization of the Arrow-Debreu theory.

A second stimulus arose from the socialist/state-ownership economics literature, as evidenced in the “socialist controversy” — the debate between Mises-Hayek and Lange-Lerner in twenties and thirties of the last century. The controversy was provoked by von Mises’s skepticism as to even a theoretical feasibility of rational allocation under socialism.

The incentives structure and information structure are thus two basic features of any economic system. The study of these two features is attributed to these two major lines, culminating in the theory of mechanism design. The theory of economic mechanism design which was originated by Hurwicz is very general. All economic mechanisms and systems (including those known and unknown, private-ownership, state-ownership, and mixed-ownership systems) can be studied with this theory.

At the micro level, the development of the theory of incentives has also been a major advance in economics in the last forty years. Before, by treating the firm as a black box the theory remains silent on how the owners of firms succeed in aligning the objectives of its various members, such as workers, supervisors, and managers, with profit maximization.

When economists began to look more carefully at the firm, either in agricultural or managerial economics, incentives became the central focus of their analysis. Indeed, delegation of a task to an agent who has different objectives than the principal who delegates this task is problematic when information about the agent is imperfect. This problem is the essence of incentive questions. Thus, **conflicting objectives and decentralized information are the two basic ingredients of incentive theory.**

We will discover that, in general, these informational problems prevent society from achieving the first-best allocation of resources that could be possible in a world where all information would be common knowledge.³ The additional costs that must be incurred because of the strategic behavior of privately informed economic agents can be viewed as one category of the transaction costs. Although they do not exhaust all possible transaction costs, economists have been rather successful during the last forty years in modeling and analyzing these types of costs and providing a good understanding of the limits set by these on the allocation of resources. This line of research also provides a whole set of insights on how to begin to take into account agents’ responses to the incentives provided by institutions.

The three words — contracts, mechanisms and institutions are to a large

³The term of first-best is relatively to the second-best. The theory of the second-best concerns what happens when one or more optimality conditions (typically the case of incomplete information) cannot be satisfied. Canadian economist Richard Lipsey and Australian economist Kelvin Lancaster showed in a 1956 paper that if one optimality condition in an economic model cannot be satisfied, it is possible that the next-best solution involves changing other variables away from the ones that are usually assumed to be optimal, see: Lipsey and Lancaster (1956), “The General Theory of Second Best”. *Review of Economic Studies* 24(1): 11-32. JSTOR 2296233.

extent synonymous. They all mean “rules of the game,” which describe what actions the parties can undertake, and what outcomes these actions would be obtained. In most cases the rules of the game are given by designer: in chess, basketball, etc. The rules are designed to achieve better outcomes. But there is one difference. While mechanism design theory may be able answer “big” questions, such as “socialism vs. capitalism,” contract theory is developed and useful for more manageable smaller questions, concerning specific contracting practices and mechanisms.

Thus, mechanism design is normative economics, in contrast to game theory, which is positive economics. Game theory is important because it predicts how a given game will be played by agents. Mechanism design goes one step further: given the physical environment and the constraints faced by the designer, what goal can be realized or implemented? What mechanisms are optimal among those that are feasible? In designing mechanisms one must take into account incentive constraints (e.g., consumers may not report truthfully how many pairs of shoes they need or how productive they are).

This part considers the design of economic mechanism in which one or many parties have private characteristics or hidden actions. The party who designs the mechanism will be called the designer or principal, while the other parties will be called agents, individuals, or participants. For the most part we will focus on the situation where the designer has no private information and the agents do. This framework is called *screening*, because the principal will in general try to screen different types of agents by inducing them to choose different bundles. The opposite situation, in which the designer has private information and agents do not, is called *signaling*, since the designer could signal his type with the design of his contract or mechanism.

We will briefly present the mechanism/contract theory in four chapters. Chapters 16 and 17 consider the principal-agent model where the principal delegates an action to a single agent with private information. This private information can be of two types: either the agent can take an action unobserved by the principal, the case of moral hazard or hidden action; or the agent has some private knowledge about his cost or valuation that is ignored by the principal, the case of adverse selection or hidden knowledge. The theory of optimal contract design considers when this private information is a problem for the principal, and what is the optimal way for the principal to cope with it. **The design of the principal’s optimal contract can be regarded as a simple optimization problem. This simple focus will turn out to be enough to highlight the various trade-offs between allocative efficiency and distribution of information rents arising under incomplete information.** The mere existence of informational constraints may generally prevent the principal from achieving allocative efficiency. We will characterize the allocative distortions that the principal

finds desirable to implement in order to mitigate the impact of informational constraints. We should acknowledge that the materials in Chapters 16 and 17 notes are mainly drawn from Laffont and Martimort (2002).

Chapter 18 will consider situations with one principal and many agents. Moreover, maintaining the hypothesis that agents adopt an individualistic behavior, those organizational contexts require a solution concept of equilibrium, which describes the strategic interaction between agents under complete information.

Chapter 16 will discuss the case of incomplete information in which asymmetric information may not only affect the relationship between the principal and each of his agents, but it may also plague the relationships between agents. As such, agents do not know each other's characteristics, and we need to consider Bayesian incentive compatible mechanism.

Chapter 17 will briefly study dynamic contract theory. We will discuss long-term incentive contracting in a dynamic principal-agent setting with one-agent and adverse selection. We will first consider the case where the principal (designer) can commit to a contract forever, and then consider what happens when she cannot commit against modifying the contract as new information arrives.

Chapter 16

Principal-Agent Theory: Hidden Information

16.1 Introduction

This chapter considers the optimal contract design with only one agent. When a principal assigns a task to an agent who has private information, the agent's utility, technology, cost, and ability are likely all private information. By misreporting the information, the agent may obtain extra benefit, and then the problem of incentive appears. We call such a principal-agent situation as hidden information or adverse selection problem.

Somes Examples

The principal-agent problem is almost everywhere. Here are some specific examples.

- (1) The department of environmental protection wants to reduce the degree of haze, but do not know the cost of pollution reduction.
- (2) The township government entrusts the collective land to farmers who will be the only one to know the farming skills and observe the local weather conditions.
- (3) Shareholders delegate the daily decisions of companies to professional managers, who know more about markets and production technologies.
- (4) Professors teach students knowledge, but do not know the students' learning ability.
- (5) A client delegates his defense to an attorney who will be the only one to know the various legal provisions and the difficulty of the case.

- (6) Venture capital companies lend funds to founders of high-tech companies who will be the only one master new technologies.
- (7) The central government delegates administrative power to local government administration, but local governments know more about local situation.
- (8) National Development and Reform Commission(NDRC) of China entrusts petroleum resources to China National Petroleum Corporation(CNPC) and China Sinopec, but NDRC does not participate in the specific camp.
- (9) An insurance company provides insurance to agents who will be the only one to know how good a driver they are.
- (10) A regulatory agency contracts for service with a public utility company without having complete information about its technology.

All of the above issues show that the difference in information between the principal and the agent has a significant influence on contract design. In order to make optimal use of resources and to allow agents have the incentive to show private information, agents need to obtain some information rent. In order to induce agents to truly display private information, the principal needs to balance the two effects of allocation efficiency and rent extraction when designing the optimal contract. The implicit assumption here is that the contractual relationship is conducted within a certain legal framework. As the contract can be enforced by the court, the agent is bound by the terms of the contract.

The main objective of this chapter is to characterize the optimal rent extraction-efficiency trade-off faced by the principal when designing his contract offer to the agent under the set of incentive feasible constraints: incentive and participation constraints. If the incentive constraint is binding at the optimal solution, it means that the adverse selection limits the efficiency of the transaction. **The main lesson of this chapter is that the optimal second-best contract calls for a distortion in the volume of trade away from the first-best, and principal must transfer some information rent to the most efficient agent.**

16.2 Basic Settings

In this section, we introduce the simplest model to discuss adverse selection. This problem was first introduced into the economics literature by Mirrlees (for Mirrlees' biography, see 17.11.2). In the literature, it is usually assumed that there are two types of participants, one is called a principal who has some sort of monopoly power in the transaction. The other is

called an agent who has private information in the transaction, and chooses the transaction contract based on the information she has. There are also some variants in the literature. For instance, the principal may also have private information, and the principal faces market competition. Different from the market equilibrium discussed before, here we focus on how information distribution affects the efficiency of market transactions.

16.2.1 Economic Environment (Technology, Preferences, and Information)

For the ease of understanding, we discuss the issue of adverse selection in a monopolistic market scenario. This problem was first proposed by Mussa and Rosen (1978), and a more deeper discussion was conducted by Maskin and Riley (1984). Consider an economy in which a monopolist offers goods to a consumer (It can also be regarded as a continuum of multiple consumers, in this case, the proportion of consumers belonging to a certain type can be considered as the probability that there is only one consumer and the consumer belongs to this type can also be considered a continuum of multiple consumers. According to the law of large numbers in probability theory, these two treatment methods are equivalent).

The consumer's evaluation of the product is private information that cannot be observed by the monopolist. The purpose of the monopolist is to get maximum surplus from the consumer, and this depends on how he distinguishes the consumer's value from the consumer's behavior. If we put this example into the principal-agent framework, the monopolist corresponds to principal, consumer corresponds to agent. Judging the role of participants in the principal-agent framework is usually based on the fact that the principals provide a series of alternative contracts, and agents choose these contracts.

Assume the consumer's utility function is

$$u(q, T, \theta) = \theta v(q) - T,$$

where q is the quantity of goods purchased by the consumer; T is consumer's transfer payment; and θ is the type of consumer's evaluation on goods with $\theta \in \{\theta_H, \theta_L\}$ and $\theta_H > \theta_L$. Let $\Delta \theta = \theta_H - \theta_L$. Assume that the probability that θ_L happens is β , which means the probability of utility function $u(q, T, \theta_L) = \theta_L v(q) - T$ is β , and the probability of utility function $u(q, T, \theta_H) = \theta_H v(q) - T$ is $1 - \beta$. The utility function above is a quasi-linear utility function, which usually satisfies the following assumptions

$$v(0) = 0, v'(q) > 0, v''(q) < 0.$$

The monopolist's objective function is

$$\pi = T - cq,$$

where, c is unit production cost of firm.

16.2.2 Outcomes Space and Contracting Variables

The contracting variables are the quantity produced q and the transfer T received by the agent. Let \mathcal{A} be the set of feasible allocations that is given by

$$\mathcal{A} = \{(q, T) : q \in \mathcal{R}_+, T \in \mathcal{R}\}. \quad (16.2.1)$$

These variables are both observable and verifiable by a third party such as a benevolent court of law.

16.2.3 Information Structure and Timing

Unless explicitly stated, two participants will proceed in the following order

- $t = 0$: the consumer discovers his own type θ ;
- $t = 1$: the monopoly producer provides contract;
- $t = 2$: the consumer accepts or rejects contract;
- $t = 3$: contract enforcement.

In the principal-agent theory, contracts are offered at the interim stage. The information structure can be divided into three categories according to the timing when the principal and the agent know his type when the principal makes his offer. It is called the *ex ante* stage when both participants do not know the type of the agent. It is called the *interim* stage when the agent knows his type while the principal does not know. When Both participants know the type of agent, it is called the *ex post* stage.

16.2.4 Complete Information Optimal Contract (Benchmark)

To analyze the influence of adverse selection on the decision more deeply, we usually use the complete information as the benchmark. In the complete information situation, the first-best outcome can be achieved.

If the monopolist knows the consumer's type, she will provide a separate contract for each type, that is, the contract (T_i, q_i) corresponds to θ_i , $i \in \{H, L\}$. The goal of the monopolist is to obtain consumer's surplus as much as she can on the basis of consumer accept the contract. Assume that the consumer does not accept the monopolist contract, his utility level is \bar{u} , for convenience, let $\bar{u} = 0$. Thus, the monopolist's goal is to solve the following maximization problem:

$$\max_{T_i, q_i} T_i - cq_i$$

subject to participation constraints:

$$\theta_i v(q_i) - T_i \geq 0,$$

or

$$T_i \leq \theta_i v(q_i).$$

By substituting constraints into the objective function:

$$\max_{T_i, q_i} \theta_i v(q_i) - cq_i.$$

From first-order conditions, we have:

$$\theta_i v'(q_i^*) = c,$$

$$T_i^* = \theta_i v(q_i^*).$$

In this case, the outcome is Pareto optimal, that is, the marginal social value of a unit of consumption is equal to the marginal cost of production. At the same time, the monopolist obtains the maximum surplus, $\pi = \beta(T_L - cq_L) + (1 - \beta)(T_H - cq_H)$, and the consumer's surplus is 0.

Implementation of the First-Best Contract

The monopolist can design the following contract in order to obtain the above profit level: first of all, these contracts require the consumer to obtain a level of utility that is not lower than the non-purchase (called reservation utility: \bar{u}). That is, $\theta_i v(q_i) - T_i \geq \bar{u}$, which we call this as the participation constraint. Second, in order to implement the above first-best contract (T_i^*, q_i^*) , $i \in \{H, L\}$, the monopolist can make the following take-it-or-leave-it offers to the agent with contract (T_i^*, q_i^*) . Since monopolies can distinguish different types under complete information situations, this form of contract will be accepted by consumer θ_i , because the utility he received is no less than his reservation utility.

Graphical Representation of Optimal Contract

Figure 16.1 depicts the indifference curves of two types of consumer. Compared to θ_L , the indifference curve of θ_H has a larger slope, at the same time, the two types of indifference curves intersect only once at most, this feature is called single crossing property or Spence-Mirrlees property, it is a very important feature in information economics.

The optimal contract under complete information is the first-best contract, which is the A^* and B^* points corresponding to figure 16.2. In this optimal contract, each consumer only obtains reservation utility, and at the same time, the slope of the optimal points all equal to the marginal cost c .

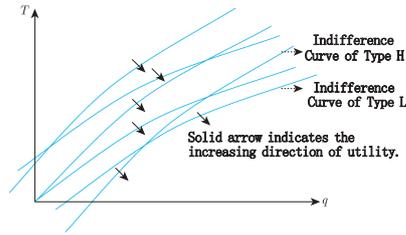


Figure 16.1: Indifference curves of two types of consumer

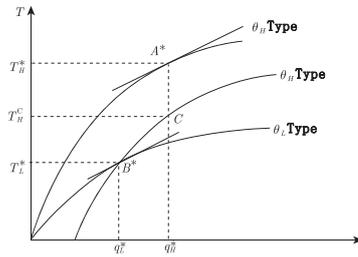


Figure 16.2: First-best contract

16.2.5 Incentive Feasible Contracts

With the result of the complete information situation as a benchmark, we can analyze the optimal contract in the case of incomplete information. We will see that under incomplete information, the optimal contract under the complete information will lead to incentive incompatibility issues.

When the monopolist cannot observe the type of consumer, the above contract $(T_i^*, q_i^*), i \in \{H, L\}$ is not implementable. From Figure 16.2, one can see that, under incomplete information, the low-efficient agent will choose point B, but the high-efficient agent (the θ_H type) will also choose point B for profit, instead of choosing point A that is optimal for A under the complete information, which means that high-efficient agent has incentive to mimic himself as agents. This is because, for high-efficient consumer (the θ_H type), the utility of the choosing (T_H^*, q_H^*) is 0, and if he chooses (T_L^*, q_L^*) , the utility level is:

$$u(T_L^*, q_L^*, \theta_H) = \theta_H v(q_L^*) - T_L^* > \theta_L v(q_L) - T_L = 0.$$

Thus, under asymmetric information, $(T_i^*, q_i^*), i \in \{H, L\}$ cannot be implemented by the “take-it-or-leave-it” contract. High-efficient consumer θ_H will imitate low-efficiency consumer θ_L .

For the monopolist to obtain higher surpluses, an appropriate contract must be designed so that different type has the incentive to choose the contract proposed for his type, this is the meaning of incentive compatibility.

Definition 16.2.1 A menu of contracts $A_i = (q_i, T_i)$, $i \in \{H, L\}$ is *incentive compatible* if it satisfies:

$$u(q_i, T_i, \theta_i) \geq u(q_j, T_j, \theta_i), \forall i, j \in \{H, L\}.$$

Incentive-compatible contracts mean that each type chooses his own contract. Under asymmetric information, when a monopolist designs a contract, the contract must satisfy the following two sets of constraints:

$$U_H = \theta_H v(q_H) - T_H \geq \theta_H v(q_L) - T_L = U_L + \Delta\theta v(q_L), \quad (16.2.2)$$

$$U_L = \theta_L v(q_L) - T_L \geq \theta_L v(q_H) - T_H = U_H - \Delta\theta v(q_H), \quad (16.2.3)$$

$$U_H = \theta_H v(q_H) - T_H \geq 0, \quad (16.2.4)$$

$$U_L = \theta_L v(q_L) - T_L \geq 0. \quad (16.2.5)$$

We call equations (16.2.2) and (16.2.3) the incentive compatibility constraints of the types θ_H and θ_L , and equations (16.2.4) and (16.2.5) the participation constraints of the types θ_H and θ_L .

Definition 16.2.2 A menu of contracts $A_i = (q_i, T_i)$, $i \in \{H, L\}$ is *incentive feasible* if it satisfies if both incentive and participation constraints (16.2.2) through (16.2.5).

Solving the asymmetric information (for monopolists) optimal contract is actually solving the following maximization problem:

$$\max_{(q_i, T_i)} \beta(T_L - cq_L) + (1 - \beta)(T_H - cq_H),$$

subject to the incentive compatibility constraints (16.2.2) and (16.2.3), and participation constraints (16.2.4) and (16.2.5).

Before formally solve the optimization problem, we first discuss the possible choice of incentive feasible contracts of the monopolist. We consider the following two special contracts.

(1) Bunching contracts or pooling contracts: the first special case of incentive feasible contract is obtained when the contracts targeted for each type coincide, that is, give the same contract $T_L = T_H = T$, $q_L = q_H = q$, and both types accept the contract. In this case, the incentive compatibility constraint is naturally established. The only constraint is the participation constraint. In this case, if the monopolist supplies for two types, $\theta_L v(q) - T \geq 0$ is the only true constraint.

(2) Shutdown of the least-efficient type: if the nonzero contract (T_H, q_H) is provided for type θ_H , the null contract $(T_L, q_L) = (0, 0)$ is provided for type θ_L , it also satisfies: $\theta_L v(q_H) - T_H \leq 0$. In such contracts, low-efficient consumer will not be served.

As with the pooling contract, the benefit of the $(0, 0)$ option is that it somewhat reduces the number of constraints since the incentive and participation constraints take the same form. The cost of such a contract may

be an excessive screening of types. Here, the screening of types takes the rather extreme form of the least-efficient type.

16.2.6 Monotonicity Constraints

Incentive compatibility constraints reduce the set of feasible allocations. Moreover, these quantities must generally satisfy a monotonicity constraint that does not exist under complete information. This condition implies Spence-Mirrlees single-crossing condition

$$\frac{\partial^2 u}{\partial \theta \partial q} > 0$$

is satisfied, which means that two different types of indifference curves intersect at most once.

Adding (16.2.2) and (16.2.3), we immediately have:

$$\Delta \theta (v(q_H) - v(q_L)) \geq 0.$$

Since $v'(q) > 0$, when $q_H \neq q_L$, there must be $q_H > q_L$, which means that the consumption of low-efficient consumer will not be higher than the consumption of high-efficient consumer, and it is necessary and sufficient for implementability.

In order to derive sufficiency, assume $q_H \geq q_L$, there exists transfers T_H and T_L such that the incentive constraints hold. It is enough to take those transfers such that:

$$\theta_L (v(q_H) - v(q_L)) \leq T_H - T_L \leq \theta_H (v(q_H) - v(q_L)). \quad (16.2.6)$$

16.2.7 Information Rents

To understand the structure of the optimal contract under asymmetric information it is useful to introduce the concept of information rent, which is the difference between the utility under asymmetric information and the utility of complete information.

Under complete information, since the monopolist has full negotiating power, she can completely obtain consumer surplus, or the consumer only gets reservation utility levels U_H^* and U_L^* :

$$U_H^* = \theta_H v(q_H^*) - T_H^* = 0,$$

$$U_L^* = \theta_L v(q_L^*) - T_L^* = 0.$$

However, under asymmetric information, if the monopolist provides products to both types, the above two conditions cannot hold simultaneously.

Take any incentive-compatible contract $(T_H, q_H; T_L, q_L)$, how much benefit (utility) would a θ_H agent get by mimicking a θ_L agent? The high-efficient agent would get:

$$\theta_H v(q_L^*) - T_L^* = \Delta\theta v(q_L^*) + U_L.$$

Even if the expected utility obtained by low-efficient type is equal to the reservation utility, that is, $U_L = \theta_L v(q_L^*) - T_L^* = 0$, high-efficient type can obtain the information rent of $\Delta\theta v(q_L^*)$. Thus, as long as the principal insists on a positive output for the low-efficient agent, $q_L^* > 0$, the principal must give up a positive rent to a high-efficient agent to have any incentive-compatible contract $(T_H, q_H; T_L, q_L)$. This information rent is generated by the informational advantage of the agent over the principal. The problem for the monopolist is thus to choose the smartest way to transfer some information rent to certain types and gets a maximum profit.

We use the notations $U_H = \theta_H v(q_H) - T_H$ and $U_L = \theta_L v(q_L) - T_L$ to denote the information rent for high-efficient type θ_H and low-efficient type θ_L , respectively.

16.2.8 The Optimal Contracts under Asymmetric Information

According to the timing of the contractual setting, the monopolist must offer a menu of contracts before knowing which type she is facing. Thus, the monopolist calculates the expected benefits of incentive compatible contract $(T_H, q_H; T_L, q_L)$, the monopolist's optimal problem then can be written as:

$$\begin{aligned} & \max_{(T_H, q_H; T_L, q_L)} (1 - \beta)(T_H - cq_H) + \beta(T_L - cq_L) \\ \text{s. t.} \quad & \theta_H v(q_H) - T_H \geq \theta_H v(q_L) - T_L, \\ & \theta_L v(q_L) - T_L \geq \theta_L v(q_H) - T_H, \\ & \theta_H v(q_H) - T_H \geq 0, \\ & \theta_L v(q_L) - T_L \geq 0. \end{aligned}$$

Using the definition of information rents $U_H = \theta_H v(q_H) - T_H$ and $U_L = \theta_L v(q_L) - T_L$, we can replace the payment T_H and T_L with information rent and output, for which the monopolist's optimization problem is to solve for $(U_H, q_H; U_L, q_L)$. The focus on information rents enables us to assess the distributive impact of asymmetric information, and the focus on outputs allows us to analyze its impact on allocative efficiency and the overall gains from trade. Thus an allocation corresponds to a volume of trade and a distribution of the gains from trade between the monopolist (principal) and the consumer (agent).

With this change of variables, the principal's objective function can be rewritten as:

$$\max(1 - \beta)(\theta_H v(q_H) - cq_H) + \beta(\theta_L v(q_L) - cq_L) - ((1 - \beta)U_H + \beta U_L) \quad (16.2.7)$$

$$\text{s.t.} \quad U_H \geq U_L + \Delta\theta v(q_L), \quad (16.2.8)$$

$$U_L \geq U_H - \Delta\theta v(q_H), \quad (16.2.9)$$

$$U_H \geq 0, \quad (16.2.10)$$

$$U_L \geq 0. \quad (16.2.11)$$

Equation (16.2.7) can be divided into two parts:

$$\underbrace{(1 - \beta)(\theta_H v(q_H) - cq_H) + \beta(\theta_L v(q_L) - cq_L)}_{\text{Expected allocation efficiency}} - \underbrace{((1 - \beta)U_H + \beta U_L)}_{\text{Expected information rent}}. \quad (16.2.12)$$

The first term denotes expected allocative efficiency, and the second term denotes expected information rent which implies that the principal is ready to accept some distortions away from efficiency in order to decrease the agent's information rent. We index the solution to this problem (16.2.7) with a superscript SB, meaning second-best.

16.2.9 The Rent Extraction-Efficiency Trade-Off

For the above optimization problem (16.2.7), a technical difficulty is how to deal with these constraints. One way is to use the Lagrangian method to incorporate constraint equations into Lagrangian equations through Lagrangian multipliers, and then to deal with the optimization problem of inequality constraints according to the Kuhn-Tucker theorem. The other way is to analyze these inequality constraints by identifying which are the equality constraints, and which are the strict inequality constraints so that they do not need to be considered. Let us first consider contracts without shutdown, i.e., $q_L > 0$. This is true when the so-called Inada condition $v'(0) = +\infty$ is satisfied and $\lim_{q \rightarrow 0} v'(q)q = 0$.

The high-efficient type's incentive compatibility constraint (16.2.8) and participation constraint (16.2.10) must be at least one binding, otherwise we can reduce the utility U_H of high-efficient type, thereby increasing monopoly profits; similarly, the low-efficient consumer's incentive compatibility constraint (16.2.9) and participation constraint (16.2.11) must be at least one of the equality constraints.

From equation (16.2.8), we know that $U_H = U_L + \Delta\theta v(q_L) > 0$, resulting in high-efficient type's participation condition (16.2.8) must be a strict inequality. In this way, the incentive compatibility constraint for high-efficient type (16.2.8) must be an equality constraint.

By the monotonicity condition $q_H > q_L$ and the equality constraints equation (16.2.8), we obtain that the low-efficient type's incentive compatibility condition (16.2.9) must be a strict inequality constraint, so that the low-efficient type's participation constraint (16.2.11) must be an equality constraint.

For this reason, in the above maximization problem (16.2.7), the essential constraints for monopolist are: the participation constraint for low-efficient type (16.2.11) must be an equality constraint; for the high-efficient type θ_H , the incentive compatibility constraint needs to be met. Thus we have:

$$U_H = \Delta\theta v(q_L), \quad (16.2.13)$$

$$U_L = 0. \quad (16.2.14)$$

Substituting the two equality constraints into (16.2.7), we have

$$\max_{q_H, q_L} (1-\beta)(\theta_H v(q_H) - cq_H) + \beta(\theta_L v(q_L) - cq_L) - (1-\beta)(\Delta\theta v(q_L)). \quad (16.2.15)$$

Solving the FOCs for q_L and q_H , we get

$$\theta_H v'(q_H^{SB}) = c, \quad (16.2.16)$$

$$\theta_L v'(q_L^{SB}) = \frac{c}{1 - \left(\frac{1-\beta}{\beta} \frac{\Delta\theta}{\theta_L}\right)} > c. \quad (16.2.17)$$

Obviously, in the above two first-order conditions, the inner solution satisfies: $q_H^{SB} = q_H^*$, $q_L^{SB} < q_L^*$.

Summarizing the above discussion, when a monopolist provides a positive amount of products to either type, there is no allocation distortion for the efficient type θ_H in the separation equilibrium, but the cost is to pay for the information rent which comes from the gains disguising low utility type; and for the low-efficient type θ_L , his consumption is lower than first-best consumption, there is exist an allocation distortion with no information rent. Formally, we have the following conclusions:

Proposition 16.2.1 *Under asymmetric information, the optimal contracts entail:*

(1) *No output distortion for the high-efficient type in respect to the first-best, $q_H^{SB} = q_H^*$; A downward output distortion for the low-efficient type, $q_L^{SB} < q_L^*$ with:*

$$\left[\theta_L - \left(\frac{1-\beta}{\beta} \right) \Delta\theta \right] v'(q_L^{SB}) = c. \quad (16.2.18)$$

(2) *Only the high-efficient type gets a positive information rent which is equal to the profit gained from mimicking a low-efficient type:*

$$U_H^{SB} = \Delta\theta v(q_L^{SB}). \quad (16.2.19)$$

(3) The second-best transfers are respectively given by:

$$T_H^{SB} = \theta_H v(q_H^*) - \Delta\theta v(q_L^{SB}), \tag{16.2.20}$$

$$T_L^{SB} = \theta_L v(q_L^{SB}). \tag{16.2.21}$$

The above conclusions are related to each other. In order to allow high-efficient type to choose the consumption designed for him, we should give him some information rent that is determined by the consumption of low-efficient type q_L^{SB} and the valuation gap of two types $\theta_H - \theta_L$. The reason why we reduce the consumption of low-efficient type is to reduce his information rent as much as we can, so as to minimize the information rent. In addition, the directories of low-efficient type depend on the valuation gap between two types.

When $\theta_H - \theta_L \rightarrow 0$, high-efficient type's information rent goes to zero, and low-efficient's consumption will go to efficient level. When $\theta_H - \theta_L \rightarrow \infty$, high-efficient type's information rent goes to infinity, and low-efficient's consumption will go to efficient level. At this point, the monopolist may adopt an exclusive contract to shut down low-efficient customers to avoid paying high information rents. In the situation of asymmetric information, monopolists face a basic trade-off when choosing contracts: (high-efficient type) information rents and (low-efficient type) consumption distortions.

The points A^{SB} and B^{SB} in Figure 16.3 specify the second-best contract for θ_H and θ_L . In the second-best contract, there is no allocation distortion in A^{SB} at which the consumer θ_H obtains an information rent of $T^* - T^{SB}$, and in B^{SB} of second-best contract, the consumption of θ_L is lower than social optimality with zero information rent, which is located on the indifference curve of θ_L 's reservation utility.

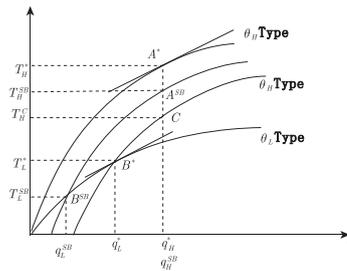


Figure 16.3: Second-best contract

The above proposition gives one of the most important conclusions of the adverse selection problem: no distortion at the top. That is, under asymmetric information, the consumption of the high-efficient agent does not appear to be distorted with respect to the first-best contact under complete information, while the consumption of the low-efficient agent may be

distorted downwards. This rule exists in almost all asymmetric information environments. For example, a capital owner (bank or others) faces a large number of financial needs, but don't know their operating efficiency, they can only set loan thresholds or differentiate interest rates based on their "declaration".

In order to prevent a high-efficient type mimics as a low-efficient type, the loan threshold or interest rate level of low-efficient type must be distorted. This is why real small corporations often do not have sufficient credit support as large corporations. They often face stricter approval and strength conditions or need to pay higher interest rates. This phenomenon is called credit rationing, see the classic paper by Stiglitz and Weiss (1981) for more details.

Remark 16.2.1 As early as 2,600 years ago, Sun Tzu already had insight into the basic idea of principal-agent theory and the above basic conclusions are given in "The Art of War". "The Art of War" by Sun Tzu is the book of ancient Chinese war books. It is the earliest, most outstanding, and most complete theory about war. Dubbed "the Bible of Military Science", it is also the earliest work on military strategies in the world. He considered the critical importance of information and its symmetry, and gave the basic conclusion of information economics: in the complete information situation, the best is first-best; when information is not symmetrical, the best is second-best. It says that: "now the enemy and know yourself and you can fight a hundred battles with no danger of defeat. If you only know yourself, but not your opponent, you may win or may lose. If you know neither yourself nor your enemy, you will always endanger yourself."

16.2.10 Shutdown Policy

When there is no positive solution in (16.2.17), ie, $\beta < \frac{\Delta\theta}{\theta_L}, q_L^{SB} = 0$, at which the monopolist will choose to close the low-efficient contract. When the low-efficient consumer's distribution probability is below a certain critical level, i.e., $\beta < \frac{\Delta\theta}{\theta_L}$, offering products to low-efficient consumer, or increasing the allocation efficiency of low-efficient consumer, becomes unworthy for the monopolist. In addition, if the difference of two types is very big and there is a low-efficient consumption, it is necessary to provide high information rent for high-efficient consumer. In this case, the monopolist may also choose to close the low-efficient consumer market, and the constraint condition becomes $T = \theta_H v(q)$. The monopolist's optimal choice is:

$$\max_q \theta_H v(q) - cq.$$

Then, we get the exclusion contract (q^c, T^c) satisfy:

$$\theta_H v'(q^c) = c, \quad T^c = \theta_H v(q^c).$$

Then, the consumption of θ_H is efficient with zero information rent, while the low-efficient type does not have any consumption.

16.3 Application

From the above discussion we know that in the case of asymmetric information, in general, the optimal outcome that one can obtain is only the second-best. Of course, there are some ways to improve the resource allocation efficiency, such as signaling, or the agent himself does not have information advantages. These are the basic results of the principal-agent model in the situation of hidden information. There are many applications for these theories.

This section proposes several classical settings where the basic model of this chapter is useful. Introducing adverse selection in each of these contexts has proved to be a significant improvement of standard microeconomic analysis.

16.3.1 Regulation

In the operation of market economy, some technical conditions make the market itself unable to operate efficiently, such as in the natural monopoly industries, the results of market competition or the occurrence of redundant construction, or the emergence of industry monopolies. At this time, the government's regulation policies may be helpful to improve the market's outcome. Unlike traditional regulatory theory, since the 1980s, the new regulatory theory has discussed the government's optimal regulatory policy under the asymmetric information environment. In this section, we use the theory of adverse selection in this chapter to discuss optimal regulation. Baron and Myerson (1982) were among the earliest contributors to incentive regulation theory.

In the Baron and Myerson (Econometrica, 1982) regulation model, the principal is a regulator (such as government) who maximizes a weighted average of the agents' surplus $S(q) - t$ and of a regulated monopoly's profit $U = t - \theta q$, where θ is marginal cost, and therefore, the smaller of the θ , the more efficient of the firms. There are two types of firm, θ_H, θ_L , where the probability of the θ_L type is ν , $\Delta\theta = \theta_H - \theta_L$, with a weight $\alpha < 1$ for the firm profit and a weight 1 for agent's net revenue. The principal's objective function is written now as $V = (S(q) - t) + \alpha U = S(q) - \theta q - (1 - \alpha)U$.

Because $\alpha < 1$, it is socially costly to give up a rent to the firm. The monopoly firm owns private information about the cost. To encourage the monopolist to truly report the cost, the principal needs to offer a second-best incentive contract. Let $(t_H, q_H; t_L, q_L)$ be the incentive contract and $U_i = t_i - \theta_i q_i$ be the utility of the θ_i type. Using the previous method,

we can get $U_H = 0, U_L = \Delta\theta q_H$. Maximizing expected social welfare under incentive compatibility and participation constraints leads to $q_L^{SB} = q_L^*$ for the high-efficient type and a downward distortion for the low-efficient type, $q_H^{SB} < q_H^*$ which is given by:

$$S'(q_H^{SB}) = \theta_H + \frac{\nu}{1-\nu}(1-\alpha)\Delta\theta. \quad (16.3.22)$$

Note that a higher value of α reduces the output distortion, because the regulator is less concerned by the distribution of rents within society as α increases. If $\alpha = 1$, the firm's rent is no longer costly and the regulator behaves as a pure efficiency maximizer implementing the first-best output in all states of nature.

Laffont and Tirole (1986, 1993) discuss a related but different incentive regulation issue. Unlike Baron and Myerson (1982), Laffont and Tirole (1986, 1993) discuss that the regulation institution can observe the cost of the enterprise, but does not know the type of the enterprise and the effort invested in reducing the cost. At this point, in order to encourage firms to reduce costs, regulation institution needs to design an optimal incentive contract.

In a public project, the value of the society is S , and the cost of the project completed by the regulated firm is $C = \theta - e$, where e is the effort invested by the firm in reducing costs, assuming that the cost function for effort is $\psi(e)$ which satisfies $\psi' > 0, \psi'' > 0$. θ is the type of firm. Suppose there are two types θ_H, θ_L , where the probability of the θ_L type is ν . Let T be the payment to the firm, the utility of the firm is $U = T - \psi(e)$. λ is the shadow cost of public funds, which means that if one unit is paid to the enterprise, the social cost is $1 + \lambda$. The regulator's goal is to maximize social welfare, i.e. maximize $S - (1 + \lambda)(T + \theta - e) + T - \psi(e) = S - (1 + \lambda)(\theta - e + \psi(e)) - \lambda U$.

The regulation institution designs an incentive compatibility contract that stipulates the cost of the firm and the compensation to the firm, $(T_H, C_H; T_L, C_L)$, which satisfies $U_i = T_i - \psi(\theta_i - C_i) \geq T_j - \psi(\theta_i - C_j)$. Using the previous principle's rent extraction and efficiency tradeoff, we get:

$$\begin{aligned} \psi'(\theta_L - C_L) &= 1, \text{ or } e_L = e^*, \\ \psi'(\theta_H - C_H) &= 1 - \frac{\lambda}{1 + \lambda} \frac{\nu}{1 - \nu} \Phi'(\theta_H - C_H) < 1, \end{aligned}$$

where $\psi \equiv \psi(e) - \psi(e - \Delta\theta)$. In other words, for the efficient firm, the effort to reduce cost under optimal incentives is socially optimal, while for the inefficiency firm, there exists a distortion in effort.

With regard to the application of the principal-agent model in regulation theory, interested readers may refer to the classic book of Laffont and Tirole (1993) for more details.

16.3.2 Financial Contracts

Asymmetric information significantly affects the financial markets. For instance, the difficulty of loaning small and medium-sized enterprises is a worldwide problem. In the case of asymmetric information, it is even worse. Freixas and Laffont (1990) discuss the issue of optimal credit decision when information is asymmetric, the principal is a bank, the agent is an investor, which has private information about project profitability.

The bank also has borrowing cost in the credit process, such as paying interest to depositors. Assume that the interest cost per unit is R . Assume the bank provides a loan of size k to an investor, and there are two types of productivity: $\Theta = \theta_H, \theta_L$, and the probabilities of taking θ_L and θ_H are $1 - \nu$ and ν respectively. The cost of the bank is Rk . Let $T(k)$ be the credit return to the bank within the agreed time. Therefore, the lender's utility function is $V = T(k) - Rk$, where we do not consider the actual factors of some financial markets, such as credit risk and default, but only focus on the issue of credit incentives. The bank gives the investor an incentive-compatible credit contract $(T_L, k_L; T_H, k_H)$.

Denote $U_i = \theta_i f(k_i) - T_i$ the utility of the investor θ_i under the incentive contract. If $(T_L, k_L; T_H, k_H)$ is an incentive compatible contract, then it needs to satisfy: $U_i = \theta_i f(k_i) - T_i \geq \theta_i f(k_j) - T_j = U_j + (\theta_i - \theta_j) f(k_j)$ and $U_i = 0$. Using the trade-off between rent extraction and efficiency in reverse selection, we easily get: for high-efficient type, the optimal capital borrowing scale is $k_H^{SB} = k_H^*$ such that $\theta_H f'(k_H^*) = R$, and for low-efficient types, there is a downward distortion in the size of the optimal capital lending:

$$k_L^{SB} < k_L^*, \theta_L f'_L(k_L^{SB}) = \frac{R}{1 - \frac{\nu \Delta \theta}{1 - \nu \theta_L}} > R = \theta_L f'_L(k_L^*).$$

In this way, the loan given to a low productivity firm is even lower when information is asymmetric. Because of economic scale, small and medium-sized firms have lower productivity compared with large-scale firms. Coupled with large repayment risks, loans are more difficult. The principal-agent model can then explain the difficulty of small and medium-sized firms loaning.

16.4 The Revelation Principle

In the above analysis, the consumer (agent) is required to report his own "type." A natural question is whether a better outcome could be achieved with a more complex contract allowing the agent possibly to choose among more options. The answer is negative. Allowing the consumer (agent) to send more general information does not obtain a better outcome but only increases the complexity of the contract.

The revelation principle introduced below ensures that there is no loss of generality in restricting the principal to offer simple menus having at most as many options as the cardinality of the type space. Those simple menus are actually examples of direct revelation mechanisms.

Definition 16.4.1 The direct revelation mechanism is a mapping $g(\cdot)$ from the type space Θ to the result space \mathcal{A} .

In the principal-agent model of this chapter, the direct revelation mechanism can be written as $g(\theta) = (q(\theta), T(\theta)), \forall \theta \in \Theta$. If the consumer (agent) reports that his type is $\tilde{\theta} \in \Theta$, then the quantity of goods purchased by the consumer is $q(\tilde{\theta})$ and the payment is $T(\tilde{\theta})$. In the direct revelation mechanism, an important concept is to "tell the truth."

Definition 16.4.2 The direct revelation mechanism $g(\cdot)$ is called a truthful display if it can guarantee that each type actually reports its type, i.e., the following incentive compatibility constraints are satisfied:

$$\theta_H v(q(\theta_H)) - T(\theta_H) \geq \theta_H v(q(\theta_L)) - T(\theta_L), \quad (16.4.23)$$

$$\theta_L v(q(\theta_L)) - T(\theta_L) \geq \theta_L v(q(\theta_H)) - T(\theta_H). \quad (16.4.24)$$

If $q_H = q(\theta_H), q_L = q(\theta_L); T_H = T(\theta_H), T_L = T(\theta_L)$, then return to the original form.

When the communication between the monopolist (principal) and the consumer (agent) is more complicated (the principal is more than just asking the agent to report its type), we can get a more general mechanism. Let M be the information space of the agent under the more general mechanism.

Definition 16.4.3 The general mechanism $\langle M, \tilde{g} \rangle$ is composed of the mapping $\tilde{g}(\cdot)$ from the information space M to \mathcal{A} :

$$\tilde{g}(m) = (\tilde{q}(m), \tilde{T}(m)), \forall m \in M. \quad (16.4.25)$$

Under such a mechanism, an agent of type θ will report the optimal information $m^*(\theta)$ determined by:

$$\theta v(\tilde{q}(m^*(\theta))) - \tilde{T}(m^*(\theta)) \geq \theta v(\tilde{q}(\tilde{m})) - \tilde{T}(\tilde{m}), \quad \forall \tilde{m} \in M. \quad (16.4.26)$$

Therefore, the mechanism $\langle M, \tilde{g}(\cdot) \rangle$ determines the allocation rule $a(\theta) = (\tilde{q}(m^*(\theta)), \tilde{T}(m^*(\theta)))$ that maps Θ to \mathcal{A} .

The question is whether we can design more complex mechanisms to make the principal more profitable. the answer is negative. Consider the general mechanism $\langle M, \tilde{g} \rangle$. For this general mechanism, as a rational person, he will seek the optimal strategy. By choosing an optimal strategy, we

16.5.1 Finite Types

We again discuss the monopolist's choice. The consumer's objective function is the same as before:

$$u(q, T, \theta_i) = \theta_i v(q) - T.$$

However, here we assume that there are at least three types:

$$\theta_n > \theta_{n-1} > \cdots > \theta_1, \quad n \geq 3.$$

Assume that the probability of taking θ_i is β_i . Consider that the monopolist offers different contracts (q_i, T_i) for different types, and the monopolist's optimal choice is given by the following maximization problem:

$$\max_{\{(q_i, T_i)\}} \sum_{i=1}^n (T_i - cq_i) \beta_i \quad (16.5.29)$$

$$\text{s.t.} \quad \theta_i v(q_i) - T_i \geq \theta_i v(q_j) - T_j, \quad \forall i, j, \quad (16.5.30)$$

$$\theta_i v(q_i) - T_i \geq 0, \quad \forall i, \quad (16.5.31)$$

where (16.5.30) are incentive compatibility constraints for type θ_i ; (16.5.31) are participation constraints for type θ_i . Below we focus on the separation of equilibrium.

Under the Spence-Mirrlees single-crossing condition, as consumer's utility to the goods increases, the consumption for them in contracts will also increase.

We can add the incentive compatibility conditions of the θ_i, θ_j to get:

$$(\theta_i - \theta_j)(v(q_i) - v(q_j)) \geq 0.$$

In this way, once $\theta_i > \theta_j$, there will be $q_i \geq q_j$.

Now we need to verify that the least significant participation constraint and other types of local downward incentive compatibility constraints that play the role of the equality constraint in the monopoly's maximizing profit contract selection. Now we need to verify that the lowest type of participation constraint and local downward incentive constraints (LDICs) are the equality constraint that work in the monopoly's maximizing profit contract choice.

Definition 16.5.1 A menu of contracts satisfies the local downward incentive constraint of type $\theta_i > \theta_1$, if

$$\theta_i v(q_i) - T_i \geq \theta_i v(q_{i-1}) - T_{i-1}. \quad (16.5.32)$$

Let $q_0 = T_0 = 0$ denote a contract that does not provide goods. Then θ_1 's participation constraint can be written in the form of a local downward incentive constraint:

$$\theta_1 v(q_1) - T_1 \geq \theta_1 v(q_0) - T_0 \equiv 0.$$

Given the participation constraint of θ_1 , due to

$$\theta_i v(q_i) - T_i \geq \theta_i v(q_1) - T_1 \geq \theta_1 v(q_1) - T_1 \geq 0,$$

the participation constraint of other types $\theta_i > \theta_1$ is naturally satisfied. If θ_1 's participation constraint is not an equality constraint, then the same tiny amount is added for all T_i , all the participation constraints and incentive compatibility constraints are established, but this will increase the monopolist's profit, so at optimal contract, θ_1 's participation constraints must be equality constraints.

We first verify that if all local downward incentive constraints of all types (except θ_1) hold, and satisfy the monotonicity condition, then all downward incentive constraints hold. That is, $\theta_i v(q_i) - T_i \geq \theta_i v(q_j) - T_j, \forall \theta_i > \theta_j$.

If $\theta_i, \dots, \theta_j > \theta_1$, the local downward incentive constraints hold, then for all $\forall \theta_k \in \{\theta_i, \dots, \theta_{j+1}\}$, we have

$$\theta_k v(q_k) - T_k \geq \theta_k v(q_{k-1}) - T_{k-1}.$$

Since $\theta_k \leq \theta_i$, from monotonicity condition $q_k \leq q_i$ and inequality (16.5.32), we have

$$\theta_i v(q_k) - T_k \geq \theta_i v(q_{k-1}) - T_{k-1}$$

for all $k \in \{i, i-1, \dots, j+1\}$. Let $k = i, i-1, \dots, j+1$, and by summation we have:

$$\theta_i v(q_i) - T_i \geq \theta_i v(q_j) - T_j.$$

Second, we verify that at the maximum profit, for all $\theta_i > \theta_1$ types, the local downward incentive constraints must be an equality constraint:

$$\theta_i v(q_i) - T_i = \theta_i v(q_{i-1}) - T_{i-1},$$

otherwise, for all $j \geq i$, we can increase T_j slightly, all incentive compatibility constraints and participation constraints are still established, but will increase the monopolist's profits.

Finally, we verify that if all local downward incentive constraints are necessarily equality constraints, and all monotonicity conditions satisfy, then all upward incentive constraints will also be established. This is because, $\theta_i v(q_i) - T_i = \theta_i v(q_{i-1}) - T_{i-1}$, and for all $k \leq i$, from $q_k \leq q_i$, we have

$$\theta_k v(q_i) - T_i \leq \theta_k v(q_k) - T_k.$$

Therefore, in a finite number of types, the monopolist's choice of contract is actually to solve the following equality constraint optimization prob-

lem:

$$\max_{\{(q_i, T_i)\}} \sum_{i=1}^n (T_i - cq_i)\beta_i \quad (16.5.33)$$

$$\text{s.t.} \quad \theta_i v(q_i) - T_i = \theta_i v(q_{i-1}) - T_{i-1}, q_0 = T_0 = 0, \quad \forall i \quad (16.5.34)$$

$$q_i \geq q_j, \text{ if } i > j. \quad (16.5.35)$$

Forming the Lagrange function, we have

$$L = \sum_{i=1}^n [T_i - cq_i]\beta_i + \sum_{i=1}^n \lambda_i [\theta_i v(q_i) - \theta_i v(q_{i-1}) - T_i + T_{i-1}].$$

We then have the following first-order conditions:

$$\begin{aligned} (\lambda_i \theta_i - \lambda_{i+1} \theta_{i+1}) v'(q_i) &= c \beta_i, \quad i < n; \\ \lambda_n \theta_n v'(q_n) &= c \beta_n; \\ \beta_i &= \lambda_i - \lambda_{i+1}, \quad i < n; \\ \beta_n &= \lambda_n. \end{aligned}$$

Therefore, for the top type consumer, there is $\theta_n v'(q_n) = c$, so there is no consumption distortion.

For other types $\theta_i < \theta_n$, there is consumption distortion

$$\theta_i v'(q_i) = c \frac{\lambda_i - \lambda_{i+1}}{\lambda_i - \lambda_{i+1} \frac{\theta_{i+1}}{\theta_i}} > c.$$

At the same time, for the lowest type of consumer, there is no information rent, but for other types there is information rent, and as the type changes from lowest to highest, its information rent also increases. By calculation we have, for $\theta_i > \theta_1$, the information rent is $\sum_{j=2}^i (\theta_j - \theta_{j-1}) v(q_{j-1})$.

From two types to more than two types, we find that the basic trade-off of monopolies are still the trade-off between information rent and allocative efficiency.

16.5.2 Continuum Types

In this section, we give a brief account of the continuum type case. Most of the principal-agent literature is written within this framework.

Reconsider the standard model with $\theta \in \Theta = [\underline{\theta}, \bar{\theta}]$. Since the revelation principle is still valid with a continuum of types, and we can restrict our analysis to direct revelation mechanisms $\{(q(\theta), T(\theta))\}$.

Assume that the type of consumer (agent) follows the density function $f(\theta)$ (distribution function is $F(\theta)$) and the range of θ is $[\underline{\theta}, \bar{\theta}]$. The monopolist (principal) chooses the contract to solve the following maximization

problem:

$$\max_{\{(q(\theta), T(\theta))\}} \int_{\underline{\theta}}^{\bar{\theta}} (T(\theta) - cq(\theta))f(\theta)d\theta \quad (16.5.36)$$

$$\text{s.t.} \quad \theta v(q(\theta)) - T(\theta) \geq \theta v(q(\hat{\theta})) - T(\hat{\theta}), \quad \forall(\theta, \hat{\theta}) \in \Theta^2, \quad (16.5.37)$$

$$\theta v(q(\theta)) - T(\theta) \geq 0, \quad \forall \theta \in \Theta. \quad (16.5.38)$$

For the participation constraint (16.5.38), the same as before, only the lowest type is required to satisfy the participation constraint, i.e.,

$$\underline{\theta}v(q(\underline{\theta})) - T(\underline{\theta}) \geq 0.$$

For incentive compatibility constraint, from the inequality (16.5.37), we can get:

$$(\theta - \hat{\theta})(v(q(\theta)) - v(q(\hat{\theta}))) \geq 0.$$

Thus, incentive compatibility alone requires that $q(\cdot)$ must be non-increasing, which implies that $q(\cdot)$ is differentiable almost everywhere. So we will restrict the analysis to differentiable functions. In this way, monotonic conditions imply: $\frac{dq(\theta)}{d\theta} \geq 0$.

Incentive compatibility conditions are equivalent to the following optimization problem:

$$\theta = \operatorname{argmax}_{\hat{\theta}} [\theta v(q(\hat{\theta})) - T(\hat{\theta})],$$

The first-order necessary condition is:

$$\theta v'(q(\theta)) \frac{dq(\theta)}{d\theta} - T'(\theta) = 0. \quad (16.5.39)$$

(16.5.39) is also called the local incentive compatibility condition.

The second-order necessary condition is:

$$\theta v''(q(\theta)) \left(\frac{dq(\theta)}{d\theta} \right)^2 + \theta v'(q(\theta)) \frac{d^2q(\theta)}{d\theta^2} - T''(\theta) \leq 0. \quad (16.5.40)$$

Differentiating (16.5.39) with respect to θ , we have

$$\theta v''(q(\theta)) \left(\frac{dq(\theta)}{d\theta} \right)^2 + v'(q(\theta)) \frac{dq(\theta)}{d\theta} + \theta v'(q(\theta)) \frac{d^2q(\theta)}{d\theta^2} - T''(\theta) = 0. \quad (16.5.41)$$

By comparing the (16.5.40) and (16.5.41), and $v'(q(\theta)) > 0$, we get that the second-order necessary condition is established as equivalent to the monotonicity condition, which is $\frac{dq(\theta)}{d\theta} \geq 0$.

We need to prove that all types must reveal their information truthfully, that is, the following constraints must be satisfied:

$$\theta v(q(\theta)) - T(\theta) \geq \theta v(q(\hat{\theta})) - T(\hat{\theta}), \quad \forall(\theta, \hat{\theta}) \in \Theta^2. \quad (16.5.42)$$

By (16.5.39), we have:

$$T(\theta) - T(\hat{\theta}) = \int_{\hat{\theta}}^{\theta} \tau v'(q(\tau)) \frac{dq(\tau)}{d\tau} d\tau = \theta v(q(\theta)) - \hat{\theta} v(q(\hat{\theta})) - \int_{\hat{\theta}}^{\theta} v(q(\tau)) d\tau, \quad (16.5.43)$$

or

$$\theta v(q(\theta)) - T(\theta) = \theta v(q(\hat{\theta})) - T(\hat{\theta}) + \int_{\hat{\theta}}^{\theta} v(q(\tau)) d\tau - (\theta - \hat{\theta})v(q(\hat{\theta})). \quad (16.5.44)$$

Since $q(\cdot)$ is non-decreasing, $\int_{\hat{\theta}}^{\theta} v(q(\tau)) d\tau - (\theta - \hat{\theta})v(q(\hat{\theta})) \geq 0$. Thus, if the local incentive constraint (16.5.39) holds, the global incentive constraint (16.5.42) also holds.

With the above settings, an infinite number of incentive constraints (16.5.42) simplifies to a differential equation and a monotonicity constraint. As such, a local analysis to the incentive is sufficient. Therefore, the condition (16.5.39) and the monotonicity condition $\frac{dq(\theta)}{d\theta} \geq 0$ characterize the true revelation mechanism.

By discussing the above incentive compatibility constraints and participation constraints, the monopolist's optimization problem can be expressed as follows:

$$\max_{\{(q(\theta), T(\theta))\}} \int_{\underline{\theta}}^{\bar{\theta}} (T(\theta) - cq(\theta)) f(\theta) d\theta \quad (16.5.45)$$

$$\text{s.t.} \quad \theta v'(q(\theta)) \frac{dq(\theta)}{d\theta} - T'(\theta) = 0, \quad \forall (\theta, \hat{\theta}) \in \Theta^2, \quad (16.5.46)$$

$$\frac{dq(\theta)}{d\theta} \geq 0, \quad \forall \theta \in \Theta, \quad (16.5.47)$$

$$\theta v(q(\underline{\theta})) - T(\underline{\theta}) \geq 0, \quad \forall \theta \in \Theta. \quad (16.5.48)$$

To solve the above optimization problem, the usual procedure is to ignore the monotonicity constraints first, and then use the standard method of incentive problem, which was first introduced by Mirrlees (1971). Define the following function:

$$U(\theta) = \theta v(q(\theta)) - T(\theta) = \max_{\hat{\theta}} \theta v(q(\hat{\theta})) - T(\hat{\theta}),$$

where $U(\theta)$ is information rent of θ type.

Using the envelope theorem, we get:

$$\frac{dU(\theta)}{d\theta} = v(q(\theta)).$$

By monotonicity, the participation constraint of lowest type is binding, meaning that his information rent is 0, that is, $U(\underline{\theta}) = 0$, which results in:

$$U(\theta) = \int_{\underline{\theta}}^{\theta} v(q(\tau)) d\tau.$$

Since $T(\theta) = \theta v(q(\theta)) - U(\theta)$, we can write the monopolist's objective function as:

$$\pi = \int_{\underline{\theta}}^{\bar{\theta}} [\theta v(q(\theta)) - \int_{\underline{\theta}}^{\theta} v(q(\theta)) d\theta - cq(\theta)] f(\theta) d\theta.$$

By an integration of parts, we have

$$\pi = \int_{\underline{\theta}}^{\bar{\theta}} \{[\theta v(q(\theta)) - cq(\theta)] f(\theta) - v(q(\theta)) [1 - F(\theta)]\} d\theta.$$

First-order conditions for $q(\theta)$ are then given by

$$\left[\theta - \frac{1 - F(\theta)}{f(\theta)} \right] v'(q(\theta)) = c.$$

Therefore, we get a basic conclusion: for the most high-efficient type $\theta = \bar{\theta}$, there is no distortion in consumption, while for other types of $\theta < \bar{\theta}$, there is downward distortion in consumption.

We further examine the characterization of the optimal contract. $h(\theta) \equiv \frac{f(\theta)}{1 - F(\theta)}$ is called the risk rate. The sufficient condition for the monotonicity constraint is $\frac{dh(\theta)}{d\theta} \geq 0$, and the risk rate is monotonically non-decreasing. Let $g(\theta) = (\theta - (h(\theta))^{-1})$, and the first-order condition (16.5.2) can be rewritten as:

$$g(\theta)v'(q(\theta)) = c.$$

Differentiating the above equation with respect to θ , we get

$$\frac{dq(\theta)}{d\theta} = -\frac{g'(\theta)v'(q(\theta))}{v''(q(\theta))g(\theta)}.$$

Thus, if $\frac{dh(\theta)}{d\theta} \geq 0$, then $g'(\theta) \geq 0$. For many continuous distributions, such as uniform distribution, normal distribution, exponential distribution, and other frequently used distributions, the monotonicity of the risk rate will be satisfied.

16.5.3 Bunching and Ironing

We now briefly discuss how to handle the optimal choice problem if the monotonicity $q(\theta)$ is not satisfied. The usual method is to iron out intervals that do not satisfy monotonicity.

Without considering the monotonicity, the previous optimal contract $q^*(\theta)$ should satisfy:

$$\left[\theta - \frac{1 - F(\theta)}{f(\theta)} \right] v'(q^*(\theta)) = c.$$

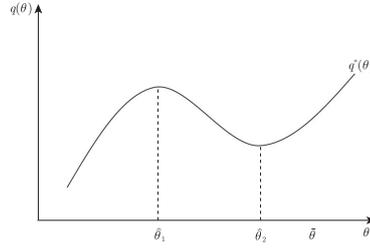


Figure 16.5: Contracts that violate monotonicity

Assume that in the interval $\theta \in [\hat{\theta}_1, \hat{\theta}_2]$, $q^*(\theta)$ does not satisfy monotonicity, as shown in Figure 16.5.

The appearance of this non-monotonic contract may be due to the fact that the risk rate $h(\theta)$ is not monotonous. At this point, the monopolist's optimal contract choice (denoted by $\bar{q}(\theta)$) is the solution to the following maximization problem with inequality constraints:

$$\begin{aligned} & \max_{\{(q(\theta), T(\theta))\}} \int_{\underline{\theta}}^{\bar{\theta}} \left[\theta v(q(\theta)) - cq(\theta) - \frac{v(q(\theta))}{h(\theta)} \right] f(\theta) d\theta \\ \text{s.t.} \quad & \frac{dq(\theta)}{d\theta} = \mu(\theta). \\ & \mu(\theta) \geq 0 \end{aligned}$$

The Hamiltonian is:

$$H(\theta, q(\theta), \mu, \lambda) = \left[\theta v(q(\theta)) - cq(\theta) - \frac{v(q(\theta))}{h(\theta)} \right] f(\theta) + \lambda(\theta)\mu(\theta).$$

Using the Pontryagin's maximum principle, the optimal solution $(\bar{q}(\theta), \bar{\mu}(\theta))$ satisfies the following conditions:

$$H(\theta, \bar{q}(\theta), \bar{\mu}(\theta), \lambda(\theta)) \geq H(\theta, q(\theta), \mu(\theta), \lambda(\theta)); \tag{16.5.49}$$

$$\frac{d\lambda(\theta)}{d\theta} = - \left[\left(\theta - \frac{1}{h(\theta)} \right) v'(\bar{q}(\theta)) - c \right] f(\theta), \tag{16.5.50}$$

$$\lambda(\underline{\theta}) = \lambda(\bar{\theta}) = 0 \quad (\text{cross-sectional conditions}) \tag{16.5.51}$$

for all continuous points.

From Kamien and Schwartz (1991, p. 133), when $H(\theta, q(\theta), \mu, \lambda)$ is a concave function with respect to $q(\theta)$, the above necessary conditions are also a sufficient conditions.

Integration by parts of (16.5.50) gets:

$$\lambda(\theta) = - \int_{\underline{\theta}}^{\theta} \left[\left(\theta - \frac{1}{h(\theta)} \right) v'(\bar{q}(\theta)) - c \right] f(\theta) d\theta.$$

The cross-sectional conditions are:

$$0 = \lambda(\underline{\theta}) = \lambda(\bar{\theta}) = - \int_{\underline{\theta}}^{\bar{\theta}} \left[\left(\theta - \frac{1}{h(\theta)} \right) v'(\bar{q}(\theta)) - c \right] f(\theta) d\theta.$$

Besides, the first-order condition for optimization requires $\mu(\theta)$ to satisfy $\mu(\theta) > 0$ when maximizing $H(\theta, q(\theta), \mu, \lambda)$, which means $\lambda(\theta) \leq 0$.

Complementary relaxation condition implies

$$\bar{\mu}(\theta)\lambda(\theta) = 0,$$

or

$$\frac{d\bar{q}(\theta)}{d\theta} \int_{\underline{\theta}}^{\theta} \left[\left(\theta - \frac{1}{h(\theta)} \right) v'(\bar{q}(\theta)) - c \right] f(\theta) d\theta = 0.$$

If $\bar{\mu}(\theta) = \frac{d\bar{q}(\theta)}{d\theta} > 0$, then $\lambda(\theta) = 0$, which implies $\frac{d\lambda(\theta)}{d\theta} = -\left[\left(\theta - \frac{1}{h(\theta)} \right) v'(\bar{q}(\theta)) - c \right] f(\theta) = 0$, and $\bar{q}(\theta) = q^*(\theta)$.

If $\lambda(\theta) < 0$, it means $\bar{\mu}(\theta) = \frac{d\bar{q}(\theta)}{d\theta} = 0$. To guarantee the continuity of $\lambda(\theta)$, find θ_1 and θ_2 on the left side of $\hat{\theta}_1$ and on the right side of $\hat{\theta}_2$, which makes $\lambda(\theta_1) = \lambda(\theta_2) = 0$.

The optimal contract $\bar{q}(\theta)$ satisfies $\frac{d\bar{q}(\theta)}{d\theta} = 0$ in the interval $[\theta_1, \theta_2]$. At the same time, in order to guarantee the continuity of $\bar{q}(\theta)$, we need to choose θ_1, θ_2 so that $q^*(\theta_1) = q^*(\theta_2)$. The new optimal contract $\bar{q}(\theta)$ is shown in Fig. 16.6.

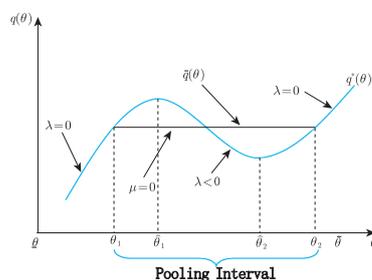


Figure 16.6: Optimal contract with bunching property

After adding the monotonicity condition, the optimal contract actually performs some kind of “iron out” on the original $q^*(\theta)$ in the interval $[\theta_1, \theta_2]$. For further discussion, see Bolton and Dewatripont (2005, pp. 91-93).

In addition, another expansion is a multi-dimension adverse selection. One of the classic problems is the monopoly pricing of multi-products. For example, in reality, many merchants often bundle products with different products. The multi-dimensional type of adverse selection provides an analytical framework for understanding similar bundled sales mechanisms. A detailed discussion of this can be found in Bolton and Dewatripont (2005, Chapter 6), as well as a review by Rochet and Stole (2003).

16.6 Ex Ante Participation Constraints

The case of contracts we consider so far is offered at the interim stage, i.e., the agent already knows his type. However, sometimes the principal and the agent can contract at the ex ante stage, i.e., before the agent discovers his type. For instance, the contracts of the firm may be designed before the agent receives any piece of information on his productivity, say a newly graduated student do not know his own productivity after taking part in the job, whether he can competent for the job.

Also, in the above example of monopoly sales, if the monopoly upstream firms provide a new production tool for the downstream firms, the downstream firms do not know the true effectiveness of this new production tool before signing a sales contract, and their interaction will have different characteristics compared with before. In this section, we characterize the optimal contract for this alternative timing under various assumptions about the risk aversion of the two players.

Unlike the previous model, before signing the contract, the buyer did not know the type of value he had for the product, the timing flow for this interaction was:

- $t = 0$: the seller provides contract;
- $t = 1$: the buyer accepts or rejects contract;
- $t = 2$: the buyer discovers his own type θ ;
- $t = 3$: contract enforcement.

To simplify the discussion, we still assume that there are only two types, $\theta \in \{\theta_H, \theta_L\}$. The agent's risk attitude will affect the contract design. For this reason, we discuss the optimal incentive compatible contract when the agent is risk neutral and risk aversion.

16.6.1 Risk Neutrality

Suppose that the principal and the agent meet and contract ex ante. If the agent is risk neutral, his ex ante participation constraint is now written as

$$(1 - \beta)U_H + \beta U_L \geq 0. \quad (16.6.52)$$

This ex ante participation constraint replaces the two interim participation constraints.

Since the principal's objective function is decreasing in the agent's expected information rent, the principal wants to impose a zero expected rent to the agent and have (16.6.52) be binding. Moreover, the principal must

structure the rents U_H and U_L to ensure that the two incentive constraints remain satisfied.

$$U_H \geq U_L + \Delta\theta v(q_L), \tag{16.6.53}$$

$$U_L \geq U_H - \Delta\theta v(q_H). \tag{16.6.54}$$

An example of such a rent distribution that is both incentive compatible and satisfies the ex ante participation constraint with an equality is:

$$\begin{aligned} U_H^* &= \beta\Delta\theta v(q_H^*) > 0; \\ U_L^* &= -(1 - \beta)\Delta\theta v(q_H^*) < 0. \end{aligned} \tag{16.6.55}$$

With such a rent distribution, the optimal contract implements the first-best outputs without cost from the principal's point of view as long as the first-best is monotonic as requested by the implementability condition. In the contract defined by (16.6.55), the agent is rewarded when he is high-efficient and punished when he turns out to be inefficient. In summary, we have the following proposition.

Proposition 16.6.1 *When the agent is risk neutral and contracting takes place ex ante, the optimal incentive contract implements the first-best outcome.*

Remark 16.6.1 The principal has in fact more options in structuring the rents U_H and U_L in such a way that the incentive compatibility constraints hold and the ex ante participation constraint (16.6.52) holds with an equality. The information rent with a risk neutral agent as shown in Figure 16.7, the rent distribution is not unique, and in fact, there are infinitely.

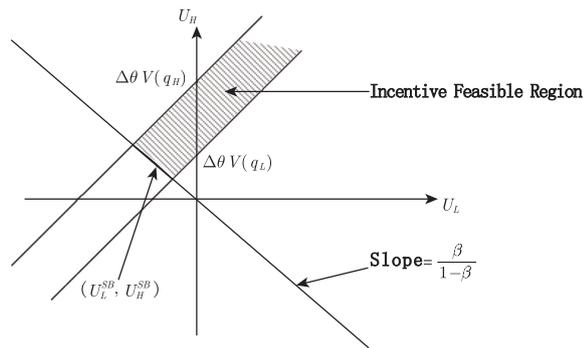


Figure 16.7: Information rent of a risk neutral agent

Consider the following contracts $\{(T_H^*, q_H^*); (T_L^*, q_L^*)\}$, where q_H^* and q_L^* are optimal production, $T_H^* = cq_H^* + T^*$, $T_L^* = cq_L^* + T^*$, with T^* being a

lump-sum payment to be defined below. By definition of q_H^* , we have

$$\begin{aligned}\theta_H v(q_H^*) - T_H^* &= \theta_H v(q_H^*) - cq_H^* - T^* > \theta_H v(q_L^*) - cq_L^* - T^* \\ &= \theta_H v(q_L^*) - T_L^*.\end{aligned}\quad (16.6.56)$$

By definition of q_L^* , we have:

$$\begin{aligned}\theta_L v(q_L^*) - T_L^* &= \theta_L v(q_L^*) - cq_L^* - T^* > \theta_L v(q_H^*) - cq_H^* - T^* \\ &= \theta_L v(q_H^*) - T_H^*.\end{aligned}\quad (16.6.57)$$

Therefore, the contract is incentive-compatible.

Note that the incentive compatibility constraints are now strict inequalities. Moreover, the fixed-fee T^* can be used to satisfy the agent's ex ante participation constraint with an equality by choosing

$$T^* = \beta(\theta_L v(q_L^*) - cq_L^*) + (1 - \beta)(\theta_H v(q_H^*) - cq_H^*).$$

This implementation of the first-best outcome amounts to having the principal selling the benefit of the relationship to the risk-neutral agent for a fixed up-front payment T^* . The agent benefits from the full value of the good and trades off the value of any production against its cost just as if he was an efficiency maximizer. We will say that the agent is residual claimant for the firm's profit. From the latter discussion, it is known that even a principal who is risk averse will get the same result.

In reality, many contracts are of such nature.

- (1) After reform and opening in China, the contract was implemented for contracting land and households, and the production team contracted the farmland to the farmers. The farmers needed to pay a certain amount of products to the government each year, and the rest was returned to the farmers. The subsequent production responsibility system is the same.
- (2) The homeowner rents out the storefront to do business, the renter pays a certain rent, and the remaining profits or losses are borne by the renter himself.
- (3) Banks lend funds to enterprises at a fixed loan interest rate, and companies bear business risks. Profits or losses are all their own.

16.6.2 Risk Aversion

The previous section has shown us that the implementation of the first-best is feasible with risk neutrality. What happens if the agent is risk-averse?

Consider now a risk-averse agent with a Von Neumann-Morgenstern

utility function $u(\cdot)$ defined on his monetary gains $v(q) - T$, such that $u' > 0$, $u'' < 0$ and $u(0) = 0$. Again, the contract between the principal and the agent is signed before the agent discovers his type. The incentive-compatibility constraints are unchanged but the agent's ex ante participation constraint is now written as:

$$\beta u(U_L) + (1 - \beta)u(U_H) \geq 0. \quad (16.6.58)$$

As usual, one can check incentive-compatibility constraint for the low-efficient agent is slack (not binding) at the optimum, and thus the principal's program reduces now to

$$\max_{\{(U_H, q_H); (U_L, q_L)\}} (1 - \beta)(\theta_H v(q_H) - cq_H - U_H) + \beta(\theta_L v(q_L) - cq_L - U_L)$$

subject to (16.6.53), (16.6.54) and (16.6.58).

High-efficient agent's incentive-compatibility constraint (16.6.53) and ex ante participation constraint (16.6.58) are binding, we have

$$\beta u(U_L) + (1 - \beta)u(U_L + \Delta\theta v(q_L)) = 0. \quad (16.6.59)$$

By (16.6.59), $U_L = U_L(q_L)$. Substituting (16.6.53) and (16.6.59) into the above objective function, the following proposition can be obtained by solving the optimization problem.

Proposition 16.6.2 *When the agent is risk-averse and contracting takes place ex ante, the optimal menu of contracts entails:*

(1) *No output distortion for the high-efficient $q_H^{SB} = q_H^*$. A downward output distortion for the low-efficient type $q_L^{SB} < q_L^*$, with*

$$\theta_L v'(q_L^{SB}) = \frac{c}{1 - \frac{(1 - \beta)[u'(U_L) - u'(U_H)] \Delta\theta}{\beta u'(U_L) + (1 - \beta)u'(U_H) \theta_L}}. \quad (16.6.60)$$

(2) *Both (16.6.53) and (16.6.58) are the only binding constraints. The high-efficient (resp. low-efficient) type gets a strictly positive (resp. negative) ex post information rent, $U_H^{SB} > 0 > U_L^{SB}$.*

The rent distribution is shown in Figure 16.8. Note that when the agent is risk neutral, the second term in the right of (16.6.60) is zero, and thus we get the same conclusion as in Proposition 16.6.1: the optimal incentive contract implements the first-best outcome.

Thus, with risk aversion, the principal can no longer costlessly structure the agent's information rents to ensure the high-efficient type's incentive compatibility constraint. Creating a wedge between U_H and U_L to satisfy (16.6.53) makes the risk averse agent bear some risk. To guarantee the participation of the risk-averse agent, the principal must now pay a risk

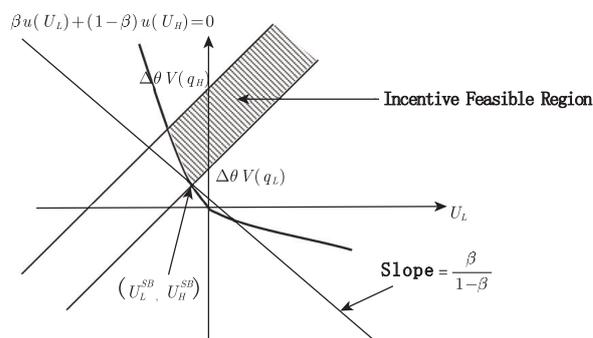


Figure 16.8: Information rent of risk averse agent

premium. Reducing this premium calls for a downward reduction in the low-efficient type's output so that the risk borne by the agent is lower. As expected, the agent's risk aversion leads the principal to weaken the incentives.

When the agent becomes infinitely risk averse, everything happens as if he had an ex post individual rationality constraint for the worst state of the world such that $U_L^{SB} = 0$. In the limit, the low-efficient agent's outputs q_H^{SB}, q_L^{SB} and the utility levels U_H^{SB}, U_L^{SB} all converge toward the same solution. So, the previous model at the interim stage can also be interpreted as a model with an ex ante infinitely risk-agent at the zero utility level.

16.6.3 Risk-Averse Principal

Consider now a risk-averse principal with a Von Neumann-Morgenstern utility function $v(\cdot)$ defined on his monetary gains from trade $T - cq$ such that $v' > 0$, $v'' < 0$ and $v(0) = 0$. Again, the contract between the principal and the risk-neutral agent is signed before the agent knows his type.

In this context, the first-best contract obviously calls for the first-best output q_H^* and q_L^* being produced. It also calls for the principal to be fully insured between both states of nature and for the agent's ex ante participation constraint to be binding. This leads us to the following two conditions that must be satisfied by the agent's rents U_H^* and U_L^* :

$$\theta_H v(q_H) - U_H^* - cq_H = \theta_L v(q_L) - U_L^* - cq_L \quad (16.6.61)$$

and

$$\beta U_L^* + (1 - \beta)U_H^* = 0. \quad (16.6.62)$$

Solving this system of two equations with two unknowns (U_H^*, U_L^*) yields

$$U_H^* = \beta([\theta_H v(q_H^*) - cq_H^*] - [\theta_L v(q_L^*) - cq_L^*]), \quad (16.6.63)$$

$$U_L^* = -(1 - \beta)([\theta_H v(q_H^*) - cq_H^*] - [\theta_L v(q_L^*) - cq_L^*]). \quad (16.6.64)$$

By the definition of q_H^*

$$U_H^* - U_L^* = [\theta_H v(q_H^*) - cq_H^*] - [\theta_L v(q_L^*) - cq_L^*] > \Delta\theta v(q_L^*).$$

By the definition of q_L^*

$$U_L^* - U_H^* = [\theta_L v(q_L^*) - cq_L^*] - [\theta_H v(q_H^*) - cq_H^*] > -\Delta\theta v(q_H^*).$$

Hence, the profile of rents U_H^*, U_L^* is incentive compatible and the first-best allocation is easily implemented in this framework. We can thus generalize the proposition for the case of risk neutral as follows:

Proposition 16.6.3 *Suppose the principal is risk-averse over the monetary gains $T - cq$, the agent is risk-neutral, and contracting takes place ex ante. Then the optimal incentive contract implements the first-best outcome.*

Remark 16.6.2 It is interesting to note that U_H^* and U_L^* obtained in (16.6.63) and (16.6.64) are also the levels of rent obtained in (16.6.56) and (16.6.57). Indeed, the lump-sum payment $T^* = \beta(\theta_L v(q_L^*) - cq_L^*) + (1 - \beta)(\theta_H v(q_H^*) - cq_H^*)$, which allows the principal to make the risk-neutral agent residual claimant for the hierarchy's profit, also provides full insurance to the principal. By making the risk-neutral agent the residual claimant for the value of trade, ex ante contracting allows the risk-averse principal to get full insurance and implement the first-best outcome despite the informational problem.

Of course this result does not hold anymore if the agent's interim participation constraints must be satisfied. In this case, the incentive compatibility constraint (16.6.54) is slack at the optimum, and then the principal's program reduces to:

$$\max(1 - \beta)\nu(\theta_H v(q_H) - cq_H - U_H) + \beta\nu(\theta_L v(q_L) - cq_L - U_L), \quad (16.6.65)$$

subject to high-efficient type's incentive constraints (16.6.53), as well as low-efficient type's participation constraints:

$$U_L \geq 0. \quad (16.6.66)$$

By substituting U_H and U_L obtained from (16.6.53) and (16.6.66) into the principal's objective function and maximizing the objective function, we have $q_H^{SB} = q_H^*$, which is the same with the result in risk-neutral buyer situation. High-efficient type buyer has no consumption distortion. However, the buyer of low utility type has a downward output distortion $q_L^{SB} < q_L^*$, where q_L^{SB} satisfies

$$\theta_L v'(q_L^{SB}) = \frac{c}{1 - \frac{(1 - \beta)\nu'(V_H)\Delta\theta}{\beta\nu'(V_L)\theta_L}}, \quad (16.6.67)$$

where $V_H^{SB} = \theta_H v(q_H^{SB}) - cq_H^{SB} - \Delta\theta v(q_L^{SB})$ and $V_L^{SB} = \theta_L v(q_L^{SB}) - cq_L^{SB}$ are the return of the principal in two natural states. By the definition of q_H^* , we have

$$\theta_H v(q_L^{SB}) - cq_L^{SB} < \theta_H v(q_H^{SB}) - cq_H^{SB} = \theta_H v(q_H^*) - cq_H^*,$$

Thus, it can be verified that $V_L^{SB} < V_H^{SB}$. In particular, it is easy to see that the distortion on the right side of equation (16.6.67) is always less than $\frac{c}{1 - \frac{(1-\beta)\Delta\theta}{\beta\theta_L}}$ in the case of a risk neutral principal. The economic implications are obvious, by increasing q_L^{SB} , the gap between V_H^{SB} and V_L^{SB} can be reduced, which will provide the principal with a certain amount of insurance and increase his prior earnings.

16.7 Extension of Classical Model

As shown in the preceding sections, the canonical principal-agent model makes several simplifying assumptions. First, there is no externality among agents, that is, an agent's behavior is assumed to have no impact on the welfare of others; second, an agent is assumed to obtain a minimal type-independent level of utility if he rejects the offer made by the principal. Under these assumptions, the optimal contract exhibits no-distortion for the "best" type agent and downward distortions for all other types.

However, as Meng and Tian (2009) showed, challenges to this "no distortion at the top" convention may arise if we relax either of these two assumptions. To introduce these results, in this section we will discuss these two "challenges" in an integrated nonlinear pricing model.

16.7.1 Network Externalities

An externality is present whenever the well-being of a consumer or the production possibilities of a firm are directly affected by the actions of others in the economy.

A special case of externality is the so-called the "network externalities" that might arise for any of the following reasons: because the usefulness of the commodity depends directly on the size of the network (e.g., telephones, emails, WeChat etc.); or because of bandwagon effect, which means the desire to be in style: to have a good because almost everyone else has it; or indirectly through the availability of complementary goods and services (often known as the "hardware-software paradigm") or of postpurchase services (e.g., for automobiles). Although "network externalities" are often regarded as positive impact on each others' consumption, it may display negative properties in some cases. For example, people has the desire to own exclusive or unique goods, which is called "Snob effect". The quantity demanded of a "snob" good is higher the fewer the people who own

it. (e.g., garment specially designed by a fashion designer, special design of watches, cars, or buildings.)

Consider a principal-agent model in which the principal is a monopolist of a network good with marginal production cost c and total output q who faces a continuum of consumers. The principal's payoff function is given by $V = t - cq$, where t is the payment received from consumer. Consumer's preference of the good is characterized by $\theta \in \Theta = \{\underline{\theta}, \bar{\theta}\}$, $\Pr(\theta = \underline{\theta}) = v$, $\Pr(\theta = \bar{\theta}) = 1 - v$, and then v can be regarded as the frequency of type $\underline{\theta}$ by the Law of Large Numbers.

A consumer of θ type is assumed to have a utility function of $U(\theta) = \theta V(q(\theta)) + \Psi(Q) - t(\theta)$, where $q(\theta)$ is the amount of the good he consumes, $Q = vq + (1 - v)\bar{q} = E[q(\theta)]$ is the total amount of consumption (network size) and $t(\theta)$ is the tariff charged by the principal. $\theta V(q(\theta))$ is the *intrinsic value* of consuming, while $\Psi(Q)$ is the *network value*. Note that we assume the network effect is homogeneous among all the consumers, namely, the network value is independent of individual preference θ and individual consumption $q(\theta)$. It is assumed that $V'(q) > 0$ and $V''(q) < 0$.

Definition 16.7.1 The network is *decreasing*, *constant*, or *increasing* if and only if $\Psi''(Q) < 0$, $\Psi''(Q) = 0$, or $\Psi''(Q) > 0$, respectively.

Remark 16.7.1 When the network capacity is large and the maintaining technology is advanced enough, an increase in one consumer's consumption will increase the marginal utilities of others, so $\Psi''(Q) > 0$. When network capacity and maintaining technology are limited, consumers are rivals to one another in the sense that an increase in one consumer's consumption will decrease the marginal utilities of others, thus $\Psi''(Q) < 0$. When network expansion benefits all the consumers with constant margin, the network value term is a linear function: $\Psi''(Q) = 0$.

The objective of the monopolist is to design a menu of incentive-compatible and self-selecting quantity-price pairs $\{q(\hat{\theta}), t(\hat{\theta})\}$ to maximize his own revenue, where $\hat{\theta} \in \Theta$ is the consumer's announcement. The network magnitude is $Q = vq + (1 - v)\bar{q}$. Under complete information, the monopolist's problem is:

$$(P2) \left\{ \begin{array}{l} \max_{\{(\underline{U}, \underline{q}); (\bar{U}, \bar{q})\}} v [\underline{\theta}V(\underline{q}) - c\underline{q}] + (1 - v) [\bar{\theta}V(\bar{q}) - c\bar{q}] + \Psi(Q) - [v\underline{U} + (1 - v)\bar{U}] \\ s.t. IR(\underline{\theta}) : \underline{U} \geq 0 \\ IR(\bar{\theta}) : \bar{U} \geq 0 \end{array} \right.$$

The first-best consumption is thus:

$$\left\{ \begin{array}{l} \underline{\theta}V'(\underline{q}^{FB}) + \Psi'(v\underline{q}^{FB} + (1 - v)\bar{q}^{FB}) = c, \\ \bar{\theta}V'(\bar{q}^{FB}) + \Psi'(v\underline{q}^{FB} + (1 - v)\bar{q}^{FB}) = c. \end{array} \right. \quad (16.7.68)$$

Under asymmetric information, two incentive compatibility constraints should be added to the above program, then we get:

$$(P3) \left\{ \begin{array}{l} \max_{\{(U, \underline{q}); (\bar{U}, \bar{q})\}} v [\underline{\theta} V(\underline{q}) - c\underline{q}] + (1-v) [\bar{\theta} V(\bar{q}) - c\bar{q}] + \Psi(Q) - [v\underline{U} + (1-v)\bar{U}] \\ s.t. IR(\underline{\theta}) : \underline{U} \geq 0 \\ IR(\bar{\theta}) : \bar{U} \geq 0 \\ IC(\underline{\theta}) : \underline{U} \geq \bar{U} - \Delta\theta V(\bar{q}) \\ IC(\bar{\theta}) : \bar{U} \geq \underline{U} + \Delta\theta V(\underline{q}) \end{array} \right.$$

The incentive compatibility constraint $IC(\bar{\theta})$ of high demand type and the participation constraint of the low-demand type $IR(\underline{\theta})$ are binding, then the consumptions in the second-best contract are characterized by the following first-order conditions:

$$\left\{ \begin{array}{l} \left(\underline{\theta} - \frac{1-v}{v} \Delta\theta \right) V'(\underline{q}^{SB}) + \Psi' (v\underline{q}^{SB} + (1-v)\bar{q}^{SB}) = c, \\ \bar{\theta} V'(\bar{q}^{SB}) + \Psi' (v\underline{q}^{SB} + (1-v)\bar{q}^{SB}) = c. \end{array} \right. \quad (16.7.69)$$

We synthesize the first-best and second-best solution by considering them as solution to the following parameterized form:

$$\max_{\{\underline{q}, \bar{q}\}} \Pi(\underline{q}, \bar{q}, \alpha) \quad (16.7.70)$$

where

$$\Pi(\underline{q}, \bar{q}, \alpha) = v[\alpha V(\underline{q}) - c\underline{q}] + (1-v)[\bar{\theta} V(\bar{q}) - c\bar{q}] + \Psi(Q).$$

Note that we have the first-best contract under complete information given in (16.7.68) when $\alpha = \underline{\theta}$, and the second-best contract under asymmetric information given in (16.7.69) when $\alpha = \underline{\theta} - \frac{1-v}{v} \Delta\theta$.

We then have the following proposition.

Proposition 16.7.1 *In the presence of network externalities and asymmetric information, the direction of distortion in consumptions depends on the sign of $\Psi''(Q)$.*

1. *If the network is mildly increasing, i.e., $\Psi''(Q) > 0$ but is not too large such that Π_{qq} is negative definite for all $\alpha \in [\underline{\theta} - \frac{1-v}{v} \Delta\theta, \underline{\theta}]$, then the consumptions of all types exhibit one-way distortion : $\underline{q}^{SB} < \underline{q}^{FB}$ and $\bar{q}^{SB} < \bar{q}^{FB}$.*
2. *If the network is decreasing, i.e., $\Psi''(Q) < 0$, then the consumptions exhibit two-way distortion: $\underline{q}^{SB} < \underline{q}^{FB}$ and $\bar{q}^{SB} > \bar{q}^{FB}$.*
3. *If the network is constant, i.e., $\Psi''(Q) = 0$, then the rules "no distortion on the top" and "one-way distortion" in canonical settings are still available: $\underline{q}^{SB} < \underline{q}^{FB}$, $\bar{q}^{SB} = \bar{q}^{FB}$.*

For all these cases, the network magnitude is downsized: $Q^{SB} < Q^{FB}$.

PROOF. The first order condition to (16.7.70) is:

$$\Pi_q(q, \alpha) = 0, \quad (16.7.71)$$

this implies,

$$\begin{cases} \alpha V'(q) + \Psi'(vq + (1-v)\bar{q}) = c, \\ \bar{\theta} V'(\bar{q}) + \Psi'(vq + (1-v)\bar{q}) = c. \end{cases} \quad (16.7.72)$$

Differentiating (16.7.71) with respect to parameter α , we get:

$$\Pi_{qq} \frac{dq}{d\alpha} + \Pi_{q\alpha} = 0 \quad (16.7.73)$$

that is,

$$\begin{pmatrix} \alpha v V''(q) + v^2 \Psi''(Q) & v(1-v) \Psi''(Q) \\ v(1-v) \Psi''(Q) & (1-v) \bar{\theta} V''(\bar{q}) + (1-v)^2 \Psi''(Q) \end{pmatrix} \begin{pmatrix} \frac{dq}{d\alpha} \\ \frac{d\bar{q}}{d\alpha} \end{pmatrix} + \begin{pmatrix} v V'(q) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving the above equations, we have

$$\begin{cases} \frac{dq}{d\alpha} = \frac{-V'(q) [\bar{\theta} V''(\bar{q}) + (1-v) \Psi''(Q)]}{\alpha V''(q) [\bar{\theta} V''(\bar{q}) + (1-v) \Psi''(Q)] + v \bar{\theta} V''(\bar{q}) \Psi''(Q)} \\ \frac{d\bar{q}}{d\alpha} = \frac{v V'(q) \Psi''(Q)}{\alpha V''(q) [\bar{\theta} V''(\bar{q}) + (1-v) \Psi''(Q)] + v \bar{\theta} V''(\bar{q}) \Psi''(Q)} \\ \frac{dQ}{d\alpha} = v \frac{dq}{d\alpha} + (1-v) \frac{d\bar{q}}{d\alpha} = \frac{-v \bar{\theta} V'(q) V''(\bar{q})}{\alpha V''(q) [\bar{\theta} V''(\bar{q}) + (1-v) \Psi''(Q)] + v \bar{\theta} V''(\bar{q}) \Psi''(Q)} \end{cases} \quad (16.7.74)$$

From the assumption that the Hessian matrix Π_{qq} is negative definite, it can be verified that the 2th diagonal element of Π_{qq} is negative, and thus

$$\bar{\theta} V''(\bar{q}) + (1-v) \Psi''(Q) < 0, \quad (16.7.75)$$

and the determinant of Π_{qq} is positive,

$$\det(\Pi_{qq}) = v(1-v) \left\{ \alpha V''(q) [\bar{\theta} V''(\bar{q}) + (1-v) \Psi''(Q)] + v \bar{\theta} V''(\bar{q}) \Psi''(Q) \right\} > 0. \quad (16.7.76)$$

The signs of derivatives in (16.7.74) can be determined, which are $\frac{dq}{d\alpha} > 0$ and $\frac{dQ}{d\alpha} > 0$, which means $q^{SB} < q^{FB}$ and $Q^{SB} < Q^{FB}$. The sign of $\frac{d\bar{q}}{d\alpha}$ and thus the distortion direction of \bar{q} depends on the sign of $\Psi''(Q)$: if $\Psi''(Q) > 0$, $\frac{d\bar{q}}{d\alpha} > 0$, then $\bar{q}^{SB} < \bar{q}^{FB}$; if $\Psi''(Q) < 0$, $\frac{d\bar{q}}{d\alpha} < 0$, then $\bar{q}^{SB} > \bar{q}^{FB}$; if $\Psi''(Q) = 0$, $\frac{d\bar{q}}{d\alpha} = 0$, then $\bar{q}^{SB} = \bar{q}^{FB}$. \square

Remark 16.7.2 $\Psi''(\cdot) > 0$ implies that the marginal value from an increase in individual consumption is higher at a higher level of others' consumption: $\frac{\partial^2 U_i}{\partial q_i \partial q_j} > 0$ for $i \neq j$. Its interpretation is that the externalities in bigger network is larger than in small network so that an agent is more eager to consume more when other agent consume more. It is consistent with the critical "strategic complementarity" assumption in Segal (1999, 2003) and Csorba (2008). This condition allows them to characterize the optimal contracts by applying monotone comparative static tools, pioneered by Topkis (1978) and Milgrom and Shannon (1994).¹

16.7.2 Countervailing Incentives

In this section we discuss another cause of the failure of "no distortion on the top" and "one-way distortion" rules: the countervailing incentive problem. We assume that the consumers can bypass the network offered by the incumbent firm and enter a competitive market including many homogenous firms. All these firms are the potential entrants of the market. Let ω denote the marginal production cost of the entrants. We assume that the goods or services offered by the entrants are incompatible with that of the incumbent monopolist,² and they have not yet formed their own consumers network. In the competitive outside market, each firm's unit charge equals its marginal cost ω , so the representative consumer's utility derived from consuming the entrants' goods is $\theta V(q) - \omega q$. Let $G(\theta) = \max_q [\theta V(q) - \omega q]$, $\underline{G} = G(\underline{\theta})$, $\overline{G} = G(\overline{\theta})$, and $\Delta G = \overline{G} - \underline{G}$. Throughout this section, we assume that the network is congestible.

The entry threat gives the consumers non-zero type-dependent reservation utilities, and thus the problem of the incumbent network supplier can be represented as

$$(P5) \left\{ \begin{array}{l} \max_{\{(\underline{U}, \underline{q}); (\overline{U}, \overline{q})\}} v [\underline{\theta} V(\underline{q}) - c\underline{q}] + (1-v) [\overline{\theta} V(\overline{q}) - c\overline{q}] + \Psi(Q) - [v\underline{U} + (1-v)\overline{U}] \\ s.t. \quad IR(\underline{\theta}) : \underline{U} \geq \underline{G} \\ \quad \quad IR(\overline{\theta}) : \overline{U} \geq \overline{G} \\ \quad \quad IC(\underline{\theta}) : \underline{U} \geq \overline{U} - \Delta\theta V(\overline{q}) \\ \quad \quad IC(\overline{\theta}) : \overline{U} \geq \underline{U} + \Delta\theta V(\underline{q}). \end{array} \right.$$

Note that (P5) is the same as (P3) except for the non-zero type-dependent

¹By Topkis (1978) and Milgrom and Shannon (1994), a twice continuously differentiable function $\Pi = \Pi(q_1, q_2, \dots, q_n; \epsilon)$ defined on a lattice Q is supermodular if and only if for all $i \neq j$, $\frac{\partial^2 \Pi}{\partial q_i \partial q_j} > 0$; furthermore, if $\frac{\partial^2 \Pi}{\partial q_i \partial \epsilon} > 0, \forall i$, then function Π has strictly increasing differences in (q, ϵ) . Let $q(\epsilon) = \max_{q \in Q} \Pi(q, \epsilon)$. Then for a supermodular function with increasing differences in (q, ϵ) , $q_i(\epsilon)$ is a strictly increasing function of ϵ for all i .

²Otherwise, the entrants can share the present network with the incumbent monopolist.

reservation utilities \underline{G} and \overline{G} . The following proposition characterizes the optimal entry-deterring contract.

Proposition 16.7.2 *The optimal entry-deterring contract depends on the marginal cost of potential entrant. Specifically, there exist positive values $\omega_1 < \omega_2 < \omega_3 < \omega_4$ such that:*

1. If $\omega > \omega_4$, then $\Delta G < \Delta\theta V(q^{SB})$, and consequently the pricing contract is: $\underline{q} = \underline{q}^{SB}$, $\overline{q} = \overline{q}^{SB}$, $\underline{U} = \underline{G}$, and $\overline{U} = \underline{G} + \Delta\theta V(\underline{q}^{SB})$.
2. If $\omega_3 \leq \omega \leq \omega_4$, then $\Delta\theta V(\underline{q}^{SB}) \leq \Delta G \leq \Delta\theta V(\underline{q}^{FB})$, and consequently the optimal consumption level \underline{q} and \overline{q} are determined by

$$\begin{cases} \underline{q} = V^{-1}\left(\frac{\Delta G}{\Delta\theta}\right) \\ \overline{\theta}V'(\overline{q}) + \Psi'(v\underline{q} + (1-v)\overline{q}) = c, \end{cases} \quad (16.7.77)$$

where $\underline{q} \in [\underline{q}^{SB}, \underline{q}^{FB}]$ and $\overline{q} \in [\overline{q}^{FB}, \overline{q}^{SB}]$. The consumers' information rents are $\underline{U} = \underline{G}$ and $\overline{U} = \overline{G}$.

3. If $\omega_2 < \omega < \omega_3$, then $\Delta\theta V(\underline{q}^{FB}) < \Delta G < \Delta\theta V(\overline{q}^{FB})$, and consequently the pricing contract is $\underline{q} = \underline{q}^{FB}$, $\overline{q} = \overline{q}^{FB}$, $\underline{U} = \underline{G}$, and $\overline{U} = \overline{G}$.
4. If $\omega_1 \leq \omega \leq \omega_2$, then $\Delta\theta V(\overline{q}^{FB}) \leq \Delta G \leq \Delta\theta V(\overline{q}^{CI})$, and consequently the optimal consumption level \underline{q} and \overline{q} are given by

$$\begin{cases} \overline{q} = V^{-1}\left(\frac{\Delta G}{\Delta\theta}\right) \\ \underline{\theta}V'(\underline{q}) + \Psi'(v\underline{q} + (1-v)\overline{q}) = c, \end{cases} \quad (16.7.78)$$

where $\underline{q} \in [\underline{q}^{CI}, \underline{q}^{FB}]$ and $\overline{q} \in [\overline{q}^{FB}, \overline{q}^{CI}]$, "CI" denotes "countervailing incentives". The consumers' information rents are $\underline{U} = \underline{G}$ and $\overline{U} = \overline{G}$.

5. If $0 < \omega < \omega_1$, then $\Delta G > \Delta\theta V(\overline{q}^{CI})$, and consequently the optimal contract is $\underline{q} = \underline{q}^{CI}$, $\overline{q} = \overline{q}^{CI}$, $\underline{U} = \overline{G} - \Delta\theta V(\overline{q}^{CI})$, $\overline{U} = \overline{G}$. \underline{q}^{CI} and \overline{q}^{CI} are given by:

$$\begin{cases} \underline{\theta}V'(\underline{q}^{CI}) + \Psi'(v\underline{q}^{CI} + (1-v)\overline{q}^{CI}) = c \\ \left(\overline{\theta} + \frac{v}{1-v}\Delta\theta\right)V'(\overline{q}^{CI}) + \Psi'(v\underline{q}^{CI} + (1-v)\overline{q}^{CI}) = c. \end{cases} \quad (16.7.79)$$

PROOF. In (P5), we have as many regimes as combinations of binding constraints among $IR(\underline{\theta})$, $IR(\overline{\theta})$, $IC(\underline{\theta})$ and $IC(\overline{\theta})$. To reduce the number of possible cases, we first give the following lemmas.

Lemma 16.7.1 *A pooling contract with $\underline{q} = \bar{q}$ and $\underline{t} = \bar{t}$ can never be optimal.*

PROOF. Suppose that the optimal contract is pooling with $\underline{q} = \bar{q} = q$ and $\underline{t} = \bar{t} = t$. There are two cases to be considered.

(i) $\bar{\theta}V'(q) > c$. Then, increase \bar{q} by ε and the transfer by $\bar{\theta}V'(q)\varepsilon$, the $\bar{\theta}$ -type can remain indifferent. Since at (q, t) the marginal rate of substitution between q and t is higher for $\bar{\theta}$ -type, this new allocation is incentive compatible. This raises the firm's revenue by $(1 - v)[\bar{\theta}V'(q) - c]\varepsilon$.

(ii) $\bar{\theta}V'(q) \leq c$ and $\underline{\theta}V'(q) < c$. Then, decrease \underline{q} by ε and adjust \underline{t} so that $\underline{\theta}$ -type can remain on the same indifference curve. Then the firm's total charge will be increased by $[c - \underline{\theta}V'(q)]\varepsilon$.

Thus, in both cases, it contradicts with the fact (q, t) is optimal contract.

□

Lemma 16.7.2 *If the two types are offered two different contracts, the two incentive constraints cannot be simultaneously bindings.*

PROOF. Suppose by way of contradiction that both ICs are binding. From $\underline{\theta}V(\underline{q}) - \underline{t} + \Psi(Q) = \underline{\theta}V(\bar{q}) - \bar{t} + \Psi(Q)$ and $\bar{\theta}V(\bar{q}) - \bar{t} + \Psi(Q) = \bar{\theta}V(\underline{q}) - \underline{t} + \Psi(Q)$, we have $\underline{q} = \bar{q}$ and $\underline{t} = \bar{t}$. But this is impossible by Lemma 1. □

Lemma 16.7.3 *The IC and IR constraints of the same type cannot be simultaneously slack.*

PROOF. If $IR(\theta)$ and $IC(\theta)$ are both slack, increase $t(\theta)$ by a tiny increment will not violate all the constraints, but the firm's charge will be increased.

□

Applying the above three lemmas, only five possible regimes are needed to be considered, which are summarized in the following table.

Table1 – The five possible regimes

Constraints	Regime1	Regime2	Regime3	Regime4	Regime5
$IR(\underline{\theta})$	B	B	B	B	S
$IR(\bar{\theta})$	S	B	B	B	B
$IC(\underline{\theta})$	S	S	S	B	B
$IC(\bar{\theta})$	B	B	S	S	S

Here "B" denotes "binding" and "S" denotes "slack".

The regimes are ordered from 1 to 5 when ΔG increases, and ΔG itself is determined by the entrants' marginal cost ω . To find how ω affects the nonlinear pricing contract of the incumbent firm, we give the following two lemmas. Lemma 16.7.4 states the the change of ΔG in different regimes, and Lemma 16.7.5 shows how ΔG is affected by ω .

Lemma 16.7.4 *The optimal pricing contracts and the utility difference ΔG in different regimes are:*

1. In regime 1, the optimal solution to (P5) is $\underline{q} = \underline{q}^{SB}$, $\bar{q} = \bar{q}^{SB}$, $\underline{U} = \underline{G}$, and $\bar{U} = \underline{G} + \Delta\theta V(\underline{q}^{SB})$. The value of utility difference satisfies $\Delta G < \Delta\theta V(\underline{q}^{SB})$.

2. In regime 2, the optimal consumption level \underline{q} and \bar{q} are determined by

$$\begin{cases} \underline{q} = V^{-1}\left(\frac{\Delta G}{\Delta\theta}\right) \\ \bar{\theta}V'(\bar{q}) + \Psi'(v\underline{q} + (1-v)\bar{q}) = c, \end{cases} \quad (16.7.80)$$

where $\underline{q} \in [\underline{q}^{SB}, \underline{q}^{FB}]$ and $\bar{q} \in [\bar{q}^{FB}, \bar{q}^{SB}]$. The consumers' information rents are $\underline{U} = \underline{G}$ and $\bar{U} = \bar{G}$. The utility difference satisfies $\Delta\theta V(\underline{q}^{SB}) \leq \Delta G \leq \Delta\theta V(\underline{q}^{FB})$.

3. In regime 3, the optimal solution to (P5) is $\underline{q} = \underline{q}^{FB}$, $\bar{q} = \bar{q}^{FB}$, $\underline{U} = \underline{G}$, $\bar{U} = \bar{G}$, and $\Delta\theta V(\underline{q}^{FB}) < \Delta G < \theta V(\bar{q}^{FB})$.

4. In regime 4, the optimal consumption level \underline{q} and \bar{q} are determined by

$$\begin{cases} \bar{q} = V^{-1}\left(\frac{\Delta G}{\Delta\theta}\right) \\ \theta V'(\underline{q}) + \Psi'(v\underline{q} + (1-v)\bar{q}) = c, \end{cases} \quad (16.7.81)$$

where $\underline{q} \in [\underline{q}^{CI}, \underline{q}^{FB}]$ and $\bar{q} \in [\bar{q}^{FB}, \bar{q}^{CI}]$. The consumers' information rents are $\underline{U} = \underline{G}$ and $\bar{U} = \bar{G}$. The utility difference satisfies $\Delta\theta V(\bar{q}^{FB}) \leq \Delta G \leq \Delta\theta V(\bar{q}^{CI})$.

5. In regime 5, the optimal contract is $\underline{q} = \underline{q}^{CI}$, $\bar{q} = \bar{q}^{CI}$, $\underline{U} = \bar{G} - \Delta\theta V(\bar{q}^{CI})$, and $\bar{U} = \bar{G}$. The utility difference $\Delta G > \theta V(\bar{q}^{CI})$. \underline{q}^{CI} and \bar{q}^{CI} are determined by:

$$\begin{cases} \theta V'(\underline{q}^{CI}) + \Psi'(v\underline{q}^{CI} + (1-v)\bar{q}^{CI}) = c \\ \left(\bar{\theta} + \frac{v}{1-v}\Delta\theta\right)V'(\bar{q}^{CI}) + \Psi'(v\underline{q}^{CI} + (1-v)\bar{q}^{CI}) = c. \end{cases} \quad (16.7.82)$$

PROOF.

- In regime 1, the constraints $IR(\theta)$ and $IC(\bar{\theta})$ are binding. Solving (P5) in the same way as (P3), we get the second-best solution

$$\left\{ \underline{q} = \underline{q}^{SB}, \bar{q} = \bar{q}^{SB}; \underline{U} = \underline{G}, \bar{U} = \underline{G} + \Delta\theta V(\underline{q}^{SB}) \right\},$$

with $\Delta G < \Delta U = \Delta\theta V(\underline{q}^{SB})$.

- In regime 2, the constraints $IR(\underline{\theta})$, $IR(\bar{\theta})$ and $IC(\bar{\theta})$ are binding. The optimal contract set is thus given by

$$\left\{ (\underline{q}, \bar{q}, \underline{U}, \bar{U}) : \Delta\theta V(\underline{q}) = \Delta G, \bar{\theta}V(\bar{q}) + \Psi'(Q) = c; \underline{U} = \underline{G}, \bar{U} = \bar{G} \right\}.$$

Substituting $IR(\underline{\theta})$ and $IR(\bar{\theta})$ into the objective function, the Lagrange function of is constructed as

$$\begin{aligned} L(\underline{q}, \bar{q}) &= v[\underline{\theta}V(\underline{q}) - c\underline{q}] + (1-v)[\bar{\theta}V(\bar{q}) - c\bar{q}] \\ &\quad + \Psi(Q) - [v\underline{G} + (1-v)\bar{G}] + \lambda[\Delta G - \Delta\theta V(\underline{q})], \end{aligned}$$

where $\lambda > 0$ is the Lagrange multiplier of the binding constraint $IC(\bar{\theta})$. Then \underline{q} and \bar{q} are determined by:

$$\begin{cases} \left(\underline{\theta} - \frac{\lambda}{v}\Delta\theta \right) V'(\underline{q}) + \Psi'(v\underline{q} + (1-v)\bar{q}) = c, \\ \bar{\theta}V'(\bar{q}) + \Psi'(v\underline{q} + (1-v)\bar{q}) = c. \end{cases} \quad (16.7.83)$$

Because $\underline{\theta} - \frac{\lambda}{v}\Delta\theta < \underline{\theta}$, from formula (16.7.74) it is easy to verify that $\underline{q} < \underline{q}^{FB}$ and $\bar{q} > \bar{q}^{FB}$. Substituting $IR(\underline{\theta})$ and $IC(\bar{\theta})$ into the objective function of (P5) and letting $\delta > 0$ be the Lagrange multiplier associate with the binding constraint $IR(\bar{\theta})$, we obtain the following Lagrange function:

$$\begin{aligned} L(\underline{q}, \bar{q}) &= v[\underline{\theta}V(\underline{q}) - c\underline{q}] + (1-v)[\bar{\theta}V(\bar{q}) - c\bar{q}] \\ &\quad + \Psi(Q) - [v\underline{G} + (1-v)(\underline{G} + \Delta\theta V(\underline{q}))] + \delta[\Delta\theta V(\underline{q}) - \Delta G]. \end{aligned}$$

The optimal consumptions \underline{q} and \bar{q} are determined by:

$$\begin{cases} \left(\underline{\theta} - \frac{1-v-\delta}{v}\Delta\theta \right) V'(\underline{q}) + \Psi'(v\underline{q} + (1-v)\bar{q}) = c, \\ \bar{\theta}V'(\bar{q}) + \Psi'(v\underline{q} + (1-v)\bar{q}) = c. \end{cases} \quad (16.7.84)$$

Because $\underline{\theta} - \frac{1-v-\delta}{v}\Delta\theta > \underline{\theta} - \frac{1-v}{v}\Delta\theta$, from expression (16.7.74) we have $\underline{q} > \underline{q}^{SB}$ and $\bar{q} < \bar{q}^{SB}$. It then suffice to show that $\Delta\theta V(\underline{q}^{SB}) \leq \Delta G \leq \Delta\theta V(\bar{q}^{FB})$.

- In regime 3, $IR(\underline{\theta})$ and $IR(\bar{\theta})$ are binding. Then the optimal contract is given by

$$\left\{ \underline{q} = \underline{q}^{FB}, \bar{q} = \bar{q}^{FB}; \underline{U} = \underline{G}, \bar{U} = \bar{G} \right\}.$$

From the slack ICs , we can verify that ΔG satisfies $\Delta\theta V(\underline{q}^{FB}) < \Delta G < \Delta\theta V(\bar{q}^{FB})$.

- In regime 4, $IR(\underline{\theta})$, $IR(\bar{\theta})$ and $IC(\underline{\theta})$ are binding. The optimal contract is:

$$\left\{ (\underline{q}, \bar{q}, \underline{U}, \bar{U}) : \Delta\theta V(\bar{q}) = \Delta G, \underline{\theta}V(\underline{q}) + \Psi'(Q) = c; \underline{U} = \underline{G}, \bar{U} = \bar{G} \right\}.$$

Substituting $IR(\underline{\theta})$ and $IR(\bar{\theta})$ into the objective function, and letting $\mu > 0$ the multiplier associate with the binding constraint $IC(\underline{\theta})$, we have the following Lagrange function:

$$\begin{aligned} L(\underline{q}, \bar{q}) = & v[\underline{\theta}V(\underline{q}) - c\underline{q}] + (1-v)[\bar{\theta}V(\bar{q}) - c\bar{q}] \\ & + \Psi(Q) - [v\underline{G} + (1-v)\bar{G}] + \mu[\Delta\theta V(\bar{q}) - \Delta G]. \end{aligned}$$

Thus, \underline{q} and \bar{q} are determined by

$$\begin{cases} \underline{\theta}V'(\underline{q}) + \Psi'(v\underline{q} + (1-v)\bar{q}) = c, \\ \left(\bar{\theta} + \frac{\mu}{1-v}\Delta\theta\right)V'(\bar{q}) + \Psi'(v\underline{q} + (1-v)\bar{q}) = c. \end{cases} \quad (16.7.85)$$

Substituting $IR(\bar{\theta})$ and $IC(\underline{\theta})$ into the objective function, and letting $\eta > 0$ be the multiplier associate with the binding constraint $IR(\underline{\theta})$, we have the Lagrange function:

$$\begin{aligned} L(\underline{q}, \bar{q}) = & v[\underline{\theta}V(\underline{q}) - c\underline{q}] + (1-v)[\bar{\theta}V(\bar{q}) - c\bar{q}] + \Psi(Q) \\ & - [v(\bar{G} - \Delta\theta V(\bar{q})) + (1-v)\bar{G}] + \eta[\Delta G - \Delta\theta V(\bar{q})]. \end{aligned}$$

Then the \underline{q} and \bar{q} are determined by:

$$\begin{cases} \underline{\theta}V'(\underline{q}) + \Psi'(v\underline{q} + (1-v)\bar{q}) = c, \\ \left(\bar{\theta} + \frac{v-\eta}{1-v}\Delta\theta\right)V'(\bar{q}) + \Psi'(v\underline{q} + (1-v)\bar{q}) = c. \end{cases} \quad (16.7.86)$$

To compare the different consumption levels, we make some comparative static analysis. To do so, let

$$\begin{cases} \underline{\theta}V'(\underline{q}) + \Psi'(v\underline{q} + (1-v)\bar{q}) = c, \\ \beta V'(\bar{q}) + \Psi'(v\underline{q} + (1-v)\bar{q}) = c. \end{cases} \quad (16.7.87)$$

If $\beta = \bar{\theta}$, it's the expression of \underline{q}^{FB} and \bar{q}^{FB} ; if $\beta = \bar{\theta} + \frac{v}{1-v}\Delta\theta$, it coincides with the countervailing incentives solution \underline{q}^{CI} and \bar{q}^{CI} .

Differentiating these two equations with respect to β leads to:

$$\begin{cases} \left[\underline{\theta} V''(\underline{q}) + v \Psi''(Q) \right] \frac{d\underline{q}}{d\beta} + (1-v) \Psi''(Q) \frac{d\bar{q}}{d\beta} = 0, \\ v \Psi''(Q) \frac{d\underline{q}}{d\beta} + [\beta V''(\bar{q}) + (1-v) \Psi''(Q)] \frac{d\bar{q}}{d\beta} = -V'(\underline{q}). \end{cases} \quad (16.7.88)$$

Thus, when $\Psi''(Q) < 0$ we have

$$\begin{cases} \frac{d\underline{q}}{d\beta} = \frac{(1-v)V'(\underline{q})\Psi''(Q)}{\beta V''(\underline{q})[\underline{\theta} V''(\underline{q}) + v\Psi''(Q)] + (1-v)\underline{\theta} V''(\underline{q})\Psi''(Q)} < 0, \\ \frac{d\bar{q}}{d\beta} = \frac{-V'(\underline{q})[\underline{\theta} V''(\underline{q}) + v\Psi''(Q)]}{\beta V''(\underline{q})[\underline{\theta} V''(\underline{q}) + v\Psi''(Q)] + (1-v)\underline{\theta} V''(\underline{q})\Psi''(Q)} > 0. \end{cases} \quad (16.7.89)$$

Because $\bar{\theta} + \frac{\mu}{1-v}\Delta\theta > \bar{\theta}$ and $\bar{\theta} + \frac{v-\eta}{1-v}\Delta\theta < \bar{\theta} + \frac{v}{1-v}\Delta\theta$, from formula (16.7.89), it can be verified that $\bar{q} > \bar{q}^{FB}$, $\underline{q} < \underline{q}^{FB}$, $\bar{q} < \bar{q}^{CI}$, and $\underline{q} > \underline{q}^{CI}$. Thus, $\Delta G = \Delta\theta V(\bar{q}) \in [\Delta\theta V(\bar{q}^{FB}), \Delta\theta V(\bar{q}^{CI})]$.

- In regime 5, $IR(\bar{\theta})$ and $IC(\underline{\theta})$ are binding constraints. Substituting $\bar{U} = \bar{G}$ and $\underline{U} = \bar{G} - \Delta\theta V(\bar{q})$ into the objective function, we obtain the following first order conditions:

$$\begin{cases} \underline{\theta} V'(\underline{q}) + \Psi'(v\underline{q} + (1-v)\bar{q}) = c, \\ \left(\bar{\theta} + \frac{v}{1-v}\Delta\theta \right) V'(\bar{q}) + \Psi'(v\underline{q} + (1-v)\bar{q}) = c. \end{cases} \quad (16.7.90)$$

It is the countervailing incentives consumption level. Note that $\bar{\theta} + \frac{v}{1-v}\Delta\theta > \bar{\theta}$, and thus, from (16.7.89) $\underline{q}^{FB} > \underline{q}^{CI}$, $\bar{q}^{FB} < \bar{q}^{CI}$. The difference of reservation utility satisfies $\Delta G > \Delta U = \Delta\theta V(\bar{q}^{CI})$.

□

Lemma 16.7.5 Suppose $V(0) = 0$, $V'(\cdot) > 0$, $V''(\cdot) < 0$, and $V(\cdot)$ satisfies the standard Inada conditions: $\lim_{q \rightarrow +\infty} V'(q) = 0$, and $\lim_{q \rightarrow 0} V'(q) = +\infty$. Then the utility difference across different states $\Delta G = \bar{G} - \underline{G}$ is a decreasing function of the marginal cost ω , $\lim_{\omega \rightarrow 0} \Delta G = +\infty$, and $\lim_{\omega \rightarrow +\infty} \Delta G = 0$.

PROOF. The first order condition of $G(\theta) = \max_q [\theta V(q) - \omega q]$ is given by $\theta V'(q^*) = \omega$. So the maximized utility derived from network bypassing is: $G(\theta) = \theta[V(q^*(\theta)) - q^*(\theta)V'(q^*(\theta))]$. Let $\Phi(q) = V(q) - qV'(q)$. Then $\Delta G = G(\bar{\theta}) - G(\underline{\theta}) = \bar{\theta}\Phi(\bar{q}^*) - \underline{\theta}\Phi(\underline{q}^*)$, and its derivative with respect to

marginal cost ω is:

$$\begin{aligned}
\frac{d\Delta G}{d\omega} &= \bar{\theta}\Phi'(\bar{q}^*)\frac{d\bar{q}^*}{d\omega} - \underline{\theta}\Phi'(\underline{q}^*)\frac{d\underline{q}^*}{d\omega} \\
&= -\bar{\theta}\bar{q}^*V''(\bar{q}^*)\frac{d\bar{q}^*}{d\omega} + \underline{\theta}\underline{q}^*V''(\underline{q}^*)\frac{d\underline{q}^*}{d\omega} \\
&= -\bar{\theta}\bar{q}^*V''(\bar{q}^*)\frac{1}{\theta V''(\bar{q}^*)} + \underline{\theta}\underline{q}^*V''(\underline{q}^*)\frac{1}{\theta V''(\underline{q}^*)} \\
&= -\bar{q}^* + \underline{q}^* < 0.
\end{aligned}$$

It is easy to verify that when the conditions $V(0) = 0$, $V'(\cdot) > 0$, $V''(\cdot) < 0$, $\lim_{q \rightarrow +\infty} V'(q) = 0$, and $\lim_{q \rightarrow 0} V'(q) = +\infty$ are satisfied, $\lim_{\omega \rightarrow 0} \Delta G = +\infty$ and $\lim_{\omega \rightarrow +\infty} \Delta G = 0$. Figure 6 depicts the relationship of ΔG and ω . \square Figure 16.9 depicts the relationship between ΔG and ω .

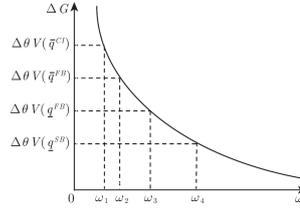


Figure 16.9: The impact of ω on ΔG

From the above lemmas, one can see that if the potential entrants' competitiveness increases, the utility differences will increase from zero to infinity. Thus, there exist positive values $\omega_i, i = 1, 2, 3, 4$, such that $\omega_1 < \omega_2 < \omega_3 < \omega_4$ corresponding to $\Delta\theta V(\bar{q}^{CI})$, $\Delta\theta V(\bar{q}^{FB})$, $\Delta\theta V(\underline{q}^{FB})$ and $\Delta\theta V(\underline{q}^{SB})$, respectively, where \bar{q}^{CI} , \bar{q}^{FB} , \underline{q}^{FB} and \underline{q}^{SB} are given in expressions (16.7.79), (16.7.68) and (16.7.69).

Thus, combining Lemmas 16.7.4 and 16.7.5, we proved Proposition 16.7.2. The proof is completed. \square

Remark 16.7.3 When $\omega > \omega_4$, the second-best contract is also entry deterring. It means that when the outside competitors are not efficient enough to give high demand consumers enough utility exceeding their information rents acquired from the present network, the outside market is only attractive to low demand consumers. The incumbent firm need not to change its pricing contract when facing the entry threat of a firm with low competitiveness.

When $\omega_3 \leq \omega \leq \omega_4$, we have $\underline{q} \in [\underline{q}^{SB}, \underline{q}^{FB}]$ and $\bar{q} \in [\bar{q}^{FB}, \bar{q}^{SB}]$. That means when the marginal cost ω decreases to the extent that the utility difference ΔG is large enough to attract high demand consumers bypassing the present network, the monopolist must give up more information rent to

him by increasing the consumption level of low demand consumers. The consumption level of high demand consumers themselves should also be lowered accordingly because of network effects. In this case, the sharper competitiveness of outside competitors makes the allocations less distorted.

When $\omega_2 < \omega < \omega_3$, asymmetric information imposes no distortion on both types' allocation. As ω decreases and ΔG increases further, \underline{q} will reach the first-best level, it is suboptimal for the monopolist to increase the high type' information rent at the cost of distorting the consumption level of the low demand consumers upward. In this case, the main task for the firm toward the high type is to prevent them from bypassing the incumbent market instead of preventing them from misreporting, the participation constraints are more difficult to be satisfied than the incentive compatibility constraints. Thus only the IRs are binding, and the first-best allocation is attained.

When $\omega_1 \leq \omega \leq \omega_2$, we have $\underline{q} \in [\underline{q}^{CI}, \underline{q}^{FB}]$ and $\bar{q} \in [\bar{q}^{FB}, \bar{q}^{CI}]$. The high difference of utilities induces the low type to pretend to be a high type, from which the countervailing incentives problem arises. The $IC(\underline{\theta})$, $IR(\underline{\theta})$ and $IR(\bar{\theta})$ in (P5) are binding. Again, the allocations of the two types will be distorted in opposite directions. But it is different from the distortions in cases 1 and 2. The monopolist then distorts \underline{q} downward to curb the rent of high demand consumers. In this case, however, information rent has to be given to low demand consumers to elicit them reporting their types truthfully. The information rent is a decreasing function of high demand consumers' consumption \bar{q} , and so \bar{q} has to be distorted upward to reduce the information rent gained by low demand consumers, while a certain amount of the low demand type's consumption is "crowded" out of the network so that \underline{q} is distorted downward.

When $0 < \omega < \omega_1$, the allocations remain at the countervailing incentives level: $\underline{q} = \underline{q}^{CI}$ and $\bar{q} = \bar{q}^{CI}$. The decrease in marginal cost ω demand further upward distortion on the consumption of high demand consumers (the consumption of low demand consumers will be distorted downward accordingly). Thus, the participation constraint of the low-type has to be slackened, which means certain amount of information rent should be given to the consumers with low willingness to pay. In this case, only $IC(\underline{\theta})$ and $IR(\bar{\theta})$ are binding constraints, the low-type get information rent $\bar{G} - \Delta\theta V(\bar{q}^{CI})$. The incumbent firm keep reducing the tariffs (\underline{t} and \bar{t} keep decreasing in this case) instead of distorting allocations to prevent high demand consumers from bypassing and low demand consumers from misreporting.

Changes in consumption with marginal cost ω are summarized in the following Figure 16.10.

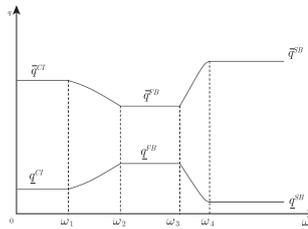


Figure 16.10: Impact of potential entrant marginal cost ω on second-best consumption

16.8 Adverse Selection in Competitive Market

In this section we mainly discuss the issue of adverse selection in a competitive market. Different from the above discussion, in the competitive market, there are many suppliers offering goods to customers. However, due to the asymmetry of information, the market efficiency and even the existence of the market will cause problems.

George Akerlof (1940- for his character, see section 15.5.2) first introduced information asymmetry in market analysis in 1970. In a used car market, the seller learned more about the car performance than the buyer. For each possible market price, used automobiles of good quality may be withdrawn from the market, and those of poor quality will enter the market. As the price drops, the quality of used automobiles in the market will become worse and worse, which may eventually lead to market collapse. Here we use the insurance market to discuss the impact of asymmetric information on market equilibrium.

In a competitive insurance market, each insured customer has his own private information, which is not known by insurance companies. For example, in health insurance, customers have information about their own health. In property insurance, there is private information about the possibility of accidents related to behavior. We introduce a simple continuous situation.

Assume the probability of an accident $p \in [p, \bar{p}] \subseteq [0, 1]$ is a private message with a density function $f(p)$. Assuming that the accidental loss is L (which is identical for all types), each customer's utility function $u(w)$ for wealth is the same, assume the utility function is strictly concave.

Under complete information: For customers of type p , in the competitive market, the premium rate is also p . The L loss premium is $I = pL$. Since the customers are risk averse, then $u(w - pL) > pu(w - L) + (1 - p)u(w)$.

Under incomplete information, there is only one type of contract in the market in the Akerlof (1970) model. Consider a simplified scenario, assuming that the contract is full loss insurance, the insurance company has no

customer's private information, the premium rate is r , the cost of insurance is $I = rL$. If market equilibrium exists, it has the following structure: there exist \hat{p} such that:

$$\begin{aligned} u(w - I(\hat{p})) &< pu(w - L) + (1 - p)u(w), \quad \forall p < \hat{p}, \\ u(w - I(\hat{p})) &> pu(w - L) + (1 - p)u(w), \quad \forall p > \hat{p}, \\ u(w - I(\hat{p})) &= \hat{p}u(w - L) + (1 - \hat{p})u(w), \\ I(\hat{p}) &= \frac{\int_{\hat{p}}^{\bar{p}} pf(p)dp}{1 - F(\hat{p})}L. \end{aligned}$$

Let $\tilde{p} = \int_{\tilde{p}}^{\bar{p}} pf(p)dp$. If $u(w - \tilde{L}) < \tilde{p}u(w - L) + (1 - \tilde{p})u(w)$, it means that $\hat{p} > \tilde{p}$ above. That is, $p \in [\tilde{p}, \hat{p})$ types cannot be covered in the insurance market, leading to market failure. In some extreme cases, only some customers with the highest probability of accidents will be covered by the market, resulting in the phenomenon of Gresham's Law: "bad money driving out good money". The following example reveals the possibility of a collapse of the old market, which is discussed more detail in Akerlof (1970).

Example 16.8.1 Suppose that in a used car market, the seller knows the quality of the car, and it is characterized by θ , which obeys the uniform distribution of $[0, \bar{\theta}]$. For a θ type vehicle, the seller's rating is θ , and the buyer's rating is $k\theta$, which satisfies $1 < k < 2$. Obviously, under complete information, all used cars (except $\theta = 0$), will be traded. If the transaction price is $p(\theta)$, after transaction, the utility of two parties is $p(\theta) - \theta$ and $k\theta - p(\theta)$ respectively, market equilibrium will be Pareto optimal. However, if the used car quality information is asymmetric, when the market price is p , the used car with $\theta \in [0, p]$ will still be on the market. At this time, the average quality is $\frac{p}{2}$. If consumers purchase, their expected utility is $\frac{kp}{2} - p < 0$, the used car market does not exist at all.

16.8.1 Discriminating Mechanism of Competitive Market under Asymmetric Information

When there is more than one type of contract, different sellers may choose different contracts to cater to different consumers. In this case, market equilibrium will have new features, such as the absence of pooling equilibrium. Rothschild and Stiglitz (1976) first introduced a contract screening mechanism in the competitive market. Below, we briefly discuss the various types of contract equilibrium in the competitive market.

For the sake of simplicity, suppose there are two types of consumer in the market. The probability of accidents is $p_L < p_H$. There are many insurance companies in the market and they can provide consumers with

different contracts. Assume that the distribution probability of the low accident type is λ . The contract form provided by the insurance company i is (I_i, D_i) , where I_i is the premium and D_i is the amount of compensation. If the consumer of type i accepts the contract, its expected utility is:

$$p_i u(w - D_i - I_i) + (1 - p_i) u(w - I_i).$$

Assume that the insurance process is divided into two steps: First, each insurance company provides insurance contracts; secondly, customers choose one of the most favorable insurance contracts. We also assume that the insurance company has the ability to fulfill its contract so that even if the insurance contract is a loss, it will be implemented. In addition, in the competitive environment, the insurance company's expected profit is zero, and imagine that the insurance companies compete in a Bertrand-like manner.

In the equilibrium, there may be two forms of pure strategy: First, pooling equilibrium, that is, all insurance companies choose the same contract which attracting all customers; second, separating equilibrium, there are two different types of insurance contracts in the market, different types choose different contracts.

We first discuss the pooling equilibrium: Assume that $\alpha = (I, D)$ is the insurance contract in pooling equilibrium. At this point, for both types, their expected benefits are:

$$U^i \equiv p_i u(w - D - I) + (1 - p_i) u(w - I).$$

Zero profit conditions imply:

$$I = (p_L \lambda + p_H (1 - \lambda))(L - D).$$

In the pooling contract, low-accident consumers subsidize high-incident consumers. At this time, there will be positive profit opportunities in the insurance market. New insurance companies can design new contracts $(\beta = \hat{D}, \hat{I})$ to attract low-incident consumers only if they meet:

$$\begin{aligned} p_H u(w - D - I) + (1 - p_H) u(w - I) &> p_H u(w - \hat{D} - \hat{I}) + (1 - p_H) u(w - \hat{I}), \\ p_L u(w - D - I) + (1 - p_L) u(w - I) &< p_L u(w - \hat{D} - \hat{I}) + (1 - p_L) u(w - \hat{I}), \\ \hat{I} &> p_L (L - \hat{D}). \end{aligned}$$

Let $W_1 = u(w - I)$, $W_2 = u(w - D - I)$, Figure 16.11 depicts such a new contract.

This is because, when $\hat{D} < D$, $\hat{I} < I$, low-accident consumers are less sensitive to deductibles, but are more sensitive to insurance costs; however, for high-accident consumers, the opposite is true. The insurance company can profit from low-level customers.

For the separating equilibrium, there are two insurance contracts in the market, one is to meet the low-accident type $\alpha^L : (D_L, I_L = p_L(L - D_L))$, and the other is to meet the high-accident type $\alpha^H : (D_H, I_H = p_H(L -$

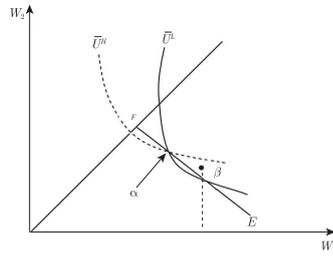


Figure 16.11: Pooling equilibrium doesn't exist

D_H)). Rothschild and Stiglitz (1976) found that separating equilibrium may not exist under certain conditions.

We know that high-incident consumers are very sensitive to deductibles, so it may be assumed that the high-accident contract is $\alpha^H = (0, I_H = p_H L)$, while the low-accident contract $\alpha^L = (D_L, I_L = p_L, L - D_L)$ satisfy:

$$U^L = p_L u(w - D_L - I_L) + (1 - p_L)u(w - I_L)$$

s.t. $u(w - I_H) \geq p_H u(w - D_L - I_L) + (1 - p_H)u(w - I_L).$

In the separating equilibrium, the low accident contract is not related to the distribution of the two types of participants in the market, but is greatly affected by $p_H - p_L$, because it is necessary to avoid the selection of low accident type contracts in order to avoid high accident types, but this makes low-incident customers face high risks. If the probability of a type of low accident in the market is high, there will be a positive profit opportunity in the market, which will allow the new insurance company to provide a pooling contract that also caters to both types of customers (the contract γ in Figure 16.12). Figure 16.12 depicts such a new contract.

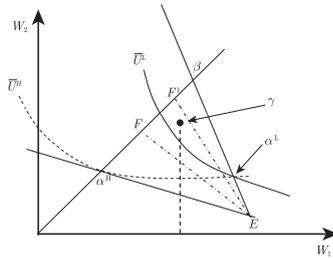


Figure 16.12: Separating equilibrium doesn't exist

For the existence of equilibrium, Wilson (1977) introduced the "anticipatory equilibrium". Under this concept, if a new contract occurs after the new contract is introduced and the old contract is withdrawn, then the

new contract is not will be introduced. Wilson proved that under such constraints, market equilibrium will exist, and pooling equilibrium may also exist. Riley (1979) introduced "reactive equilibrium". Under this concept, if a series of chain reactions will be triggered after the introduction of a new contract and the original new contract loss money, the new contract will not be introduced.

Similarly, under such constraints, market equilibrium will also exist. Hellwig (1987) and Engers and Fernandez (1987) provide the basis for game theory on the above mentioned anticipatory equilibrium and reactive equilibrium respectively. For the new equilibrium's ability to interpret real-world problems, Kreps (1990, p. 645) believes that this is closely related to the market regulatory environment. If the regulatory agency does not allow the initial contract (even if it loses money) to exit, the reactive equilibrium will be closer to reality. If the regulatory agency allows the loss of the original contract to withdraw from the market, then the anticipatory equilibrium will be closer to reality.

16.8.2 Signal transmission under asymmetric information

We discussed the signal mechanism of education in the previous chapter on game theory. Different employees show their ability information through diplomas. Below we use an example (Tirole, 1988) to discuss how producers can show their quality information through some mechanism in the presence of asymmetric information.

Let s be the quality of the product, the utility level of the consumer at price p is $\theta_s - p$. Assuming $s \in \{0, 1\}$, the unit production cost of enterprise is c_s ($c_0 < c_1$). We mainly discuss the monopoly situation. Assuming that there are two period repeated purchases of the product, the quality of the product cannot be changed and consumers will understand the quality of the product after using the product. Let δ be the time discount rate. The monopolist can display quality information by price, and p_1 shows high-quality price information. The profits of high-quality companies are:

$$\pi_1 = (p_1 - c_1) + \delta(\theta - c_1).$$

In order to avoid the imitation of low-quality companies, it is required that:

$$p_1 \leq c_0,$$

which implies:

$$\pi_1 \leq \delta(\theta - c_1) - (c_1 - c_0).$$

If $\delta(\theta - c_1) > (c_1 - c_0)$, there is a high-quality revelation equilibrium. The monopolist asks $p_1 = c_0$ for overall profit by obtaining a monopoly profit in the second phase to make up for the first-stage profit loss. If $\delta(\theta - c_1) \leq (c_1 - c_0)$, there is no separating equilibrium of high-quality goods. $\delta(\theta -$

c_1) is the benefit obtained by high-quality firms in revelation equilibrium, and $c_1 - c_0$ is the cost of showing high quality, for this reason, separating equilibrium can exist if and only if high-quality companies show that their type of benefits outweigh their costs.

In the separating equilibrium, we found that companies established their reputations through initial price concessions. In real life, some businesses choose low prices to attract customers when they open a business, the logic behind is related to this. However, there are many ways for companies to display signals. For example, companies can display their types through extravagant advertising expenditures. For example in China, they show their ability through high bidding in CCTV news broadcast.

Let A is a extravagant advertisement expenditure, assuming that it is a signal of quality, when $A = \theta - c_0$, we find that low-quality companies have no incentive to imitate because

$$\pi_0 \leq (\theta - c_0) - (\theta - c_0) = 0,$$

while the profits of high-quality companies under advertising investment are:

$$\pi_1 = (\theta - c_1) + \delta(\theta - c_1) - (\theta - c_0) = \delta(\theta - c_1) - (c_1 - c_0).$$

Therefore, when $\delta(\theta - c_1) > (c_1 - c_0)$, the advertisement becomes a signal for displaying the quality information of company.

In addition, the company can also display the quality information through some warrants, after-sales services, etc, the mechanism behind which is that high-quality companies have higher returns than low-quality companies in terms of display signals, such as achieving by repeated purchases.

16.9 Further Extension

The main theme of this chapter was to determine how the fundamental conflict between rent extraction and efficiency could be solved in a principal-agent relationship with adverse selection. In the model discussed in this chapter, because there is only one incentive constraint and one participation constraint, this conflict is relatively easy to understand. Here we mention several possible directions for extension. We can consider two-dimensional inverse selection models, models that involve random participation constraints, limited liquidity constraints, or models of supervision of the agent by the principal. For a detailed discussion of these topics and their applications, interested readers can see Laffont and Martimort (2002).

16.10 Biographies

16.10.1 Ludwig Mises

Ludwig von Mises (1881-1973), the third generation head of the Austrian School, a member of the Pilgrimage Hill Society. He enrolled at the University of Vienna in 1900, where he was greatly influenced by Carl Menger (1840 - 1921) and got his doctoral degree in law and economics in 1906. From 1909 to 1934, he was an economist at the Vienna Chamber of Commerce. After the First World War, he worked for the Austrian Industrial Commission; in 1921 he also served as a legal advisor to a government agency, responsible for drafting the terms of the final-war treaty - to resolve pre-war private debt problem between belligerents. On New Year's Day in 1927, the Institute for Economics (Business Cycle) he founded was formally established and Hayek became the first director.

In 1934 - 1940, he moved to Geneva as a professor at the Geneva Institute of International Studies. In 1940 he moved to New York in USA. At pre, Keynesianism in the American academic world was prevalent, Mises's liberalism was clearly out of the mainstream, and he was not employed by any academic organization. In 1945, through the recommendation of the Lawrence Fertig & William Volker Foundation, Mises entered New York University, but he could only serve as a visiting professor. In 1949, Mises published "Human Behavior". Even so, he was only able to find a visiting professor position till until retirement in 1969.

For a long time, even though Mises's ideas have not been accepted by mainstream economists, his ideological influence and knowledge contribution to the 20th century human society cannot be ignored. To a certain extent, it could be said that the history of economic thought of human society in the 20th century can not tell a complete story if Mises is missing. In 2000, America's "Freedom" magazine referred to Mises as "the century figure of libertarianism".

Mises has made many theoretical contributions in understanding the basic principles of human economic and social operations. The reason why Mises occupied such an important position in the history of contemporary human society is mainly because Mises made many remarkable theoretical contributions in understanding the basic principles of human economic and social operations. In addition to his theoretical contributions in inflation, economic cycles, economic epistemology and methodology, and his own unique cataliactics (i.e., a theory of the way the free market system reaches exchange ratios and prices) and praxeology, his main theoretical contributions lie in the early 1920s. He presented such a major theoretical insight with a discerning eye: In the absence of a market price mechanism, the impossibility of economic calculations will result in the infeasibility of the centrally planned economy.

16.10.2 Leonid Hurwicz

Leonid Hurwicz (1917-2008), the father of mechanism design theory, was awarded the Nobel Prize in Economics in 2007 for "laying a theoretical foundation for mechanism design". Despite his pioneering achievements in many areas of economics, Hurwicz did not earn any degree in economics. Professor Leonid Hurwicz was a Jew. He was born in Poland in 1917 and came to the United States during the Second World War. The highest degree he received is a law degree equivalent to a master's degree in Poland.

In the outbreak of World War II, Hurwicz was in Switzerland. He did not return to Poland but went to the United States. He said, "If I stay in Poland, I could very well be a victim of the Auschwitz concentration camp". After coming to the United States, he did not go to get a doctoral degree, but instead became an assistant to the economist Paul Samuelson, and he promoted directly to a full professor from an assistant professor.

As early as the middle of the 20th century, Hurwicz had begun to think about research projects derived from general equilibrium theory, social selection theory and game theory. In the early 1960s, his article entitled "Optimization and Information Efficiency in Resource Allocation" kicked off the prelude to mechanism design theory. Hurwicz's mechanism design theory is closely linked with major economic issues such as market regulation, market screening and public goods provision. Subsequently, many economists have involved in this research. Both Myerson and Maskin, who were young at the time, were active members.

Later, Hurwicz wrote the well-known papers such as "Revealed Preference without Continuity of Demand" and "System of Information Dispersion", and gradually improved the theoretical basis. In 1973, Hurwicz published the paper "Mechanism Design Theory of Resource Allocation" in *American Economic Review*, which proposed two core issues in the theoretical framework of mechanism design - incentive compatibility principle and revelation principle, laid the framework of the mechanism design theory. Moreover, as an extension and application of the general mechanism design theory, it provides a basic analytic framework for the two subfields: auction theory and matching theory.

The mechanism design theory can be regarded as the most important development in the economics field in the past half century. It aims to systematically analyze resource allocation institution and process, reveal the important role of information, communication, incentives and economic man's processing capabilities in decentralized resource allocation, and enables us to identify sources of market failure. In fact, almost all areas of modern economic theory are influenced by the mechanism design theory pioneered by Hurwicz. His "incentive compatibility" has now become a core concept in economics.

The two most objective realities in the real economic world are the in-

dividual's personal interest-seeking and asymmetric nature of individual private information. In simple terms, incentive compatibility is to be compatible with individual rationality and collective rationality, that is, policies can achieve their own subjective and objective effects for others. Information asymmetry can lead to market failure, so some incentive mechanism should be designed to induce economic people to reveal their true information. Incentives are ubiquitous, science, sociology and even marriage and family are all available. The revelation principle allows participants to report the truth, thereby eliminating information asymmetry and achieving second-best results.

Hurwicz had close connect with Lange, a representative of the Lausanne school, and Hayek, a representative of the Austrian school. His mechanism design theory is also influenced by these two aspects of economic theory and methodology, and rigorously unifies these two seemingly very different schools of thought into mechanism design theory. In the later period, Hurwicz also interacted academically with representatives of the new institutional economics group, such as Douglass C. North (1920-2015, whose character was seen in the ?? section), trying to establish a dialogue channel and a connection path between the two theories.

Hurwicz also did a lot of other pioneering work: in the late 1940s, he laid the groundwork for the identification problem of dynamic econometric models; as early as 1947, he first proposed and defined the concept of rational expectation in neoclassical macroeconomics. The school of rational expectations has become the mainstream of today's macroeconomics; Hurwicz also made important work on the existence of the utility function from the existence of the demand function, which is an important result from the perspective of political economy. Utility is a basic concept in the consumption theory of modern microeconomics and is the basis of the modern economics. Traditional political economics believes that utility is an idealistic concept. It does not exist. He and Kenneth J. Arrow also made pioneering work on the stability of the general equilibrium of competitive markets.

In addition, Hurwicz attaches great importance on the preciseness of expressing economic problem. According to Hurwicz, one of the biggest problems in many traditional economic theories is that it is too arbitrary to express concepts. The greatest significance of the axiomatic approach lies in the certainty and clarity of the expression of theory, which makes the discussion and criticism have a universal research paradigm and analysis framework. This is also a strong feeling he got from the socialist economic calculations in the 20th and 30s of the last century, because he felt that the two sides of the discussion are incommensurable in some theoretical expressions. This easily confuses the issue. According to Hurwicz, the study of the economic system can adopt either an empirical scientific approach or a normative scientific approach. Whatever the method, as long as it is analytical, a crucial first step is to carry out an standardized concept of e-

onomy. Thus, it also prompted him to create a mechanism design theory that is very general and allows different mechanisms to be compared under a framework.

Professor Hurwicz is also a knowledgeable professor of mathematics and statistics. He has also studied linguistics and is very humor in speaking and teaching. According to each person's economic knowledge and the difference in training, the profound problem can be explained very clearly in a very popular or rigorous language.

On April 14, 2007, the University of Minnesota hosted the 90th Birthday Party for Professor Leonid Hurwicz. This became a gathering of masters of economics, including Eric Maskin and Roger Myerson who won the Nobel Prize with Hurwicz in economics in 2007, and Arrow who won the 1972 Nobel Prize for economics, and McFadden who won the 2000 Nobel Prize for economics and many other economics masters. They come together to celebrate the great birthday. The author of this book also made a special trip to congratulate for his 90th birthday. Six months later, Hurwicz and Eric Maskin and Roger Myerson jointly won the Nobel Prize in Economics for their contribution to the creation and development of mechanism design theory. At the same time, it also created a record that Hurwicz became the oldest Nobel Prize winner in history. After learning that he had won the Nobel Prize in Economics, with humor for his later life, he said at his home in Minneapolis, "I thought my time had passed. For the Nobel Prize, I am really too old. But this bonus is indeed good for a retired old man."

16.11 Exercises

Exercise 16.1 Consider the principal-agent model with unobserved cost function of agent. Different cost function $C(q, \theta)$ represent different types. Assume $C_q > 0$, $C_{\theta} > 0$, $C_{qq} > 0$ and $C_{q\theta} > 0$, where q is the perfectly observed output of agent. Assume two types of agent, $\theta \in \{\theta_L, \theta_H\}$, with the probability of occurrence ν and $1 - \nu$ and $\Delta\theta \equiv \theta_H - \theta_L > 0$. Output is evaluated at $S(q)$ by principal, where $S' > 0$, $S'' < 0$ and $S(0) = 0$. Let T denote the transfer from the principal to the agent. The payoff function of the agent is $T - C(q, \theta)$. The principal and the risk-neutral agent signed the contract during the mid-term phase.

1. Write out the principal's optimization problem satisfying incentive compatibility.
2. What is the optimal payment of T_H and T_L and economic rent for both types?
3. Find out the first-order conditions for the issue of principal's incentive compatibility. What are the meanings of these conditions?
4. Let $S(q) = q$, $C(q, \theta) = \theta q^2/2$, $\theta_L = 1$, $\theta_H = 2$, $\nu = 2/3$. Solve the optimal contract.

Exercise 16.2 (Risk averse principal) Under the same setup as the previous question, suppose that the principal is an individual who is risk averse and defines the monetary benefits of the transaction $S(q) - t$ of VNM utility function $v(\cdot)$, satisfying $v' > 0$, $v'' < 0$ and $v(0) = 0$.

1. Write down the principal's optimization problem satisfying incentive compatibility constraints.
2. What is the optimal payment of T_H and T_L and economic rent for both types?
3. Find the first-order conditions of the principal's optimal problem. Comparing the output distortions of the risk-averse principal and the risk-neutral principal, which distortion of the output is greater? why?
4. Suppose $v(x) = \frac{1-e^{-rx}}{r}$, find out the first order condition for the optimal problem. What happens when $r \rightarrow 0$. Does the solution converge to the distortion of the situation with the risk neutral principal and the participation constraint of intermediate stage agent. What happens when r tends to infinity? Can the first-best output be achieved?

Exercise 16.3 (Shutdown contract) Under the setting of Exercise 16.1, assume that the cost function is $C(q, \theta) = \theta q$. Solve the optimal contract with shutdown property. (shutdown contract)

1. Under what conditions, the optimal contract will shut down the inefficient party.
2. Assume that the reservation utility is U_0 . Prove that the conditions for closing the contract are $\frac{v}{1-v} \Delta \theta \bar{q}^{SB} + U_0 \geq S(\bar{q}^{SB}) - \bar{\theta} \bar{q}^{SB}$.

Exercise 16.4 (State-dependent fixed costs) Under the same settings as Exercise 16.1, assume that the cost function is given by $C(q, \theta) = \theta q + F(q)$, and the fixed cost satisfies $F(\theta_L) > F(\theta_H)$, where higher marginal costs are associated with lower fixed costs and vice versa.

1. Write down the principal's optimization problem satisfying incentive compatibility.
2. What is the optimal payment of T_H and T_L and economic rent for both types?
3. Find the first-order conditions of the principal's incentive compatibility problem.
4. Discuss the different intervals of the solution, based on the different positions of $F(\theta_L) - F(\theta_H)$, $\Delta \theta q_H$, and $\Delta \theta q_L$.

Exercise 16.5 (Three types of agent) Under the setting of Exercise 16.1, assume there are three possible types θ of agent $\{\theta_1, \theta_2, \theta_3\}$, and $\theta_3 - \theta_2 = \theta_2 - \theta_1 = \Delta\theta$, with probability v_1, v_2, v_3 . The direct revelation mechanisms are $\{(t_1, q_1), (t_2, q_2), (t_3, q_3)\}$. Other settings is the same as exercise 16.1. The cost function is $C(q, \theta) = \theta q$.

1. Write down the economic rents, participation constraints, and incentive compatibility constraints for the three types.
2. Simplify participation constraints and incentive compatibility constraints.
3. If $v_2 > v_1 v_3$, prove that the monotonicity conditions are strictly satisfied and solve the optimal solution.
4. If $v_2 \leq v_1 v_3$, solve the optimal contract.

Exercise 16.6 (Agent with several types) In the above question, there are three possible types of agent. Now go further. Assume that the type of agent is n kinds of possibilities, $\theta_n > \dots > \theta_2 > \theta_1$. The probability of each type θ_i is β_i , refer to the multi-type question in the 16.5 section of the textbook.

1. Write the Spence-Mirrlees single crossing condition for this problem.
2. Write down the local downward incentive compatibility constraints for this problem.
3. Proof: The single-crossing condition implies the sufficiency of monotonic and local incentive constraints.
4. Proof: At the optimal solution, all local downward incentive constraints (LDICs) are compact.
5. Proof: All LDICs are compact, with the monotonicity condition, guarantee that all local upward incentive constraints (LUICs) are satisfied.
6. Solve the simplified optimal contract problem.

Exercise 16.7 (Continuous type agent) In the above question, the types of agent are finite. It is now assumed that the type of agent is continuous, that is, θ no longer takes a finite number of values, but is defined on the interval $[\underline{\theta}, \bar{\theta}]$, and the density function is $f(\theta)$. Due to the revelation principle, we only consider the direct revelation mechanism $\{q(\theta), T(\theta)\}$.

1. Write the Spence-Mirrlees single crossover condition for this problem.
2. Write down the local downward incentive compatibility constraint for this problem.
3. Refer to the contents of the 16.5.2 section, write down monotonic conditions, local incentive compatibility constraints, and Hamiltonian function of optimal contract problems.

4. Solve the optimization problem and explain the results. Compare the optimal contract problem with the continuous type and the two types of problems in Exercise 16.1.

Exercise 16.8 Consider a screening model. The seller's cost function is $C(q) = q^{1/2}$. If a buyer consumes q units of goods, his utility function is $\theta \ln q$. The buyer's type θ is private and subject to a uniform distribution on $[0, 1]$.

1. Write down a monopolist optimization problem.
2. Solve the optimization problem $(q(\theta), t(\theta))$.
3. Find the optimal non-linear pricing $t(\theta)$ that implements the direct mechanism.

Exercise 16.9 A monopolist wants to sell a single item to a consumer. The latter's willingness to pay for this item is t_1 or $t_2 (> t_1)$. The seller knows that the buyer is subject to liquidity and the maximum payment for the item is $t \in \{t_1, t_2\}$. The buyer's willingness to pay is private information, and the seller only knows the probability $\mu(t_i)$ of type t_i . This item is worth $c < t$ to the seller. Suppose both the buyer and the seller are risk-neutral. Let v denote the probability of the transaction occurring, h denote the payment to the seller, and t_i is the value of the buyer. The buyer's revenue function is:

$$vt_i - h,$$

the seller's revenue function is:

$$h - vc.$$

In addition, regardless of the type, the buyer's reservation utility are all 0.

1. Give the participation constraints, incentive compatibility constraints and liquidity constraints required by direct mechanism.
2. Find the direct mechanism for maximizing the expected returns of sellers.

Exercise 16.10 (Bundle and iron out) In the agent problem with continuous types, it is generally assumed that the risk rate condition, that is, the risk rate (hazard rate) $h(\theta) = \frac{f(\theta)}{1-F(\theta)}$ is increasing about θ .

1. Prove that if the risk rate condition holds, the monotonicity condition is satisfied in the implementation of the continuous type optimal contract.
2. If the monotonicity condition does not hold, then in most cases we need to re-solve the optimization problem. Refer to the contents of the 16.5.3 section of the textbook, redefine the Hamiltonian function and the corresponding constraints.

3. Use Pontryagin maximum principle to re-solve the optimal contract.

Exercise 16.11 Suppose two persons consider trading some kind of asset at price of p . This asset can only be used as a store of wealth. Person 1 currently owns this asset. Everyone's evaluation of this asset is only known to him, and each person only cares about the expected value of the asset after one year. Assume that only when both sides think that the transaction can make their situation better, they are willing to trade at the price of p . Prove the probability of a transaction is zero.

Exercise 16.12 Consider the following process. First, the nature determines the type of worker. It is continuously distributed over the interval $[\underline{\theta}, \bar{\theta}]$. Once the worker knows his type, he can choose whether to participate in an exam that does not cost. The exam accurately reflects his ability. Finally, after observing whether the worker took the test and the performance of the worker who took the test, the two companies started bidding on the service of the worker. Prove that in any subgame perfect Nash equilibrium of this model, all worker types take the exam, and the company does not provide any salary greater than $\underline{\theta}$ for any worker who does not take the exam.

Exercise 16.13 (Borrowing with adverse selection) Assume there is a continuum of risk-neutral borrowers. The borrower does not have personal wealth and assumes limited liability. A borrower with a ν ratio (known as type 1) has an investment project with a definitive return of h for one unit investment. Borrowers with a $1 - \nu$ ratio (known as type 2) have a random independent investment project, with θ probability to get h ($\theta \in (0, 1)$) return for one unit investment, and with $1 - \theta$ probability to get return of 0. If the borrower does not apply for a loan, he has the external opportunity with u utility. In this economy there is only one risk-neutral bank that provides the loan. The financial cost of the loan is r . The bank provide lending contracts to maximize her expected profit. For the sake of simplicity, assume that investing in any project is socially valid: $\theta h > r + u$.

1. Explain why you only look at the menu of contracts (r_1, P_1) and (r_2, P_2) will not lose its generality, where P is the probability of obtaining a loan, and r_i is the borrower who claims to be of type i pay to the bank after successful investment.
2. Write down the bank's menu (r_1, P_1) and (r_2, P_2) to optimize the expected profit when satisfying borrower participation constraints and incentive compatibility constraints (for simplicity, assume that if a borrower applies for the loan, he loses the external opportunity of getting utility u).
3. Prove that the loan allocation in the optimal contract is non-random (ie, P_i can only be 0 or 1, for $i = 1, 2$). Write down the characteristics of the optimal contract.

Exercise 16.14 (Bribery game) Consider a regulatory agency to provide a service to citizens with a fixed period of delay. Under the normal operation of the regulatory agency, citizens can get the utility of u_0 (depending on their valuation of time). Officials who devote extra effort can shorten the delay period. Officials can shorten the delay period of q at the cost of $\frac{(q-Q)^2}{2}$, where Q is a constant. Assume that there is ν (or $1 - \nu$) type 1 (type 2) citizens rating q as $\theta_L q$ ($\theta_H q$). Citizens are willing to bribe officials to shorten the delay period. Write down the optimal bribe contract that officials provide to citizens.

Exercise 16.15 Monopolists can produce a product at different quality levels. The cost required to produce a unit with a mass of s is s^2 . Consumers buy up to one unit of product. If it consumes a product with unit mass s , its utility function is $u(s|\theta) = s\theta$. The monopolist can determine the price and quality of the product. Consumers observe the quality and price of the product and decide what quality product to buy.

1. Solve the optimal solution.
2. Suppose seller cannot observe θ , and $\text{prob}(\theta = \theta_H) = 1 - \beta$, $\text{prob}(\theta = \theta_L) = \beta$, where $\theta_H > \theta_L > 0$. Solve the second optimal solution and information rent of consumer.
3. Suppose that θ obeys a uniform distribution on the interval $[0, 1]$. Find the second optimal solution.

Exercise 16.16 (Labor contract) Consider the following settings: A firm faces a worker. The worker's utility function is $U^A = u(c) - \theta l$, where c is consumption and l is labor supply. $\theta \in \{\theta_L, \theta_H\}$ is a parameter that only workers know. $\theta_L < \theta_H$, $u(\cdot)$ is an incremental concave function. The ratio of worker who has a lower negative utility ($\theta = \theta_L$) is ν . The agent's optimal choice must satisfy the budget constraint $c \leq T$, where t is the payment he received from the employer. The utility function of the employer is $U^P = f(l) - T$, where $f(l)$ is a production function with decreasing returns to scale.

1. Suppose that the employer knows θ . Write out the solution for the employer to maximize his utility in satisfying the worker's participation constraints. This solution is called as the first-best (first best) solution.
2. Suppose the employer can observe and verify the labor supply but cannot observe U and θ . Prove that the first-best solution can not be achieved. The employer can now provide the contract menus (T_L, l_L) and (T_H, l_H) , where (T_H, l_H) is the contract selected by type θ_H , (T_L, l_L) is the contract selected by type θ_L . Find the optimal contract and compare the second-best solution with the first-best solution.

3. Suppose that the worker with lower negative utility effect of the effort has an external opportunity that can bring him the utility of V . Compare the first-best solution with the second-best solution in this case. Notice that the solution will depend on V . Consider respectively $l_H^{SB} \Delta\theta \geq qV$ (SB denote "second best"), $l_H^{SB} \Delta\theta \leq V \leq l_H^{FB} \Delta\theta$ ($l_H^* \Delta\theta \leq qV \leq ql_L^* \Delta\theta$, denote "the first-best") and $V \geq ql_L^* \Delta\theta$. What are the compact constraints in each situation? What type of distortion is needed?

Exercise 16.17 (Labor contract with adverse selection) Consider a principal-agent relationship in which the principal is the employer and the agent is the worker. For worker of type θ ($\theta \in \{\theta_L, \theta_H\}$), the work negative utility for production y is $\psi(\theta y)$. In other words, the worker of type θ must work on l units ($l = \theta y$), and the resulting negative utility is $\psi(l)$. If he gets a compensation of t from his employer, the net utility is $U = t - \psi(\theta y)$, and the employer's utility function is $V = y - t$.

1. Refer to revelation principle, write down the characteristic of direct mechanism.
2. Suppose that $\nu((1 - \nu))$ is the probability that the worker is of type $\theta_L(\theta_H)$. Find the contract maximizing the expected utility of the employer (while satisfying the workers' incentive compatibility constraints and participation Constraints).

Exercise 16.18 (Information and Incentive) An agent (natural monopoly manufacturers) produces q output with the variable cost function θq ($\theta \in \{\theta_L, \theta_H\}$, $\Delta\theta = \theta_H - \theta_L$). The utility obtained by the principal from production is $S(q)$ ($S' > 0, S'' < 0$) and the transfer to the agent is t . The utility function of the principal is $V = S(q) - T$, and the agent's utility function is $U = T - \theta q$. In addition, the agent's current utility is standardized to 0.

1. When the principal has complete information on θ , write down the principal's first-best contract.
2. Suppose that θ is the private information of the agent and $\nu = Pr(\theta = \theta_L)$. Write down the optimal contract for the principal that satisfies the agent's participation constraints in the intermediate phase (Suppose that the value of the project is large enough, the principal is always willing to have a positive output.)
3. Suppose that with information technology the principal can obtain signals $\sigma \in \{\sigma_L, \sigma_H\}$. Suppose

$$\nu = Pr(\sigma = \sigma_L | \theta = \theta_L) = Pr(\sigma = \sigma_H | \theta = \theta_H) \geq \frac{1}{2}q.$$

Write down the principal's belief about the agent's efficiency after updating $\nu = Pr(\theta = \theta_L | \sigma = \sigma_L)$; $\nu = Pr(\theta = \theta_H | \sigma = \sigma_H)$. Solve the optimal contract for each σ .

4. Prove that the increase in μ will lead to an increase in the expected utility of the principal.

Exercise 16.19 A government agency signs a purchase contract with a company. The marginal cost of an enterprise producing q units is c , so its profit is $P - cq$. The cost of the business is private information, which may be high cost or low cost ($0 < c_L < c_H$). The government's prior belief about the cost is $\text{Prob}(c = c_L) = \beta$, and the company's reservation utility is 0.

1. Let $B(q)$ represent the revenue function of the government when it obtains q units. What is the government's second-best contract?
2. Compare the second-best contract with the first-best contract.
3. Suppose c obeys uniform distribution over $[0, 1]$. Solve the first-best contract and the second-best contract.

Exercise 16.20 (Lemon market) Consider a used car market. There are many sellers in the market. Each seller has a used car ready to sell. It will be sold with the quality of the car's quality $\theta \in [0, 1]$, and assumes that θ obeys uniform distribution over $[0, 1]$. If a seller of type θ sells his own used car at price P , then his utility is $u_s(p, \theta)$; if he is not sold, the utility obtained is 0. If a buyer buys a used car, his utility function is $\theta - p$; if not, it is 0. The quality of the used car is the seller's private information and the buyer did not know beforehand. Assume that there are not enough used cars on the market for all possible buyers.

1. Prove that for competitive equilibrium under asymmetric information, $E(\theta|p) = p$.
2. Prove that if $u_s(p, \theta) = p - \frac{\theta^2}{2}$, then any $p \in (0, 1/2)$ is an equilibrium price.
3. If $u_s(p, \theta) = p - \sqrt{\theta}$, find the equilibrium price and indicate which used cars will be traded in equilibrium.
4. Suppose $u_s(p, \theta) = p - \theta^3$. Solve the equilibrium price and find how many equilibriums there are in this case.
5. Are the above results Pareto efficient? If possible, propose a Pareto improvement plan.

Exercise 16.21 (Insurance Market) Consider the following insurance market model. There are two types: high risk and low risk. The initial wealth of each consumer is W , but it is possible to lose L due to fire. For high-risk and low-risk consumers, the probability of fire is p_H and p_L respectively, where $p_H > p_L$. Both types are pursuing the largest expected utility, and the utility function is $u(w)$, which satisfies neoclassical properties (such as differentiability, convexity, monotonicity,

etc.). There are two insurance companies and all are risk-neutral. Any insurance contract consists of a premium of M and a compensation of R insurance company pays according to claims.

1. Suppose that each consumer purchases up to one insurance contract. Prove that the insurance contract specifies the wealth of the insured in "no loss" state and "in loss" state.
2. Suppose that each insurance company provides a contract at the same time, and each company can provide any finite number of insurance contracts. What is the subgame perfect equilibrium for this problem? Does the equilibrium necessarily exist?

Exercise 16.22 (Pollution regulation) Investigate a manufacturer whose revenue is R . The manufacturer makes x units of pollution in production. The damage caused by the pollution is $D(x)$, $D'(x) > 0$, $D''(x) \geq 0$. The cost function of the manufacturer is $C(x, \theta)$, $C_x < 0$, $C_{xx} > 0$. θ is the parameter the manufacturer only knows, $\theta \in \{\theta_L, \theta_H\}$, $\nu = Pr(\theta = \theta_L)$ is public knowledge.

1. The first-best of pollution $x^*(\theta)$ under complete information is given by:

$$D'(x) + C_x(x, \theta) = 0.$$

Prove that if the regulator does not have to meet the manufacturer's participation constraint, he can implement it by giving the manufacturer a transfer payment equal to the damage of the pollution or a constant $x^*(\theta)$.

2. Suppose that manufacturer can now refuse to accept regulation (in this case the manufacturer utility is 0), and that the regulator's objective function is as follows:

$$W = -D(x) - (1 + \lambda)T + T - C(x, \theta),$$

where, T is the transfer payment from the regulator to the manufacturer, and $(1 + \lambda)$ is the opportunity cost of social fund spending. For any θ , the regulator must satisfy the manufacturer's participation constraints:

$$T - C(x, \theta) \geq q_0.$$

Write down the decision rule $\hat{x}(\theta)$ ($\theta \in \{\theta_L, \theta_H\}$) that maximizes W under full information, and compare with problem (1).

3. Suppose that $C_\theta < 0$ and $C_{x\theta} < 0$. Find the menu of contracts (T_L, x_L) and (T_H, x_H) that maximizes the expected W value under participation constraints and incentive compatibility constraints.
4. Suppose θ is distributed in the interval $[\theta_L, \theta_H]$ according to the distribution function $F(\theta)$ and the density function $f(\theta)$, satisfy:

$$\frac{d}{d\theta} \left(\frac{1 - F(\theta)}{f(\theta)} \right) < 0,$$

and $C_{\theta\theta x} \leq q_0$. Answer the same question as above.

Exercise 16.23 (Life insurance demand and adverse selection) A group of consumers have income of y_0 on the 1 period and no income on the 2 period. Each consumer knows his/her probability of death π_i . Insurance companies can neither observe the probability of death of consumers nor observe the behavior of consumers. Insurers offer insurance at a price of p : If a consumer purchases x units of insurance, the insurance company will receive px of income for the period and pay x when the consumer die. The consumer can choose to purchase the insurance amount of x and save the amount of s . Savings of the current period are added to the income of the consumer when he is alive in the next period. Assume there are many consumers. The goal of consumers is to maximize the income of each period to increase the consumption of family members. Consumer's utility function is Bernoulli's form.

1. If a consumer's utility function is:

$$u(c_1, c_2a, c_2d) = \log c_1 + (1 - \pi_i) \log c_2a + \pi_i \log c_2d.$$

Calculate the demand price elasticity of life insurance.

2. Suppose that the competition between insurance companies means that consumers' expectation payments are equal to insurance premiums. Prove that p is greater than the average of π_i .
3. Suppose that $C_\theta < 0$ and $C_{x\theta} < 0$. Find the menu of contracts (T_L, x_L) and (T_H, x_H) that maximizes the expected W value under participation constraints and incentive compatibility constraints.
4. If θ is distributed in the interval $[\theta_L, \theta_H]$ according to the distribution function $F(\theta)$ and the density function $f(\theta)$, satisfy:

$$\frac{d}{d\theta} \left(\frac{1 - F(\theta)}{f(\theta)} \right) < 0,$$

and $C_{\theta\theta x} \leq q_0$ answer the same question as above.

Exercise 16.24 (Controlling self-managing manufacturer) A manufacturer's production function is $y = \theta l^{1/2}$, where l is the number of workers (here considered as a continuous variable) and $\theta > 0$ is the parameter only known to the manufacturer. The fixed cost of production is A , so p represents the price of the manufacturer's product in the competitive market.

1. Suppose that the manufacturer is managed by the worker, and the objective function is:

$$U^{SM} = \frac{py - A}{l}.$$

Calculate the optimal number of workers for this worker self-managing manufacturer.

2. Let w represent the wage rate under the perfectly competitive labor market, i.e., w is the opportunity cost of labor in this economy. What is the optimal allocation of labor? What happens if w is too large? Why is the scale of self-managed manufacturer generally not optimal?
3. Suppose the government knows θ . Investigate the situation where w is small enough to make the self-managed company relatively reasonable. Calculate the unit product tax τ that can restore the first-best labor allocation. Prove that the same goal can be achieved by imposing a fixed tax of T on the manufacturer (assuming that the size of the manufacturer is negligible compared to the entire economy).
4. Suppose that the government does not know θ , only know that θ may be θ_L or θ_H , $\Delta\theta = \theta_H - \theta_L > 0$. The government uses a mechanism $(l(\tilde{\theta}), t(\tilde{\theta}))$ specifying a labor input of $l(\tilde{\theta})$ and a transfer payment of $t(\tilde{\theta})$ for the type of manufacturer report $\tilde{\theta}$. The manufacturer's objective function is now:

$$U^{SM} = \frac{p\tilde{\theta}(l(\tilde{\theta}))^{1/2} + t(\tilde{\theta}) - A}{l(\tilde{\theta})}.$$

Write down a regulatory mechanism that leads to an honest report type.

5. Suppose that $\nu = Pr(\theta = \theta_L)$. Government want to maximize

$$U^G = p\theta^{1/2} - wl.$$

Prove that even if the transfer payment is costless to the government, the first-best result cannot be implemented. At this point, suppose the opportunity cost of the manufacturer is 0. What is the optimal regulatory mechanism?

Exercise 16.25 (Insurance contract) In a continuum economy, the economic individual's production function is $q = \theta l$, where θ is the productivity parameter and the probability density function is $f(\theta), \theta \in [\underline{\theta}, \bar{\theta}]$. The utility function of the economic individual is $u = u(c) - l$, which is a concave function.

1. If it is an autarkic; economic environment, solve the distribution of output and consumption in the economy.
2. Suppose that the insurance contract is signed in advance, that is, all economic individuals sign the insurance contract before they know their productivity parameter θ . If θ and l can be observed afterwards, solve the optimal insurance contract. If only θ can be observed afterwards, solve the optimal insurance contract.
3. In the previous question, if θ and l are not observable afterwards, and $f(\theta)/[1 - F(\theta)]$ monotonically increases, what is the optimal insurance contract?

Exercise 16.26 Consider an educational investment signaling model. There are two possible types of employee: $\theta \in \{\theta_H, \theta_L\}$, satisfying $\theta_H > \theta_L$. Given $i \in \{H, L\}$, the ex ante probabilities of this employee's type θ_i is β_i . The reservation utility for all employees is $\bar{u} = 0$. Each type θ can produce θ for the enterprise. Companies are willing to hire an employee at the salary level of w if and only if that employee's expected productivity can at least offset wages. Type θ can get e year of education at a cost of $c(e, \theta) = \frac{e}{\theta}$. The educational investment cost function $c(e, \theta)$ satisfies the single crossover property for (e, θ) , that is, if $e > e'$, then $c(e, \theta_L) - c(e', \theta_L) > c(e, \theta_H) - c(e', \theta_H)$. Given a salary level of w and an education level of e , type θ has a reward function of $u(w, e|\theta) = w - c(e, \theta)$. Consider the following sequential actions:

- The employee observe his own type that is his private information;
- The employee chooses education investment level;
- The company observes the level of the employee's education, but cannot observe his type;
- The employee proposes a salary offer to the company;
- The company either rejects the offer or accepts the offer and employs the employee at the salary level.

It is assumed that education investment can promote low-type employee to high-type employee. Specifically, suppose that the probability of type θ_L investing in a e year education to become type θ_H is $p(e)$, which satisfies the following properties: $p(0) = 0$, $\lim_{e \rightarrow \infty} p(e) = 1$, $p' > 0$, $p'(0) = \infty$, $\lim_{e \rightarrow \infty} p'(e) = 0$ and $p'' < 0$. Once type θ_L is invested in education and converted to type θ_H , he can observe his type change before entering the labor market. First, it is assumed that there is no asymmetric information between the company and its employees. Answer the following questions:

1. Write down the optimization problem that can find the optimal salary and the optimal education investment level.
2. Solve this problem. In particular, prove that the educational investment level of type θ_H is 0, while the education investment level of the low-type employee is strictly positive.
3. Suppose that investment in education now becomes more efficient, that is, the probability that type θ_L invests e in education and becomes type θ_H with the probability $q(e)$. It satisfies $q(e) > p(e)$ for any $e > 0$. What is the optimal education investment level at this time? What is the salary of type θ_L at this time? And give an intuitive explanation.

Exercise 16.27 (Educational investment signaling model 1) Assume that there is asymmetric information between the company and its employees, and adopts a refined Bayesian pure strategic equilibrium solution concept. Consider separating equilibrium to satisfy $e_H \neq e_L$. Answer the following questions:

1. In any segregated equilibrium, the type θ_L employee selects the education level $e_L = e_{FB}$, where FB represents the first-best.
2. Describe the education investment level of type θ_H , which satisfies $e_H \neq e_{FB}$.
3. Explain whether separation equilibrium always exists.
4. Suppose that investment in education is now becoming more efficient. That is, the probability that a worker of type θ_L invests in e year education and becomes type θ_H is $q(e)$. It satisfies $q(e) > p(e)$ for any $e > 0$. Does the employee of type θ_H change the level of education investment in equilibrium as described above? Explain your conclusions.

Exercise 16.28 (Educational investment signaling model 2) Consider the mixed equilibrium. It is satisfied that $e_H = e_L = e^*$. Answer the following questions.

1. Given educational investment e^* under mixed equilibrium, what is the salary of employee?
2. Describe the mixed equilibrium e^* .
3. Does the mixed equilibrium $e^* = 0$ always exist? If $e^* = e_{FB}$, does the mixed equilibrium always exist?
4. Suppose that investment in education now becomes more efficient. That is, the probability that a worker of type θ_L invests in e year education and becomes type θ_H is $q(e)$. It satisfies $q(e) > p(e)$ for any $e > 0$. Describe the influence of the change on the mixed equilibrium e^* . Explain your conclusions.
5. Given the assumption of effective education investment, is there mixed equilibrium satisfying intuitive criterion? (intuitive criterion)

Exercise 16.29 (Supervision cost) There are two types of economic individuals in the economy, banks and firms respectively, indexed by $i = 1, 2, 3, \dots$. Each bank receives one unit of investment endowment on every period, which can be invested or lend to firm. If the bank makes investment, it can get t_i units of certain benefit. A firm does not have any endowment on every period. But, every firm has an investment project, which can get a random benefit of $w_i \in [0, \bar{w}]$ with one unit investment, where w_i is independently identical distribution. The density function is $f(w)$ and the distribution function is $F(w)$. f is continuous and differentiable,

which is public information. For every firm i , w_i can be observed without cost. The bank need to pay γ units of effort cost to observe the investment benefit of a firm. The firm and bank sign loan contract on ex ante, the interest rate being x . The firm makes signal to the bank $w^s \in [0, \bar{w}]$ after observing the return of w . If $w^s \in S \subset [0, \bar{w}]$, the bank make supervision; if $w^s \notin S$, there is no supervision. According to the contract, the bank lend one unit of investment to the firm, the firm should return $R(w)$ unit if $w^s \in S$, where $0 \leq R(w) \leq w$. If $w^s \notin S$, the firm should return $K(w)$ unit. In order to guarantee that all borrowing demand is satisfied, assume the ratio of bank is $1/2 < \alpha < 1$.

1. Prove that $R(w) = w$ and $K(w) = x$ are the optimal payment plan.
2. For the optimal payment plan, rewrite the profit of bank and firm (a function of x), and prove that the profit of firm is monotonously decreasing with x .
3. Write down the optimization problem of firm, including the participation constraint of bank.
4. Suppose the profit function of bank is concave, prove there exist Nash equilibrium in the credit allocation of the economy.
5. Compare Stiglitz with Weiss(1981), all firms are homogenous in advance. Why there still exist credit allocation?

Exercise 16.30 (Status verification with cost) Suppose there exist two types of agent, with different cost function $C(q, \theta)$. Assume $C_q > 0$, $C_\theta > 0$, $C_{qq} > 0$ and $C_{q\theta} > 0$, where q is the output observed perfectly by the principal. $\theta \in \{\theta_L, \theta_H\}$, with the occurrence probabilities of v and $1 - v$, and $\Delta\theta \equiv \theta_H - \theta_L > 0$. The output q is valued at $S(q)$ by the principal, where $S' > 0$, $S'' < 0$ and $S(0) = 0$. Let T denote the transfer payment from principal to agent. The benefit function of agent is $t - C(q, \theta)$ and the cost function is $C(q, \theta) = \theta q$.

The principal has an audit technology. The principal can observe the true type of agent with the probability of p by paying a cost of $c(p)$. For the audit cost, assume $c(0) = 0$, $c' > 0$, $c'' > 0$. To insure the existence of inner point solution, suppose that the cost function satisfy Inada conditions. The incentive mechanism include four variables: payment $t(\bar{\theta})$, output $q(\bar{\theta})$, the probability of observing true type by auditing $p(\bar{\theta})$, and if $\tilde{\theta}$ the agent reported is different with the true θ , the principal can exert a punishment function $P(\theta, \tilde{\theta})$ on agent. In the equilibrium, revelation principle indicate that the report is true.

1. Write down the participation constraint and incentive compatibility constraints of the problem.
2. Suppose the punishment is exogenous, $P(\underline{\theta}, \bar{\theta}) \geq l$, and $P(\bar{\theta}, \underline{\theta}) \geq l$. Solve the optimal contract.
3. Suppose the punishment is endogenous, $P(\underline{\theta}, \bar{\theta}) \geq \bar{t} - \underline{\theta}\bar{q}$, $P(\bar{\theta}, \underline{\theta}) \geq \underline{t} - \bar{\theta}\underline{q}$. Solve the optimal contract.

4. Compare the difference of the above two results.

Exercise 16.31 (Financial contract) Assume that there are two types of participants, with bank acting as principal and company acting as agent. The profit of the investment project operated by the company is θ . Among them, a high profit of $\bar{\theta}$ is obtained with probability of v , and a low profit of $\underline{\theta}$ is obtained with probability of $1 - v$. The principal has a random auditing technique with a probability of $p(\theta)$. The cost of using this technique is $c(p)$.

1. Prove that if the type of profit reported by the agent is a high type, the optimal probability of auditing is 0.
2. Write down the incentive compatibility and participation constraints for this problem, as well as the principal's objective function.
3. Solve the optimal contract.
4. Under the optimal contract arrangement, will there be credit rationing in the economy?

Exercise 16.32 (Terminated threat) Assume that there are two types of participants, with bank acting as principal and company acting as agent. The profit of the investment project operated by the company is θ , where a high profit of $\bar{\theta}$ is obtained with probability of v and a low profit of $\underline{\theta}$ is obtained with probability of $1 - v$. The agent does not have any endowment at the beginning. If he needs to invest, he must invest I units. Suppose $v\bar{\theta} + (1 - v)\underline{\theta} > I$ and $\underline{\theta} > I$. Assume that the relationship of the contract lasts for two periods, and the profit type θ in both periods is independent and the time discount is 1. After knowing the type of agent's report in the first period, the principal can terminate the contract relationship with the agent at the end of the first period. The probability is \bar{p} and \underline{p} .

1. Write down the incentive compatibility and participation constraints for this two-stage problem.
2. Suppose the agent has limited liability, $\bar{t} \leq \bar{\theta}$ and $\underline{t} \leq \underline{\theta}$. Solve the optimal contract.
3. Compare the difference between the contract and the one with the random auditing technique.

16.12 Reference

Books and Monographs:

Bolton, P. and Dewatripont, M. (2005). *Contract Theory*, MIT Press.

- Cheung, Steven N. S. (1969). *The Theory of Share Tenancy*, University of Chicago Press.
- Laffont, J. J., and Martimort, D. (2002). *The Theory of Incentives: The Principal-Agent Model*, Princeton and Oxford: Princeton University Press, Chapters 4-5.
- Laffont, J. J., and Tirole, J. (1993). *The Theory of Incentives in Procurement and Regulation*, Cambridge: MIT Press.
- Luenberger, D. (1995). *Microeconomic Theory*, McGraw-Hill, Inc., Chapter 12.
- Mas-Colell, A., Whinston, M.D. and Green, J. (1995). *Microeconomic*, Oxford University Press, Chapter 14.
- Tirole, J. (2006) *The Theory of Corporate Finance*, Princeton University Press.
- Varian, H. R. (1992). *Microeconomic Analysis, Third Edition*, W.W. Norton and Company, Chapter 25.
- Wolfstetter, E. (1999). *Topics in Microeconomics—Industrial Organization, Auctions, and Incentives*, Cambridge Press, Chapters 8-10.

Papers:

- Alikakos, N. and Bates, P. (1984). "Estimates for the Eigenvalues of the Jordan Product of Hermitian Matrices", *Linear Algebra and its Application*, Vol. 57, 41-56.
- Akerlof, G. (1976). "The Market for Lemons: Quality Uncertainty and the Market Mechanism", *Quarterly Journal of Economics*, Vol. 89, 488-500.
- Baker, G. P. (1992). "Incentive Contracts and Performance Measurement", *Journal of Political Economy*, Vol. 100, 598-614.
- Green, L. R, and Stokey, N. (1983). "A Comparison of Tournaments and Contracts", *Journal of Political Economy*, Vol. 91, 349-364.
- Grossman, S., and Hart, O. (1983). "An Analysis of the Principal Agent", *Econometrica*, Vol. 51, 7-45.
- Guesnerie, R. (1981). "On Taxation and Incentives, Further Reflections on the Limits of Redistribution", Discussion paper No. 89, Sonderforschungsbereich 21, University of Bonn.
- Hammond, P. (1979). "Straightforward Individual Incentive Compatibility in Large Economies", *Review of Economic Studies*, Vol. 46, 263-282.

- Holmstrom, B. (1979). "Moral Hazard and Observability", *Bell Journal of Economics*, Vol. 10, 74-91.
- Holmstrom, B., and Milgrom, P. (1987). "Aggregation and Linearity in the Provision of Intertemporal Incentives", *Econometrica*, Vol. 55, 303-328.
- Holmstrom, B., and Milgrom, P. (1991). "Multitask Principal-Agent Analyses: Incentive Contracts, Asset Ownership, and Job Design", *Journal of Law, Economics and Organization*, Vol. 7, 24-52.
- Holmstrom, B. (1982). "Managerial Incentive Problems—A Dynamic Perspective". In *Essays in Economics and Management in Honor of Lars Wahlbeck*. Helsinki: Swedish School of Economics. (See also *Review of Economic Studies*, Vol. 66, No. 1, 169-182, 1999.)
- Holmstrom, B., and Tirole, J. (1994). "Financial Intermediation, Loanable Funds, and the Real Sector", *American Economic Review*, Vol. 84, 972-991.
- Laffont, J. J. (1994). "The New Economics of Regulation Ten Years After", *Econometrica*, Vol. 62, 507-538.
- Lazear, E. P., and Rosen, S. (1981). "Rank-Order Tournaments as Optimum Labor Contracts", *Journal of Political Economy*, Vol. 89, 841-864.
- Li, J. and Tian, G. (2003). "Optimal Contracts for Central Banks Revised", Working Paper, Texas A&M University.
- Meng, D., and Tian, G. (2013). "Multi-task Incentive Contract and Performance Measurement with Multidimensional Types", *Games and Economic Behavior*, Vol. 77, 377-404.
- Rochet, J.-C. (1985). "The Taxation Principle and Multi-time Hamilton-Jacobi Equations", *Journal of Mathematical Economics*, Vol. 14, 113-128.
- Rogerson, W. P. (1985). "The First-order Approach to Principal-agent Problems", *Econometrica*, Vol. 53, 1357-1367.
- Shapiro, C., and Stiglitz, J. (1984). "Equilibrium Unemployment as a Worker Discipline Device," *American Economic Review*, Vol. 74, 433-444.
- Stiglitz, J. (1974). "Incentives and Risk Sharing in Sharecropping", *Review of Economic Studies*, Vol. 41, 219-255.

Chapter 17

Principal-Agent Theory: Moral Hazard

17.1 Introduction

In the previous chapter, we stressed that the delegation of tasks creates an information gap between the principal and his agent when the latter learns some piece of information relevant to determining the efficient volume of trade. The principal thus needs to design appropriate incentive contracts to induce the agent to tell the truth. However, it is costly to provide incentives to agents to truly reveal their characteristics. So that the first-best result in general cannot be achieved and only the second-best result is possible. In this way, the optimal contract obtained with asymmetric information is different from the optimal contract with complete information. The basic problem in reverse selection is to make a trade-off between the extraction of rent and the efficiency of allocation.

Adverse selection is not the only problem of information asymmetry. In many cases, the principal usually has no control over the agent's actions and the agent's behavior is also unobservable or expensive to observe. In fact, due to the unobservability of its behavior, agents are always inevitably acting in ways that are beneficial to themselves and unfavorable to the principal. We call this type of hidden action moral hazard. Here are some examples of this.

- (1) The bank does not know the credit rating of the borrowing company, nor does it know whether the company will abuse the money after obtaining the loan or do its best to risk aversion.
- (2) The employer does not know the employee's ability to work, nor does he know that if the employee is lazy, does not care for machinery and equipment or does not favor the employ-

- er during working hours.
- (3) Drivers will not be particularly careful after buying car insurance.
 - (4) The public sector is not particularly concerned with job performance and innovation. Nevertheless, the profit or loss is national, but the risk is their own and they would not take risks to innovate.
 - (5) Duplicity: say yes and mean no. Saying one thing and doing another one.
 - (6) Not taking good care of public property like protecting themselves'. For example, before the reform and opening up in China, public bicycles would be broken in a year or two. Individual bicycles are still like new for years.
 - (7) Many government officials are not acting in position and are reluctant to take risks. In any case, more work will result in more mistakes, less work will result in fewer mistakes and no work will result in no mistake. Why should they do it?
 - (8) The big pots before the reform and opening up in China made workers and peasants have no enthusiasm.
 - (9) The teacher does not know whether the student is listening or not. Even if the student is staring at the teacher. We don't know whether he has heard it or not. Maybe he is thinking about other things.
 - (10) If there is no assessment or pressure of students, students will not work hard.

The leading candidates for such moral hazard actions are effort variables. In this chapter, we represent moral hazard with the agent's effort. The basic analysis of moral hazard issues is as follows. The agent's level of effort will affect the client's revenue, but the level of effort is not observable. As the effort will bring some negative effects (effort cost) to the agent, it may cause the agent not to work hard. At the same time, the outcomes of effort are uncertain. Effort may bring good outcomes (such as high output) and may also bring bad outcomes (such as low output). Therefore, the outcomes (such as output level) is a random variable. But the level of effort will affect the probability of the outcome (such as high or low output). For instance, the output of a field depends on the amount of time that the tenant has spent selecting the best crops or the quality of their harvesting. Similarly, the probability that a driver has a car accident depends on how safely he drives, which also affects his demand for insurance. Also, a regulated firm may have to perform a costly and nonobservable investment to reduce its cost of producing a socially valuable good.

Therefore, the principal needs to design appropriate contracts to give agents incentive to work hard instead of being lazy. What we want to examine is what kind of mechanism will make agents work hard. In the case of complete information, it's easy to make agents work hard through a strict reward and punishment system. In the case of hidden information, incentives are needed and the principal needs to pay a certain amount of information rent to motivate agents to work hard. For example, the benefits, bonus and other benefits that companies provide in order to make employees work hard are information rents. However, it is obviously impossible for such an incentive contract to be based on unobservable efforts, but only on the performance of the agent.¹ These performances are affected by the level of effort of the agents and they are also disturbed by some random factors.²

What the principal needs to do is to design a wage contract based on observable performance so as to induce the agent's best effort level to maximize their own interests. In this process, the principal needs to face the tradeoff between the two effects: incentives and risks. On the one hand, high-intensity incentive contracts can induce agents to work hard and thus increase the benefits of the principal. This is called the incentive effect. On the other hand, since the performance indicators used include noise factors, the performance-based incentives will be amplified by high incentives. The uncertainty caused by noise increases the risk that agents need to take. In this way, the risk-neutral principal must provide insurance for the risk-averse agent by paying more risk premium, which is a risk effect.³ What the principal needs to do is to make the optimal balance between incentive and risk to determine the optimal incentive intensity.

It should be emphasized that one of the differences between adverse selection and moral hazard is that for the principal, the uncertainty in the case of adverse selection is exogenous, but in the case of moral hazard it is endogenous. Therefore, the probability of moral hazard is different from the probability of natural state. Its size is determined by the degree of effort of the agent. For instance, careful driving will reduce the probability of car accidents. This uncertainty is critical to understanding the contractual issues of the moral hazard. If the correspondence between the agent's effort and performance is completely determined, there is no difficulty for the principal and the court to infer the agent's level of effort based on the observed outcomes. Even if the agent's effort cannot be directly observed,

¹For instance, the research achievements of researchers, the number of products produced by workers on the production line, etc.

²For instance, agricultural output depends not only on the efforts of farmers, but also on climate, hydrology and many other uncontrollable human factors.

³If the relationship between a performance indicator and effort is weak and stochastic, it is undoubtedly to guide agents to engage in a gambling activity with high-intensity incentives on such performance indicators.

it may also be indirectly stipulated by the contract because the outcome itself is observable and verifiable.

We will discuss the properties of incentive schemes that induce a positive and costly effort. Such schemes must thus satisfy an incentive compatibility constraint and the agent's participation constraint. Among such schemes, the principal prefers the one that implements the positive level of effort at minimal cost. By minimizing the cost of the incentive scheme, it is possible to derive the second-best cost for the implementation of the level of effort. In general, the second-best cost is greater than the first-best cost at the observable level of effort. An allocative inefficiency emerges as the outcome of the conflict of interests between the principal and the agent.

17.2 Basic Model

In this section, we will discuss the interaction between principal and agent by establishing the simplest form of moral hazard. The form of interaction is a contract in which the principal does not know the actual action of the agent. In the interaction, the principal has a task delegated to the agent and the agent needs to devote energy and bear the cost of the effort. After the task is completed, the agent's income comes from the principal's payment to the agent. There may be a potential conflict of interest between the principal and the agent (the principal wants the agent to do more work and the agent wants less work). However, the principal can use the design of the contract to ease their differences of interest and motivate agents to work hard. Encouraging the agent to work hard is the core issue to be solved by moral hazard.

17.2.1 Model Setting

Consider a simplest case where the completion of a task has two possible outcomes: q , $q = 1$ on success and $q = 0$ on failure. The returns to the principal are \bar{S} and \underline{S} respectively where $\bar{S} > \underline{S}$. The agent has two possible actions e which are working hard or lazy, corresponding to $e = 1, 0$. In general, the outcomes of the task depend on other external factors in addition to the effort. Therefore, we assume that the agent's action do not directly affect the outcome, but only affect the probability of the outcome.

Let π_e denote the probability that the agent succeeds when the agent selects the action e . The harder the agent is, the higher the probability of successful task is (i.e. $\pi_1 > \pi_0$). Let $\Delta\pi = \pi_1 - \pi_0$ denote the increase in the probability that the agent will choose to work harder to succeed.

Note that effort improves production in the sense of first-order stochastic dominance, namely, $\Pr(\tilde{q} \leq q^*|e)$ is decreasing with e for any given

production q^* . Indeed, we have $\Pr(\tilde{q} \leq \underline{q} | e = 1) = 1 - \pi_1 < 1 - \pi_0 = \Pr(\tilde{q} \leq \underline{q} | e = 0)$ and $\Pr(\tilde{q} \leq \bar{q} | e = 1) = 1 = \Pr(\tilde{q} \leq \bar{q} | e = 0)$.

In addition, the cost of the agent's choice for effort is denoted as $\psi = c(1)$ and $c(0) = 0$. The cost is zero when no effort is made. The principal cannot directly observe the agent's action. The compensation to the agent can only rely on the outcome of q . For this reason, the compensated contract is in the form of $t(q)$. $\bar{t} = t(1)$ and $\underline{t} = t(0)$ are the compensations for success and failure respectively.

Assume that the principal is risk-neutral and his utility function is $V(y) = y$. The agent's utility function for compensation and effort is separable: $U(t, e) = u(t) - c(e)$, where $u' > 0, u'' \leq 0$. $h = u^{-1}$, which will be useful in the below. Obviously, h is a strictly increasing convex function: $h' > 0, h'' \geq 0$. Under such a utility function setting, the agent chooses to work hard and his effort cost expressed in currency is $h(\psi)$.

If the agent's effort level is $e = 1$, the principal's expected utility function can be written as:

$$\pi_1(\bar{S} - \bar{t}) + (1 - \pi_1)(\underline{S} - \underline{t}). \quad (17.2.1)$$

If the agent's effort level is $e = 0$, the principal's expected utility function can be written as:

$$\pi_0(\bar{S} - \bar{t}) + (1 - \pi_0)(\underline{S} - \underline{t}). \quad (17.2.2)$$

For the agent, if he chooses $e = 1$, his expected utility is:

$$\pi_1 u(\bar{t}) + (1 - \pi_1) u(\underline{t}) - \psi. \quad (17.2.3)$$

We assume that agents choosing to work hard is socially dominant, namely, $\pi_1 \bar{S} + (1 - \pi_1) \underline{S} - h(\psi) > \pi_0 \bar{S} + (1 - \pi_0) \underline{S}$, or

$$\Delta\pi(\bar{S} - \underline{S}) > h(\psi). \quad (17.2.4)$$

Equation (17.2.4) means that the profit of effort is more than the cost of effort. Then the agent choosing to work hard is socially optimal. In this way, the question we have to answer is that in the case of moral hazard, can the principal always choose the appropriate contract so that the agent's choice is consistent with the social optimal?

The moral hazard game time sequence above can be represented by Figure 17.1.

17.2.2 Benchmark Case

In order to study the optimal contract under incomplete information, we first need to consider the basic case of the optimal contract under complete information. In this case, the agent's actions can be put into the contract and can be executed by a third party, such as a judicial agency. The problem of

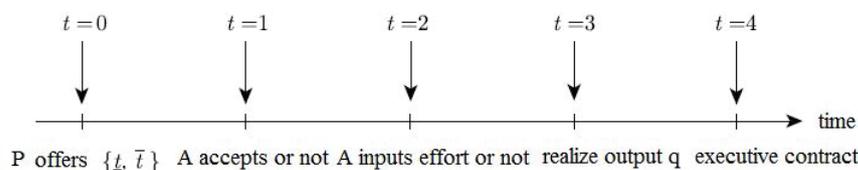


Figure 17.1: Contract time sequence under moral hazard

the principal is to design an optimal contract. The contract sets the action of the agent and the expected utility obtained by the agent which is not less than the reservation utility. For simplicity, let reservation utility be 0. Under condition (17.2.4), we want $e = 1$ to be optimal. The contract is the solution to the following problem:

$$\begin{aligned} & \max_{\{\underline{t}, \bar{t}\}} \pi_1(\bar{S} - \bar{t}) + (1 - \pi_1)(\underline{S} - \underline{t}) \\ \text{s.t.} \quad & \pi_1 u(\bar{t}) + (1 - \pi_1)u(\underline{t}) - \psi \geq 0. \end{aligned}$$

Since the action can be implemented by the contract, as long as the contract provides that if the agent does not choose $e = 1$, the agent will be given a serious punishment. Then the agent will abide by the contract. The above inequality constraint is called participation constraint. Because if the agent accepts the contract, his expected utility cannot be lower than the reservation utility. Obviously in this optimization problem, this constraint must be an equality constraint (sometimes called a binding constraint). Let λ be the Lagrangian multiplier involved in the constraint, in the binding constraints optimization problem:

$$L(\lambda, \bar{t}, \underline{t}) = \pi_1(\bar{S} - \bar{t}) + (1 - \pi_1)(\underline{S} - \underline{t}) + \lambda[\pi_1 u(\bar{t}) + (1 - \pi_1)u(\underline{t}) - \psi].$$

The first-order conditions for \bar{t} and \underline{t} are:

$$-\pi_1 + \lambda \pi_1 u'(\bar{t}^*) = 0, \quad (17.2.5)$$

$$-(1 - \pi_1) + \lambda(1 - \pi_1)u'(\underline{t}^*) = 0. \quad (17.2.6)$$

From the formula (17.2.5) and the formula (17.2.6), it can be immediately derived:

$$\underline{t}^* = \bar{t}^*,$$

Thus the agent can avoid all risks in the optimal contract. Since the participation constraint must be satisfied and the first-best contract is $\underline{t}^* = \bar{t}^* = h(\psi)$. That is, the transfer payment to the agent is exactly equal to the monetary cost of labor. In this case, as long as the agent works hard,

whether the outcome is good or bad, the principal should give the agent the same remuneration since the difference is caused by natural factors. For the agent, the expected return is:

$$\pi_1 \bar{S} + (1 - \pi_1) \underline{S} - h(\psi).$$

Why is this the first-best contract? This is because if the contract only requires the agent to choose $e = 0$, then the contract at this time is a solution to the following optimization problem:

$$\begin{aligned} & \max_{\{(\bar{t}, \underline{t})\}} \pi_0 (\bar{S} - \bar{t}) + (1 - \pi_0) (\underline{S} - \underline{t}) \\ \text{s.t.} \quad & \pi_0 u(\bar{t}) + (1 - \pi_0) u(\underline{t}) \geq 0. \end{aligned}$$

Similar to the reasoning above, we get the contract $\underline{t}' = \bar{t}' = 0$ and the agent's expected return is

$$\pi_0 \bar{S} + (1 - \pi_0) \underline{S}.$$

Under condition (17.2.4), it is clear that the principal stipulates that the agent's action in the first-best contract is $e = 1$. Therefore, the contract is Pareto effective in the case of complete information. Figure 17.2 depicts the level of effort under the first-best contract.

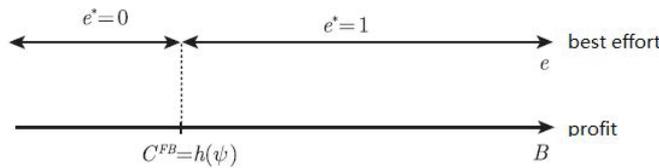


Figure 17.2: the level of effort under the first-best contract

17.2.3 Incentive Feasible Contracts

If the principal cannot directly observe the agent's action, the principal and the agent are unable to agree on the agent's action in the contract. If there is only one interaction between the principal and the agent, the agreed action in the contract is still unenforceable as long as the agent's action cannot be verified by a third party even if the principal can observe the agent's action.

In this situation, the contract can only be based on verifiable outcomes, that is, whether the task is successful. At this point, the agent's contractual form of avoiding all risks in the complete information situation is not enforceable. Because if $\underline{t}^* = \bar{t}^*$, the agent's compensation does not depend on the outcome. The agent has no incentive to choose the action $e = 1$. In

fact, intuition is very simple. The difference in outcomes is not known if it is caused by hard work or natural (uncertainty) when the action is not observable. As such, the principal should not give the agent the same amount of compensation. Otherwise, the agent has no incentive to choose $e = 1$ because of this asymmetric information. Thus, an implementable contract needs to introduce a new constraint which is similar to the adverse selection:

$$\pi_1 u(\bar{t}) + (1 - \pi_1)u(\underline{t}) - \psi \geq \pi_0 u(\bar{t}) + (1 - \pi_0)u(\underline{t}). \quad (17.2.7)$$

We call the formula (17.2.7) the incentive compatibility constraint. Under this condition, even if the principal does not specify the agent's actions, the agent will have the incentive to select the principal's desired action. Note that in the above incentive compatibility constraint (17.2.7), the complete information contract $\bar{t} = \underline{t}$ cannot satisfy the incentive compatibility condition.

In addition, the agent must satisfy the participation constraint in selecting the action, namely:

$$\pi_1 u(\bar{t}) + (1 - \pi_1)u(\underline{t}) - \psi \geq 0. \quad (17.2.8)$$

Note that the agent's participation constraint must be established before the output variations are realized.

The principal hopes that each level of effort token by the agent corresponds to a group of contracts that guarantees that the agent's moral hazard incentive compatibility constraint and participation constraint are established.

Definition 17.2.1 An incentive feasible contract satisfies the incentive compatibility and participation constraints (17.2.7) and (17.2.8).

17.2.4 Risk Neutrality and First-Best Implementation

If the agent is risk-neutral, we have (up to an affine transformation) $u(t) = t$ for all t and $h(u) = u$ for all u . The principal who wants to induce effort must thus choose the contract that solves the following problem:

$$\max_{\{(\bar{t}, \underline{t})\}} \pi_1(\bar{S} - \bar{t}) + (1 - \pi_1)(\underline{S} - \underline{t}) \quad (17.2.9)$$

$$\text{s. t.} \quad \pi_1 \bar{t} + (1 - \pi_1)\underline{t} - \psi \geq \pi_0 \bar{t} + (1 - \pi_0)\underline{t}; \quad (17.2.10)$$

$$\pi_1 \bar{t} + (1 - \pi_1)\underline{t} - \psi \geq 0. \quad (17.2.11)$$

Incentive compatibility constraint (17.2.10) can be written as:

$$\Delta \pi \bar{t} \geq \Delta \pi \underline{t} + \psi. \quad (17.2.12)$$

First, we can verify that in the above optimization problem, (17.2.11) must be binding. Because one of (17.3.14) and (17.2.11) must be binding, otherwise we can reduce \bar{t} to increase the principal's expected utility. Secondly, if (17.2.11) is not binding, we can reduce the same amount of \bar{t} and \underline{t} by a small amount. Under the above two constraints, the agent's expected utility can be increased. Of course, (17.3.14) may not be binding. The shaded area in Figure 17.3 depicts the area where incentives are feasible. Any point of the binding constraint (17.2.11) is risk-neutral second-best contract. Because objective function (17.2.9) can be written as:

$$\max_{\{(\bar{t}, \underline{t})\}} \pi_1 \bar{S} + (1 - \pi_1) \underline{S} - [\pi_1 \bar{t} + (1 - \pi_1) \underline{t}].$$

namely:

$$\min_{\{(\bar{t}, \underline{t})\}} \pi_1 \bar{t} + (1 - \pi_1) \underline{t}.$$

We pay attention to the equilibrium points that the incentive compatibility constraint (17.2.11) and participation constraint (17.3.14) satisfy, that is, the contract corresponding to the intersection of the two lines in Figure 17.3.

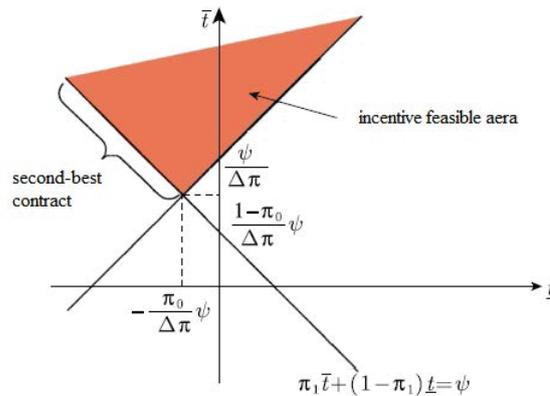


Figure 17.3: Optimal contract when the agent is risk-neutral

The principal makes an expected payment $\pi_1 \bar{S} + (1 - \pi_1) \underline{S} - \psi$. This result is consistent with the complete information situation and the expected return of the agent is 0. But the reward for different outcomes is not the same. This is different from the complete information situation although the expectation is equal to 0. So we have the following proposition:

Proposition 17.2.1 *Moral hazard is not an issue with a risk-neutral agent despite the nonobservability of effort. The first-best level of effort is still implemented.*

Remark 17.2.1 One may find the similarity of these results with those described last chapter. In both cases, when contracting takes place *ex ante*, the incentive constraint, under either adverse selection or moral hazard, does not conflict with the *ex ante* participation constraint with a risk-neutral agent, and the first-best outcome is still implemented.

Remark 17.2.2 Inefficiencies in effort provision due to moral hazard will arise when the agent is no longer risk-neutral. There are two alternative ways to model these transaction costs. One is to maintain risk neutrality for positive income levels but to impose a limited liability constraint, which requires transfers not to be too negative. The other is to let the agent be strictly risk-averse. In the following, we analyze these two contractual environments and the different trade-offs they imply.

17.3 Second-best Contract under Limited Liability and Risk Aversion

17.3.1 Second-best Contract under Limited Liability

In the previous section, we found that the imposition of the incentive compatibility constraint still results in the first-best outcome of the contract or does not change the agent's behavior (as opposed to the complete information condition) in the agent's risk-neutral situation. However, the incentive compatibility constraint requires that the agent's compensation difference must be different between the success and failure of the task: $\bar{t} - \underline{t} \geq \frac{\psi}{\Delta\pi}$. This means that the agent will be punished for failure.

In the optimal contract shown in Figure 17.3, this penalty has at least one lower bound: $\underline{t} \geq -\pi_0/\Delta\pi\psi$. However, due to some legal or ethereal cultural factors, the penalty still exists in an upper bound in reality. For example, in a limited liability company, the investor's loss limit in the case of failure is limited to the amount of investment and do not bear unlimited liability. In the following discussion what is limited liability (or a limited degree of punishment for the agent), the agent is still risk-neutral.

Let l be the upper bound of punishment which satisfies $\underline{t}, \bar{t} \in [-l, \infty)$. These constraints together with the incentive compatibility and participation constraints may prevent the principal from implementing the first-best level of effort even if the agent is risk-neutral. Indeed, when the principal

wants to induce a high effort, her program can be written as

$$\max_{\{\bar{t}, \underline{t}\}} \pi_1(\bar{S} - \bar{t}) + (1 - \pi_1)(\underline{S} - \underline{t}) \quad (17.3.13)$$

$$\text{s. t.} \quad \pi_1 \bar{t} + (1 - \pi_1) \underline{t} - \psi \geq \pi_0 \bar{t} + (1 - \pi_0) \underline{t}; \quad (17.3.14)$$

$$\pi_1 \bar{t} + (1 - \pi_1) \underline{t} - \psi \geq 0; \quad (17.3.15)$$

$$\bar{t} \geq -l; \quad (17.3.16)$$

$$\underline{t} \geq -l. \quad (17.3.17)$$

The second-best contract can be analyzed with the following figures. Similar to the market price floor (such as the minimum wage), if the transfer obtained without limited liability under the failure outcome does not exceed the upper bound of penalty, i.e., $-\pi_0/\Delta\pi\psi \geq -l$, then the limited liability constraint is not binding, so that we still have first-best outcome although the set of first-best outcomes is shrinking. This situation can be illustrated by Figure 17.4: We have seen that because of the existence of the upper bound of penalty, the set of incentives feasible contracts has been restricted. But, since the agent's participation constraints are still satisfied, it still results in first-best outcomes in the range of optimal contracts.

If the transfer obtained without limited liability under the failure outcome exceeds the penalty upper bound which means $-\pi_0/\Delta\pi\psi < -l$, the limited liability constraint is binding, we can only have second-best contract. We graphically show this in Figure 17.5, in which the region of incentive feasible contract is restricted. In this new region, the agent's participation constraint (17.2.11) becomes inequality constraints and the principal's optimal decision is to minimize the expected information rent in the viable area, i.e.,

$$\min_{\{\bar{t}, \underline{t}\}} \pi_1 \bar{t} + (1 - \pi_1) \underline{t}$$

subject to the limited liability constraint and incentive compatibility constraint.

As such, $\underline{t}^{SB} = -l$, at which the limited liability affects the transfer (penalty) for failure and it is equal to the upper bound of the penalty. Then $\bar{t}^{SB} = -l + \frac{\psi}{\Delta\pi}$, and thus the expected utility of the principal is $\pi_1 \bar{S} + (1 - \pi_1) \underline{S} - (\psi \frac{\pi_1}{\Delta\pi} - l)$ and the agent's expected utility is $\pi_1/\Delta\pi\psi - l > 0$. We can further obtain that the principal will provide incentives for the agent to choose $e = 1$ if and only if $\Delta\pi\Delta S \geq \psi + \pi_0/\Delta\pi\psi - l$.

We see that only when the failure state occurs, the limited liability (upper bound of punishment) constraint may be binding. Because exerting effort requires a difference between \bar{t} and \underline{t} , there must be $\bar{t} > -l$ when $\underline{t} \geq -l$. When the upper bound of penalty is smaller than the payment under the original failure, the principal's punishment for the agent is limited. In this case, the agent can only pay a penalty of l . When the task is successful, the agent gets a reward of $-l + \psi/\Delta\pi$. Therefore, the agent

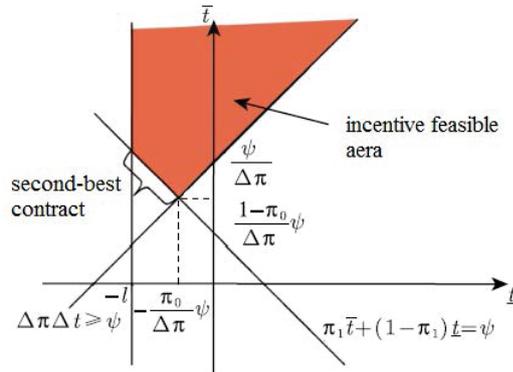


Figure 17.4: First-best outcomes still prevail when $l \geq \pi_0 \psi / \Delta \pi$

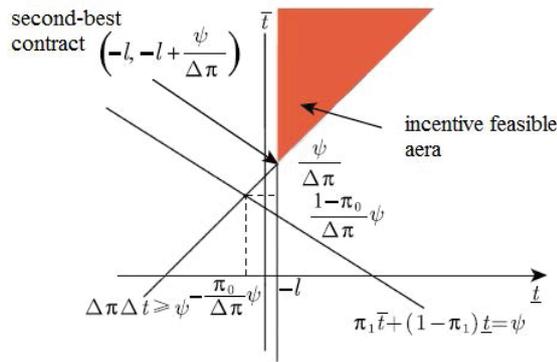


Figure 17.5: Only second-best outcomes prevail when $0 \leq l < \pi_0 \psi / \Delta \pi$

obtains a non-negative expected rent $EU^{SB} > 0$, which means the optimal contract is second-best, but not first-best. This rent derives from the additional payments made by the principal to the agent due to the combined effect of moral hazard and limited liability. With the continuous increase of l , the degree of protection of the agent with limited liability is getting smaller and smaller. The conflict between moral hazard and limited liability constraints is getting smaller. When $l > \pi_0 \psi / \Delta \pi$, there will be no conflict between them.

To sum up the above discussion, we have the following propositions.

Proposition 17.3.1 *With limited liability, the optimal contract inducing effort from the agent entails:*

- (1) For $l > \frac{\pi_0}{\Delta \pi} \psi$, only the incentive compatibility and participation constraints are binding so that the set of first-best transfers

is given in Figure 17.4, and one of them is given by $(\underline{t}^*, \bar{t}^*) = (-\pi_0/\Delta\pi\psi, (1-\pi_0)\psi/\Delta\pi)$. The agent has no expected limited liability rent which is $EU^{SB} = 0$.

(2) For $0 \leq l \leq \frac{\pi_0}{\Delta\pi}\psi$, the limited liability and incentive compatibility constraints are binding. Second-best transfers are given by

$$\underline{t}^{SB} = -l, \quad (17.3.18)$$

$$\bar{t}^{SB} = -l + \frac{\psi}{\Delta\pi}. \quad (17.3.19)$$

Moreover, the agent's expected limited liability rent EU^{SB} is non-negative:

$$EU^{SB} = \pi_1 \bar{t}^{SB} + (1-\pi_1) \underline{t}^{SB} - \psi = -l + \frac{\pi_0}{\Delta\pi} \psi \geq 0. \quad (17.3.20)$$

17.3.2 Second-best Contract under Risk Aversion

We now discuss the optimal contract under when the agent is risk-averse. In this case, there is a fundamental trade-off: efficiency and risk. We have seen that in the case of hidden information, if the agent does not bears any risk, the compensation contract that is independent of the outcome will not satisfy the incentive compatibility constraints. In this way, the agent needs to take a certain risk in order to satisfy incentive compatibility constraints. However, for the risk-averse agent, taking risks will reduce his expected returns. Since the agent's compensation must also satisfy the participation constraints, the cost for the risk will eventually be transferred to the principal. Therefore, the problem of the principal is to balance the efficiency from incentives and the cost of risk costs from incentives.

The principal's program is then written as:

$$\max \pi_1(\bar{S} - \bar{t}) + (1 - \pi_1)(\underline{S} - \underline{t}) \quad (17.3.21)$$

$$\text{s.t.} \quad \pi_1 u(\bar{t}) + (1 - \pi_1)u(\underline{t}) - \psi \geq \pi_0 u(\bar{t}) + (1 - \pi_0)u(\underline{t}), \quad (17.3.22)$$

$$\pi_1 u(\bar{t}) + (1 - \pi_1)u(\underline{t}) - \psi \geq 0. \quad (17.3.23)$$

If this is a concave plan, the first-order Kuhn-Tucker condition is the necessary and sufficient condition for the optimal solution. But this is not the case since the concave function $u(\cdot)$ appears on both sides of the formula (17.3.22). However, the following change of variables makes the new program is a concave plan. Define $\bar{u} = u(\bar{t})$, $\underline{u} = u(\underline{t})$, so $\bar{t} = h(\bar{u})$, $\underline{t} = h(\underline{u})$.

In this way, the issue of the principal can be written as follows:

$$\max_{\{\bar{u}, \underline{u}\}} \pi_1(\bar{S} - h(\bar{u})) + (1 - \pi_1)(\underline{S} - h(\underline{u})) \quad (17.3.24)$$

$$\text{s.t.} \quad \pi_1 \bar{u} + (1 - \pi_1)\underline{u} - \psi \geq \pi_0 \bar{u} + (1 - \pi_0)\underline{u}, \quad (17.3.25)$$

$$\pi_1 \bar{u} + (1 - \pi_1)\underline{u} - \psi \geq 0. \quad (17.3.26)$$

Since the objective function of the above problem is strictly concave in \bar{u} and \underline{u} because $h(\cdot)$ is strictly convex, and the constraints (17.3.25) and (17.3.26) are linear. The first-order condition of the Lagrangian equation is the sufficient condition for the constrained optimal solution.

Let λ and μ be the Lagrange multipliers for the constraints (17.3.25) and (17.3.26) respectively. The Lagrangian equation is :

$$\begin{aligned} L(\bar{u}, \underline{u}; \lambda, \mu) &= \pi_1(\bar{S} - h(\bar{u})) + (1 - \pi_1)(\underline{S} - h(\underline{u})) \\ &\quad + \lambda[\pi_1\bar{u} + (1 - \pi_1)\underline{u} - \psi - \pi_0\bar{u} - (1 - \pi_0)\underline{u}] \\ &\quad + \mu[\pi_1\bar{u} + (1 - \pi_1)\underline{u} - \psi]. \end{aligned}$$

Thus, the first-order conditions for \bar{u} and \underline{u} are given by

$$\frac{1}{u'(\bar{t}^{SB})} = \mu + \lambda \frac{\Delta\pi}{\pi_1}, \quad (17.3.27)$$

$$\frac{1}{u'(\underline{t}^{SB})} = \mu - \lambda \frac{\Delta\pi}{1 - \pi_1}, \quad (17.3.28)$$

where \bar{t}^{SB} and \underline{t}^{SB} are second-best transfer payments.

Since the incentive compatibility constraint (17.3.25) is binding, $\bar{t}^{SB} \neq \underline{t}^{SB}$ and $\lambda \neq 0$. In addition, because $u'(\cdot) > 0$, which means $\mu \neq 0$, the participation constraint (17.3.26) is also binding. Also, by the Kuhn-Tucker theorem, since $\lambda > 0, \mu > 0$ and u is a concave function, we have $\bar{t}^{SB} > \underline{t}^{SB}$. The compensation when the task is successful is greater than the compensation when it fails.

Since the incentive compatibility constraint (17.3.25) and the participation constraint (17.3.26) are all equality constraints, we obtain the following equation by solving the binary equations:

$$\bar{u} = \psi + (1 - \pi_1) \frac{\psi}{\Delta\pi}, \quad (17.3.29)$$

$$\underline{u} = \psi - \pi_1 \frac{\psi}{\Delta\pi}, \quad (17.3.30)$$

and thus we have

$$\bar{t}^{SB} = h\left(\psi + (1 - \pi_1) \frac{\psi}{\Delta\pi}\right), \quad (17.3.31)$$

$$\underline{t}^{SB} = h\left(\psi - \pi_1 \frac{\psi}{\Delta\pi}\right). \quad (17.3.32)$$

We then have the following propositions.

Proposition 17.3.2 *When the agent is strictly risk-averse, the optimal (second-best) contract that induces effort makes both the agent's participation and incentive*

constraints binding. This contract does not provide full insurance. Moreover, second-best transfers are given by

$$\bar{t}^{SB} = h\left(\psi + (1 - \pi_1)\frac{\psi}{\Delta\pi}\right) = h\left(\frac{1 - \pi_0}{\Delta\pi}\psi\right) \quad (17.3.33)$$

and

$$\underline{t}^{SB} = h\left(\psi - \pi_1\frac{\psi}{\Delta\pi}\right) = h\left(-\frac{\pi_0}{\Delta\pi}\psi\right). \quad (17.3.34)$$

Figure 17.6 shows the second-best solution for risk aversion.

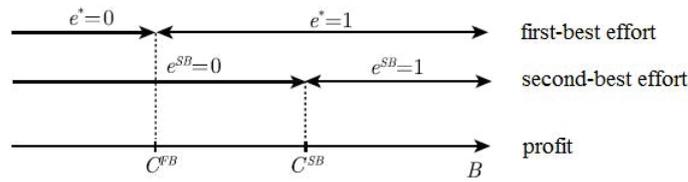


Figure 17.6: second-best solution for risk aversion

17.3.3 Basic Trade-off: Risk Aversion and Incentive

The second-best cost of inducing a high effort $e = 1$ is:

$$\begin{aligned} C^{SB} &= \pi_1 h\left(\psi + (1 - \pi_1)\frac{\psi}{\Delta\pi}\right) + (1 - \pi_1)h\left(\psi - \frac{\pi_1\psi}{\Delta\pi}\right) \\ &= \pi_1 h\left(\frac{1 - \pi_0}{\Delta\pi}\psi\right) + (1 - \pi_1)h\left(-\frac{\pi_0}{\Delta\pi}\psi\right). \end{aligned} \quad (17.3.35)$$

Since $h(\cdot)$ is strictly convex, we have $C^{SB} > h(\psi) = C^{FB}$ according to the Janssen inequality. The principal's return when he induces the agent to exert $e = 1$ is still

$$B = \Delta\pi\Delta S.$$

Thus, in the risk-averse situation, the principal will induce the agent to choose $e = 1$ if and only if $B = \Delta\pi\Delta S \geq C^{SB} > h(\psi) = C^{FB}$. With moral hazard, inducing high-level efforts is more difficult than complete information. Figure 17.7 compares the first-best contract and second-best contract under the complete and incomplete information.

When B belongs to the interval $[C^{FB}, C^{SB}]$, the second-best effort level is zero and therefore it is strictly less than the first-best effort level. Due to the combined effect of moral hazard and agent's risk aversion, the agent's efforts are distorted downwards.

The following proposition summarizes the basic trade-off under moral hazard.

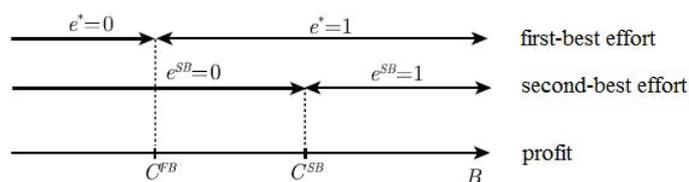


Figure 17.7: Second-best effort level in the context of moral hazard and risk aversion

Proposition 17.3.3 *With moral hazard and risk aversion, there is a trade-off between inducing effort and providing insurance to the agent (avoiding risk), and the principal induces a positive effort from the agent less often than when effort is observable.*

17.4 Contract Theory at Work

This section elaborates on the basic moral hazard model discussed above in a number of settings that have been discussed extensively in the contracting literature.

17.4.1 Efficiency Wage

In the labor market, a common observation is that in some firms, the wages of employees are often higher than the competitive wages in the labor market. This wage is called efficiency wage. Here, the exceeding part refers to the deduction of possible human capital factors. In this regard, a macroeconomic question is what prevents firms from adjusting their wages to market equilibrium wages. Study on these economic phenomena is sometimes called the Keynesian School, which discusses the microscopic basis of some non-classical economic equilibrium phenomena, of which efficiency wages are one of them, "involuntary unemployment" is closely related to it.

Shapiro and Stiglitz (AER, 1984) first discussed the microeconomic logic behind efficient wages. Their research framework is to consider this problem in a dynamic environment. Here we consider a simplified static version where we refer to the Laffont and Martmort (2002) model settings.

Suppose a firm employs a worker to perform a task, and both the owner and the employee are risk neutral. We standardized the equilibrium wages outside the market to 0. The firm wants the worker to work hard, assuming that the level of effort is discrete, i.e. $e \in \{0, 1\}$, and that the level of effort determines the outcome of the task, assuming there are only two scenarios, $y \in \{0, 1\}$. When the task succeeds, $y = 1$, the firm's added value is \bar{V} ; when it fails, the added value is \underline{V} with $0 \leq \underline{V}$, denoting $\Delta V = \bar{V} - \underline{V}$. The

success probability of a task depends on the effort input of the employee, and let π_e probability of succeed when the effort input is e with $1 > \pi_1 > \pi_0 > 0$, denoting $\Delta\Pi = \pi_1 - \pi_0$. Assume the effort function is $\psi(1) = \psi$ and $\psi(0) = 0$, which can not be directly observed by the firm. The firm wants the worker to work hard, but the most likely punishment is dismissal. At this point, the agent can only be rewarded for a good performance and cannot be punished for a bad outcome, since they are protected by limited liability.

To induce effort, the principal (firm) must find an optimal compensation scheme $\{(\underline{t}, \bar{t})\}$ that is the solution to the program below:

$$\max_{\{(\underline{t}, \bar{t})\}} \pi_1(\bar{V} - \bar{t}) + (1 - \pi_1)(\underline{V} - \underline{t}) \quad (17.4.36)$$

subject to

$$\pi_1\bar{t} + (1 - \pi_1)\underline{t} - \psi \geq \pi_0\bar{t} + (1 - \pi_0)\underline{t}, \quad (17.4.37)$$

$$\pi_1\bar{t} + (1 - \pi_1)\underline{t} - \psi \geq 0, \quad (17.4.38)$$

$$\underline{t} \geq 0. \quad (17.4.39)$$

The problem is completely isomorphic to the one analyzed earlier. The limited liability constraint is binding at the optimum, and the firm chooses to induce a high effort when $\Delta\pi\Delta V \geq \frac{\pi_1\psi}{\Delta\pi}$. As such, we obtain a second-best optimal contract, at which $\underline{t}^{SB} = 0$ and $\bar{t}^{SB} = \frac{\psi}{\Delta\pi} > 0$. At this point, the expected wage is greater than the market equilibrium wage, and the reason why the efficiency wage is greater than the market equilibrium wage is that the firm needs to give the worker an incentive to work hard. The expected utility of the firm is $\pi_1\bar{t} + (1 - \pi_1)\underline{t} - \psi = \frac{\pi_0\psi}{\Delta\pi} > 0$. The incentive cost of the firm is $\frac{\pi_1\psi}{\Delta\pi}$, and only when $\Delta\pi\Delta V > \frac{\pi_1\psi}{\Delta\pi}$, the firm has incentive to induce the worker to work hard.

The positive wage $\bar{t}^{SB} = \frac{\psi}{\Delta\pi}$ is often called an efficiency wage because it induces the agent to exert a high (efficient) level of effort. To induce production, the principal must give up a positive share of the firm's profit to the agent.

17.4.2 Sharecropping

The theoretical framework of moral hazard is also widely applied to development economics. The sharing of rent in the agricultural economy is an incentive mechanism. If the landlord employs someone to work on the farm, the farmer with the fixed income does not have the incentive to do his best. If the landlord has full control over the work situation of the farmer, he can naturally command him accordingly. However, to obtain such sufficient information, the landlord will have to spend a lot of effort to supervise the monitoring personally. If this is not done, the landlord and the farmer

will have different information and production will not be able to be fully efficient.

Another extreme approach is to rent the land at a fixed amount which can give farmers a very large incentive. However, one should not forget that agriculture is also a high-risk business. The poorest farmers may not be able to bear this uncertainty at all. Therefore, the sharecropping arrangement arises. This approach attenuates the incentives for work rather than eliminates them and bears some of the risk. Similar ideas can also be applied to health insurance. Most of the health insurance has so-called co-insurance (co-pay) which the risk is partially shared. But the patient still has some incentive to save. These examples all show that the diversity of information is common in the economic system, resulting in inefficiency. And it also prompts us to protect the under-informed party through contractual arrangements or informal consensus.

The idea of sharecropping originated from the doctoral dissertation of Steven Cheung (1969). Stiglitz (RES, 1974) extends Cheung's idea for a general setup of moral hazard problem. In sharecropping model examined by Stiglitz (RES, 1974), the principal is the landlord and the agent is the tenant. The tenant has only limited responsibilities. As such the optimal contract provided by Stiglitz is second-best. Through the following discussion, we can see that the linear contract is not a second-best contract. This outcome is different from one in Steven Cheung (1969). In reality, the rent sharing does not reach the optimal contract. The rent sharing is more beneficial to tenants but not to landowners.

We will discuss a simplified version of the Stiglitz (1974) model: By exerting an effort e in $\{0, 1\}$, the tenant increases (decreases) the probability $\pi(e)$ (resp. $1 - \pi(e)$) that a large \bar{q} (resp. small \underline{q}) quantity of an agricultural product is produced. The price of this good is normalized to one so that the principal's stochastic return on the activity is also \bar{q} or \underline{q} , depending on the state of nature.

We assume that the agent is risk neutral and protected by limited liability. When he wants to induce effort, the principal's optimal contract must solve

$$\max_{\{(\underline{t}, \bar{t})\}} \pi_1(\bar{q} - \bar{t}) + (1 - \pi_1)(\underline{q} - \underline{t}) \quad (17.4.40)$$

subject to

$$\pi_1\bar{t} + (1 - \pi_1)\underline{t} - \psi \geq \pi_0\bar{t} + (1 - \pi_0)\underline{t}, \quad (17.4.41)$$

$$\pi_1\bar{t} + (1 - \pi_1)\underline{t} - \psi \geq 0, \quad (17.4.42)$$

$$\underline{t} \geq 0. \quad (17.4.43)$$

The optimal contract therefore satisfies $\underline{t}^{SB} = 0$ and $\bar{t}^{SB} = \frac{\psi}{\Delta\pi}$. This is again akin to an efficiency wage. The expected utilities obtained respec-

tively by the principal and the agent are given by

$$EV^{SB} = \pi_1 \bar{q} + (1 - \pi_1) \underline{q} - \frac{\pi_1 \psi}{\Delta \pi}. \quad (17.4.44)$$

and

$$EU^{SB} = \frac{\pi_0 \psi}{\Delta \pi}. \quad (17.4.45)$$

The flexible second-best optimal contract described above has sometimes been criticized as not corresponding to the contractual arrangements observed in most agrarian economies. Contracts often take the form of simple linear schedules linking the tenant's production to his compensation. In particular, in the reality, the tenant's contract usually has the form of sharing. $t = \alpha q$, where α is the share of tenants' harvest. Next we will discuss the optimal share ratio under the sharecropping. Since $\underline{q} > 0$ and $\underline{t} = \alpha \underline{q} > 0$, the limited liability constraint naturally satisfies in the following discussion.

The optimal rent sharing contract is:

$$\max_{\alpha} (1 - \alpha)(\pi_1 \bar{q} + (1 - \pi_1) \underline{q}) \quad (17.4.46)$$

subject to

$$\alpha(\pi_1 \bar{q} + (1 - \pi_1) \underline{q}) - \psi \geq \alpha(\pi_0 \bar{q} + (1 - \pi_0) \underline{q}), \quad (17.4.47)$$

$$\alpha(\pi_1 \bar{q} + (1 - \pi_1) \underline{q}) - \psi \geq 0, \quad (17.4.48)$$

The formula (17.4.47) and the formula (17.4.48) are incentive compatibility constraint and participation constraint when the incentive tenant chooses to take effort. Since $\pi_0 > 0$, $\bar{q} > 0$ and $\underline{q} > 0$. If the formula (17.4.47) holds, the formula (17.4.48) must also hold. Therefore, only the incentive compatibility constraint (17.4.47) is binding. From this we get that the second-best contract for sharecropping is $\alpha^{SB} = \frac{\psi}{\Delta \pi \Delta q}$. Since the condition for selecting an incentive compatibility contract is $\Delta q \Delta \pi > \psi$, the optimal sharing ratio needs to satisfy $\alpha^{SB} < 1$.

This sharing rule also yields the following expected utilities to the principal and the agent, respectively

$$EV_{\alpha} = \pi_1 \bar{q} + (1 - \pi_1) \underline{q} - \left(\frac{\pi_1 \bar{q} + (1 - \pi_1) \underline{q}}{\Delta q} \right) \frac{\psi}{\Delta \pi} \quad (17.4.49)$$

and

$$EU_{\alpha} = \left(\frac{\pi_1 \bar{q} + (1 - \pi_1) \underline{q}}{\Delta q} \right) \frac{\psi}{\Delta \pi}. \quad (17.4.50)$$

Since there is no need to satisfy the limited constraints, this second-best sharecropping contract is different from the previous second-best contract for the principal. Indeed, comparing (17.4.44) and (17.4.49) on the one

hand and (17.4.45) and (17.4.50) on the other hand, we observe that the constant sharing rule benefits the agent but not the principal since the second-best contract brings more expected utility to the principal than the constant sharecropping contract. A linear contract is less powerful than the optimal second-best contract.

The constant sharecropping contract is an inefficient way to extract rent from the agent even if it still provides sufficient incentives to exert effort. With a linear sharing rule, the agent always benefits from a positive return on his production, even in the worst state of nature. This positive return yields to the agent more than what is requested by the second-best optimal contract in the worst state of nature, namely zero. Punishing the agent for a bad performance is thus found to be rather difficult with a linear sharing rule.

17.4.3 Financial Contracts: Credit Rationing

Information asymmetry in financial markets has two important issues, one is the adverse selection discussed in the previous chapter, the other is the moral hazard of borrowers to be discussed here. The discussion is about the credit rationing in financial market which was studied by Holmstrom and Tirole (1994) on credit rationing. It is also referred to the discussion of Laffont and Martimort (2002).

Moral hazard is an important issue in financial markets. In Holmstrom and Tirole (AER, 1994), it is assumed that a risk-averse entrepreneur wants to start a project that requires an initial investment worth an amount I . The entrepreneur has no cash of his own and must raise money from a bank or any other financial intermediary. The return on the project is random and equal to \bar{V} (resp. \underline{V}) with probability $\pi(e)$ (resp. $1 - \pi(e)$), where the effort exerted by the entrepreneur e belongs to $\{0, 1\}$. We denote the spread of profits by $\Delta V = \bar{V} - \underline{V} > 0$. The financial contract consists of repayments $\{(\bar{z}, \underline{z})\}$, depending upon whether the project is successful or not.

To induce effort from the borrower, the risk-neutral lender's program is written as

$$\max_{\{(\bar{z}, \underline{z})\}} \pi_1 \bar{z} + (1 - \pi_1) \underline{z} - I \quad (17.4.51)$$

subject to

$$\begin{aligned} & \pi_1 u(\bar{V} - \bar{z}) + (1 - \pi_1) u(\underline{V} - \underline{z}) - \psi \\ & \geq \pi_0 u(\bar{V} - \bar{z}) + (1 - \pi_0) u(\underline{V} - \underline{z}), \end{aligned} \quad (17.4.52)$$

$$\pi_1 u(\bar{V} - \bar{z}) + (1 - \pi_1) u(\underline{V} - \underline{z}) - \psi \geq 0. \quad (17.4.53)$$

Note that the project is a valuable venture if it provides the bank with a positive expected profit.

With the change of variables, $\bar{t} = \bar{V} - \bar{z}$ and $\underline{t} = \underline{V} - \underline{z}$, the principal's program takes its usual form. Then, the second-best cost of implementing a positive effort is given by $C^{SB} = \pi_1 h(\psi + (1 - \pi_1) \frac{\psi}{\Delta\pi}) + (1 - \pi_1) h(\psi - \frac{\pi_1 \psi}{\Delta\pi})$, where $h = u^{-1}$. At the same time, we have $\Delta\pi \Delta V \geq C^{SB}$ so that the lender wants to induce a positive effort level even in a second-best environment.

The lender's expected profit is worth:

$$V_1 = \pi_1 \bar{V} + (1 - \pi_1) \underline{V} - C^{SB} - I. \quad (17.4.54)$$

At this point, the initial investment of I has an important impact on whether the principal will lend to the firm, because the principal will finance the project only if the expected profit from the project is positive $V_1 > 0$. From 17.4.54, this requires the investment to be low enough, and typically we must have

$$I < I^{SB} = \pi_1 \bar{V} + (1 - \pi_1) \underline{V} - C^{SB}. \quad (17.4.55)$$

But, under complete information and no moral hazard, the project would instead be financed as soon as

$$I < I^* = \pi_1 \bar{V} + (1 - \pi_1) \underline{V} \quad (17.4.56)$$

Comparing (17.4.55) with (17.4.56), we obtain $I^{SB} < I^*$. This implies that in an asymmetric environment, the borrower's moral hazard implies that the borrower's credit scale is smaller than that in a complete information environment, i.e., for I in $[I^{SB}, I^*]$, some projects are financed under complete information but no longer under moral hazard. This result can be seen as similar to the form of credit rationing.

Finally, note that the optimal financial contract offered to the risk-averse and cashless entrepreneur does not satisfy the limited liability constraint $\underline{t} \geq 0$. Indeed, we have $\underline{t}^{SB} = h\left(\psi - \frac{\pi_1 \psi}{\Delta\pi}\right) < 0$. To be induced to make an effort, the agent must bear some risk, which implies a negative payoff in the bad state of nature. Adding the limited liability constraint, the optimal contract would instead entail $\underline{t}^{LL} = 0$ and $\bar{t}^{LL} = h\left(\frac{\psi}{\Delta\pi}\right)$. This contract has sometimes been interpreted in the corporate finance literature as a debt contract, with no money being left to the borrower in the bad state of nature and the residual being pocketed by the lender in the good state of nature.

Finally, note that

$$\begin{aligned} \bar{t}^{LL} - \underline{t}^{LL} &= h\left(\frac{\psi}{\Delta\pi}\right) \geq \bar{t}^{SB} - \underline{t}^{SB} \\ &= h\left(\psi + (1 - \pi_1) \frac{\psi}{\Delta\pi}\right) - h\left(\psi - \frac{\pi_1 \psi}{\Delta\pi}\right). \end{aligned} \quad (17.4.57)$$

since $h(\cdot)$ is strictly convex and $h(0) = 0$.⁴ This inequality shows that the debt contract has less incentive power than the optimal incentive contract. Indeed, it becomes harder to spread the agent's payments between both states of nature to induce effort if the agent is protected by limited liability by the agent, who is interested only in his payoff in the high state of nature, only rewards are attractive.

17.5 Extensions to the Basic Model

In this section, we will extend the basic model. In the basic model, there are two output outcomes and two agent's actions. If the outcome or action is finite or continuous, what will the new features of the second-best contracts be? In this section, we focus on situations where agents are risk averse.

17.5.1 More than Two Outcomes, Two Actions

We now extend our previous 2×2 model to allow for more than two levels of performance. We consider a production process where n possible outcomes can be realized. Those performances can be ordered so that $q_1 < q_2 < \dots < q_i < \dots < q_n$. We denote the principal's return in each of those states of nature by $S_i = S(q_i)$. In this context, a contract is a n -tuple of payments $\{t_1, \dots, t_n\}$. Also, let π_{ik} be the probability that production q_i takes place when the effort level is e_k . We assume that π_{ik} for all pairs (i, k) with $\sum_{i=1}^n \pi_{ik} = 1$. Finally, we keep the assumption that only two levels of effort are feasible. i.e., e_k in $\{0, 1\}$. We still denote $\Delta\pi_i = \pi_{i1} - \pi_{i0}$.

We say that the agent's effort to make $e = 1$, relative to $e = 0$, will result in better performance (in the random sense). Using mathematical definitions, the following performance distribution can be used to monotonically follow the action. The nature of the rate, that is, the effort of the agents to make the high-performance outcomes more likely to occur.

Definition 17.5.1 *The probabilities of success satisfy the monotone likelihood ratio property (MLRP) if $\frac{\pi_{i1} - \pi_{i0}}{\pi_{i1}}$ is nondecreasing in i .*

Suppose now that the agent is strictly risk-averse. The optimal contract that induces effort must solve the program below:

$$\max_{\{t_1, \dots, t_n\}} \sum_{i=1}^n \pi_{i1} (S_i - t_i) \quad (17.5.58)$$

⁴It can be easily seen that, when h is differentiable, by the mean-value theorem, $h(\frac{\psi}{\Delta\pi}) - h(\psi + (1 - \pi_1)\frac{\psi}{\Delta\pi}) = h'(\delta_1)(\frac{\pi_1\psi}{\Delta\pi} - \psi) > h'(\delta_2)(\frac{\pi_1\psi}{\Delta\pi} - \psi) = h(0) - h(\psi - \frac{\pi_1\psi}{\Delta\pi})$, where $\delta_1 > \psi + (1 - \pi_1)\frac{\psi}{\Delta\pi} > 0 > \delta_2$.

subject to

$$\sum_{i=1}^n \pi_{i1} u(t_i) - \psi \geq \sum_{i=1}^n \pi_{i0} u(t_i) \quad (17.5.59)$$

and

$$\sum_{i=1}^n \pi_{i1} u(t_i) - \psi \geq 0, \quad (17.5.60)$$

where the latter constraint is the agent's participation constraint.

Using the same change of variables as before, it should be clear that the program is again a concave problem with respect to the new variables $u_i = u(t_i)$. The objective function is:

$$\max_{\{u_1, \dots, u_n\}} \sum_{i=1}^n \pi_{i1} (S_i - h(u_i)), \quad (17.5.61)$$

where $h(\cdot) = u^{-1}$. The constraints are:

$$\sum_{i=1}^n \pi_{i1} u_i - \psi \geq \sum_{i=1}^n \pi_{i0} u_i, \quad (17.5.62)$$

$$\sum_{i=1}^n \pi_{i1} u_i - \psi \geq 0. \quad (17.5.63)$$

Let μ and λ be the Lagrange multipliers of the participate constraint (17.5.63) and the incentive compatibility constraint (17.5.62), the first-order conditions of the principal's program are written as:

$$\frac{1}{u'(t_i^{SB})} = \mu + \lambda \left(\frac{\pi_{i1} - \pi_{i0}}{\pi_{i1}} \right) \quad \forall i \in \{1, \dots, n\}. \quad (17.5.64)$$

Multiplying each of these equations by π_{i1} and summing over i yields $\mu = E_q \left(\frac{1}{u'(t_i^{SB})} \right) > 0$, where E_q denotes the expectation operator with respect to the distribution of outputs induced by effort $e = 1$.

Multiplying (17.5.64) by $\pi_{i1} u(t_i^{SB})$, summing all these equations over i , and taking into account the expression of μ obtained above yields

$$\lambda \left(\sum_{i=1}^n (\pi_{i1} - \pi_{i0}) u(t_i^{SB}) \right) = E_q \left(u(\tilde{t}_i^{SB}) \left(\frac{1}{u'(\tilde{t}_i^{SB})} - E \left(\frac{1}{u'(\tilde{t}_i^{SB})} \right) \right) \right). \quad (17.5.65)$$

Using the slackness condition $\lambda \left(\sum_{i=1}^n (\pi_{i1} - \pi_{i0}) u(t_i^{SB}) - \psi \right) = 0$ to simplify the left-hand side of (17.5.65), we finally get

$$\lambda \psi = cov \left(u(\tilde{t}_i^{SB}), \frac{1}{u'(\tilde{t}_i^{SB})} \right). \quad (17.5.66)$$

By assumption, $u(\cdot)$ and $u'(\cdot)$ covary in opposite directions. Moreover, a constant wage $t_i^{SB} = t^{SB}$ for all i does not satisfy the incentive constraint, and thus t_i^{SB} cannot be constant everywhere. Hence, the right-hand side of (17.5.66) is necessarily strictly positive. Thus we have $\lambda > 0$, and the incentive constraint is binding.

Coming back to (17.5.64), we observe that the left-hand side is increasing in t_i^{SB} since $u(\cdot)$ is concave. For t_i^{SB} to be nondecreasing with i , MLRP must hold again. Then higher outputs are also those that are the more informative ones about the realization of a high effort. Hence, the agent should be more rewarded as output increases.

17.5.2 Two Levels of Performance, Continuous Actions

We now discuss the second-best contract with continuous action and two levels of performance. Assume that the agent's effort is in a continuous set, $a \in [0, \infty)$. There are two possible outcomes, $q \in \{0, 1\}$. $q = 1$ represents a good outcome, otherwise it is a bad outcome. The outcome depends on the effort and $p(a) \equiv \text{prob}(q = 1|a) \in (0, 1)$ indicates the probability of a good outcome occurring when the effort is a . Suppose that there is $p'(a) > 0$ which means that if the agent chooses a higher level of effort, the probability of a good outcome is also higher. To simplify the discussion, we assume that the agent's effort to select a is $\Psi(a) = a$. Let w be the principal's compensation to the agent. Under incomplete information, the principal's payment to the agent depends only on the outcome q .

The utility function of the principal is $V(q - w) = q - w$, that is, the principal is risk-neutral. The agent's utility function is $u(w) - \Psi(a)$, which satisfies $u'(\cdot) > 0$ and $u''(\cdot) \leq 0$. So the agent is risk averse.

In the following, we consider the choice of the principal's contract when the information is complete and incomplete.

If the agent's actions can be observed by the principal and can be verified by a third party, then the principal's optimal contract choice is the following maximization problem:

$$\max_{a, w_0, w_1} p(a)(1 - w_1) + (1 - p(a))(-w_0) \quad (17.5.67)$$

$$\text{s.t. } p(a)u(w_1) + (1 - p(a))u(w_0) - a \geq 0, \quad (17.5.68)$$

The formula (17.5.68) is the agent's participation constraint. Let λ be the Lagrangian multiplier of the participation constraint. We get the optimal first-order condition:

$$\frac{1}{u'(w_1)} = \lambda = \frac{1}{u'(w_0)}, \quad (17.5.69)$$

Thus there is $w_1 = w_0$. The agent is fully insured and the principal bears all risks.

We now discuss the second-best contract when information is incomplete. Since the agent's actions cannot be observed by the principal, the agent's actions must satisfy the incentive compatibility condition. The principal contract choice is then the following maximization problem:

$$\max_{a, w_0, w_1} p(a)(1 - w_1) + (1 - p(a))(-w_0) \quad (17.5.70)$$

$$\text{s.t.} \quad p(a)u(w_1) + (1 - p(a))u(w_0) - a \geq 0, \quad (17.5.71)$$

$$a = \operatorname{argmax}_{a'} p(a')u(w_1) + (1 - p(a'))u(w_0) - a', \quad (17.5.72)$$

where (17.5.72) is the incentive compatibility condition in a continuous situation.

From the (17.5.72), we obtain:

$$p'(a)(u(w_1) - u(w_0)) = 1. \quad (17.5.73)$$

Let λ and μ be the Lagrange multipliers of the formula (17.5.68) and (17.5.73) respectively. From the first-order conditions of w_0 and w_1 , we get:

$$\frac{1}{u'(w_1)} = \lambda + \mu \frac{1}{u'(w_1)}, \quad (17.5.74)$$

$$\frac{1}{u'(w_0)} = \lambda - \mu \frac{1}{u'(w_0)}. \quad (17.5.75)$$

When the incentive compatibility constraint (17.5.73) is binding, we get $w_1 > w_0$ which means that the agent's greater effort will receive higher compensation and the agent will assume some risk of the second-best contract.

17.5.3 Continuous Performance, Continuous Actions

In this subsection, we discuss the case that performance and actions are both continuous. This approach is widely used in the literature. In the literature, this case is also widely used. In continuous situation, you can see the trade-off between efficiency and risk in second-best contracts better.

We assume that the agent's effort $a \in [0, \infty)$ (in the probability sense) determines the principal's output $q = a + \epsilon$ where $\epsilon \in (-\infty, \infty)$ obeys the normal distribution. It can be understood as an external factor that influences output with an expectation of 0 and a variance of σ^2 . The principal is risk averse and the agent's utility function for income is $u(w) = -e^{-rw}$ which is the constant-coefficient absolute risk aversion utility function. Assume that the principal's compensation for the agent is a linear function of the outcome: $w(q) = \alpha + \beta q$. We examine how α, β are chosen in the second-best optimal contract. In addition, we assume that the private cost of choosing a for the agent is $c(a)$.

Under the linear contract, the agent's compensation also obeys a normal distribution. And the expectation is $\bar{w} = \alpha + \beta a$ and the variance is $\sigma^2(w) = \beta^2 \sigma^2$. In the constant-coefficient absolute risk aversion utility function, its expected utility is $-e^{-(\bar{w} - \frac{r\sigma^2(w)}{2})}$.

Then, under linear compensation contracts, the agent's expected utility for compensation is:

$$-e^{-(\alpha + \beta a - \frac{r(\alpha\beta)^2 \sigma^2}{2})}. \quad (17.5.76)$$

The agent's incentive compatibility constraint is

$$a = \operatorname{argmax}_{a'} \alpha + \beta a' - \frac{r\beta^2 \sigma^2}{2} - c(a'),$$

Thus, its first-order condition is given by

$$c'(a) = \beta. \quad (17.5.77)$$

Therefore, the principal's second-best contract is to solve the following optimization problems:

$$\max_{a, \alpha, \beta} (1 - \beta)a - \alpha \quad (17.5.78)$$

$$\text{s.t.} \quad \alpha + \beta a - \frac{r\beta^2 \sigma^2}{2} - c(a) \geq \bar{u}, \quad (17.5.79)$$

$$c'(a) = \beta. \quad (17.5.80)$$

Here, utility function $-e^{(-\bar{u})}$ corresponds to $-\bar{u}$ is the agent's reservation utility. Obviously, the participation constraint (17.5.79) must be binding, otherwise a higher profit can be obtained by lowering α . Substituting participation constraint and incentive compatibility constraint into the objective function, the above optimization problem becomes:

$$\max_a a - \frac{rc'(a)^2 \sigma^2}{2} - c(a),$$

from which, we have the first-order condition:

$$1 - rc'(a)c''(a)\sigma^2 - c'(a) = 0,$$

and thus

$$\beta = c'(a) = \frac{1}{1 + rc''(a)\sigma^2},$$

where β is the marginal payment of the agent's effort outcomes in the second-best optimal contract which has a direct influence on the agent's incentive for the effort. So this coefficient is called Incentive Strength, which depends on the risk metric σ^2 and the degree of risk aversion of the agent. We can think of $r \rightarrow 0$ as a risk-neutral. According to (17.5.76), if $r = 0$, the

utility of the consumer depends only on the expectation of income. When the agent is risk-neutral or the outcome has no risk, the incentive intensity is 1. As the agent's risk aversion increases and the risk of the outcome increases, the incentive intensity becomes smaller and smaller. And this is precisely a fundamental trade-off under moral hazard: trade off between efficiency and risk of motivation.

For general continuous performance and continuous actions, second-best contracts do not necessarily have linear features. Holmstrom and Milgrom (1987) give the conditions that ensure the linear contracts to be second-best contracts. Interested readers can refer to their paper.

In reality, incentive contracts for managers usually contain more factors. We use the results from the above continuous case to discuss an incentive contract with multiple factors. Consider the following case: A manager who is trying to invest a will affect the company's current profits and the company's stock price as well. These two ways of influence are different because the factors affecting the stock price exceed the corporate accounting statements. These factors affect the future profitability of the company.

Let $q = a + \epsilon_q$ be the current profit level of the company where $\epsilon_q \sim N(0, \sigma_q^2)$. $p = a + \epsilon_p$ is the stock price of the company, where $\epsilon_p \sim N(0, \sigma_p^2)$. $Cov(\epsilon_q, \epsilon_p) = \sigma_{pq}$ is the covariance between stock price and current profit (external) influence factor. If $\sigma_{pq} = 0$, it means that these two factors are independent.

The agent's utility function for income is $u(w) = -e^{-rw}$ and the principal is risk-neutral. We consider a linear contract for manager compensation $w = \alpha + p\beta_p + q\beta_q$. We focus on how the manager's compensation is dependent on the current profits and the company's stock price in the second-best contract under incomplete information.

To simplify the discussion, we assume that the private cost of the manager's effort is $c(a) = \frac{ca^2}{2}$. The shareholder is the principal and her problem is:

$$\begin{aligned} & \max_{a, \alpha, \beta_p, \beta_q} q - (\alpha + p\beta_p + q\beta_q) \\ \text{s. t.} \quad & E[-e^{-(\alpha + p\beta_p + q\beta_q - c(a))}] \geq -e^{-\bar{u}}, \\ & a \in \operatorname{argmax}_{a'} E[-e^{-(\alpha + p(a')\beta_p + q(a')\beta_q - c(a'))}], \end{aligned}$$

where $E[\cdot]$ is the expected value for ϵ_p and ϵ_q . Use the certainty equivalence principle of income, the above optimization problem can be rewritten as:

$$\max_{a, \alpha, \beta_p, \beta_q} (1 - \beta_p - \beta_q)a - \alpha \quad (17.5.81)$$

$$\text{s. t.} \quad (\beta_p + \beta_q)a - \frac{r[\beta_p^2\sigma_p^2 + \beta_q^2\sigma_q^2 + 2\beta_p\beta_q\sigma_{pq}]}{2} - \frac{ca^2}{2} \geq \bar{u}; \quad (17.5.82)$$

$$a \in \operatorname{argmax}_{a'} (\beta_p + \beta_q)a' - \frac{r[\beta_p^2\sigma_p^2 + \beta_q^2\sigma_q^2 + 2\beta_p\beta_q\sigma_{pq}]}{2} - \frac{ca'^2}{2}. \quad (17.5.83)$$

Solving the incentive compatibility constraint (17.5.83), we get:

$$a = \frac{\beta_p + \beta_q}{c}.$$

Substituting it into the objective function (17.5.81) and participation constraint (17.5.82) and noting that the participation constraint is binding, we can simplify the principal's optimization problem as:

$$\begin{aligned} \max_a & (1 - \beta_p - \beta_q) \frac{\beta_p + \beta_q}{c} - \alpha & (17.5.84) \\ \text{s. t.} & (\beta_p + \beta_q) \frac{\beta_p + \beta_q}{c} - \frac{r[\beta_p^2 \sigma_p^2 + \beta_q^2 \sigma_q^2 + 2\beta_p \beta_q \sigma_{pq}]}{2} - \left(\frac{\beta_p + \beta_q}{c}\right)^2 \frac{c}{2} = 0 & (17.5.85) \end{aligned}$$

Simplifying the first-order conditions of β_p and β_q , we get:

$$\begin{aligned} \beta_p^* &= \frac{\sigma_q^2 - \sigma_{pq}}{\sigma_q^2 + \sigma_p^2 - 2\sigma_{pq}} \frac{1}{1 + rc\Omega}, \\ \beta_q^* &= \frac{\sigma_p^2 - \sigma_{pq}}{\sigma_q^2 + \sigma_p^2 - 2\sigma_{pq}} \frac{1}{1 + rc\Omega}, \end{aligned}$$

where $\Omega = \frac{\sigma_q^2 \sigma_p^2 - \sigma_{pq}^2}{\sigma_q^2 + \sigma_p^2 + 2\sigma_{pq}}$. The β_p^* and β_q^* obtained above are the marginal compensation for the managers of share price and current profit in the second-best contract.

When $\sigma_{pq} = 0$, β_p^* and β_q^* in the second-best contract become:

$$\begin{aligned} \beta_p^* &= \frac{\sigma_q^2}{\sigma_q^2 + \sigma_p^2 + rc2\sigma_q^2\sigma_p^2}, \\ \beta_q^* &= \frac{\sigma_p^2}{\sigma_q^2 + \sigma_p^2 + rc2\sigma_q^2\sigma_p^2}. \end{aligned}$$

When the agent is risk-neutral (i.e. $r = 0$), β_p^* and β_q^* in the second-best contract become:

$$\begin{aligned} \beta_p^* &= \frac{\sigma_q^2 - \sigma_{pq}}{\sigma_q^2 + \sigma_p^2 - 2\sigma_{pq}}, \\ \beta_q^* &= \frac{\sigma_p^2 - \sigma_{pq}}{\sigma_q^2 + \sigma_p^2 - 2\sigma_{pq}}, \end{aligned}$$

and thus $\beta_p^* + \beta_q^* = 1$. Along with σ_q^2 (σ_q^2) increases, β_q^* becomes smaller (larger) and β_p^* becomes larger (smaller). That is, if the profit risk is greater, the compensation to the manager reduces the sensitivity to profit and increases sensitivity to stocks, and *vas versa*.

When $\epsilon_p = \epsilon_q + \zeta$ with $\zeta \sim N(0, \sigma_\zeta^2)$, and ζ and ϵ_q are independent so that $\sigma_{pq} = \sigma_q^2$, we then have $\beta_p^* = 0$ and $\beta_q^* = \frac{1}{1+r c \sigma_q^2}$. Thus the stock will not enter the manager's compensation contract because the profit is a sufficient statistics of the stock. Introducing the stock will not increase the manager's incentives, but it will bring greater risk. Holmstrom (1979) proved the conclusion of sufficient statistics in similar incentive contracts.

17.6 Relative Performance Incentive of Multi-Agents

In reality, there are usually multiple agents inside an organization which will bring about some new incentive problems. The compensation of an agent depends not only on its own performance but also on the performance of others which is often referred to as relative performance incentives in the literatures. There are many literatures that use incentive theory to analyze incentives for government-level institutions. In addition to fiscal incentives, local officials are also concerned about their promotion. And an officer's promotion depends on his performance relative to other officials at the same level which is a common incentive method in hierarchical organizations. In this section we will discuss the logic of promotion or tournaments in organizational incentives and compare the differences between it and the incentive of sharing.

17.6.1 Tournament Model

We first introduce the tournament model of Lazear and Rosen (1981). They first considered a simple case: the principal allows two risk-neutral agents to complete their tasks. The performance of the task depends on their respective level of effort and some external factors. Lazear and Rosen revealed that in a risk-neutral situation, tournament can result in the first-best compensation contract.

Suppose the output function of the two agents is $q_i = a_i + \epsilon_i$, $i = 1, 2$, where a_i is effort and ϵ_i is iid. The distribution function is $F(\cdot)$ and the mean is 0. The variance is σ^2 . Assume that each agent's cost function of effort is $c(a_i) = \frac{c a_i^2}{2}$.

Optimal effort a^* then satisfies:

$$a^* = \frac{1}{c}.$$

Consider a tournament: if an agent's output performance is higher than another's, he will get a high compensation, otherwise he will get a low compensation. Let W_1 and W_2 with $W_1 > W_2$ be the high and low compensations respectively, and $\Delta W = W_1 - W_2$. Assume that the two agents are

symmetric, that is, they have the same cost and the same reservation utility. Their participation constraints are:

$$W_1 P_i + W_2 (1 - P_i) - \frac{ca_i^2}{2} \geq \bar{u},$$

where P_i is the probability that the participant (performance) wins, defined as:

$$P_i = \text{prob}(q_i > q_j) = \text{prob}(\epsilon_j - \epsilon_i < a_i - a_j) = H(a_i - a_j),$$

Here $H(\cdot)$ is the distribution function of $\epsilon_j - \epsilon_i$ with a mean of 0, a variance of $2\sigma^2$, and denote its density function by $h(\cdot)$. Under such a contract, the first-best choice for the agent i is given by:

$$\Delta W \frac{\partial P_i}{\partial a_i} - ca_i = \Delta W h(a_i - a_j) = ca_i.$$

In symmetric equilibrium $a_i^T = a^T$, we have

$$a^T = \frac{\Delta W h(0)}{c}.$$

Thus, when

$$\Delta W = \frac{1}{h(0)}, \quad (17.6.86)$$

we have $a^T = a^*$. At the same time, the agent's participation constraint is:

$$W_1 \frac{1}{2} + W_2 \frac{1}{2} - \frac{1}{2c} \geq \bar{u}. \quad (17.6.87)$$

Therefore, the tournament contract achieves the outcomes of the optimal contract when $W_1 = \frac{h(0)+c}{2ch(0)} + \bar{u}$, $W_2 = \frac{h(0)-c}{2ch(0)} + \bar{u}$, although the form of the contract was very different.

17.6.2 Tournament Contract under Risk Aversion

Now consider a tournament contract when the agent is risk averse and compare it to a piece rate contract or an individual performance contract. The following discussion is from Green and Stockey (1983) and refers to the discussion of Bolton and Dewatripont (2005).

There are two agents and they need to undertake two tasks. There are only two outcomes, namely $q_i \in \{0, 1\}$, corresponding to "failure" and "success". Assume that the two tasks are similar and their outcomes depend on a common state. There are two states. One has no external shock. The probability of success of the task depends on the agent's effort level of a_i . The other has an external shock. Once it occurs, the task will fail regardless of what the agent invests in. Assume that the principal knows whether there

is an external shock. The probability of a shock is $1 - \xi$. In the absence of an external shock, the agent's effort a_i has the following relationship to the probability of success of the task:

$$\Pr(q_i = 1|a_i) = \xi a_i,$$

The agent's effort cost function is:

$$c(a_i) = \frac{ca_i^2}{2}.$$

The agents' utility functions are same:

$$u(w) - c(a),$$

$u' > 0$ and $u'' < 0$. The principal is risk-neutral.

Let us discuss the tournament first. Since the outcome of the task and the common shock are observable, the feasible contract contains five possible compensations $w = (w_{11}, w_{10}, w_{01}, w_{00}, w_c)$ where w_{ij} is the compensation for the agent i when his output is q_i and the output of his opponent j is q_j . w_c is the compensation obtained by the agent when an external shock occurs. In the following we discuss the optimal incentive contract for agent i .

The incentive constraint for agent i is:

$$a_i = \operatorname{argmax}_a \max\{\xi[a(1 - a_j)u(w_{10}) + aa_ju(w_{11}) + (1 - a)(1 - a_j)u(w_{00}) + (1 - a)a_ju(w_{01})] + (1 - \xi)u(w_c) - \frac{1}{2}ca^2\}. \quad (17.6.88)$$

From the optimized first-order condition, the above incentive constraint (17.6.88) can be written as:

$$\xi[(1 - a_j)u(w_{10}) + a_ju(w_{11}) - (1 - a_j)u(w_{00}) - a_ju(w_{01})] = ca_i. \quad (17.6.89)$$

If the principal wants the agent i to choose a_i , his problem is to minimize the execution cost:

$$\begin{aligned} \min_w \quad & \xi[a_i(1 - a_j)w_{10} + a_i a_j w_{11} + (1 - a_i)(1 - a_j)w_{00} + (1 - a_i)a_j w_{01}] + (1 - \xi)w_c \\ \text{s.t.} \quad & \xi[a_i(1 - a_j)u(w_{10}) + a_i a_j u(w_{11}) + (1 - a_i)(1 - a_j)u(w_{00}) \\ & + (1 - a_i)a_j u(w_{01})] + (1 - \xi)u(w_c) - \frac{1}{2}ca_i^2 \geq \bar{u}, \end{aligned} \quad (17.6.90)$$

$$\xi[(1 - a_j)u(w_{10}) + a_ju(w_{11}) - (1 - a_j)u(w_{00}) - a_ju(w_{01})] = ca_i. \quad (17.6.91)$$

Let λ and μ be the Lagrange multipliers of the participate constraint (17.6.90) and the incentive compatibility constraint (17.6.91). The first-order conditions for w_{10} and w_{11} are respectively given by:

$$\xi a_i(1 - a_j) = \xi(1 - a_j)[\lambda a_i + \mu]u'(w_{10}), \quad (17.6.92)$$

$$\xi a_i a_j = \xi a_j[\lambda a_i + \mu]u'(w_{11}). \quad (17.6.93)$$

In contrast with (17.6.92) and (17.6.93), we get $w_{10} = w_{11}$.

Similarly, we can get the first-order conditions for w_{00} and w_{01} :

$$\xi(1 - a_i)(1 - a_j) = \xi(1 - a_j)[\lambda(1 - a_i) - \mu]u'(w_{00}), \quad (17.6.94)$$

$$\xi(1 - a_i)a_j = \xi a_j[\lambda(1 - a_i) - \mu]u'(w_{01}). \quad (17.6.95)$$

In contrast with (17.6.94) and (17.6.95), we get $w_{00} = w_{01}$.

The first-order conditions for w_c is:

$$(1 - \xi) = (1 - \xi)\lambda u'(w_c). \quad (17.6.96)$$

In contrast with (17.6.96) and (17.6.94), due to $\mu > 0$, we have $w_c \neq w_{00}$.

Comparing (17.6.94) with (17.6.92) and (17.6.96) with (17.6.92), we have: $w_{10} \neq w_{00}, w_c \neq w_{10}$.

17.6.3 Comparison of Tournament and Individual Performance Contract

Let us compare tournament and individual performance contract. For the tournament, it is mainly to compare the performances of agent i and agent j , q_i and q_j . If $q_i > q_j$, agent i 's reward is W ; if $q_i < q_j$, agent i 's reward is L ; if $q_i = q_j$, agent i 's reward is T . According to the previous discussion of the optimal contract, this means $W = w_{10}, L = w_{01}, T = w_{00}, T = w_{11}, T = w_c$.

For the individual performance contract, since this compensation method only depends on its own performance. $W_1 = w_{10} = w_{11}, W_0 = w_{01} = w_{00} = w_c$ according to the previous optimal contract.

We then obtain the following conclusions:

Conclusion 1: When the common shock can be separated (i.e. $w_c \neq w_{ij}$) and there is no public shock (i.e. $\xi = 1$), then for any relative performance incentive, we have $w_{10} \neq w_{11}, w_{00} \neq w_{01}$ or both. Moreover, the relative performance and tournament contract are second-best and the individual performance contract is first-best for the principal.

Conclusion 2: When $\xi < 1$, there is a common shock such that the individual performance contract is second-best for the principal. This is because the individual performance contract requires $w_c = w_{00}$, but the first-best contract requires $w_c \neq w_{00}$.

Conclusion 3: When $\xi < 1$, the tournament may be better than the individual performance contract for the principal.

Conclusion 1 and 2 above can be derived from the characteristics of tournament and individual performance contract.

In the following, we discuss the possibility of Conclusion 3. Assume that the first-best case for the principal is to make both agents choose $a_1 = a_2 = 1$. In the tournament, given agent j to choose $a_j = 1$, the cost for the principal to induce the agent i to choose $a_i = 1$ is given by

$$\begin{aligned} & \min_w \xi[a_i T + (1 - a_i)L] + (1 - \xi)T \\ \text{s.t.} \quad & \xi[a_i u(T) + (1 - a_i)u(L)] + (1 - \xi)u(T) - \frac{1}{2}ca_i^2 \geq \bar{u}, \\ & \xi[u(T) - (1 - a_j)u(L)] = ca_i. \end{aligned}$$

To induce the agent i to choose $a_i = 1$, the principal sets:

$$u(T) = u(L) + \frac{c}{\xi}.$$

When $a_i = a_j = 1$, agents i and j will receive a payment of T . In this case, the agent completely avoids the risk and there is no insurance cost.

However, for individual performance contract, if the principal wants to induce the agents to choose $a_1 = a_2 = 1$, the agent's compensation must be: $W_1 = w_{11} = w_{10} > W_0 = w_{01} = w_{00}$. Otherwise the agents will not have the incentive to choose $a_i = 1$ which means that the agents' payment is risky. Therefore, under a common shock, tournament is better than individual performance contract for principal.

The above is just a comparison of the utility of the two contract forms through a simple case. For more general situations, see Lazear and Rosen (1981) and Green and Stockey (1983).

17.7 Performance Incentive of Multi-Task

In many organizations, agents often perform multiple tasks at the same time. For example, for teachers, not only do they need to impart knowledge to students, they also need to cultivate students' imagination and creativity. But these different aspects of the task are very different in terms of measurement. The imparting of knowledge can be reflected in students' achievement. However, students' imagination and creativity are not well recognized. How do we form effective incentives for teachers in these two tasks? In the literature, the study of this problem is called multi-task principal-agent. Holmstrom and Milgrom (1991) conducted a deep study of this issue. We will discuss the incentive problem with multiple tasks through a simple model.

Assume that one agent is engaged in two tasks and the output of each task depends on respective effort:

$$q_i = a_i + \epsilon_i, i = 1, 2.$$

To simplify the discussion, assume $\epsilon_i \sim N(0, \sigma_i^2)$, $\text{cov}(\epsilon_1, \epsilon_2) = 0$, where ϵ_i is understood as a measurement error.

Assume that the cost of efforts exerted by the agent is:

$$\Psi(a_1, a_2) = \frac{1}{2}(c_1 a_1^2 + c_2 a_2^2) + \delta a_1 a_2,$$

where $0 \leq \delta \leq \sqrt{c_1 c_2}$. If $\delta = 0$, this means that the two efforts are completely independent. If $\delta = \sqrt{c_1 c_2}$, the two efforts can be completely substituted.

Assume that the principal is risk-neutral and the agent has constant-coefficient absolute risk aversion. Then her utility function is:

$$u(w; a_1, a_2) = -e^{-r[w - \Psi(a_1, a_2)]},$$

where w is the principal's monetary compensation to the agent.

Since the principal can only observe the two outputs of the agent, the principal formulates a linear incentive contract for the output:

$$w = \alpha + \beta_1 q_1 + \beta_2 q_2.$$

By the agent's deterministic equivalence principle, the agent's deterministic equivalent net income is:

$$\alpha + \beta_1 a_1 + \beta_2 a_2 - \frac{r}{2}(\beta_1^2 \sigma_1^2 + \beta_2^2 \sigma_2^2) - \frac{1}{2}(c_1 a_1^2 + c_2 a_2^2) - \delta a_1 a_2.$$

From the incentive compatibility condition of the agents above, the first-order condition of a_i is obtained:

$$\beta_i = c_i a_i + \delta a_j, i \neq j, i, j = 1, 2,$$

and thus

$$a_i = \frac{\beta_i c_j - \delta \beta_j}{c_i c_j - \delta^2}.$$

The principal's problem is to choose α, β_1, β_2 to implement the following optimization problem:

$$\max a_1(1 - \beta_1) + a_2(1 - \beta_2) - \alpha \quad (17.7.97)$$

$$\text{s.t.} \quad a_i = \frac{\beta_i c_j - \delta \beta_j}{c_i c_j - \delta^2}, i \neq j, i, j = 1, 2, \quad (17.7.98)$$

$$\begin{aligned} \alpha + \beta_1 a_1 + \beta_2 a_2 - \frac{r}{2}(\beta_1^2 \sigma_1^2 + \beta_2^2 \sigma_2^2) \\ - \frac{1}{2}(c_1 a_1^2 + c_2 a_2^2) - \delta a_1 a_2 \geq \bar{w}, \end{aligned} \quad (17.7.99)$$

where \bar{w} is the certainty equivalence income under the agent's reservation utility. Optimization means that the participation constraint (17.7.99) must be binding. Substituting a_1 and a_2 of incentive compatibility constraint

(17.7.98) and α of participation constraint (17.7.99) into the objective function (17.7.97), we have the unconstrained optimization problem:

$$\begin{aligned} \max_{(\beta_1, \beta_2)} & \left(\frac{\beta_1 c_2 - \delta \beta_2 + \beta_2 c_1 - \delta \beta_1}{c_1 c_2 - \delta^2} \right) - \frac{r}{2} (\beta_1^2 \sigma_1^2 + \beta_2^2 \sigma_2^2) \\ & - \frac{1}{2} c_1 \left(\frac{\beta_1 c_2 - \delta \beta_2}{c_1 c_2 - \delta^2} \right)^2 - \frac{1}{2} c_2 \left(\frac{\beta_2 c_1 - \delta \beta_1}{c_1 c_2 - \delta^2} \right)^2 - \delta \left(\frac{\beta_1 c_2 - \delta \beta_2}{c_1 c_2 - \delta^2} \right) \left(\frac{\beta_2 c_1 - \delta \beta_1}{c_1 c_2 - \delta^2} \right), \end{aligned}$$

The first-order conditions for β_1 and β_2 are then given by

$$\beta_1 = \frac{c_2 - \delta + \delta \beta_2}{c_2 + r \sigma_1^2 (c_1 c_2 - \delta^2)}, \quad (17.7.100)$$

$$\beta_2 = \frac{c_1 - \delta + \delta \beta_1}{c_1 + r \sigma_2^2 (c_1 c_2 - \delta^2)}, \quad (17.7.101)$$

from which we obtain β_1^* and β_2^* in the second-best optimal contract:

$$\beta_1^* = \frac{1 + (c_2 - \delta) r \sigma_2^2}{1 + r c_2 \sigma_2^2 + r c_1 \sigma_1^2 + r^2 \sigma_1^2 \sigma_2^2 (c_1 c_2 - \delta^2)}, \quad (17.7.102)$$

$$\beta_2^* = \frac{1 + (c_1 - \delta) r \sigma_1^2}{1 + r c_2 \sigma_2^2 + r c_1 \sigma_1^2 + r^2 \sigma_1^2 \sigma_2^2 (c_1 c_2 - \delta^2)}. \quad (17.7.103)$$

When the two tasks are independent (i.e. $\delta = 0$), we have $a_i = \frac{\beta_i}{c_i}$ according to the agent's incentive condition (17.7.98). Based on the conditions (17.7.102) and (17.7.103) of the second-best contract, we get: $\beta_i^* = \frac{1}{1 + r c_i \sigma_i^2}$. The result of this is that the principal takes a separate second-best contract for each of the agent's tasks.

In addition, as σ_2^2 increases, the incentive intensity of the second task will decrease $\frac{\partial \beta_2^*}{\partial \sigma_2^2} < 0$. Since the derivative of β_2^* with respect to σ_2 in formula (17.7.103) is less than 0, the incentive for the first task will also decrease. For interior solutions, we have $\frac{\partial \beta_1^*}{\partial \sigma_2^2} < 0$. Thus, if there is a substitutivity for efforts, there will be complementarity between multiple tasks. A decrease in the incentive intensity of a task means that the incentive intensity of the other task will also decrease. Holmstrom and Milgrom (1991) push this logic to the extreme and find that under certain conditions, if the principal cares about the performance of the two tasks, the principal's second-best contract does not provide any incentive for all tasks when the measurement error of a certain task tends to be infinite ($\sigma_i^2 \rightarrow \infty$) or cannot be measured.

This shows that the strength of incentives that the principal needs to face is the trade-off between efficiency and risk. Incentive performance with high-energy incentives will amplify the uncertainty caused by noise and increase the risk that agents must bear. If a relationship between performance indicators and effort is weak and highly random, the high-intensity incentives on such performance indicators undoubtedly means that the guiding agent engages in activities that are tantamount to gambling.

17.8 Implicit Incentive and Career Concern

In the labor market, the incentives of the laborers come from explicit incentives for job performance. For example, the better the performance, the higher the compensation. On the other hand, it also comes from the evaluation of the laborers in the market, especially the market's assessment of their abilities. It directly influences his future value in the labor market. The latter is called implicit incentive or career motivation. Holmstrom (1982, 1999) analyzed the logic of implicit incentive. We will discuss the implicit incentive in the principal-agent problem through a simple example.

Assume that there is no incentive contract between the principal and the agent. The agent pays attention to the current income and future earnings. Consider a model of two periods. The output of the agent in each period depends on effort, capabilities, and external uncontrollable factors:

$$q_t = \theta + a_t + \epsilon_t, t = 1, 2,$$

where θ is the intrinsic ability of the agent and does not change over time. Assume that the ability and effort of the agent cannot be observed. This model differs from the previous moral hazard model in that there is an information asymmetry about type. To simplify the discussion, assume that the principal and the agent are risk neutral. The agent's effort cost is $\Psi(a_t)$ which satisfies $\Psi'(a_t) > 0$ and $\Psi''(a_t) > 0$.

In the future period 2, all employers in the market will observe the agent's output q_i . Since in period 2, the agent does not have incentive for working, which means $a_2 = 0$, the market wage to the agent is $w_2(q_1) = E(\theta|q_1)$. The higher the capacity, the greater the output. Therefore, on the posterior estimate of the agent's ability, the higher the output of the first-stage agent, the higher the estimation of his ability in the market. In a purely strategic equilibrium, suppose that the market expects the agent to invest a^* in period 1. The ability's posterior belief expectation is:

$$w_2(q_1) = E(\theta|q_1) = q_1 - a^* = \theta + a_1 - a^*.$$

Let δ be the agent's discount rate for the future. The longer the future period, the greater the discount rate and there may be $\delta > 1$.

The agent's goal for the first period is:

$$\max_{a_1} = w_1 + \delta w_2(q_1) - \Psi(a_1) = w_1 + \delta(\theta + a_1 - a^*) - \Psi(a_1).$$

Since there is no explicit incentive, w_1 is not related to q_1 and the first-order condition for a_1 is:

$$\Psi'(a^*) = \delta.$$

The optimal labor input is:

$$\max_{a_1} a_1 - \Psi(a_1).$$

In this way, $\Psi'(a^{FB}) = 1$. When $\delta > 1$, the agent's effort will exceed the social optimal effort even if there is no incentive contract between the principal and the agent under the implicit incentive.

17.9 Incentive Intensity and Efficiency-Tradeoff between Distortions

In the previous principal-agent problem, the incentive intensity of the agent is balanced between efficiency and risk, which is based on an implicit hypothesis that there is a performance measure about the principal's interest (target). However, this assumption does not necessarily hold in reality. Although for listed companies, the stock price can be considered as an objective measure of the interests of shareholders. But for non-listed companies, government agencies, non-profit organizations and other institutions, there is usually no objective measure. Even if there is interest in an organization, it cannot be contracted and incorporated into the contract. Therefore, performance measurement is not an objective measure of the organization's goals.

For example, for government agencies, social welfare can be regarded as the government's goal, but it is difficult to measure. Economic output such as GDP is easy to measure, but it is not equal to social welfare. In this situation, how can the principal-agent problem be analyzed and discussed? Baker (1992) took the lead in introducing the analytic framework of this issue in the economics literature. He found that the consideration of the agent's incentive intensity for such a situation is a trade-off between incentive and distortion.

Assume that the principal's goal is denoted by $V(a, \epsilon)$ where a is the agent's effort whose cost function is assumed to be $C(a) = \frac{\epsilon}{2}a^2$ and ϵ is an external influence factor. $V(a, \epsilon)$ cannot be specified in the contract. Assume that $P(a, \epsilon)$ is a performance measure that can be contracted. For the sake of discussion, we assume that the principal and the agent are risk-neutral. Also assume that the incentive contract provided by the principal is $w(P) = \alpha + \beta P$. If $P(a, \epsilon) = V(a, \epsilon)$, the incentive intensity under the second-best contract is: $\beta = 1$ since the agent does not have a risk cost. What is the incentive intensity under the second-best contract if $P(a, \epsilon) \neq V(a, \epsilon)$?

First of all, we standardize performance measurement so that the effort's marginal contribution to the principal's goal is consistent with the contribution to performance:

$$E[P_a(a, \epsilon)] = E[V_a(a, \epsilon)].$$

Under the above incentive contract, the agent's utility is:

$$u(w, a) = w - C(a) = \alpha + \beta P(a, \epsilon) - C(a).$$

Assume that the agent's reservation utility is \bar{u} . Then, the incentive compatibility and participation constraints of the agent are:

$$\alpha + \beta P(a, \epsilon) - C(a) \geq \bar{u}, \quad (17.9.104)$$

$$\beta P_a(a^*, \epsilon) = C'(a^*). \quad (17.9.105)$$

The principal's goal is:

$$\max_{\alpha, \beta} E[V(a^*, \epsilon) - \alpha + \beta P(a^*, \epsilon)]$$

subject to the incentive compatibility and participation constraints (17.9.104) and (17.9.105), which leads to:

$$\beta^* = \frac{E[V_a a_\beta^*]}{P_a a_\beta^*}. \quad (17.9.106)$$

By (17.9.105), we have:

$$a_\beta^* = \frac{P_a}{c - \beta P_{aa}}, \quad (17.9.107)$$

where $P_{aa} = \frac{\partial^2 P}{\partial a \partial a}$. P is the second-order partial derivative of a . Substituting (17.9.107) into (17.9.106), we obtain:

$$\beta^* = \frac{E[V_a P_a]}{P_a^2}. \quad (17.9.108)$$

Without loss of generality, suppose that $E[p_a] = E[V_a] = 1$ for a^* . Then (17.9.108) can be written as:

$$\beta^* = \frac{\text{cov}(V_a, P_a) + 1}{\text{var}(P_a) + 1} = \frac{\rho \sigma_{V_a} \sigma_{p_a} + 1}{\sigma_{P_a}^2 + 1}, \quad (17.9.109)$$

where $\text{var}(P_a)$ is the variance of P_a , ρ is the correlation coefficient of V_a and P_a , and σ_{V_a} and σ_{p_a} are the standard deviations of V_a and P_a respectively.

If V_a and P_a are not completely related, then the incentive intensity mainly depends on the correlation between incentive performance V_a and P_a . The higher the correlation, the more incentive intensity the principal give to the agent. At the same time if $\rho < 0$, β^* may even be negative under certain conditions.

For a more in-depth discussion on the impact of the difference between contract performance and the value of the organization, see the discussion in Baker (1992).

17.10 A Mixed Model of Moral Hazard and Adverse Selection

So far, when we discuss asymmetric information between a principal and an agent in the previous and current chapters, we only allow either adverse selection or moral hazard presented, but not both. However, in many cases, a principle may know information neither about agent's action (such as his efforts) nor his characteristic (such as his risk aversion or cost). In this section, we consider the case where both action and characteristic of an agent is unknown to the principal. In doing so, we introduce a mixed model of moral hazard and adverse selection, which was recently studied in Meng and Tian (GEB, 2013). This model investigates the optimal wage contract design when both efforts and risk aversion of the agent are unobservable.

From the previous section on performance incentives under multi-task, we saw that using high-powered incentives amplifies the uncertainty brought about by noise and increases the risk that agents have to take, but it is assumed that the noise parameters can be observed by the principal. In this section, we assume that the risk aversion coefficient of an agent is his private information. The general theoretical model in this section reveals that in an innovative economy, the risk (insurance) effect should dominate the incentive effect because of the greater risk of innovative activities, so the low-powered incentive contract is reasonable and relatively optimal. When moral hazard and adverse selection coexist, the incentive intensity of the optimal contract will be further reduced. This provides a new explanation for the ubiquitous phenomenon of low-powered incentives, and proves the necessity of low-powered incentives for innovation-driven.

In the previous sections, we also assumed that the effort e and the performance output \tilde{q} are all one-dimensional. Now we turn to a more general situation that effort and performance are multidimensional and continuous. We will first consider the benchmark case where agent's efforts are unobservable, and then consider the case where either risk aversion or cost is also unobservable.

17.10.1 Optimal Wage Contract with Unobservable Efforts Only

Consider a principal-agent relationship in which the agent controls n activities that influence the principal's payoff. The principal is risk neutral and her gross payoff is a linear function of the agent's effort vector e :

$$V(e) = \beta'e + \eta, \quad (17.10.110)$$

where the n -dimensional vector β characterizes the marginal effect of the agent's effort e on $V(e)$, and η is a noise term with zero mean. The agent

chooses a vector of efforts $e = (e_1, \dots, e_n)' \in \mathbf{R}_+^n$ at quadratic personal cost $\frac{e'Ce}{2}$, where C is a symmetric positive definite matrix. The diagonal element C_{ii} reflects the agent's task-specific productivities, while the sign of off-diagonal elements C_{ij} indicates the relationship between two tasks i and j , which are substitute (resp. complementary, independent) if $C_{ij} > 0$ (resp. $< 0, = 0$). The agent's preferences are represented by the negative exponential utility function $u(x) = -e^{-rx}$, where r is the agent's absolute risk aversion and x is his compensation minus personal cost.

It is assumed that there is a linear relation between the agent's efforts and the expected levels of the performance measures:

$$P_i(e) = b_i'e + \varepsilon_i, i = 1, \dots, m, \quad (17.10.111)$$

where $b_i \in \mathbf{R}^n$ captures the marginal effect of the agent's effort e on the performance measure $P_i(e)$; $B = (b_1, \dots, b_m)'$ is an $m \times n$ matrix of performance parameters, and it is assumed that the matrix B has full row rank m so that every performance measure cannot be replaced by the other measures; and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)'$ is an $m \times 1$ vector of normally distributed variables with mean zero and variance-covariance matrix Σ .

Definition 17.10.1 (Orthogonality) A performance system is said to be orthogonal if and only if $b_i'C^{-1}b_j = 0$ and $Cov(\varepsilon_i, \varepsilon_j) = 0$, for $i \neq j$, that is, $B'C^{-1}B$ and Σ are both diagonal matrices.

Definition 17.10.2 (Cost-adjusted Correlation) The cost-adjusted correlation between two performance measures i and j is the ratio of the cost-adjusted inner product of their vectors of sensitivities divided by the covariance of the error terms:

$$\rho_{ij}^c = \frac{b_i'C^{-1}b_j}{\sigma_{ij}} \quad (17.10.112)$$

which measure performance not only about the agent's task-specific abilities and interaction among tasks, but also what degree two performance measures are aligned with each other. If n tasks are technologically independent and identical, i.e., $C = cI$, then we use the concept *correlation* $\rho_{ij} \equiv b_i'b_j/\sigma_{ij}$ to measure the degree of alignment between two performance measures.

Definition 17.10.3 (Cost-adjusted Congruence) The cost-adjusted congruence of a performance measure $P_i = b_i'e + \varepsilon_i$ is defined as

$$\Gamma_i = \frac{b_i'C^{-1}\beta}{\sqrt{b_i'C^{-1}b_i}\sqrt{\beta'C^{-1}\beta}}. \quad (17.10.113)$$

A performance measure with nonzero cost-adjusted congruence is called *congruent*; a performance measure with unit cost-adjusted congruence is said to be *perfectly congruent*. We assume that there exists at least one congruent measure, i.e., $BC^{-1}\beta \neq 0$.

The principal compensates the agent's effort through a linear contract:

$$W(e) = w_0 + w'P(e), \quad (17.10.114)$$

where $P(e) = (P_1(e), \dots, P_m(e))'$, w_0 denotes the base wage, and $w = (w_1, \dots, w_m)'$ the performance wage. Under this linear compensation rule, the principal's expected profit is $\Pi_p = \beta'e - w_0 - w'Be$, and the agent's certainty equivalent is

$$CE_a = w_0 + w'Be - \frac{1}{2}e'Ce - \frac{r}{2}w'\Sigma w. \quad (17.10.115)$$

The principal's problem is to design a contract (w_0, w) that maximizes her expected profit Π_p while ensuring the agent's participation and eliciting his optimal effort. The optimization problem of the principal is thus formulated as:

$$\begin{cases} \max_{\{w_0, w, e\}} \beta'e - w_0 - w'Be \\ \text{s.t. } IR : w_0 + w'Be - \frac{1}{2}e'Ce - \frac{r}{2}w'\Sigma w \geq 0 \\ IC : e \in \arg \max_{\tilde{e}} \left[w_0 + w'B\tilde{e} - \frac{1}{2}\tilde{e}'C\tilde{e} - \frac{r}{2}w'\Sigma w \right]. \end{cases}$$

The *IR* constraint ensures that the principal cannot force the agent into accepting the contract, and here the agent's reservation utility is normalized to zero; the *IC* constraint represents the rationality of the agent's effort choice.

We now consider the effort choosing problem of the agent for a given incentive scheme (w_0, w) . Since the objective is concave by noting that the second-order derivative of CE_a with respect to e is a negative definite matrix $-C$, the maximizer can be determined by the first-order condition: $Ce = B'w$. After replacing e with $e^* = C^{-1}B'w$ and substituting the *IR* constraint written with equality into the principal's objective function, the principal's optimization problem simplifies to:

$$\max_{w \in \mathbf{R}^m} \left[\beta'C^{-1}B'w - \frac{1}{2}w' \left(BC^{-1}B' + r\Sigma \right) w \right].$$

The optimal wage contract and effort to be elicited are therefore:

$$w^p = \left[BC^{-1}B' + r\Sigma \right]^{-1} BC^{-1}\beta \quad (17.10.116)$$

$$w_0^p = \frac{rw^{p'}\Sigma w^p - w^{p'}BC^{-1}B'w^p}{2} \quad (17.10.117)$$

$$e^p = C^{-1}B'w^p. \quad (17.10.118)$$

The resulting surplus of the principal is⁵

$$\Pi^p = \frac{1}{2} \beta' C^{-1} B' [BC^{-1} B' + r\Sigma]^{-1} BC^{-1} \beta. \quad (17.10.119)$$

If the performance evaluation system is an orthogonal system and the n tasks are technically identical and independent ($C = cI$), then we have

$$w_i = \frac{\mathbf{b}'_i \beta}{\mathbf{b}'_i \mathbf{b}_i + r c \sigma_i^2} = \frac{|\mathbf{b}_i| |\beta| \cos(\widehat{\mathbf{b}_i, \beta})}{|\mathbf{b}_i|^2 + r c \sigma_i^2},$$

which implies that the more sensitive the principal's revenue to the agent's effort is (the larger the norm of vector β_i), the greater the incentive intensity (i.e. w_i), and the greater the uncertainty of performance indicators (σ_i^2), the lower the incentive intensity (i.e. w_i).

Thus, a higher incentive pay could induce the agent to implement a higher effort, but it will also expose the agent to a higher risk. It therefore requires a premium to compensate the risk-averse agent for the risk he bears. The optimal power of incentive is therefore determined by the trade-off between incentive and insurance. Moreover, the results above show that in multi-task agency relationships, the degree of congruity of available performance measures and the agent's task-specific abilities also affects the power and distortion of incentive contract.

17.10.2 Optimal Wage Contract with Unobservable Efforts and Risk Aversion

The pure moral hazard incentive contract stated above relies crucially on the agent's attitude towards risk. In the following, we assume that risk aversion r is also private information of the agent, and its distribution function $F(r)$ and density function $f(r)$ supported on $[r, \bar{r}]$ are common knowledge to all parties. The principal then has to offer a contract menu $\{w_0(\hat{r}), w(\hat{r})\}$ contingent on the agent's reported "type" \hat{r} to maximize her expected payoff.

A contract $\{w_0(\hat{r}), w(\hat{r})\}$ is said to be implementable if the following incentive compatibility condition is satisfied:

$$w_0(r) + \frac{1}{2} w(r)' [BC^{-1} B' - r\Sigma] w(r) \geq w_0(\hat{r}) + \frac{1}{2} w(\hat{r})' [BC^{-1} B' - r\Sigma] w(\hat{r}). \quad (17.10.120)$$

Let $U(r, \hat{r}) \equiv w_0(\hat{r}) + \frac{1}{2} w(\hat{r})' [BC^{-1} B' - r\Sigma] w(\hat{r})$,⁶ and $U(r) \equiv U(r, r)$, then

⁵Superscript "p" denotes "pure moral hazard".

⁶Substituting $e^* = C^{-1} B' w$ into expression (17.10.115) yields $U = w_0 + \frac{1}{2} w' (BC^{-1} B' - r\Sigma) w$.

the implementability condition of $\{U(r), w(r)\}$ is stated equivalently as:

$$\exists w_0 : [\underline{r}, \bar{r}] \rightarrow \mathbf{R}_+, \forall (r, \hat{r}) \in [\underline{r}, \bar{r}]^2, U(r) = \max_{\hat{r}} \left\{ w_0(\hat{r}) + \frac{1}{2} w(\hat{r})' [BC^{-1}B' - r\Sigma] w(\hat{r}) \right\}, \quad (17.10.121)$$

which is in turn equivalent to the following very similar condition

$$\exists w_0 : \mathbf{R}^m \rightarrow \mathbf{R}_+, \forall r \in [\underline{r}, \bar{r}], U(r) = \max_w \left\{ w_0(w) + \frac{1}{2} w' [BC^{-1}B' - r\Sigma] w \right\}. \quad (17.10.122)$$

It is possible to show that $U(\cdot)$ is continuous, convex⁷ (thus almost everywhere differentiable), and satisfies the envelop condition:

$$U'(r) = -\frac{1}{2} w' \Sigma w. \quad (17.10.123)$$

Conversely, if (17.10.123) holds and $U(r)$ is convex, then

$$U(r) \geq U(\hat{r}) + (r - \hat{r})U'(\hat{r}) = U(\hat{r}) - \frac{1}{2}(r - \hat{r})w'(\hat{r})\Sigma w(\hat{r}),$$

which implies the incentive compatibility condition $U(r) \geq U(r, \hat{r})$. Formally, we have

Lemma 17.10.1 *The surplus function $U(r)$ and performance wage function $w(r)$ are implementable if and only if:*

- (1) *envelop condition (17.10.123) is satisfied;*
- (2) *$U(r)$ is convex in r .*

Substituting $U(r)$ into the principal's expected payoff, we get

$$\begin{aligned} \Pi &= \int_{\underline{r}}^{\bar{r}} [\beta' e^* - w_0(r) - w(r)' B e^*] f(r) dr \\ &= \int_{\underline{r}}^{\bar{r}} \left\{ \beta' C^{-1} B' w(r) - \frac{1}{2} w(r)' [BC^{-1}B' + r\Sigma] w(r) - U(r) \right\} f(r) dr. \end{aligned}$$

The principal's optimization problem is therefore:

$$\max_{U(r), w(r)} \Pi, \text{ s.t.: } U(r) \geq 0, U'(r) = -\frac{1}{2} w(r)' \Sigma w(r), U(r) \text{ is convex.} \quad (17.10.124)$$

The following proposition summarizes the solution of the principal's problem.

⁷One way to define the convex functions is through representing them as maximum of the affine functions, that is, $s(x)$ is convex if and only if

$$s(x) = \max_{a, b \in \Omega} (a \cdot x + b)$$

for some $a \in \mathbf{R}^n$, $b \in \mathbf{R}$ and some $\Omega \subset \mathbf{R}^{n+1}$. In this example $a = -\frac{1}{2} w' \Sigma w$, $b = w_0(w) + \frac{1}{2} w' BC^{-1} B' w$, and thus $U(r) = \max_{(a, b) \in \mathbf{R}_- \times \mathbf{R}_+} (ar + b)$ is a convex function in r .

Proposition 17.10.1 *If $\Phi(r)$ is nondecreasing⁸, then the optimal wage contract is given by*

$$w^h(r) = \left[BC^{-1}B' + \Phi(r)\Sigma \right]^{-1} BC^{-1}\beta \quad (17.10.125)$$

$$w_0^h(r) = \frac{1}{2} \int_r^{\bar{r}} w^h(\tilde{r})' \Sigma w^h(\tilde{r}) d\tilde{r} - \frac{1}{2} w^h(r)' \left[BC^{-1}B' - r\Sigma \right] w^h(r) \quad (17.10.126)$$

where $\Phi(r) \equiv r + \frac{F(r)}{f(r)}$.⁹

PROOF. Using the envelop condition $U'(r) = -\frac{1}{2}w'\Sigma w$, the participation constraint $U(r) \geq 0$ simplifies to $U(\bar{r}) \geq 0$. Incentive compatibility implies that only the participation constraint of the most risk averse type can be binding, i.e., $U(\bar{r}) = 0$. We therefore get

$$U(r) = \int_r^{\bar{r}} \frac{1}{2} w(\tilde{r})' \Sigma w(\tilde{r}) d\tilde{r}. \quad (17.10.127)$$

The principal's objective function becomes

$$\Pi = \int_{\underline{r}}^{\bar{r}} \left\{ \beta' C^{-1} B' w(r) - \frac{1}{2} w(r)' \left[BC^{-1}B' + r\Sigma \right] w(r) - \int_r^{\bar{r}} \frac{1}{2} w(\tilde{r})' \Sigma w(\tilde{r}) d\tilde{r} \right\} f(r) dr$$

which, by an integration of parts, gives

$$\int_{\underline{r}}^{\bar{r}} \left\{ \beta' C^{-1} B' w(r) - \frac{1}{2} w(r)' \left[BC^{-1}B' + \left(r + \frac{F(r)}{f(r)} \right) \Sigma \right] w(r) \right\} f(r) dr.$$

Maximizing pointwise the above expression, we get

$$w^h(r) = \left[BC^{-1}B' + \Phi(r)\Sigma \right]^{-1} BC^{-1}\beta$$

and

$$w_0^h(r) = \frac{1}{2} \int_r^{\bar{r}} w^h(\tilde{r})' \Sigma w^h(\tilde{r}) d\tilde{r} - \frac{1}{2} w^h(r)' \left[BC^{-1}B' - r\Sigma \right] w^h(r).$$

The only work left is to verify the convexity of $U(r)$. Notice that

$$U''(r) = -(D_r w^h)' \Sigma w^h = \Phi'(r) w^h(r)' \Sigma \left[BC^{-1}B' + \Phi(r)\Sigma \right]^{-1} \Sigma w^h(r).$$

⁸This condition is weaker than and could be implied by the monotone hazard rate property: $\frac{d}{dr} \left[\frac{F(r)}{f(r)} \right] \geq 0$.

⁹Superscript "h" denotes "hybrid model of moral hazard and adverse selection".

The second equality comes from the fact that the derivative of w^h with respect to r is¹⁰

$$\begin{aligned} D_r w^h &= - \left[BC^{-1}B' + \Phi(r)\Sigma \right]^{-1} \Phi'(r)\Sigma \left[BC^{-1}B' + \Phi(r)\Sigma \right]^{-1} BC^{-1}\beta \\ &= -\Phi'(r) \left[BC^{-1}B' + \Phi(r)\Sigma \right]^{-1} \Sigma w^h. \end{aligned}$$

It is clear that $U''(r) \geq 0$ because $\Phi'(r) \geq 0$ and the matrix $\Sigma \left[BC^{-1}B' + \Phi(r)\Sigma \right]^{-1} \Sigma$ is positive definite. The proof is completed. \square

The following conditions prove to be sufficient for the emergence of low-powered incentives.

Condition 17.10.1 Σ is diagonal.

Condition 17.10.2 Matrix $BC^{-1}B'$ is diagonal.

Condition 17.10.3 Matrices $BC^{-1}B'$ and Σ commute: $BC^{-1}B'\Sigma = \Sigma BC^{-1}B'$.¹¹

Condition 17.10.4 The following inequality holds:

$$2r\lambda_m^2 + \rho > 0 \quad (17.10.128)$$

where

$$\begin{aligned} \rho &= \max \left\{ \min_{i=1,m} \lambda_i \mu_m \frac{(\sqrt{k_\lambda} + 1)^2 - k_\mu(\sqrt{k_\lambda} - 1)^2}{2\sqrt{k_\lambda}}, \min_{i=1,m} \mu_i \lambda_m \frac{(\sqrt{k_\mu} + 1)^2 - k_\lambda(\sqrt{k_\mu} - 1)^2}{2\sqrt{k_\mu}} \right\} \\ &= \begin{cases} \lambda_m \mu_m \frac{(\sqrt{k_\lambda} + 1)^2 - k_\mu(\sqrt{k_\lambda} - 1)^2}{2\sqrt{k_\lambda}} & \text{if } \sqrt{k_\mu} \leq \frac{\sqrt{k_\lambda} + 1}{\sqrt{k_\lambda} - 1}, k_\mu \geq k_\lambda \\ \lambda_m \mu_m \frac{(\sqrt{k_\mu} + 1)^2 - k_\lambda(\sqrt{k_\mu} - 1)^2}{2\sqrt{k_\mu}} & \text{if } \sqrt{k_\mu} \leq \frac{\sqrt{k_\lambda} + 1}{\sqrt{k_\lambda} - 1}, k_\mu < k_\lambda \\ \lambda_1 \mu_m \frac{(\sqrt{k_\lambda} + 1)^2 - k_\mu(\sqrt{k_\lambda} - 1)^2}{2\sqrt{k_\lambda}} & \text{if } \sqrt{k_\mu} > \frac{\sqrt{k_\lambda} + 1}{\sqrt{k_\lambda} - 1}, k_\mu \geq k_\lambda \\ \lambda_m \mu_1 \frac{(\sqrt{k_\mu} + 1)^2 - k_\lambda(\sqrt{k_\mu} - 1)^2}{2\sqrt{k_\mu}} & \text{if } \sqrt{k_\mu} > \frac{\sqrt{k_\lambda} + 1}{\sqrt{k_\lambda} - 1}, k_\lambda < k_\mu \end{cases} \end{aligned}$$

λ_i, μ_i are the i -th eigenvalues of $\Sigma, BC^{-1}B'$ respectively in a descending enumeration. $k_\lambda = \frac{\lambda_1}{\lambda_m}$ and $k_\mu = \frac{\mu_1}{\mu_m}$ denote the spectral condition number of Σ and $BC^{-1}B'$ respectively.

Condition 17.10.5 There exists a positive number λ such that $BC^{-1}B' = \lambda\Sigma$.

¹⁰Let A be a nonsingular, $m \times m$ matrix whose elements are functions of the scalar parameter α , then

$$\frac{\partial A^{-1}}{\partial \alpha} = -A^{-1} \frac{\partial A}{\partial \alpha} A^{-1}.$$

¹¹That is, $BC^{-1}B'\Sigma$ is symmetric.

Condition 17.10.1 requires that the error terms of performance measures are stochastically independent. It assumes off the possibility that different measures are affected by common stochastic factor. Condition (17.10.2) states that $b'_i C^{-1} b_j = 0$ for all $i \neq j$. Intuitively, it requires that different performance measures respond in distinct ways to the agent's effort when cost is incorporated. Condition (17.10.4) holds when agent is sufficiently risk averse or when either matrix $BC^{-1}B'$ or Σ is well-conditioned¹². Several important special cases are:

- the performance measures system is orthogonal. In this case conditions (17.10.1) to (17.10.3) are all satisfied;
- Σ is a scalar matrix, in which case (17.10.1), (17.10.3) and (17.10.4) are satisfied;
- $BC^{-1}B'$ is a scalar matrix, in which case conditions (17.10.2), (17.10.3) and (17.10.4) are satisfied.

Condition (17.10.5) emphasizes that the covariance matrix Σ is a transformation of the measure-cost efficiency matrix $BC^{-1}B'$. That is to say, correlation between any pair of performance measures i and j is constant: $\rho_{ij} = \lambda$.

By comparing the wage contract obtained in the hybrid model to the benchmark pure moral hazard model, we find that the principal will reduce the power of incentives offered to the agent.

Theorem 17.10.1 1. *Given any one of the conditions 17.10.1 to 17.10.4, there exists an $i \in \{1 \cdots m\}$, such that $|w_i^h(r)| < |w_i^p(r)|$ for all $r \in (r, \bar{r}]$;*

2. *If both condition 17.10.1 and condition 17.10.2 are satisfied, then $|w_i^h(r)| < |w_i^p(r)|$ for all $r \in (r, \bar{r}]$ and all $i \in \{1 \cdots m\}$.*

3. *Let $\omega_i, i \in \mathcal{K} \equiv \{1, 2, \dots, k\}$ denote k distinct generalized eigenvalues of $BC^{-1}B'$ relative to Σ , $\mathcal{V}_i \equiv \mathcal{N}(BC^{-1}B' - \omega_i \Sigma)$ be the eigenspace corresponding to ω_i , and \mathcal{V}_i^\perp be its orthogonal complement. Suppose that $BC^{-1}\beta \notin \bigcup_{i \in \mathcal{K}} \mathcal{V}_i^\perp$, then there exists a positive number $k \in (0, 1)$ such that $w^h = kw^p$ if and only if condition 17.10.5 is met.*

PROOF. Since the proof is long and complicated, it is omitted here and can be found in Meng and Tian (2013). \square

When the risk aversion parameter is unobservable to the principal, the less risk-averse agent gains information rent by mimicking the more risk-averse one. The amount of information rent gained by an agent depends on the performance wage of agents with larger risk aversion, and therefore the

¹²Matrices with condition numbers near 1 are said to be well-conditioned, while matrices with high condition numbers are said to be ill-conditioned.

basic tradeoff between efficiency and rent extraction leads to low-powered incentive for all but the least risk-averse types. Under conditions (17.10.1) to (17.10.4), wage vector w is shortened in different quadratic-form norms compared with the pure moral hazard case. Under condition (17.10.5), the wage vector that minimizes the cost of effort $e'Ce = w'BC^{-1}B'w$ points in the same direction as the wage vector that minimizes the risk premium $rw'\Sigma w$. Consequently, the efficiency-rent tradeoff alters only the overall intensity of wage vector, not its relative allocation among performance measures.

17.10.3 Optimal Wage Contract with Unobservable Efforts and Cost

In this subsection we assume that the cost parameter is private information instead of risk aversion. To avoid the complicated multidimensional mechanism design issue, we assume that $C = cI$, that is, the tasks are technologically identical and independent. $\delta = \frac{1}{c}$ is assumed to be distributed on the support $[\underline{\delta}, \bar{\delta}]$, according to a cumulative distribution function $G(\delta)$ and density $g(\delta)$. The timeline of this problem is analogous to that in Section 17.10.2 except that the agent is now required to report $\hat{\delta}$. A contract menu $\{w_0(\delta), w(\delta)\}$ is said to be implementable if the following incentive compatibility condition is satisfied:

$$w_0(\delta) + \frac{1}{2}w(\delta)'[\delta BB' - r\Sigma]w(\delta) \geq w_0(\hat{\delta}) + \frac{1}{2}w(\hat{\delta})'[\delta BB' - r\Sigma]w(\hat{\delta}), \forall (\delta, \hat{\delta}) \in [\underline{\delta}, \bar{\delta}]^2. \quad (17.10.129)$$

Let $U(\delta, \hat{\delta}) \equiv w_0(\hat{\delta}) + \frac{1}{2}w(\hat{\delta})'[\delta BB' - r\Sigma]w(\hat{\delta})$, and $U(\delta) \equiv U(\delta, \delta)$. Then $\{U(\delta), w(\delta)\}$ is called implementable if

$$\exists w_0 : [\underline{\delta}, \bar{\delta}] \rightarrow \mathcal{R}_+, \forall (\delta, \hat{\delta}) \in [\underline{\delta}, \bar{\delta}]^2, U(\delta) = \max_{\hat{\delta}} \left\{ w_0(\hat{\delta}) + \frac{1}{2}w(\hat{\delta})'[\delta BB' - r\Sigma]w(\hat{\delta}) \right\}. \quad (17.10.130)$$

or equivalently,

$$\exists w_0 : \mathcal{R} \rightarrow \mathcal{R}_+, \forall \delta \in [\underline{\delta}, \bar{\delta}], U(\delta) = \max_{w \in \mathcal{R}^m} \left\{ w_0(w) + \frac{1}{2}w'[\delta BB' - r\Sigma]w \right\}. \quad (17.10.131)$$

$U(\delta)$ is necessarily continuous, increasing and convex in δ ¹³ and satisfies the envelop condition:

$$U'(\delta) = \frac{1}{2}w'BB'w. \quad (17.10.132)$$

¹³In this case, let $a = \frac{1}{2}w'BB'w$, $b = w_0(w) - \frac{1}{2}w'\Sigma w$, then $U(\delta) = \max_{a,b} (a\delta + b)$ is convex in δ .

Conversely, similar to the case with unobservable risk aversion, the convexity of $U(\delta)$ and envelop condition (17.10.132) implies

$$U(\delta) \geq U(\hat{\delta}) + (\delta - \hat{\delta})U'(\hat{\delta}) = U(\hat{\delta}) + \frac{1}{2}(\delta - \hat{\delta})w'BB'w = U(\delta, \hat{\delta}),$$

which in turn implies the implementability of contract. We summarize the above discussion in the following lemma.

Lemma 17.10.2 *The surplus function $U(\delta)$ and wage function $w(\delta)$ are implementable if and only if*

- (1) $U'(\delta) = \frac{1}{2}w'BB'w$;
- (2) $U(\delta)$ is convex in δ .

The second-best δ -contingent contract solves the following optimization problem:

$$\begin{cases} \max_{w(\delta), U(\delta)} \int_{\underline{\delta}}^{\bar{\delta}} \left\{ \delta \beta' B' w(\delta) - \frac{1}{2} w(\delta)' [\delta BB' + r\Sigma] w(\delta) - U(\delta) \right\} g(\delta) d\delta \\ \text{s.t: } U(\delta) \geq 0, U'(\delta) = \frac{w'BB'w}{2}, U(\delta) \text{ is convex} \end{cases}.$$

Proposition 17.10.2 *With unobservable cost, if $\delta H(\delta)$ is nonincreasing¹⁴, then the optimal wage is given by*

$$w^h(\delta) = \left(H(\delta)BB' + \frac{r\Sigma}{\delta} \right)^{-1} B\beta \quad (17.10.133)$$

$$w_0^h(\delta) = \frac{1}{2} \int_{\underline{\delta}}^{\delta} w^h(\tilde{\delta})' BB' w^h(\tilde{\delta}) d\tilde{\delta} - \frac{1}{2} w^h(\delta)' [\delta BB' - r\Sigma] w^h(\delta) \quad (17.10.134)$$

where $H(\delta) \equiv 1 + \frac{1-G(\delta)}{\delta g(\delta)}$.

PROOF. Again, the proof is omitted and referred to Meng and Tian (2013).
□

The following conditions justify the adoption of low-powered incentives in the case with unobservable cost parameter.

Condition 17.10.6 BB' is a diagonal matrix.

Condition 17.10.7 Matrices BB' and Σ commute: $BB'\Sigma = \Sigma BB'$.¹⁵

¹⁴This assumption is a bit stronger than the usual monotone inverse hazard rate condition. It holds for any nondecreasing $g(\cdot)$.

¹⁵Again, it is true if $BB'\Sigma$ is symmetric.

Condition 17.10.8 The following inequality holds:

$$2\nu_m^2 + \frac{r}{\delta}\eta > 0.$$

$$\eta = \max \left\{ \min_{i=1,m} \lambda_i \nu_m \frac{(\sqrt{k_\lambda} + 1)^2 - k_\nu(\sqrt{k_\lambda} - 1)^2}{2\sqrt{k_\lambda}}, \min_{i=1,m} \nu_i \lambda_m \frac{(\sqrt{k_\nu} + 1)^2 - k_\lambda(\sqrt{k_\nu} - 1)^2}{2\sqrt{k_\nu}} \right\}$$

$$= \begin{cases} \lambda_m \nu_m \frac{(\sqrt{k_\lambda} + 1)^2 - k_\nu(\sqrt{k_\lambda} - 1)^2}{2\sqrt{k_\lambda}} & \text{if } \sqrt{k_\nu} \leq \frac{\sqrt{k_\lambda} + 1}{\sqrt{k_\lambda} - 1}, k_\nu \geq k_\lambda \\ \lambda_m \nu_m \frac{(\sqrt{k_\nu} + 1)^2 - k_\lambda(\sqrt{k_\nu} - 1)^2}{2\sqrt{k_\nu}} & \text{if } \sqrt{k_\nu} \leq \frac{\sqrt{k_\lambda} + 1}{\sqrt{k_\lambda} - 1}, k_\nu < k_\lambda \\ \lambda_1 \nu_m \frac{(\sqrt{k_\lambda} + 1)^2 - k_\nu(\sqrt{k_\lambda} - 1)^2}{2\sqrt{k_\lambda}} & \text{if } \sqrt{k_\nu} > \frac{\sqrt{k_\lambda} + 1}{\sqrt{k_\lambda} - 1}, k_\nu \geq k_\lambda \\ \lambda_m \nu_1 \frac{(\sqrt{k_\nu} + 1)^2 - k_\lambda(\sqrt{k_\nu} - 1)^2}{2\sqrt{k_\nu}} & \text{if } \sqrt{k_\nu} > \frac{\sqrt{k_\lambda} + 1}{\sqrt{k_\lambda} - 1}, k_\nu < k_\lambda \end{cases}$$

represents the lower bound of eigenvalues of Jordan product $BB'\Sigma + \Sigma BB'$; λ_i, ν_i are the i -th eigenvalues of Σ and BB' respectively in a descending enumeration. $k_\lambda = \frac{\lambda_1}{\lambda_m}$ and $k_\nu = \frac{\nu_1}{\nu_m}$ denote the spectral condition number of Σ and BB' respectively.

Condition 17.10.9 There exists a positive number k such that $BB' = k\Sigma$.

In the special case where performance measures system is orthogonal, conditions (17.10.1), (17.10.6) and (17.10.7) are satisfied. If Σ (resp. BB') is a scalar matrix, then conditions (17.10.1) (resp. (17.10.6)) and (17.10.8) are both satisfied. Besides, condition (17.10.8) could hold even for nondiagonal BB' and Σ , provided either of them is well-conditioned or $\frac{r}{\delta}$ is sufficiently small.

Theorem 17.10.2 1. Given any of conditions (17.10.1), (17.10.6), (17.10.7), (17.10.8), there exists at least one $i \in \{1, \dots, m\}$, such that $|w_i^h(\delta)| < |w_i^p(\delta)|$ for all $\delta \in [\underline{\delta}, \bar{\delta})$;

2. If both conditions (17.10.1) and (17.10.6) are satisfied, namely, the performance measure system is orthogonal, then $|w_i^h(\delta)| < |w_i^p(\delta)|$ for all $\delta \in [\underline{\delta}, \bar{\delta})$ and all i ;

3. Let $\tau_i, i \in \mathcal{L} \equiv \{1, 2, \dots, l\}$ denote l distinct generalized eigenvalues of BB' relative to Σ , $\mathcal{U}_i \equiv \mathcal{N}(BB' - \tau_i \Sigma)$ be the eigenspace corresponding to τ_i , \mathcal{U}_i^\perp be its orthogonal complement. Suppose that $B\beta \notin \bigcup_{i \in \mathcal{L}} \mathcal{U}_i^\perp$, then there exists a positive number $s \in (0, 1)$ such that $w^h = sw^p$ if and only if condition (17.10.9) is met.

PROOF. Again, it is omitted here and referred to Meng and Tian (2013). \square

When the agent possesses private information on his own cost, a more efficient agent (the agent with higher δ) would accrue information rent by

mimicking his less efficient counterpart. To minimize agency costs, optimality requires a downward distortion of the power of inefficient types' incentive wage. Theorem 17.10.2 gives various conditions ensuring low-powered incentives. If the performance measure sensitivities are orthogonal to each other ($b'_i b_j = 0$ for $i \neq j$), or error terms are uncorrelated ($\sigma_{ij} = 0$ for $i \neq j$), or either BB' or Σ is well-conditioned (k_ν or k_λ is close to one), or the agent is nearly risk neutral (r is very small), or the agent is highly efficient (δ is very large), then the power of incentives will be lowered for at least one performance measure. For an orthogonal system with all its performance measures congruent ($b'_i \beta \neq 0$ for all i), the wage vector in hybrid model is shorter than but points in the same direction as its pure moral hazard counterpart if and only if all the measures share the same signal-to-noise ratio ($b'_i b_i / \sigma_i^2 \equiv k$ for all i).

17.11 Character biography

17.11.1 Oskar Lange

Oskar Ryszard Lange (1904-1965), Polish economist, politician and diplomat. He earned a doctorate in law in 1928 and taught statistics and economics at the University of Krakow, Poland. He taught statistics and economics at the University of Michigan and The University of Chicago during from 1938 to 1945. From 1945 to 1946, he was the Polish ambassador to the United States. From 1946 to 1947, he served as representative of the Polish Security Council to the United Nations. After 1948 he was a professor who taught statistics and economics at the University of Warsaw, Poland. When the Polish Communist Party and the Socialist Party merged into the Unification Workers' Party, he was elected as a member of the Central Committee.

Lange's economic theory is of great significance for observing and studying socialist economic relations and exploring the theory of modern socialist economic operation. Lange wrote many books in his life. The most influential ones are "Socialist Economic Theory"(1936-1937), "Price Elasticity and Employment" (1944), "Introduction to Econometrics" (1958), "Socialist Political Economics" (Volume 1) (1958), "Optimum Decision" (1964) and "Introduction to Economic Cybernetics" (1965) and so on. Lange proposed the decentralized model of the socialist economy which is the famous "Lange model" for the first time in the controversy between Mises (1881-1973) and Hayek (1899-1992) in the mid 1930s. Through analysis of the model, he believes that in the socialist economy, prices are not arbitrarily established, but are as objective as the market prices in the free competition system. It maintains that although the means of production are nationalized, the prices of consumer goods and labor are still priced through

the market, while the price of the means of production is simulated by the planning agency, following the same "trial and error method" as the competitive market mechanism. Lange introduced the role of the market mechanism into the socialist economy and created a precedent for the analysis of the operation of the market mechanism in the socialist economy. During the Second World War, Lange studied the issue of price and employment in the capitalist economy. He started from the general equilibrium theory and analyzed the currency effect. He pointed out that price elasticity can lead to the automatic maintenance or restoration of supply and demand balance of production factors only under special conditions. However, he believes that the possibility of achieving this special condition in the capitalist economy since the beginning of the First World War in 1914 is very small. Lange also did a lot of pioneering work in applying econometrics to planning the socialist national economy and applying the cybernetics method to economic research. He regarded the scientificization of the plan as the theme of the theory of socialist economic operation which formed the modern scientific planning theory.

17.11.2 James Mirrlees

James A. Mirrlees(1936-) who is the founder of the incentive theory and has made a significant contribution to the study of information economics theory. He received the 1996 Nobel Prize in Economics with William Vickrey in 1996. Morris was born in Minnef, Scotland which is home to Adam Smith in 1936. In He received a master degree in mathematics from the University of Edinburgh in 1957 and a doctorate in economics from the University of Cambridge in 1963. From 1963 to 1968, Morris was an assistant lecturer and lecturer in economics at Cambridge University and a researcher at Trinity College of Cambridge University. He also served as a consultant for the Pakistan Economic Development Institute in Karachi during this period. In 1969, he was officially hired as a professor at Oxford University. At the age of 33, he became the youngest professor of economics at Oxford University. Professor Morris taught at Oxford from 1969 to 1996, served as the professor of economics at Edgeworth and a fellow at Nuffield College.

Mirrlees has been active in the economics profession since the 1960s, and he is known for incentive research in economic theory. In the 1970s, together with Stiglitz, Roth, Spencer, and others, he initiated the study of principal-agent theory and achieved great results. The current method of modelling principal-agents is what Morris pioneered. Mirrlees published three papers in 1974, 1975, and 1976 respectively. The "Notes on Welfare Economics, Information and Uncertainty", "Moral Risk Theory and Unobserved Behavior" and "Optimal Structure of Incentives and Authorities in an Organization" laid the basic model framework of principal-agent theory. The framework pioneered by Mirrlees was further developed by Holm-

strom et al and in the literatures, it was called the Mirrlees-Holmstrom model approach. Mirrlees has also made remarkable achievements in economic growth and development. He once co-edited the book "Economic Growth Model" with Stern and co-authored "Project Signing and Plan for Developing Countries" with Little. In 1975, he published the paper "On the Pure Theory of the Underdeveloped Economy that Uses the Relationship between Consumption and Productivity" which conducted a utilitarian analysis of economic policies, especially growth theory, explored the impact of uncertainty on moderate growth, non-renewable resources theory, inseparable growth theory and irreplaceable theorem of durable goods. In the area of development economics, Mirrlees proposed a cost-benefit analysis method, established a development model for the low-income economies and studied the effectiveness and results of international aid policies.

17.12 Exercises

Exercise 17.1 (Debit and Credit under Moral Hazard) *A cashless entrepreneur wants to borrow money to implement an investment project. With an investment of 1 unit, if he puts in an effort of $\bar{e} > 0$, he will get q unit with probability \bar{p} ; if he doesn't put any effort, he will get q with probability $\underline{p} > 0$ ($\bar{p} > \underline{p}$). Let ψ indicate the entrepreneur's cost of effort \bar{e} . In addition, normalize the current utility of entrepreneur to 0, suppose $\underline{p}q < r$. For a monopolistic bank, the cost of capital is r . When the project is successful, each unit of the loan is repaid $z - x$.*

1. Describe the principal-agent issue faced by the bank.
2. Work out the optimal loan contract that the bank maximizes its expected profit under the entrepreneurial's incentive compatibility constraints and participation constraints.

Exercise 17.2 (Risk-averse Principal and Moral Hazard) *Assume that the risk-averse principal delegates the task to a risk-neutral agent. The agent's effort is recorded as e and gets \underline{q} with probability $1 - e$ or \bar{q} with probability e , $\underline{q} < \bar{q}$. The risk-averse agent's utility function is $v(q - t)$ that t is the transfer payment to the agent and $v(\cdot)$ is a CARA VNM utility function. The cost of agent's effort is $\psi(e)$ ($\psi' > 0, \psi'' > 0$).*

1. Assume that e is not observable and calculate the optimal contract when the agent is risk-neutral.
2. With limited liability constraint, calculate the second-best effort level.
3. Analysis two extreme cases:(a) the principal is unlimited risk-averse; (b) the principal is risk-neutral. Explain your answers.

Exercise 17.3 (*More than Two Levels of Effort*) Consider the moral hazard model in the textbook and extend the model by allowing more than two levels of effort. Consider the more general case, with n levels of production $q_1 < q_2 < \dots < q_n$ and K levels of effort with $0 = e_0 < e_1 < \dots < e_{K-1}$. We still make the normalization $\psi_0 = 0$, and assume that ψ_K is increasing in K . Let π_{ik} denote the probability of producing q_i when the effort level is e_k . Other conditions are the same as the moral hazard model in the textbook.

1. Write down the participation constraints and incentive compatibility constraints for this problem.
2. Solve the optimization problem with complete information.
3. Solve the second-best problem in the moral hazard model.

Exercise 17.4 (*Continuum of Effort Levels*) Consider the moral hazard model in the textbook and extend the model by allowing the model with a continuum of effort levels. We reparameterize the model by assuming that $\pi(e) = e$, for all $e \in [0, 1]$. Hence, the agent's effort level equals the probability of a high performance. The disutility of effort function $\psi(e)$ is increasing and convex in e with $\psi(0) = 0$. Moreover, to ensure interior solutions, we assume that the Inada conditions $\psi'(0) = 0$ and $\psi'(1) = +\infty$ both hold. Other conditions are the same as the moral hazard model in the textbook. Let us finally consider a risk-neutral agent with zero initial wealth who is protected by the limited liability constraints.

1. Write down the participation constraints and incentive compatibility constraints for this problem.
2. Solve the optimization problem with complete information.
3. Solve the second-best problem in the moral hazard model.

Exercise 17.5 (*The First-Order Approach*) Let us consider the case where the risk-averse agent may exert a continuous level of effort $e \in [0, \bar{e}]$ and by doing so incurs a disutility $\psi(e)$ which is increasing, convex and $\psi(0) = 0$. To avoid corner solutions, we will also assume that the Inada conditions $\psi'(0) = 0$ and $\psi'(\bar{e}) = +\infty$ are satisfied. Other conditions are the same as the moral hazard model in the textbook. The agent's performance is $q \in [\underline{q}, \bar{q}]$ with the conditional distribution $F(q|e)$ and the density function $f(q|e)$. We assume that $F(\cdot|e)$ is twice differentiable with respect to e . Principal's payment is $t(q)$.

1. Write down the participation constraints and incentive compatibility constraints for this problem.
2. Prove that when the monotone likelihood ratio property (MLRP) holds, the principal's payment $t(q)$ is increasing in q .

3. Prove that when the convexity of the distribution function condition (CDFC) holds which means $F_{ee}(q|e) > 0$, the principal's value function $U(e)$ is concave in e .
4. Prove that when MLRP and CDFC both hold, the solution to the optimal contract can be obtained from the first-order condition.
5. Verify that the moral hazard model with the Continuum in the textbook meets MLRP and CDFC.
6. What are your intuitive explanations for MLRP and CDFC?

Exercise 17.6 Consider a loan relationship with moral hazard. The risk-neutral borrower wants to borrow I of funds from the lender to support a risk-free project with a return of V . The project has damage to third party with probability $1 - e$. The borrower's effort e for safety care costs $\psi(e)$ and h is a compensation for the third party damage. A loan contract is (\underline{t}, \bar{t}) , which \underline{t} (\bar{t}) is the repayment to the bank when the borrower is (isn't) environmental damage.

1. Suppose e is observable. Calculate the first-best level of effort e for safety care.
2. Suppose e is unobservable, the bank is competitive, and the borrower has sufficient repayment ability. Prove that if the bank can compensate the third party h in the accident, the first-best outcome can still be implemented.
3. Assume that in the accident, the bank must compensate the third party $c < h$ and use w to represent the borrower's initial asset. Prove that when w gradually gets smaller, the first-best outcome can no longer be implemented.
4. Calculate the second-best effort level to maximize the borrower's expected return under the conditions of satisfying the bank's zero-profit constraint, borrower's incentive compatibility constraint and limited liability.
5. Prove that raising bank's repayment obligation c will reduce the expected welfare level.
6. Prove that when the banking industry is a monopoly industry, this result is no longer valid.

Exercise 17.7 (Risk-averse Agent with Hidden Actions) Consider the principal-agent problem of hidden actions. Suppose $h(u) = u + \frac{ru^2}{2}$, where $r > 0, u > -\frac{1}{r}$. Equivalently, $u(x) = \frac{-1 + \sqrt{1+2x}}{r}$, where $x > -\frac{1}{2r}$.

1. Find out the second-best transfer payment required by the principal to incentive the agent to pay a high level of effort.

2. Find out the second-best cost required by the principal to incentive the agent to pay a high level of effort.
3. Find the agent's optimal second-best utilities of inducing a high effort for the principal's incentive-compatible optimization problem \bar{u}^{SB} and \underline{u}^{SB} .
4. Find the optimal second-best agency cost AC which is the principal's loss between his first-best and second-best expected profit when he implements a positive effort and is defined as the difference between the second-best cost and the first-best cost.

Exercise 17.8 A risk averse individual has a utility function of $u(\cdot)$ and an initial wealth of w_0 . The risk he faces is the potential loss of x for an accident. In a competitive insurance market, risk-neutral insurers can provide him with a net payout of $R(x)$ (excluding insurance premiums). Assume that the distribution of x is dependent on the degree of effort e to prevent accidents and is not continuous at $x = 0$ where $f(0, e) = 1 - p(e)$. When $x > 0$, $f(x, e) = p(e)g(x)$. Assume $p''(e) > 0 > p'(e)$, the individual's effort cost $\psi(e)$ is an increasing convex function and separable from the utility function. Find the first-best and second-best insurance contracts.

Exercise 17.9 (Insurance Contract) Consider the moral hazard problem in an insurance model. The consumer's VNM utility function is $u(w) = \sqrt{w}$ and the initial wealth is $w_0 = 500$. Assume there are two possible loss levels: $l = 0$ and $l = 200$. There are also two levels of consumer's effort: $e = 0$ and $e = 1$. The cost of effort is $\psi(e)$ where $\psi(0) = 0$ and $\psi(1) = 1/3$. The corresponding probability distribution of wealth loss and effort is as follows:

1. Prove that the above probability distribution satisfies the monotonic likelihood ratio property.
2. Assuming there is only one insurance company, otherwise, consumers can only choose the way of self-insurance. Calculate the consumer's level of reservation utility.
3. If there is no insurance company, what is the level of consumer's effort?
4. Prove that if the information is symmetric, letting the consumer choose a high level of effort is optimal for the insurance company.
5. If the information is asymmetric, prove that the results in the previous question will make the consumer choose low effort levels.
6. Find out the solution to the optimal contract under information asymmetry.

	$l = 0$	$l = 51$
$e = 0$	$1/4$	$3/4$
$e = 1$	$3/4$	$1/4$

Exercise 17.10 (*Information Learning*) Consider the following principal-agent issues: A risk-neutral principal confronts an unfunded risk-neutral agent protected by limited liability. The principal can obtain information on the quality of the risky project and make a decision on whether to invest in the risky project. Suppose there is a risk-free project, the principal gets return 0 with probability of 1. And there is a risky project, in the absence of information, the principal gets return \bar{S} with probability of ν and return \underline{S} with probability of $1 - \nu$. Suppose $\nu\bar{S} + (1 - \nu)\underline{S} = 0$. By paying the cost of ψ , the agent can get a signal $\sigma \in \{\underline{\sigma}, \bar{\sigma}\}$. This signal can provide information about useful information for future return on risk project. Assume $\Pr(\bar{\sigma}|\bar{S}) = \Pr(\underline{\sigma}|\underline{S}) = \theta$ where $\theta \in [\frac{1}{2}, 1]$ is the degree of accuracy of the signal.

1. As the benchmark, assume that the principal uses information gathering technology by himself. Prove that this project will only be implemented when he observes $\bar{\sigma}$. Write down the optimal information learning conditions.
2. Suppose now that the agent decides whether to implement the risky project, and the principal adopts a contract $(\bar{t}, \underline{t}, t_0)$ to motivate the agent. $\bar{t}(\underline{t})$ is the transfer payment received by the agent when the agent chooses to implement the risky project and $\bar{S}(\underline{S})$ is realized. t_0 is the transfer payment received when the agent selects the risk-free project. Write down the incentive constraints that guarantees risk project is implemented only when $\bar{\sigma}$ is observed.
3. Write down incentive compatibility constraints that motivate agents to learn information.
4. Find out the contract for the agent and the \bar{t} that prompted the agent to learn the information.
5. Find out the second-best rule followed by the principal.

Exercise 17.11 Consider a principal-agent problem with three exogenous states of nature: $\theta_1, \theta_2, \theta_3$ and two effort levels: e_H, e_L . The level of output and the corresponding probability are as follows:

nature state	θ_1	θ_2	θ_3
probability	0.4	0.3	0.3
output of e_H	20	20	1
output of e_L	20	1	1

The principal is risk neutral, while the agent has utility function $\sqrt{w} - \psi(e)$. The cost of effort is normalized to 0 for e_L and 1 for e_H . The agent's reservation expected utility is 1.

1. Derive the first-best contract.

2. Derive the second-best contract when only output levels are observable.
3. Assume the principal can buy for a price of 1 an information system that allows the parties to verify whether state of nature θ_3 happened or not. Will the principal buy this information system?

Exercise 17.12 Consider the following moral hazard model. The principal is risk-neutral, and the agent's preference is defined in terms of his mean and variance of his income and his own effort e . The agent's expected utility is $E(w) - \phi \text{Var}(w) - g(e)$ where $g'(0) = 0$, $g'(e)$, $g''(e)$ and $g'''(e) > 0$ are true for all $e > 0$. And the profit based on e is subject to the mean of e with a variance of σ^2 normal distribution.

1. Consider the linear compensation scheme $w(\pi) = \alpha + \beta\pi$. Prove that given $w(\pi)$, e and σ^2 , agent's expected utility is $\alpha + \beta E - \phi\beta^2\sigma^2 - g(e)$.
2. Derive the first-best contract when e is observable.
3. Derive the first-best linear compensation scheme when e is unobservable and the effect of the changing of β and σ^2 .

Exercise 17.13 The principal needs to hire an agent. The work which the agent receives may be good (G) or bad (B). The principal does not know the type of work and only knows that the probability of the work for G type is $p \in (0, 1)$. The agent knows the type of work the agent can choose to invest high effort H or low effort L after being hired. Performance of Work G with high effort is 4 and is 2 with low effort. Performance of work B with high effort is 2 and is 0 with low effort. The agent's high effort will cost 1 unit and low effort has no cost. The agent's reservation utility is 0. The principal observes the performance and then gives the agent a payment.

1. Give the first-best contract given by the principal to the agent.
2. How does the conclusion relate to p ? Please explain.

Exercise 17.14 Consider the following moral hazard model. The principal is risk averse. His utility function is $u(w) = \sqrt{w}$ and the reservation utility is 0. The agent's effort level has three conditions: $E = \{e_1, e_2, e_3\}$. There are two possible profit outcomes: $\pi_H = 10$ and $\pi_L = 0$. The probability distribution is $f(\pi_H|e_1) = 2/3$, $f(\pi_H|e_2) = 1/2$ and $f(\pi_H|e_3) = 1/3$ respectively. In the agent's effort cost function, $g(e_1) = 5/3$, $g(e_2) = 8/5$ and $g(e_3) = 4/3$.

1. Derive the first-best contract when levels of effort are observable.
2. Prove that if the effort levels aren't observable, then e_2 will not be implemented. How large is $g(e_2)$ when e_2 is enforceable?
3. When the levels of effort are not observable, find the first-best contract.

Exercise 17.15 Go on with the previous question. Assume that the relationship between the principal and the agent continues for two periods. The type of work is established before the contract is signed and remains unchanged in both periods. The principal gives the agent a contract for each period. The order is: the agent first knows the type of work, then the principal provides the contract of period 1. The agent chooses the effort level and the performance and payment are realized. The principal provides the contract of period 2, the agent chooses the level of effort and performance and payment are realized. Assume that at the first period, the principal always wants to make the agent put high effort when work is B. And the optimality requires that the two types of agents in the 2 period put high effort on the remaining two situations (G type chooses H effort and G type chooses L effort).

1. Write down the first-best contract for both outcomes.
2. Assume that at the beginning of the two periods, the principal can make commitments for the first period and the second period. For the two situations discussed before, discuss how this commitment ability affects the principal's return. Under what circumstances is the promise optimal?
3. Some organizations regularly perform job rotations for their employees, and this practice is often criticized as sacrificing human capital at specific positions. According to the answers of previous questions, explain why job rotation is a good idea.

Exercise 17.16 Suppose a company hires two types of workers θ_H and θ_L to produce a product. The proportion of workers of type θ_L is λ . When a type θ worker pays T dollars to consume x units of goods, the utility is $u(x, T) = \theta v(x) - T$ where $v(x) = \frac{1-(1-x)^2}{2}$. The company is the sole producer of this product, and its unit production cost is $C > 0$.

1. Consider a non-discriminatory monopolist. Derive the monopolist's pricing strategy. Prove that when θ_L or λ is large enough, the monopolist will provide products to these two types of consumers.
2. Consider the following monopolist. The monopolist can distinguish between these two types of consumers, but it can only request a uniform price of p_i for each type of θ_i . Describe the optimal pricing of the monopolist.
3. Calculate the optimal nonlinear pricing.

Exercise 17.17 (Debt Financing) An entrepreneur has two projects, each of which requires an investment of 6 at $t = 0$. The cash flow generated of the first project is $C_1 \in \{10, 80\}$ at $t = 1$. The second project generates a cash flow of $C_2 \in \{0, 90\}$. The probability of obtaining high cash flow in both cases is v , and e is the entrepreneurial effort level. The entrepreneur's effort cost is $50a^2$. He can choose

three levels of effort: $e \in \{0, 1, 2\}$. The company does not have any assets. All people are risk-neutral and there is no discounting.

1. If the entrepreneur has the ability to invest in himself, what level of effort will he choose for each of the two projects?
2. Assuming that the entrepreneur has financing constraints, all project funds need to be financed through debt. In both projects, how large debt D will he choose? If he can get an unconditional loan, which project will he eventually choose?
3. Can the entrepreneur replace debt financing with issuing equities with a share ratio of s to improve his situation?

Exercise 17.18 (Insurance Contract) Consider a risk averse consumer whose utility function is $u(\cdot)$. The consumer's initial wealth is W and he faces the risk of losing L with probability p where $W > L > 0$. The insurance company's contract is (c_1, c_2) where c_1 is the amount of wealth the consumer has in the absence of a loss and c_2 is the amount of wealth in the event of a loss. In the event that the loss does not occur, he pays the insurance company a premium of $W - c_1$ and in the event of a loss, the compensation he receives from the insurance company is $c_2 - (W - L)$.

1. Assume that the insurance company is a risk-neutral monopolist. Calculate the optimal contract when the consumer's probability of loss θ is observable.
2. If the insurance company cannot see θ and the parameter θ may take $\{\theta_H, \theta_L\}$, $p(\theta_L) = \lambda$. Calculate the insurance company's optimal contract problem at this time. Does the insurance purchase amount have the property of distribution?

Exercise 17.19 (Political Economics of Regulation) Consider an enterprise that implements projects with values of S_1 and S_2 , respectively. The enterprise can invest effort e to reduce production costs. For project i , the enterprise's production cost is $C_i = \beta - e_i$. $\beta \in \{\underline{\beta}, \bar{\beta}\}$, $\nu = \Pr(\beta = \underline{\beta})$. Efforts to reduce costs will bring the enterprise $\psi(e_1, e_2) = \frac{1}{2}(e_1^2 + e_2^2) + \gamma e_1 e_2$, $\gamma > 0$. Regulator compensates the enterprise t with observable cost C_1 and C_2 . The utility of enterprise is $U = t - \psi(e_1, e_2)$ and the social welfare is $S_1 + S_2 - (1 + \lambda)(t + C_1 + C_2) + U$.

1. What is the optimal mechanism under complete information?
2. What is the optimal regulatory mechanism when β is the private information of the enterprise?
3. Assume that the regulatory mechanism is determined by a majority vote. There are two types of individuals: shareholders or non-shareholders of regulated enterprise. Let α denote the proportion of shareholders. If $\alpha >$

$1/2$, the goal of regulation is to maximize shareholders' objective function $\alpha(S_1 + S_2 - (1 + \lambda)(t_1 + C_1 + t_2 + C_2)) + U$. If $\alpha < 1/2$, the goal of regulation is to maximize non-shareholders' objective function $(1 - \alpha)(S_1 + S_2 - (1 + \lambda)(t_1 + C_1 + t_2 + C_2))$. Calculate the optimal regulatory mechanism for incomplete information in two situations.

Exercise 17.20 (Bolton, 2005) The probability of success for a project is a . a is also a risk-neutral agent's effort and the effort cost is a^2 . The project can be rewarded with R for success and 0 for failure. There are two values for parameter R : the probability of taking X is v and the probability of taking 1 is $1 - v$. In order to implement this project, the agent needs to borrow I from the principal. The sequence of events is as follows:

- First, the principal provides the agent with a debt contract with a value of D_0 and the agent chooses to accept or reject it.
- The nature determines the value of R when the project is finished. Both the principal and the agent can observe this value and the principal can choose to reduce the debt from D_0 to D_1 .
- The agent chooses an effort level of a and a is not observable to the principal.
- Project will success or failure. If successes, the agent pays the minimum of R and the denomination of D_1 .

Answer the following questions:

1. Find out the game's subgame perfect Nash equilibrium.
2. When is $D_1 < D_0$?

Exercise 17.21 (Regulation of a Risk Averse Enterprise) Consider that a regulator wants to implement a public project which worth S . An enterprise can put in a cost of $C = \beta - e$ to implement this project where $\beta \in \{\underline{\beta}, \bar{\beta}\}$ is a parameter of efficiency. e is the effort level brings negative effect of $\psi(e)$ ($\psi' > 0, \psi'' > 0, \psi''' > 0$) to the enterprise. The cost C can be observed by the regulator and the regulator gives the enterprise transfer payment t with the price of $1 + \lambda$. The enterprise is risk-averse and the utility function is $u(t - \psi(e))$ ($u' > 0, u'' < 0$). The regulator cannot observe e or β , but $\nu = \Pr(\beta = \underline{\beta})$ is public knowledge.

1. Consider the display mechanism $\{t(\bar{t}) = \bar{t}, C(\bar{\beta}) = \bar{C}; t(\underline{\beta}) = \underline{t}, C(\underline{\beta}) = \underline{C}\}$. Write down the enterprise's incentive compatibility constraints and participation constraints.
2. The expected social welfare is defined as:

$$W = S - (1 + \lambda)[\nu(\underline{t} + \underline{C}) + (1 - \nu)(\bar{t} + \bar{C})] + u^{-1}(\nu(\underline{\pi}) + (1 - \nu)u(\bar{\pi})),$$

where $\pi = \underline{t} - \psi(\underline{\beta} - \underline{C}), \bar{\pi} = \bar{t} - \psi(\bar{\beta} - \bar{C})$. Explain the social welfare function. Suppose it is concave for $\underline{\pi}$ and $\bar{\pi}$. And find out the optimal regulatory mechanism for the principal to provide the contract in the middle stage.

3. Compare the conclusion in the previous question with the situation of the risk-neutral enterprise's.
4. Consider a special case $v(x) = \frac{1}{\rho}(1 - e^{-\rho x})$. Prove the effort level of type $\bar{\beta}$ increases with ρ .

Exercise 17.22 (Information Gathering before Signing Contract) Consider a principal-agent problem: An agent produces a commodity of q output at a cost of θq , $\theta \in \{\underline{\theta}, \bar{\theta}\}$, $\bar{\theta} > \underline{\theta}$. Let t denote the transfer payment to the agent. The agent's utility is $U = t - \theta q$. The principal's utility is $V = S(q) - t$, $S' > 0$, $S'' < 0$.

- On the first day, the principal proposes a list of contracts $(\underline{t}, \underline{q}), (\bar{t}, \bar{q})$.
- On the second day, the agent decides whether to learn θ at a cost of ψ . Let e denote this decision: if he learns, then $e = 1$, if not, then $e = 0$. e is a moral risk variable that the principal cannot observe.
- On the third day, the agent decides whether to accept the contract.
- On the fourth day, the agent knows θ (if he decides not to study on the second day).
- On the fifth day, the contract was executed.

Consider two contract sets: the C_1 type causes the agent to select $e = 1$ and the C_2 type causes the agent to select $e = 0$.

1. Write down the optimization problem for the principal to maximize his expected utility under satisfying the agent's incentive compatibility constraints and budget constraints, whether it is a contract of C_1 or C_2 type.
2. Prove that the principal can limit the contract to $\underline{t} - \underline{\theta}\underline{q} \neq 0, \bar{t} - \bar{\theta}\bar{q} \neq 0$ to achieve a utility lower bound. Derive a meaningful contract from this conclusion which requires $\bar{t} - \bar{\theta}\bar{q} \leq 0$. Prove that the principal can always imitate a contract in C_2 with a contract in C_1 .
3. Find the optimal contract in the C_1 type contract. (Discuss the scope of ψ . And distinguish whether the participation constraint is established, the moral hazard constraint is established and the two constraints are established at the same time.)

Exercise 17.23 A risk-neutral principal employs a risk-averse agent. For the principal, the agent's effort level e is not observable and the principal's profit level has n possible outcomes which are $q_1 < q_2 < \dots < q_n$. These n outcomes are observable and verifiable. The probability of realization of profit q_i is $\pi_i(e)$ and the corresponding agent's income is $I = (I_1, I_2, \dots, I_n)$. The agent's reservation utility is \bar{u} , and the agent's utility function is $u(I, e) = -e^{-r(I-e)}$.

1. Prove that the first-best effort level is independent of the reservation utility \bar{u} .
2. Proof that the second-best effort level is independent of the reservation utility \bar{u} .
3. Proof that the optimal salary plan can be expressed as $(I_1+k, I_2+k, \dots, I_n+k)$ where k is only a function of the reservation utility \bar{u} .
4. Assume that the agent's utility function is $V(I) - e$, $V' > 0$ and $V'' < 0$. Are the results of the previous three questions still correct? Explain the reason.
5. Assume that the agent's utility function is $V(I) - e$, $V' > 0$, $V'' < 0$ and $n = 2$. There are only two levels of effort which are working hard e_H and lazy e_L . In the second-best outcome, the principal wants to motivate the agent to take the effort level e_H . Prove that the first-best incentive plan makes the agent have no difference in the choice of working hard and laziness.
6. Continue with the previous question and prove that the first-best incentive plan satisfies $I_2 > I_1$, $q_2 - q_1 > I_2 - I_1$.

Exercise 17.24 In a two-period question which $t = 1, 2$. The agent chooses the effort level $e_t = H$ or $e_t = L$ and the profit is $q_t = 1$ or $q_t = 0$. The harder the agent works, the greater the probability of achieving a high profit. In particular, $\Pr(q_t = 1 | e_t = H) = p_H, \Pr(q_t = 1 | e_t = L) = p_L < p_H$. The agent's effort cost is $c(e_t)$ where $c = c(H) > c(L) = 0$. The agent's utility function is $u(s, e_1, e_2) = -e^{-r(sc(e_1)-c(e_2))}$ where s is the salary that the principal pays to the agent. The agent's reservation utility is a zero deterministic equivalent level and the agent can observe q_1 before determining the action e_2 . The principal is risk-neutral and his profit is $q_1 + q_2 - s$. The salary of s depends on the profit of each period and is paid at the end of the second period.

1. Assume that the principal hopes that the agent will work hard in two periods to. Prove that the contract that achieves the minimum cost of (H, H) has the following form: $s(q_1, q_2) = \alpha(q_1 + q_2) + \beta$.
2. If the agent can not observe q_1 before determining the action e_2 . Is the contract still optimal? Explain the reason.

Exercise 17.25 A monopolist sells its products at the price of p and the production process only requires labor. In order to produce q units of product, workers must work hard and the cost of the effort is $\frac{q^2}{2}\theta$. The number of workers is standardized to 1 and the skill level of the workers is represented by θ . And the principal cannot see θ . Assume that the workers with proportion λ are highly skilled (i.e. $\theta = \theta_H$) and the rest are low-skilled (i.e. $\theta = \theta_L < \theta_H$). The principal can hire any number of workers. The reservation utility of each worker is 0. The principal's profit is the product's revenue minus the wages paid to the worker.

1. Solve the first-best contract.
2. Assume that the output q is observable and verifiable and that only the salary w can be written into the contract (i for each worker is the same, $T_i = w_i q_i$). Calculate the first-best solution w . Is w the same for high-skilled and low-skilled workers? Does the first-best w depend on the proportion of highly skilled workers?
3. Assume any contract (q_i, T_i) is possible. Find the first-best contract and compare it with the result of the previous question. You may be able to assume that the proportion of high-skilled workers is quite low and it is not optimal for firms to employ only highly skilled workers.

Exercise 17.26 Assume that the principal is risk-neutral. The agent chooses the effort levels e_H and e_L to achieve three possible outputs of $q_H > q_M > q_L > 0$. If the agent selects e_H , the utility level is $2\sqrt{x} - 5$. If e_L is selected, the utility level is $2\sqrt{x}$. If the agent does not work, the utility level is 3. If the agent chooses to work, his minimum wage is w , $0 \leq w \leq 5$. The principal cannot observe the agent's effort level, but he can observe the realized output, $P(q = q_H|e_H) = \frac{1}{2}, P(q = q_M|e_H) = \frac{1}{2}, P(q = q_L|e_H) = 0, P(q = q_H|e_L) = \frac{1}{4}, P(q = q_M|e_L) = 0, P(q = q_L|e_L) = \frac{3}{4}$. Assume that $q_H - q_M$ and $q_M - q_L$ are large enough. Calculate the first-best contract for the principal.

Exercise 17.27 Define monotonic likelihood ratio and first-order stochastic dominance.

1. Prove that under the two outcomes, the above two concepts are equivalent.
2. Give an example to show that in the general case, the above two concepts are different. And prove that if the monotonic likelihood ratio is satisfied, the first-order random dominance is also satisfied.

Exercise 17.28 Consider a moral hazard model. The principal is risk-neutral and the agent is risk averse, The agent can choose two levels of effort e_H and e_L . The cost of high effort is c and the cost of low effort is 0. There are two types of outcomes which are x_H and x_L , $p(x_L|a_L) > p(x_L|a_H)$. The agent's utility function is $u(w, c_i) = \ln w - c_i$ and the agent's reservation utility is 0.

1. Describe the principal-agent problem of implementing a high level of effort.
2. Solve the first-best salary.

Exercise 17.29 (Supervisory Cost) Consider a financing contract where the parties are a risk-neutral entrepreneur with wealth constraints and a wealthy risk-neutral investor. The investment cost I is required at $t = 0$. The project generates a random gain of $\pi(\theta, I) = 2 \min\{\theta, I\}$ when $t = 1$, where θ is a natural state and obeys uniform distribution on $[0, 1]$.

1. Describe the characteristics of the first-best investment level I^{FB} .
2. Assume that at $t = 1$, the realized return can only be observed by the entrepreneur and the investor must pay a cost of $K > 0$ to observe $\pi(\theta, I)$. Consider that the return cannot exceed the return minus the cost of supervision and the investor's expected profit is 0. Solve the first-best contract.
3. Prove that the second-best investment level is lower than the first-best investment level I^{FB} .

Exercise 17.30 (Forged Output) Consider a risk-averse entrepreneur whose utility function is $u(\cdot)$ and the entrepreneur's output q is a random variable distributed over the interval $[0, \bar{q}]$. The entrepreneur hopes to diversify the risk by signing a contract with a risk-neutral financier whose initial wealth is $w \geq \bar{q}$. The contract provides that the payment to the entrepreneur depends on the level of the output. The entrepreneurs can observe the output. The financier can also observe output unless the entrepreneur tampers with the accounts. After observing the output q , the entrepreneur can falsify an output report R which costs $\psi(q, R) = \frac{1}{4}(qR) + \frac{1}{2}c(qR)^2$, $c > 0$. Assume that the entrepreneur is protected by limited liability and his reservation utility is higher than $\bar{q}/2$.

1. Characterize the first-best contract.
2. If there is no output forgery in the contract (i.e., for all $q \in [0, \bar{q}]$, $R(q) = q$). Prove that the first-best contract will lead to falsifying. According to the first-best contract, calculate the equilibrium solution of the entrepreneur's falsified output.
3. What is the first-best contract without falsifying? Prove that the falsification-free first-best contract is linear to the output q .

17.13 Reference

Books and Monographs:

Huang Youguang and Zhang Dingshen.(2008). *Advanced Microeconomics*, Shanghai Gezhi Press.

- Ping Xinqiao.(2001).*Eighteen Lectures on Microeconomics*,Peking University Press.
- Bolton, P. and Dewatripont, M. (2005). *Contract Theory*, MIT Press.
- Cheung, Steven N. S. (1969). *The Theory of Share Tenancy*, University of Chicago Press.
- Laffont, J. J., and Martimort, D. (2002). *The Theory of Incentives: The Principal-Agent Model*, Princeton and Oxford: Princeton University Press, Chapters 4-5.
- Laffont, J. J., and Tirole, J. (1993). *The Theory of Incentives in Procurement and Regulation*, Cambridge: MIT Press.
- Luenberger, D. (1995). *Microeconomic Theory*, McGraw-Hill, Inc., Chapter 12.
- Mas-Colell, A., Whinston, M.D. and Green, J. (1995). *Microeconomic*, Oxford University Press, Chapter 14.
- Tirole, J.(2006) *The Theory of Corporate Finance*, Princeton University Press.
- Varian, H. R. (1992). *Microeconomic Analysis, Third Edition*, W.W. Norton and Company, Chapter 25.
- Wolfstetter, E. (1999). *Topics in Microeconomics—Industrial Organization, Auctions, and Incentives*, Cambridge Press, Chapters 8-10.

Papers:

- Alikakos, N. and Bates, P. (1984). "Estimates for the Eigenvalues of the Jordan Product of Hermitian Matrices", *Linear Algebra and its Application*, Vol. 57, 41-56.
- Akerlof, G. (1976). "The Market for Lemons: Quality Uncertainty and the Market Mechanism", *Quarterly Journal of Economics*, Vol. 89, 488-500.
- Baker, G. P. (1992). "Incentive Contracts and Performance Measurement", *Journal of Political Economy*, Vol. 100, 598-614.
- Green, L. R, and Stokey, N. (1983). "A Comparison of Tournaments and Contracts", *Journal of Political Economy*, Vol. 91, 349-364.
- Grossman, S., and Hart, O. (1983). "An Analysis of the Principal Agent", *Econometrica*, Vol. 51, 7-45.

- Guesnerie, R. (1981). "On Taxation and Incentives, Further Reflections on the Limits of Redistribution", Discussion paper No. 89, Sonderforschungsbereich 21, University of Bonn.
- Hammond, P. (1979). "Straightforward Individual Incentive Compatibility in Large Economies", *Review of Economic Studies*, Vol. 46, 263-282.
- Holmstrom, B. (1979). "Moral Hazard and Observability", *Bell Journal of Economics*, Vol. 10, 74-91.
- Holmstrom, B., and Milgrom, P. (1987). "Aggregation and Linearity in the Provision of Intertemporal Incentives", *Econometrica*, Vol. 55, 303-328.
- Holmstrom, B., and Milgrom, P. (1991). "Multitask Principal-Agent Analyses: Incentive Contracts, Asset Ownership, and Job Design", *Journal of Law, Economics and Organization*, Vol. 7, 24-52.
- Holmstrom, B. (1982). "Managerial Incentive Problems—A Dynamic Perspective". In *Essays in Economics and Management in Honor of Lars Wahlbeck*. Helsinki: Swedish School of Economics. (See also *Review of Economic Studies*, Vol. 66, No. 1, 169-182, 1999.)
- Holmstrom, B., and Tirole, J. (1994). "Financial Intermediation, Loanable Funds, and the Real Sector", *American Economic Review*, Vol. 84, 972-991.
- Laffont, J. J. (1994). "The New Economics of Regulation Ten Years After", *Econometrica*, Vol. 62, 507-538.
- Lazear, E. P., and Rosen, S. (1981). "Rank-Order Tournaments as Optimum Labor Contracts", *Journal of Political Economy*, Vol. 89, 841-864.
- Li, J. and Tian, G. (2003). "Optimal Contracts for Central Banks Revised", Working Paper, Texas A&M University.
- Meng, D., and Tian, G. (2013). "Multi-task Incentive Contract and Performance Measurement with Multidimensional Types", *Games and Economic Behavior*, Vol. 77, 377-404.
- Rochet, J.-C. (1985). "The Taxation Principle and Multi-time Hamilton-Jacobi Equations", *Journal of Mathematical Economics*, Vol. 14, 113-128.
- Rogerson, W. P. (1985). "The First-order Approach to Principal-agent Problems", *Econometrica*, Vol. 53, 1357-1367
- Shapiro, C., and Stiglitz, J. (1984). "Equilibrium Unemployment as a Worker Discipline Device", *American Economic Review*, Vol. 74, 433-444.

Stiglitz, J. (1974). "Incentives and Risk Sharing in Sharecropping", *Review of Economic Studies*, Vol. 41, 219-255.

Chapter 18

General Mechanism Design: Contracts with Multi-Agents

18.1 Introduction

This chapter introduces the general mechanism design theory, which has two branches. One is the implementation theory or mechanism design theory, which is the Incentive mechanism design theory of "advance to benefit". In the implementation of the incentive mechanism design theory, people wish to find a mechanism for the compatibility of interest incentives, a mechanism that makes personal interests consistent with the desired social optimal outcome. The other is the realization theory, which mainly focuses on the information efficiency of the mechanism. One of the main indicators is that, the smaller the mechanism information space is, the smaller the operation (transition) cost of the mechanism is. Therefore, this chapter mainly discusses the mechanism implementation theory, and then briefly discusses the mechanism realization theory.

In the previous chapters on the principal-agent theory, we have introduced basic models to explain the core of the principal-agent theory with complete contracts. The reverse selection problem analyzes the trade-offs between allocation efficiency and information rents. The moral hazard problem analyzes the trade-offs between incentives and risks. In the principal-agent theory, the principal does not know the basic economic characteristics or actions of the agent. Therefore, the principal needs to induce the agent to maximize its own benefits by designing a certain optimal contract. We summarize such problems as a constrained optimization problem of the principals' interests under the conditions of the agent's participation constraints and incentive compatibility constraints. The basic conclusion is that, there is no first-best outcome under incomplete information generally. The optimal outcome is only the second-best, which allocation will be distorted downwards, compared with the first-best outcome.

The biggest difference between the principal-agent model and the general mechanism design problem studied in this chapter is the agents' number. The principal-agent model usually only considers one agent faced by the principal. Since the model involves only one agent, we do not need to consider the strategic interaction between agents, which is the game between agents.

In a society made up with multiple economic people, there are usually two ways to achieve a certain outcome. The first way is to determine the result directly through a social choice rule. In this way, the social planner (also called the designer) only needs to decide on one or more alternatives that meet certain selection criteria according to different "environments". But when the information of the "environment" is dispersed in each individual, this direct approach hardly works in most situations. This is because the central planner can only know what the "environment" is based on information reported by the economic people. And every economic person has incentives to lie on their own information or disguise in this process, making "do not say lies do not make big events" becoming a creed for people to live. For example, if the government taxes based on the preferences of residents to provide a certain public good or services, then everyone may report lower utility in order to pay less tax. Even worse, some actual beneficiaries may claim damaged by the public goods to obtain compensation. What is more common to see is that many people only talk but never do, and many corrupt officials are serving for money when they say serving for people. Incentive problems arise when the social planner cannot distinguish between things that are indeed different so that free-ride problem may appear. A free rider can improve his welfare by not telling the truth about his own unobservable characteristics.

The second way is an indirect method, which is adopted in order to avoid the drawbacks of the first way, that is to induce a certain result or goal through some kind of incentive mechanism. In this way, central (or social) planner first needs to design certain rules of the game. And each economic person will make his own optimal response under such rules and send out information which will become the basis to choose the final result. The theory of mechanism design studies how to design the game rules to induce the economic people to send appropriate information willingly through their autonomous decision and achieve the established goal eventually, under the premise of the certain goal. It is in this sense that Hurwicz(1972) considered any mechanism to be an information communication and processing system. If these two methods "reach the same route", which means that the results selected by a certain social selection rule are exactly the same as those induced by an incentive mechanism in any "environment", and the social selection rule can be said to be "executed" by this mechanism. Then it is said that the social selection rule can be "implemented" by this mechanism. Therefore, the mechanism design theory is also

referred as "implementation theory" when considering incentive issues.

In the general incentive mechanism design problem with multiple agents, there is often a game between agents which can lead to different equilibriums. One of the most fundamental contributions of the mechanism theory has been shown that the free-rider problem may or may not occur, depending on the kind of game (mechanism) that agents play and other game theoretical solution concepts. Let's start with the Smart Pig Game and the design of its feeding program, which introduced in Chapter 6, as an example to illustrate.

Example 18.1.1(the Smart Pig Game and the Incentive mechanism design)

The Smart Pig Game tells a story about two pigs, one big pig and one small pig, live together in a pigsty. At one end of the pigsty there is a pedal, and the pedal is connected with an organ which can open the feed. As long as the pedal is stepped, there will be 10 units of food appearing at the other end of the pigsty. But any pig who steps the pedal will pay a physical cost equivalent to 2 units of food. The big pig eats faster and small pig eats slower. Each pig can choose to "step" or "do not step" on the pedal. There are four possible outcomes: (1) Two pigs all do not step the pedal, and they get no food, the profit is 0. (2) The big pig goes to step the pedal and the small pig does not step the pedal. Although the big pig is delayed, it eats faster and gets 6 units of food. While the small pig eats slower and gets 4 units of food. (3) The small pig goes to step the pedal and the big pig does not step the pedal. when the small pig comes back, the big pig has already eaten 9 units of food. So the small pig only ate 1 unit of food, but has to pay 2 units cost, the profit is -1. (4) Two pigs go to step the pedal and come back to eat at the same time. The big pig eats faster and gets 7 units of food while the small pig only eats slower and gets 3 units of food. After deducting their physical cost, big pig earns 5 units of food and small pig earns 1 unit of food. Then we can get the following revenue matrix. See Table 18.1.

		small pig	
		step	not step
big pig	step	5,1	4,4
	not step	9, -1	0, 0

Table 18.1: the Smart Pig Game

Its Nash equilibrium is that the small pig does not step and the big pig step the pedal. For the small pig, whether or not the big pig steps the pedal, it is better to not step than to step on the pedal. So not stepping is a dominant strategy. On the other hand, the big pig knows that the small

pig does not step the pedal, so stepping is better than not stepping on the pedal.

Under the above feeding program, this "Smart Pig Game" tells us that: Whoever steps on the pedal will benefit the whole society, but more work is not necessarily have more rewarding. It profoundly reflects that if the system design is not appropriate, there will be free ride problems everywhere in economic and social life. Regardless of whether the big pig is stepping or not stepping on the pedal, the small pig chooses not stepping; and given the small pig not stepping, the big pig is better to step. The big pig chooses to step subjectively for its own interests, but the small pig objectively also enjoys the benefit. So the small pig is called a "free rider" in economics. Of course, as a rational person, nobody is willing to benefit others all the time. I am afraid that the big pig may not be willing to work in the long run, and as a result, all of them are lazy (such as the pre-reform planning economy era) or change jobs elsewhere! Therefore, we need to redesign or reform the rules of the game.

The "Smart Pig Game" gives the best strategy that doing nothing to the weak (small pig) in a competition. But for the society, as far as the small pig keeps free ride, the allocation of social resources can not be optimal, since the small pig fails to participate in the competition. And the reason why the above results arisen is the unappropriate design of the system and rules. Whether free ride problems occur or not depends on the design of the incentive mechanism. And the design or reform requires a theoretical guidance in order to form incentive compatibility, otherwise it may not achieve intended results. Different institutional arrangements and different rules can lead to the different behaviors of economic people, leading to different choices and results, such as:

Reduction plan: Feed only half of the original (poor substance).

As a result, the big pig and the small pig may all not step the pedal. For example, if the small pig steps the pedal, the big pig will eat all the food; if the big pig steps the pedal, the small pig will eat 4 units of food, so that the big pig has negative returns. In this way, whoever steps on the pedal means negative income, thus no one has the motivation to step the pedal. In the era of planned economy with poor resources, everyone, big pig or small pig, does not step the pedal, and wants to take advantage of others to create a beautiful communist society.

Increment plan: Feed double of the original (abundant substance). As a result, one pig can not finish all the food, leaving enough food for the other pig. In this case, the big pig and the small pig are all likely to step the pedal. And whoever wants to eat will step the pedal, since the other can not

finish all the food. The pigs are similar to people living in a communist society with abundant resources or some European countries with high social welfare system, so their competitive awareness is not strong.

Reduction and displacement plan: Feed only half of the original, and move the feeding mouth next to the pedal at the same time. As a result, the big pig and small pig may desperately rush to step the pedal, since more work means more harvest. Regardless of the big pig who are entrepreneurial or the small pig who are working, as long as there are limited profits and scarce food in the competitive industry, they have to adapt to the laws of the jungle and the "Vicious" competition. Therefore, we must innovate to get rid of the situation. Without innovation, it is not possible to survive. And the result is mentioned previously, competition—innovation—monopoly—competition such a repeated dynamic cycle, which means that the market competition tends to be balanced. However, innovation breaks the equilibrium. Such game continuously carried out in the market motivates enterprises to pursue innovation. And through this game process, the market economy maintains long-term vigor, increasing social welfare and promoting economic development.

The differences in the above plans illustrate the crucial importance of proper system design. In order to avoid the free ride phenomenon and make the most efficient allocation of resources, it is necessary to design a appropriate incentive mechanism. If the incentive mechanism is not appropriate, there will be such a situation where the small pig does not run and the big pig always runs. And this kind of situation is really common in reality. If most things are completed by a few people, then most people will just enjoy it and do nothing. There are so many phenomenon in our society just like the small pig and big pig story: the people are big pigs and the government is small pig; the private enterprises are big pigs and the country-owned enterprises are small pigs; the reformers are big pigs and the conservatives are small pigs; the innovators are big pigs and the followers are small pigs. The initiative of creating social wealth in country-owned enterprises is absent, the profit comes mainly from monopoly, and the loss is borne by the government. So the country-owned enterprises have no motivation for reform and innovation. While private enterprises must face competition and pursue innovation, or else they will be eliminated. So they have to be ahead of each other, with incentives for reform and innovation. By the way, they also solve the social unemployment problem and create more social wealth. Private companies have to step the pedal (efficiency

and innovation), and this is also their only way out. The biggest pigs, of course, are ordinary people. They are more passive and have no choice but to stampede harder.

If a system has something wrong, reform is needed. But reform always goes with risks and costs. From ancient times until now, there are always some people who play the role of the big pig, running around for reform and bearing the cost of reform, while others, like the small pig, do not make efforts but enjoy the benefits, even possible to say sarcastic words and criticize reformers. If everyone in this society wants to take free ride and does not want to step forward, the old system, which is not good, will be locked down for a long time. This may explain why some systems are unreasonable but long-lasting.

This example tells us the importance of incentive system design. The system design is decisive and bad rules can have a negative incentive impact on a country or unit. This also shows that Deng Xiaoping's incisive saying: "a good system can prevent bad people from acting arbitrarily, while a bad system can prevent good people from doing good deeds adequately or even go to the opposite". It requires the designers consider clearly and carefully about the foresight, adaptability and efficiency of the rules. As mentioned in Chapter 1, the most successful example of the design of incentive and compatibility mechanisms that align individual and social interests was the establishment of the basic Constitution of the United States, which led to the United States becoming the most powerful country in the world for more than 100 years and there is no sign of decline so far.

The discussion of incentive problem mentioned earlier was caused by the argument about the feasibility of socialist economic mechanism, which leads to the theory of mechanism design. Actually, examples of incentive mechanism design that takes strategic interactions among agents exist for a long time. An early example is the Biblical story of the famous judgement of Solomon for determining who is the real mother of a baby by incentive mechanism design. Two women came before the King, disputing who was the mother of a child. The King's problem was that although two women know who the real mother was but the King didn't. The King's solution used a method of threatening to cut the lively baby in two and give half to each, then watched their reaction. One woman was willing to give up the child, but another woman agreed to cut in two. The King then made his judgement and decision: The first woman was the mother who did not kill the child and gave the baby to the first woman. Actually, Solomon's solution is problematic, because if both women claim to cede their children to each other, Solomon will not be able to determine who is the mother. (The results of this chapter show that this mechanism can not be executed by Nash.) In ancient China, there were similar but less easily falsified cases. For example, Bao Gong's method of judging baby's mother: put the child

in a circle and let two women pull the child, who pulled the child out had the child. In fact, even if the designer knows the real information, there still exists incentive problems, like pie-splitting problem. How to divide a cake into several portions is the fairest way? The issue is also closely related to how to prevent government officials from rent-seeking through their power. It can be seen that no matter whether the information is symmetric or not, there will always some problems in incentive mechanism design. The solution is nothing more than 'to be reasonable, to be emotional, and to be beneficial'.

So what is the incentive compatibility problem? Suppose the mechanism designer (or client) has a goal, that is always called a social goal, which can be Pareto optimal allocation of resources, or the fair allocation of resources with rational allocation of individuals in a sense, or a goal that departments, institutions, or business owners want to achieve. The designer thinks this goal is good and wants to achieve. Therefore, is it possible to motivate each participant (consumers, companies, families, Basic institutions, etc) to do what they are wanted to do? In other words, what rules or institutional arrangements should be adopted to align the actual result of the egoistic behavior of each member of economic activity consist with a given social or collective goal? Or what rules should be put in place to enable everyone to pursue personal interests and achieve established social goals at the same time? Incentive mechanism design theory is mainly to answer such a question. It should be noticed that the designer in the question is an abstract designer, not necessarily a person. Depending on the problem, the designer can be one person, a group of people, legislatures (such as the United States Congress and the Chinese people's Congress), government departments, policy makers, reformers, managers, heads of departments, economists who propose various economic models, or even all participants who agree to follow established rules of the game, or some other institutions that make rules. Herwitz argues that the United States Congress or other legislatures is equivalent to the designer and design a new mechanism.

Designers know which social goals are good and worthwhile to achieve. For example, they think it is good to allocate resources efficiently, distribute fairly, and reduce losses of companies. The task of economists or reformers is to lay down specific rules to implement this goals. In fact, some very specific economic policy issues always need to be strictly described in very abstract mathematical models. When we think some schemes cannot be implemented, we should ask what is holding it back. Of course, an obvious limitation or obstacle is the material and technical conditions. In addition, there is another factor: incentive compatibility problem. If an economic mechanism is not incentive compatible, there will be distortions in incentives, which will lead to inconsistency between individual behaviors and social goals, resulting in so-called "there always Countermeasures for the

policies" or "the monk recited the scriptures" phenomenon to prevent a policy or system from playing its intended role. Without the enthusiasm and initiative of individuals, social goals can not be achieved (or at least too far from the ideal state). Why do individual or corporate behavior outcomes often differ from the goals that regulators want to achieve? And why can't many of the central government's good policies and reforms implement? It is because that these regulations, policies or reforms are not compatible with individual incentives. Under the rules, individuals or businesses that do not follow the social goals set by designers can benefit more. Under the rules, individuals or companies can benefit more if they do not follow the social goals. So what mechanisms (or rules) should we adopt to align everyone's behavior (either self-interested or not) with social goals? This is one of the core questions of incentive mechanism design theory to be introduced in this chapter.

One of the basic conclusions of the implementation theory of mechanism design is that it is not easy to design an individual incentive compatibility mechanism that leads to a given social goal, and the degree of difficulty depends on the type of agent game (mechanism) and the concept of solution of other game theory. After the mechanism is given, the agents will play games. And if "certain" equilibrium is exactly the same as that of the pre-given social choice in any environment, then it is said that the social choice can be carried out in a "certain" equilibrium manner. The social choice can be carried out in a "certain" equilibrium manner. The "certain" equilibrium can be dominant equilibrium, Nash equilibrium, subgame perfect equilibrium, Bayesian equilibrium, Bayesian perfect equilibrium, repeated suboptimal equilibrium and so on. In this way, the theory of mechanism design not only studies whether a certain social rule can be "executed" by a certain mechanism, but also studies how it is executed. When considering the mechanism design, First, we hope that the executable social choice rules have some desirable properties, such as non-autocracy, Pareto optimal, and so on. Secondly, we hope that the executed solution of the game is as strongly as possible. The strongest solution is the dominant equilibrium, where each economic person does not need the information of others when making the decision, so the amount of information required is the least.

From the earlier part of this book, we know that when in the dominant equilibrium, every agent has no incentive to deviate. And more importantly, the dominant equilibrium of a mechanism is equivalent to inducing participants to tell the truth. The revelation principle, first proposed by Gibbard(1973), points out that any social selection rule that can be implemented in a dominant way by a 'direct mechanism' strategy that telling the truth. The direct mechanism refers to the same mechanism of information space and individual type space. In other words, if a social selection rule can be executed honestly by the dominant strategy (or dominant strategy

with incentive compatibility and strategy-proof), there is a direct mechanism where each agent required to report on his own type directly. The results of this mechanism are exactly the same as those selected by the social selection rules, and under the guidance of the mechanism, each agent will declare its type truthfully, whether others tell the truth or not. The revelation principle allows us to search for desirable social selection rules only in the class of superior strategic which is honest execution. Unfortunately, several famous negation propositions make finding such a mechanism useless.

In the first two chapters, the principal-agent model of a single agent can induce people to speak the truth in the second-best contract mechanism. But when the participants have more than two individuals, the impossibility theorem, proposed by Gibbard(1973) and Satterthwaite(1975), tells us that: the only truth-telling social choice mechanism is dictatorial, That means the dictator's optimal decision is the optimal decision of all other people. However, such a mechanism is clearly impossible to exist for profit-driven individuals. Hervitz Hurwicz (1972) then came up with a surprising result for the restrictive neo-classical economic environment. Herwitz proved that in the neoclassical private goods economy with at least two agents, there is no social choice rule which satisfies both Pareto optimal and individual rational constraints, and can be executed honestly by dominant strategy. And Liad and Roberts (1974) subsequently proved a similar conclusion in an economic environment containing public goods. According to the first fundamental theorem of welfare economics, under certain conditions, Valas (Lindal) equilibrium is Pareto optimal. Thus, according to the conclusion of these impossibility theorems, neither Valas correspondence nor Lindal correspondence can be carried out honestly by a dominant strategy. The basic conclusions of Gibed's-Herbert Savitz's-Liad Robert's impossibility theorem is that we have to choose between telling the truth and Pareto efficiency (or the first-best result).

Based on the enlightenment of Gibbed's-Herbert Savitz's-Savitz's-Liad Robert's impossibility theorems, the subsequent studies are carried out in two directions. One direction of research is to design a mechanism to motivate each individual to tell the truth, but to give up the requirement that leads to Pareto efficiency and to limit the domain of definition to discuss the implementation of a social goal. One of the prerequisites for the Gibbed-Sateswett 's impossibility theorem is that the domain is not restricted. If this assumption is relaxed, can truth-telling social selection rules be found in a given domain? The answer is yes, and the most famous result is the VCG(Vickrey-Clark-Grovess) mechanism in quasilinear preference environment. This mechanism can induce agents to tell the truth and perform the effective result of Pareto. Goering and Lafonne (1977) further proved that the VCG mechanism is the only one mechanism that can induce agents to tell the truth and lead to pareto optimal result in quasilinear environ-

ment. In many cases, when considering the design of a more microscopic incentive mechanism at the managerial level, one can ignore the first-best or Pareto efficiency, and so one can expect the truth-telling behavior.

Another direction of study is to replace the dominant equilibrium with the concept of a weaker behavioral solution to enforce a social choice rule, which means to abandon the requirement that everyone must tell the truth, but to implement the established social optimal results, such as Pareto efficiency. When information about agent characteristics is shared among them, but the mechanism designer does not know the information (like that in Solomon's case), the related concept is the Nash equilibrium. In this situation, people are not required to tell the truth, their information belongs to the general information space, and a Nash equilibrium can be designed, resulting in Pareto efficient allocation.

Hervitz (1979a) proved that when there are at least three agents, the Valas (Lindahl) correspondence can be full Nash implementation. Since then, according to the new classical economic environment, a lot of expansion have been done to design the Nash executable mechanism with good properties. The author of this book have done a lot of work in this area with collaborators. See specifically: Tian(2010,2009a, 2009b, 2006, 2005, 2004, 2000a, 2000b,2000c, 2000d, 1999a, 1999b, 1997, 1996a, 1996b, 1994a, 1994b,1994c, 1993, 1992, 1991, 1990, 1989, 1988), Li, Nakmura and Tian(1995), Tian and Li(1995a,1995b) and so on. This chapter will introduce the dominance and Nash implementation in the neoclassical economic environment.

The original discussion on the implementation of the general form of social choice rule is Maskin(1977). Maskin (1977) proved that a necessary condition for any social choice rule to be fully implemented by Nash was to satisfy Maskin monotonicity. But the Maskin's monotonicity is only a necessary condition but not a sufficient condition of complete Nash execution. When there are at least three agents, the Maskin's monotonicity and the condition of no veto are the sufficient conditions for Nash's execution. Since Maskin (1977) , a large number of literature on Nash execution has been produced. Although Maskin(1977) has given the necessary and sufficient conditions for Nash execution, there are two obvious shortcomings: first, the conclusion is obtained in the multi-person environment; secondly, for the two-person game environment, which is important in the field of mechanism design, the conclusion is not true. For this reason, many documents, such as Moulino(1983), Moore and Repullo(1990), Danilov(1992) and so on, have discussed the necessary and sufficient conditions for Nash execution. Moore and Repullo(1990), Dutta and Senn(1991) discussed the necessary and sufficient conditions for Nash execution in the case of two people. General speaking, the implementation of the two-person situation is more difficult than the multi-person situation. The section 4 of this chapter describes in detail the sufficient conditions, the necessary conditions, the sufficient and necessary conditions for Nash execution.

Maskin monotonicity is difficult to satisfied in the general economic environment. For example, Saijo(1987) proved that if individual preference is complete and transferable, only the constant rule can satisfy Maskin monotonicity. Therefore, the concept of Nash's execution must be relaxed a little more. Some literatures introduce the concept of virtual implementation on the basis of exact implementation. The so-called virtual implementation refers to that the "distance" between the result of a mechanism's execution and the result of a social selection rule is close enough. This "distance" is defined in some classical literatures, such as Arbreu and Senn(1991),Duggan(1997). In this way, without satisfying the Maskin monotonicity condition, a social choice rule can be Nash executed. Arbreu and Senn(1991) proved that any social choice rule can be implemented by Nash in a multi-person environment, while in the case of two persons, a social selection rule can be implemented by a similar Nash only if it satisfies certain intersection property. A famous example is that King Solomon's Rule cannot be Nash implemented because it does not satisfy Maskin monotonicity, but can be Nash approximated execution. Duggan(1997) extended the concept of Nash approximated execution to the incomplete information environment and defined the Bayesian approximated execution and its conditions. The implementation of mechanism design under incomplete information is discussed in the next chapter.

This chapter mainly discusses the implementation theory of general incentive mechanism design under various equilibrium behaviors under complete information. And the last section briefly introduces the realization theory of information dispersion mechanism design. It is about the information efficiency of the mechanism, which is the minimum amount of information required by an economic mechanism to achieve a goal. In this theory, we only pay attention to the information requirement of the economic mechanism (the cost of operating information),but not the incentive problem, that means individual self-interest (personal rationality) is not required to be consistent with established goals (collective rationality). In Hurwicz(1972, 1986b)'s theory of execution (as a part of the basic literatures of mechanism design theory), designers are mainly concerned with the scale of mechanism information space and hope to find an economic system with minimum operating cost. In the great debate of socialism, Hayek and other people believed that planned economy was difficult to achieve, because all market elements had to be determined by the market and the amount of information needed was too large, although planned economy can achieve effective allocation of resources. And Herwitz proved this result. In 1972, Herwitz et al also proved that the market mechanism was the one that required the least amount of information, Of all known or unknown mechanisms that enable efficient resource allocation.

18.2 Basic Settings

Theoretical framework of the incentive mechanism design consists of five components: (1) economic environments (fundamentals of economy); (2) social choice goal to be reached; (3) economic mechanism that specifies the rules of game; (4) description of solution concept on individuals' self-interested behavior, and (5) implementation of a social choice goal (incentive-compatibility of personal interests and the social goal at equilibrium).

18.2.1 Economic Environments

$e_i = (Z_i, w_i, \succsim_i, Y_i)$: economic characteristic of agent i which consists of outcome space, initial endowment if any, preference relation, and the production set if agent i is also a producer;

$e = (e_1, \dots, e_n)$: an economy;

E : The set of all priori admissible economic environments.

$U = U_1 \times \dots \times U_n$: The set of all admissible utility functions.

$\Theta = \Theta_1 \times \Theta_2 \cdots \times \Theta_n$: The set of all admissible parameters $\theta = (\theta_1, \dots, \theta_I) \in \Theta$ that determine types of parametric utility functions $u_i(\cdot, \theta_i)$, and so it is called the space of types or called the state of the world.

Remark 18.2.1 Here, E is a general expression of economic environments. However, depending on the situations facing the designer, the set of admissible economic environments under consideration sometimes may be just given by $E = U$, $E = \Theta$, or by the set of all possible initial endowments, or production sets.

The designer is assumed that he does not know individuals' economic characteristics. The individuals may or may not know the characteristics of the others. If they know, it is called the **complete information** case, otherwise it is called the **incomplete information** case.

For simplicity (but without loss of generality), when individuals' economic characteristics are private information (the case of incomplete information), we assume preferences are given by parametric utility functions. In this case, each agent i privately observes a type $\theta_i \in \Theta_i$, which determines his preferences over outcomes. The state θ is drawn randomly from a prior distribution with density $\varphi(\cdot)$ that can also be probabilities for finite Θ . Each agent maximizes von Neumann-Morgenstern expected utility over outcomes, given by (Bernoulli) utility function $u_i(y, \theta_i)$. Thus, **Information structure** is specified by

- (1) θ_i is privately observed by agent i ;
- (2) $\{u_i(\cdot, \cdot)\}_{i=1}^n$ is common knowledge;
- (3) $\varphi(\cdot)$ is common knowledge.

In this chapter, we will first discuss the case of complete information and then the case of incomplete information.

18.2.2 Social Goal

Given economic environments, each agent participates economic activities, makes decisions, receives benefits and pays costs on economic activities. The designer wants to reach some desired goal that is considered to be socially optimal by some criterion.

Let

$Z = Z_1 \times \dots \times Z_n$: the outcome space (For example, $Z = X \times Y$).

$A \subseteq Z$: the feasible set.

$F : E \rightarrow \rightarrow A$: the social goal or called social choice correspondence in which $F(e)$ is the set of socially desired outcomes at the economy under some criterion of social optimality.

Examples of Social Choice Correspondences:

$P(e)$: the set of Pareto efficient allocations.

$I(e)$: the set of individual rational allocations.

$W(e)$: the set of Walrasian allocations.

$L(e)$: the set of Lindahl allocations.

$FA(e)$: the set of fare allocations.

When F becomes a single-valued function, denoted by f , it is called a social choice function.

Examples of Social Choice Functions:

Solomon's goal.

Majority voting rule.

18.2.3 Economic Mechanism

Since the designer lacks the information about individuals' economic characteristics, he needs to design an appropriate incentive mechanism (contract or rules of game) to coordinate the personal interests and the social goal, i.e., under the mechanism, all individuals have incentives to choose actions which result in socially optimal outcomes when they pursue their personal interests. To do so, the designer informs how the information he

collected from individuals is used to determine outcomes, that is, he first tells the rules of games. He then uses the information or actions of agents and the rules of game to determine outcomes of individuals. Thus, a mechanism consists of a message space and an outcome function. Let

M_i : the message space of agent i .

$M = M_1 \times \dots \times M_n$: the message space in which communications take place.

$m_i \in M_i$: a message reported by agent i .

$m = (m_1, \dots, m_n) \in M$: a profile of messages.

$h : M \rightarrow Z$: outcome function that translates messages into outcomes.

$\Gamma = \langle M, h \rangle$: a mechanism

That is, a mechanism consists of a message space and an outcome function.

Remark 18.2.2 A mechanism is often also referred to as a game form. The terminology of game form distinguishes it from a game in game theory in number of ways. (1) Mechanism design is normative analysis in contrast to game theory, which is positive economics. Game theory is important because it predicts how a given game will be played by agents. Mechanism design goes one step further: given the physical environment and the constraints faced by the designer, what goal can be realized or implemented? What mechanisms are optimal among those that are feasible? (2) The consequence of a profile of message is an outcome in mechanism design rather than a vector of utility payoffs. Of course, once the preference of the individuals are specified, then a game form or mechanism induces a conventional game. (3) The preferences of individuals in the mechanism design setting vary, while the preferences of a game takes as given. This distinction between mechanisms and games is critical. Because of this, an equilibrium (dominant strategy equilibrium) in mechanism design is much easier to exist than a game. (4) In designing mechanisms one must take into account incentive constraints in a way that personal interests are consistent to the goal that a designer want to implement it.

Remark 18.2.3 In the implementation (incentive mechanism design) literature, one requires a mechanism be incentive compatible in the sense that personal interests are consistent with desired socially optimal outcomes even when individual agents are self-interested in their personal goals without paying much attention to the size of message. In the realization literature originated by Hurwicz (1972, 1986b), a sub-field of the mechanism literature, one also concerns the size of message space of a mechanism, and tries to find economic system to have small operation cost. The smaller a

message space of a mechanism, the lower (transition) cost of operating the mechanism. For the neoclassical economies, it has been shown that competitive market economy system is the unique most efficient system that results in Pareto efficient and individually rational allocations (cf, Mount and Reiter (1974), Walker (1977), Osana (1978), Hurwicz (1986b), Jordan (1982), Tian (2004, 2005)).

18.2.4 Solution Concept of Self-Interested Behavior

A basic assumption in economics is that individuals are self-interested in the sense that they pursue their personal interests. Unless they can be better off, they in general does not care about social interests. As a result, different economic environments and different rules of game will lead to different reactions of individuals, and thus each individual agent's strategy on reaction will depend on his self-interested behavior which in turn depends on the economic environments and the mechanism.

Let $b(e, \Gamma)$ be the set of equilibrium strategies that describes the self-interested behavior of individuals. Examples of such equilibrium solution concepts include Nash equilibrium, dominant strategy, Bayesian Nash equilibrium, etc.

Thus, given E , M , h , and b , the resulting equilibrium outcome is the composite function of the rules of game and the equilibrium strategy, i.e., $h(b(e, \Gamma))$.

18.2.5 Implementation and Incentive Compatibility

In which sense can we see individuals's personal interests do not have conflicts with a social interest? We will call such problem as implementation problem. The purpose of an incentive mechanism design is to implement some desired socially optimal outcomes. Given a mechanism Γ and equilibrium behavior assumption $b(e, \Gamma)$, the implementation problem of a social choice rule F studies the relationship of the intersection state of $F(e)$ and $h(b(e, \Gamma))$, which can be illustrated by the following diagram.

We have the following various definitions on implementation and incentive compatibility of F .

A Mechanism $\langle M, h \rangle$ is said to

- (i) fully implement a social choice correspondence F in equilibrium strategy $b(e, \Gamma)$ on E if for every $e \in E$
 - (a) $b(e, \Gamma) \neq \emptyset$ (equilibrium solution exists),
 - (b) $h(b(e, \Gamma)) = F(e)$ (personal interests are fully consistent with social goals);

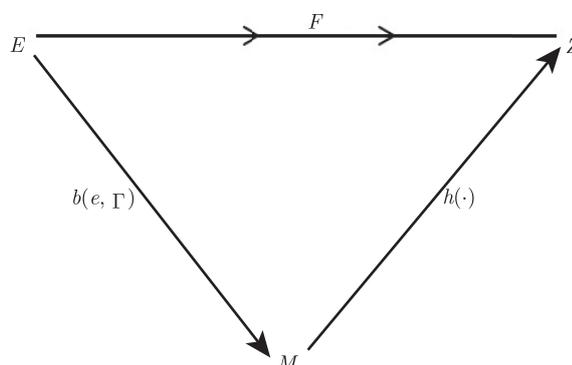


Figure 18.1: Diagrammatic Illustration of Mechanism design Problem.

(ii) strongly implement a social choice correspondence F in equilibrium strategy $b(e, \Gamma)$ on E if for every $e \in E$

- (a) $b(e, \Gamma) \neq \emptyset$,
- (b) $h(b(e, \Gamma)) \subseteq F(e)$;

(iii) implement a social choice correspondence F in equilibrium strategy $b(e, \Gamma)$ on E if for every $e \in E$

- (a) $b(e, \Gamma) \neq \emptyset$,
- (b) $h(b(e, \Gamma)) \cap F(e) \neq \emptyset$.

A Mechanism $\langle M, h \rangle$ is said to be $b(e, \Gamma)$ incentive-compatible with a social choice correspondence F in $b(e, \Gamma)$ -equilibrium if it (fully or strongly) implements F in $b(e, \Gamma)$ -equilibrium.

Note that we did not give a specific solution concept yet when we define the implementability and incentive-compatibility. As shown in the following, whether or not a social choice correspondence is implementable will depend on the assumption on the solution concept of self-interested behavior. When information is complete, the solution concept can be dominant equilibrium, Nash equilibrium, strong Nash equilibrium, subgame perfect Nash equilibrium, undominated equilibrium, etc. For incomplete information, equilibrium strategy can be Bayesian Nash equilibrium, undominated Bayesian Nash equilibrium, etc.

18.3 Examples

Before we discuss some basic results in the mechanism theory, we first give some economic environments which show that one needs to design a mechanism to solve the incentive compatible problems.

Example 18.3.1 (A Public Project) A society is deciding on whether or not to build a public project at a cost c . The cost of the public project is shared by individuals. Let s_i be the share of the cost by i so that $\sum_{i \in N} s_i = 1$.

The outcome space is then $Y = \{0, 1\}$, where 0 represents not building the project and 1 represents building the project. Individual i 's value from use of this project is r_i . In this case, the net value of individual i is 0 from not having the project built and $v_i = r_i - s_i c$ from having a project built. Thus agent i 's valuation function can be represented as

$$v_i(y, v_i) = y r_i - y s_i c = y v_i.$$

Example 18.3.2 (Continuous Public Goods Setting) In the above example, the public good could only take two values, and there is no scale problem. But, in many cases, the level of public goods depends on the collection of the contribution or tax. Now let $y \in R_+$ denote the scale of the public project and $c(y)$ denote the cost of producing y . Thus, the outcome space is $Z = R_+ \times R^n$, and the feasible set is $A = \{(y, z_1(y), \dots, z_n(y)) \in R_+ \times R^n : \sum_{i \in N} z_i(y) = c(y)\}$, where $z_i(y)$ is the share of agent i for producing the public goods y . The benefit of i for building y is $r_i(y)$ with $r_i(0) = 0$. Thus, the net benefit of not building the project is equal to 0, the net benefit of building the project is $r_i(y) - z_i(y)$. The valuation function of agent i can be written as

$$v_i(y) = r_i(y) - z_i(y).$$

Example 18.3.3 (Allocating an Indivisible Private Good) An indivisible good is to be allocated to one member of society. For instance, the rights to an exclusive license are to be allocated or an enterprise is to be privatized. In this case, the outcome space is $Z = \{y \in \{0, 1\}^n : \sum_{i=1}^n y_i = 1\}$, where $y_i = 1$ means individual i obtains the object, $y_i = 0$ represents the individual does not get the object. If individual i gets the object, the net value benefitted from the object is v_i . If he does not get the object, his net value is 0. Thus, agent i 's valuation function is

$$v_i(y) = v_i y_i.$$

Note that we can regard y as n -dimensional vector of public goods since $v_i(y) = v_i y_i = v^i y$, where v^i is a vector where the i -th component is v_i and the others are zeros, i.e., $v^i = (0, \dots, 0, v_i, 0, \dots, 0)$.

From these examples, a socially optimal decision clearly depends on the individuals' true valuation function $v_i(\cdot)$. For instance, we have shown previously that a public project is produced if and only if the total values of all individuals is greater than its total cost, i.e., if $\sum_{i \in N} r_i > c$, then $y = 1$, and if $\sum_{i \in N} r_i < c$, then $y = 0$.

Let V_i be the set of all valuation functions v_i , let $V = \prod_{i \in N} V_i$, let $h : V \rightarrow Z$ is a decision rule. Then h is said to be efficient if and only if:

$$\sum_{i \in N} v_i(h(v)) \geq \sum_{i \in N} v_i(h(v')) \quad \forall v' \in V.$$

18.4 Dominant Strategy and Truthful Revelation Mechanisms

The most robust and strongest solution concept of describing self-interested behavior is dominant strategy. The dominant strategy identifies situations in which the strategy chosen by each individual is the first-best, regardless of choices of the others. The beauty of this equilibrium concept is in the weak rationality it demands of the agents: an agent need not forecast what the others are doing. An axiom in game theory is that agents will use it as long as a dominant strategy exists.

For $e \in E$, a mechanism $\Gamma = \langle M, h \rangle$ is said to have a dominant strategy equilibrium m^* if for all i

$$h_i(m_i^*, m_{-i}) \succsim_i h_i(m_i, m_{-i}) \text{ for all } m \in M. \quad (18.4.1)$$

Denote by $D(e, \Gamma)$ the set of dominant strategy equilibria for $\Gamma = \langle M, h \rangle$ and $e \in E$.

Under the assumption of dominant strategy, since each agent's optimal choice does not depend on the choices of the others and does not need to know characteristics of the others, the required information is least when an individual makes decisions. Thus, if it exists, it is an ideal situation. Another advantage is that bad equilibria are usually not a problem. If an agent has two dominant strategies they must be payoff-equivalent, which is not a generic property.

In a given game, dominant strategies are not likely to exist. However, since we are designing the game, we can try to ensure that agents do have dominant strategies.

When the solution concept is given by dominant strategy equilibrium, i.e., $b(e, \Gamma) = D(e, \Gamma)$, a mechanism $\Gamma = \langle M, h \rangle$ is said to

- (i) fully implement a social choice correspondence F in dominant equilibrium strategy $D(e, \Gamma)$ on E if for every $e \in E$
 - (a) $D(e, \Gamma) \neq \emptyset$ (equilibrium solution exists),
 - (b) $h(D(e, \Gamma)) = F(e)$ (personal interests are fully consistent with social goals);
- (ii) strongly implement a social choice correspondence F in dominant equilibrium strategy $D(e, \Gamma)$ on E if for every $e \in E$

- (a) $D(e, \Gamma) \neq \emptyset$,
- (b) $h(D(e, \Gamma)) \subseteq F(e)$;

(iii) implement a social choice correspondence F in dominant equilibrium strategy $D(e, \Gamma)$ on E if for every $e \in E$

- (a) $D(e, \Gamma) \neq \emptyset$,
- (b) $h(D(e, \Gamma)) \cap F(e) \neq \emptyset$.

The above definitions have applied to general (indirect) mechanisms, there is, however, a particular class of game forms which have a natural appeal and have received much attention in the literature. These are called direct or revelation mechanisms, in which the message space M_i for each agent i is the set of possible characteristics E_i . In effect, each agent reports a possible characteristic but not necessarily his true one.

A mechanism $\Gamma = \langle M, h \rangle$ is said to be a *revelation or direct mechanism* if $M = E$.

Example 18.4.1 The optimal contracts we discussed in Chapter 13 are revelation mechanisms.

Example 18.4.2 The Groves mechanism we will discuss below is a revelation mechanism.

The most appealing revelation mechanisms are those in which truthful reporting of characteristics always turns out to be an equilibrium. It is the absence of such a mechanism which has been called the “free-rider” problem in the theory of public goods.

A revelation mechanism $\langle E, h \rangle$ is said to implement a social choice correspondence F truthfully in $b(e, \Gamma)$ on E if for every $e \in E$,

- (a) $e \in b(e, \Gamma)$;
- (b) $h(e) \subset F(e)$.

That is, $F(\cdot)$ is truthfully implementable in dominant strategies if truth-telling is a dominant strategy for each agent in the direct revelation mechanism.

Truthfully implementable in dominant strategies is also called *dominant strategy incentive compatible, strategy proof* or *straightforward*.

Although the message space of a mechanism can be arbitrary, the following Revelation Principle tells us that one only needs to use the so-called revelation mechanism in which the message space consists solely of the set of individuals’ characteristics, and it is unnecessary to seek more complicated mechanisms. Thus, it will significantly reduce the complicity of constructing a mechanism.

Theorem 18.4.1 (Revelation Principle) *Suppose a mechanism $\langle M, h \rangle$ implements a social choice rule F in dominant strategy. Then there is a revelation mechanism $\langle E, g \rangle$ which implements F truthfully in dominant strategy.*

Proof. Let d be a selection of dominant strategy correspondence of the mechanism $\langle M, h \rangle$, i.e., for every $e \in E$, $m^* = d(e) \in D(e, \Gamma)$ such that $h(d(e)) \in F(e)$. Since $\Gamma = \langle M, h \rangle$ implements social choice rule F , such a selection exists by the implementation of F . Since the strategy of each agent is independent of the strategies of the others, each agent i 's dominant strategy can be expressed as $m_i^* = d_i(e_i)$.

Define the revelation mechanism $\langle E, g \rangle$ by $g(e) \equiv h(d(e))$ for each $e \in E$. We first show that the truth-telling is always a dominant strategy equilibrium of the revelation mechanism $\langle E, g \rangle$. Suppose not. Then, there exists a message e' and an agent i such that

$$u_i[g(e'_i, e'_{-i})] > u_i[g(e_i, e'_{-i})].$$

However, since $g = h \circ d$, we have

$$u_i[h(d(e'_i), d(e'_{-i}))] > u_i[h(d(e_i), d(e'_{-i}))],$$

which contradicts the fact that $m_i^* = d_i(e_i)$ is a dominant strategy equilibrium. This is because, when the true economic environment is (e_i, e'_{-i}) , agent i has an incentive not to report $m_i^* = d_i(e_i)$ truthfully, but have an incentive to report $m'_i = d_i(e'_i)$, a contradiction.

Finally, since $m^* = d(e) \in D(e, \Gamma)$ and $\langle M, h \rangle$ strongly implements a social choice rule F in dominant strategy, we have $g(e) = h(d(e)) = h(m^*) \in F(e)$. Hence, the revelation mechanism implements F truthfully in dominant strategy. The proof is completed.

Thus, by the Revelation Principle, we know that, if truthful implementation rather than full or strong implementation is all that we require, we need never consider general mechanisms. In the literature, if a revelation mechanism $\langle E, h \rangle$ truthfully implements a social choice rule F in dominant strategy, the mechanism Γ is sometimes said to be *strongly individually incentive-compatible* with a social choice correspondence F . In particular, when F becomes a single-valued function f , $\langle E, f \rangle$ can be regarded as a revelation mechanism. Thus, if a mechanism $\langle M, h \rangle$ implements f in dominant strategy, then the revelation mechanism $\langle E, f \rangle$ is incentive compatible in dominant strategy, or called strongly individually incentive compatible.

Remark 18.4.1 Notice that the Revelation Principle may be valid only for implementation. It does not apply to full or strong implementation. The Revelation Principle specifies a correspondence between a dominant strategy equilibrium of the original mechanism $\langle M, h \rangle$ and the true profile

of characteristics as a dominant strategy equilibrium, and it does not require the revelation mechanism has a unique dominant equilibrium so that the revelation mechanism $\langle E, g \rangle$ may also exist non-truthful strategy equilibrium that does not corresponds to any equilibrium. Thus, in moving from the general (indirect) dominant strategy mechanisms to direct ones, one may introduce undesirable dominant strategies which are not truthful. More troubling, these additional strategies may create a situation where the indirect mechanism is a full or strong implantation of a given F , while the direct revelation mechanism is not. Thus, even if a mechanism fully or strongly implements a social choice function, the corresponding revelation mechanism $\langle E, g \rangle$ may only implement, but not fully or strongly implement F .

However, if agents have strict preferences, i.e., $\forall a, b \in \mathcal{A}$, $a = b$ if and only if $a \sim_i^e b, \forall i \in n$, any two different outcomes cannot be indifferent. Thus the Revelation Principle is valid for full implementation, but not just for implementation.

The following result shows that, although (Γ, θ) may have more than one dominant strategy equilibrium, under condition of strict preferences, the resulting equilibrium outcome will be unique, and thus the revelation mechanism $\langle E, g \rangle$ by $g(e) \equiv h(d(e))$ fully implements the social choice correspondence in dominant strategy.

Proposition 18.4.1 *Suppose agents have strict preferences. Then any dominant strategy outcome of $\Gamma = \langle M, h \rangle$ that implements a social choice goal F in dominant strategy is unique, and thus the revelation mechanism $\langle E, g \rangle$ by $g(e) \equiv h(d(e))$ strongly implements F in dominant strategy.*

PROOF. Suppose $\Gamma = \langle M, h \rangle$ implements F in dominant strategy, but the equilibrium is not unique. Then, there is $m_i^*, m_i^{*'} \in \mathcal{D}_i(\Gamma, \theta)$ such that $m_i^* \neq m_i^{*'}$. By dominant strategy, for $\forall m_{-i} \in M_{-i}$, we have

$$h(m_i^*, m_{-i}) \succsim_i^e h(m_i^{*'}, m_{-i})$$

$$h(m_i^{*'}, m_{-i}) \succsim_i^e h(m_i^*, m_{-i})$$

Thus, $h(m_i^*, m_{-i}) \sim_i^e h(m_i^{*'}, m_{-i})$. By strict preferences, we have $h(m_i^*, m_{-i}) = h(m_i^{*'}, m_{-i})$. Repeating the process for each i , we have $h(m^*) = h(m^{*'}), \forall (m^*, m^{*'}) \in \mathcal{D}(\Gamma, \theta)$. Therefore, dominant strategy is unique.

Thus, for all $m^* \in \mathcal{D}(e, \Gamma)$, $g(e) \equiv h(m^*(e))$ is a single-valued mapping. So the revelation mechanism $\langle E, g \rangle$ strongly implements F in dominant strategy. \square

As a direct corollary of Proposition 18.4.1, any social choice goal that can be full implemented in dominant strategy must be a single-valued social choice function. Formally, we have

Corollary 18.4.1 *Suppose agents have strict preferences. Any social choice goal F that can be fully implemented in dominant strategy (i.e., $h(\mathcal{D}(e, \Gamma)) = F(e)$) must be a single-valued social choice function.*

PROOF. Since $\Gamma = \langle M, h \rangle$ fully implements F in dominant strategy, we have $h(\mathcal{D}(e, \Gamma)) = F(e)$. Then, by Proposition 18.4.1, we have $h(m^*) = h(m^{*'}), \forall (m^*, m^{*'}) \in \mathcal{D}(\Gamma, \theta)$. Thus, for $\forall \theta \in \Theta, f(\theta) = h(\mathcal{D}(\Gamma, \theta))$ is a single-valued function. □

18.5 Gibbard-Satterthwaite Impossibility Theorem

Social planners want the social choice rules to be enforced to have some good quality, such as non-autocracy; at the same time, due to the need for as little information as possible, the game solution to execute the rule is expected to be as strong as possible, such as dominant equilibrium. The revelation principle is very useful for designing superior strategy mechanism. If people want social choice target f to execute according to the dominant strategy, then we only need to prove that the display mechanism $\langle E, f \rangle$ is strongly excited compatible.

But unfortunately, it is generally impossible to get such a result. The Gibbard-Satterthwaite impossibility theorem in twelfth chapter tells us that such mechanism cannot exist except dictatorial mechanism without restrictions on the economic environment. Let's revisit this theorem from the view of mechanism design.

Theorem 18.5.1 (Gibbard-Satterthwaite Impossibility Theorem) *If the result space \mathcal{A} contains at least three options $|\mathcal{A}| \geq 3$, the domain of social choice correspondence $F : E \rightrightarrows \mathcal{A}$ contains all strict preferences, which is $E^s \subseteq E$, and $F(\cdot)$ is an upper-to-top map that can be executed realistically according to the dominant strategy (surjection onto), Then F must be autocratic.*

This theorem shows that if the choices of individuals are unrestricted and each individual is required to tell the truth, then such an incentive mechanism does not exist. Before the theorem is proved, the following lemma is given.

Lemma 18.5.1 *If the domain contains all strict preferences $E = E^s$, the result space \mathcal{A} contains at least three options $|\mathcal{A}| \geq 3$, and the social choice rule $F : E^s \rightrightarrows \mathcal{A}$ satisfies **uniformity** and **Maskin monotonicity**, then that must be autocratic.*

PROOF.

See tables 18.1 to 18.7

- Consider the two options $a, b \in \mathcal{A}$. Suppose $a \succ_i^e x \succ_i^e b, \forall x \in \mathcal{A}, \forall i \in I$. According to consistency, we get $F(e) = a$. In the preference order of economic man 1, move b forward one position at a time. According to Maskin monotonicity, as long as b does not exceed a , then option a is still selected. But when b exceed a , according to uniformity, the possible outcome is a or b . If the selected option is still a , the same operation is performed on the preference ordering of economic person 2, and so on. There must be an economic person who, in his preference, changes the social choice from a to b when preference a and b reverse, then mark that person as j . And also mark the state as e^1 before preference a and b reversing due to the preference order of j , mark the state after that as e^2 . In the state of e^1 and e^2 , move the preference order a to the end for each agent $i < j$, while move the preference order a to the second to last position for each agent $i > j$. The preference of the economic person j remains unchanged, and the new state is recorded as $e^{1'}$ and $e^{2'}$.
- because of $F(e^2) = b$ and $L_i(b, e^2) = L_i(b, e^{2'}), \forall i \in I$, according to Maskin monotonicity, we get $F(e^{2'}) = b$. And compare $e^{2'}$ with $e^{1'}$, only at the preference ordering of j , a and b reverse, which means $F(e^{1'}) = a$ or b . If $F(e^{1'}) = b$, since $L_i(b, e^{1'}) = L_i(b, e^1), \forall i \in I$, we have $F(e^1) = b$ according to Maskin monotonicity, and come up with the contradiction, so we must have $F(e^{1'}) = a$.
- suppose $c \neq a, b$ is another option, define the state e^3 due to Table 18.6. since $L_i(a, e^3) = L_i(a, e^1), \forall i \in I$, we have $F(e^3) = a$ according to Maskin monotonicity.
- At the state e^3 , reserve the preference a and b for each agent $i > j$, and get state e^4 . According to monotonicity, $F(e^4) = a$ or b . Obviously, $c \succ_i^{e^4} b, \forall i \in I$, and according to consistency, we must have $F(e^4) = a$.
- Obviously, for any state $\alpha \in E^s$, as long as a is at the top of the list of economic man j 's preferences, then $L_i(a, e^4) \subseteq L_i(a, \alpha), \forall i \in I$, and we can have $F(\alpha) = a$ according to Maskin monotonicity. It is clear that individual j is the autocrat of the plan a .

From the proof above, we can see that under the strict preference, there is a dictator for any option a , but there can be no more than one dictator. Suppose at the state e , j is the dictator for option a , then we have $F(e) = a$; and suppose j' is another dictator for another option a' , then we have $F(e) = a'$, it is contradictory. \square \square

Lemma 18.5.2 If the social choice rule $F : E^s \rightrightarrows \mathcal{A}$ is real execution of dominant strategy and upper mapping to (surjection, onto), then it must satisfy consistency and

individual	preference \succ^{θ^1}			
1	b	a	...	*
2	b	a	...	*
...
j-1	b	a	...	*
j	a	b	...	*
j+1	a	b
...
n	a	b

Table 18.2: the preference of State e_1

individual	preference $\succ^{\theta^{1'}}$				
1	b	*	...	*	a
2	b	*	...	*	a
...
j-1	b	*	...	*	a
j	a	b	...	*	*
j+1	*	*	...	a	b
...
n	*	*	...	a	b

Table 18.3: the preference of state e'_1 **PROOF.**

- **Maskin monotonicity:** suppose state $e, e' \in E^s$ satisfies $F(e) = a$, and $L_i(a, e) \subseteq L_i(a, e'), \forall i \in I$. If $F(e'_1, e_{-1}) \neq a$, according to honest execution of dominant strategy and strict preference, there must be $a = F(e) \succ_i^e F(e'_1, e_{-1})$, that is $F(e'_1, e_{-1}) \in L_1(a, e)$, thus $F(e'_1, e_{-1}) \in L_1(a, e')$, that is $a = F(e_1, e_{-1}) \succ_i^{e'} F(e'_1, e_{-1})$. This contradicts the nature of the honest execution of $F(\cdot)$'s dominant strategy, so there must be $F(e'_1, e_{-1}) = a$. The rest may be deduced by analogy.

$$F(e'_1, e_{-1}) = F(e'_1, e'_2, e_3, \dots, e_n) = \dots = F(e') = a,$$

so we prove the Maskin monotonicity of $F(\cdot)$.

- **consistency:** since $F(\cdot)$ is the upper mapping (surjection), for any $a \in \mathcal{A}$, there exists $e \in E^s$ so that $F(e) = a$. Suppose β shows such a situation: option a is at the top of the preference list for every agent. Obviously, $L_i(a, e) \subseteq L_i(a, \beta), \forall i \in I$, we get $F(\beta) = a$ according to Maskin monotonicity, so consistency is proved.

individual	preference \succ^{θ^2}			
1	b	a	...	*
2	b	a	...	*
...
j-1	b	a	...	*
j	b	a	...	*
j+1	a	b
...
n	a	b

Table 18.4: the preference of state e_2

individual	preference $\succ^{\theta^{2'}}$				
1	b	*	...	*	a
2	b	*	...	*	a
...
j-1	b	*	...	*	a
j	b	a	...	*	*
j+1	*	a	b
...
n	*	*	...	a	b

Table 18.5: the preference of state e_2'

□

□

Now prove the Gibbard-Satterthwaite Impossibility Theorem:

PROOF. According to the above two Lemmas, $\exists j$ for $\forall e \in E^s$, there is $F(e) = M_j(e)$, among $M_j(e) = \{x \in \mathcal{A} | x \succ_i^e b, \forall b \in \mathcal{A} \setminus \{x\}\}$ represents the most preferred set for economic person j . Now we only need to prove that $\forall e \in E \setminus E^s, F(e) = M_j(e)$. Using countervailing method, suppose $\exists e \in E \setminus E^s, a = F(e)$ but $a \notin M_j(e)$, then $\exists b \in M_j(e), b \succ_j^e a$. Consider another state $e' \in E^s$ satisfying (i) $a \succ_i^{e'} b \succ_i^{e'} c, \forall i \neq j, c \neq a, b$; (ii) $b \succ_j^{e'} a \succ_j^{e'} c, \forall c \neq a, b$. Then $L_i(a, e) \subseteq L_i(a, e'), \forall i \in I$, and $F(e') = b$. According to real execution of dominant strategy, $F(e_1, e_{-1}) \succ_i^e F(e'_1, e_{-1})$, so $F(e_1, e_{-1}) \succ_i^{e'} F(e'_1, e_{-1})$. And on the other hand, according to the property of real execution of dominant strategy again, $F(e'_1, e_{-1}) \succ_i^{e'} F(e_1, e_{-1})$, so $F(e'_1, e_{-1}) = F(e_1, e_{-1}) = a$, and so on, we can get $F(e') = a$. Therefore $F(e') = a \neq M_j(e') = b$, which is contradiction. So for $\forall e \in E \setminus E^s, F(e) = M_j(e)$. The theorem is proved. □ □

In short, Gibbard-Satterthwaite Impossibility Theorem is a rather disappointing result. And the result is essentially equivalent to Arrow Impossi-

individual	preference \succ^{θ^3}						
1	*	*	*	...	c	b	a
2	*	*	*	...	c	b	a
...
j-1	*	*	*	...	c	b	a
j	a	c	b	...	*	*	*
j+1	*	*	*	...	c	a	b
...
n	*	*	*	...	c	a	b

Table 18.6: the preference of state e_3

individual	preference \succ^{θ^4}						
1	*	*	*	...	c	b	a
2	*	*	*	...	c	b	a
...
j-1	*	*	*	...	c	b	a
j	a	c	b	...	*	*	*
j+1	*	*	*	...	c	b	a
...
n	*	*	*	...	c	b	a

Table 18.7: the preference of state e_4

bility Theorem. However, when the economic environment is constrained in the conditions of Gibbard-Satterthwaite Impossibility Theorem, for example, by relaxing the two conditions "there are at least three options" and the unconstrained domain ($E^s \subseteq E$), its conclusion may not hold true. Let's start with the counterexample of relaxing the condition of "there are at least three options".

Example 18.5.1 A simple majority vote with only two candidates Suppose there are N voters casting their votes on two candidates, a and b , and each voter reported the preference (only need to name the candidate for your support), then vote on the basis of the following simple majority rule:

$$F(e) = \begin{cases} a & \text{if } \#\{i \in I | a \succ_i^e b\} > \#\{i \in I | b \succ_i^e a\}, \\ b & \text{if } \#\{i \in I | a \succ_i^e b\} \leq \#\{i \in I | b \succ_i^e a\}. \end{cases}$$

It can be seen that every voter will truly declare their preferences in this process, regardless of whether others are telling the truth or not. Therefore, $F(\cdot)$ can be real executed by dominant strategy.

Another example to be discussed below is that the VCG mechanism defined on a quasilinear utility function is a truth-telling mechanism that leads to efficient decision (maximization of social surplus). But the VCG mechanism does not generally lead to Pareto effective results. If the Pareto efficiency and individual maximization allocation are still adhered to, Hurwitz got a stronger impossibility theorem: even if further restricted to the neo-classical economic environment, the Pareto efficiency and the truth-telling are incompatible.

18.6 Hurwicz Impossibility Theorem

The Gibbard-Satterthwaite impossibility theorem is a very negative result. This result is essentially equivalent to Arrow's impossibility result. However, as we will show, when the admissible set of economic environments is restricted, the result may be positive as the Groves mechanism defined on quasi-linear utility functions. Unfortunately, the following Hurwicz's impossibility theorem shows the Pareto efficiency and the truthful revelation is fundamentally inconsistent even for the class of neoclassical economic environments.

Theorem 18.6.1 (Hurwicz Impossibility Theorem, 1972) *For the neoclassical private goods economies, there is no mechanism $\langle M, h \rangle$ that implements Pareto efficient and individually rational allocations in dominant strategy. Consequently, any revelation mechanism $\langle M, h \rangle$ that yields Pareto efficient and individually rational allocations is not strongly individually incentive compatible. (Truth-telling about their preferences is not Nash Equilibrium).*

Proof: By the Revelation Principle, we only need to show that any revelation mechanism cannot implement Pareto efficient and individually rational allocations truthfully in dominant equilibrium for a particular pure exchange economy. In turn, it is enough to show that truth-telling is not a Nash equilibrium for any revelation mechanism that yields Pareto efficient and individually rational allocations for a particular pure exchange economy.

Consider a private goods economy with two agents ($n = 2$) and two goods ($L = 2$),

$$w_1 = (0, 2), w_2 = (2, 0)$$

$$circu_i(x, y) = \begin{cases} 3x_i + y_i & \text{if } x_i \leq y_i \\ x_i + 3y_i & \text{if } x_i > y_i. \end{cases}$$

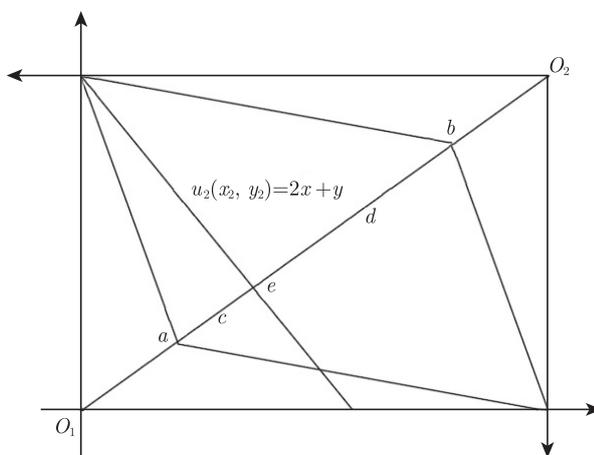


Figure 18.2: An illustration of the proof of Hurwicz's impossibility theorem.

Thus, feasible allocations are given by

$$A = \left\{ [(x_1, y_1), (x_2, y_2)] \in R_+^4 : \begin{aligned} x_1 + x_2 &= 2 \\ y_1 + y_2 &= 2 \end{aligned} \right\}.$$

Let U_i be the set of all neoclassical utility functions, i.e. they are continuous and quasi-concave, which agent i can report to the designer. Thus, the true utility function $\text{circ}_{i \in U_i}$. Then,

$$\begin{aligned} U &= U_1 \times U_2 \\ h &: U \rightarrow A. \end{aligned}$$

Note that, if the true utility function profile \hat{u}_i were a Nash Equilibrium, it would satisfy

$$\hat{u}_i(h_i(\hat{u}_i, \hat{u}_{-i})) \geq \hat{u}_i(h_i(u_i, \hat{u}_{-i})) \quad (18.6.2)$$

We want to show that \hat{u}_i is not a Nash equilibrium. Note that,

- (1) $P(e) = \overline{O_1 O_2}$ (contract curve);
- (2) $IR(e) \cap P(e) = \overline{ab}$;
- (3) $h(\hat{u}_1, \hat{u}_2) = d \in \overline{ab}$.

Now, suppose agent 2 reports his utility function by cheating:

$$u_2(x_2, y_2) = 2x + y \quad (18.6.3)$$

Then, with u_2 , the new set of individually rational and Pareto efficient allocations is given by

$$IR(e) \cap P(e) = \bar{ae} \quad (18.6.4)$$

Note that any point between a and e is strictly preferred to d by agent 2. Thus, an allocation determined by any mechanism which is IR and Pareto efficient allocation under (\hat{u}_1, u_2) is some point, say, the point c in the figure, between the segment of the line determined by a and e . Hence, we have

$$\hat{u}_2(h_2(\hat{u}_1, u_2)) > \hat{u}_2(h_2(\hat{u}_1, \hat{u}_2)) \quad (18.6.5)$$

since $h_2(\hat{u}_1, u_2) = c \in \bar{ae}$. Similarly, if d is between \bar{ae} , then agent 1 has incentive to cheat. Thus, no mechanism that yields Pareto efficient and individually rational allocations is incentive compatible. The proof is completed.

Thus, the Hurwicz's impossibility theorem implies that Pareto efficiency and the truthful revelation about individuals' characteristics are fundamentally incompatible. However, if one is willing to give up Pareto efficiency, say, one only requires the efficient provision of public goods, is it possible to find an incentive compatible mechanism which results in the efficient provision of a public good and can truthfully reveal individuals' characteristics? The answer is positive. For the class of quasi-linear utility functions, the so-called Vickrey-Clark-Groves Mechanism can be such a mechanism.

18.7 Vickrey-Clark-Groves Mechanisms

From Chapter ?? on public goods, we have known that public goods economies may present problems by a decentralized resource allocation mechanism because of the free-rider problem. Private provision of a public good generally results in less than an efficient amount of the public good. Voting may result in too much or too little of a public good. Are there any mechanisms that result in the "right" amount of the public good? This is a question of the incentive compatible mechanism design.

Again, we will focus on the quasi-linear environment, where all agents are known to care for money. In this environment, the results are more positive, and we can even implement the efficient decision rule (the efficient provision of public goods). For simplicity, let us first return to the model of discrete public good.

18.7.1 Vickrey-Clark-Groves Mechanisms for Discrete Public Good

Consider a provision problem of a discrete public good. Suppose that the economy has n agents. Let

c : the cost of producing the public project.

r_i : the maximum willingness to pay of i .

s_i : the share of the cost by i .

$v_i = r_i - s_i c$: the net value of i .

The public project is determined according to

$$y = \begin{cases} 1 & \text{if } \sum_{i=1}^n v_i \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

From the discussion in Chapter ??, it is efficient to produce the public good, $y = 1$, if and only if

$$\sum_{i=1}^n v_i = \sum_{i=1}^n (r_i - g_i) \geq 0.$$

Since the maximum willingness to pay for each agent, r_i , is private information and so is the net value v_i , what mechanism one should use to determine if the project is built? One mechanism that we might use is simply to ask each agent to report his or her net value and provide the public good if and only if the sum of the reported value is positive. The problem with such a scheme is that it does not provide right incentives for individual agents to reveal their true willingness-to-pay. Individuals may have incentives to underreport their willingness-to-pay.

Thus, a question is how we can induce each agent to truthfully reveal his true value for the public good. The so-called Vickrey-Clark-Groves (VCG) mechanism gives such a mechanism.

Suppose the utility functions are quasi-linear in net increment in private good, $x_i - w_i$, which have the form:

$$\begin{aligned} \bar{u}_i(x_i - w_i, y) &= x_i - w_i + r_i y \\ \text{s.t. } x_i + g_i y &= w_i + t_i, \end{aligned}$$

where t_i is the transfer to agent i . Then, we have

$$\begin{aligned} u_i(t_i, y) &= t_i + r_i y - g_i y \\ &= t_i + (r_i - g_i) y \\ &= t_i + v_i y. \end{aligned}$$

- Groves Mechanism:

In a Groves mechanism, agents are required to report their net values. Thus the message space of each agent i is $M_i = \Re$. The Groves mechanism is defined as follows:

$$\Gamma = (M_1, \dots, M_n, t_1(\cdot), t_2(\cdot), \dots, t_n(\cdot), y(\cdot)) \equiv (M, t(\cdot), y(\cdot)), \text{ where}$$

- (1) $b_i \in M_i = R$: each agent i reports a “bid” for the public good, i.e., report the net value of agent i which may or may not be his true net value v_i .
- (2) The level of the public good is determined by

$$y(b) = \begin{cases} 1 & \text{if } \sum_{i=1}^n b_i \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

- (3) Each agent i receives a side payment (transfer)

$$t_i(b) = \begin{cases} \sum_{j \neq i} b_j & \text{if } \sum_{i=1}^n b_i \geq 0 \\ 0 & \text{otherwise.} \end{cases} \quad (18.7.6)$$

Then, the payoff of agent i is given by

$$\phi_i(b) = \begin{cases} v_i + t_i(b) = v_i + \sum_{j \neq i} b_j & \text{if } \sum_{i=1}^n b_i \geq 0 \\ 0 & \text{otherwise.} \end{cases} \quad (18.7.7)$$

We want to show that it is optimal for each agent to report the true net value, $b_i = v_i$, regardless of what the other agents report. That is, truth-telling is a dominant strategy equilibrium.

There are two cases to be considered.

Case 1: $v_i + \sum_{j \neq i} b_j > 0$. Then agent i can ensure the public good is provided by reporting $b_i = v_i$. Indeed, if $b_i = v_i$, then $\sum_{j \neq i} b_j + v_i = \sum_{i=1}^n b_j > 0$ and thus $y = 1$. In this case, $\phi_i(v_i, b_{-i}) = v_i + \sum_{j \neq i} b_j > 0$.

Case 2: $v_i + \sum_{j \neq i} b_j \leq 0$. Agent i can ensure that the public good is not provided by reporting $b_i = v_i$ so that $\sum_{i=1}^n b_i \leq 0$. In this case, $\phi_i(v_i, b_{-i}) = 0 \geq v_i + \sum_{j \neq i} b_j$.

Thus, for either cases, agent i has incentives to tell the true value of v_i . Hence, it is optimal for agent i to tell the truth. There is no incentive for agent i to misrepresent his true net value regardless of what other agents do.

The above preference revelation mechanism has a major fault: the total side-payment may be very large. Thus, it is very costly to induce the agents to tell the truth.

Ideally, we would like to have a mechanism where the sum of the side-payment is equal to zero so that the feasibility condition holds, and consequently it results in Pareto efficient allocations, but in general it is impossible by Hurwicz’s impossibility theorem. However, we could modify the above mechanism by asking each agent to pay a “tax”, but not receive payment. Because of this “waster” tax, the allocation of public goods is still not Pareto efficient.

The basic idea of paying a tax is to add an extra amount to agent i 's side-payment, $d_i(b_{-i})$ that depends only on what the other agents do.

General Groves Mechanism (Vickrey-Clark-Groves Mechanism): Ask each agent to pay additional tax, $d_i(b_{-i})$.

In this case, the transfer is given by

$$t_i(b) = \begin{cases} \sum_{j \neq i} b_j + d_i(b_{-i}) & \text{if } \sum_{i=1}^n b_i \geq 0 \\ d_i(b_{-i}) & \text{if } \sum_{i=1}^n b_i < 0. \end{cases}$$

The payoff to agent i now takes the form:

$$\phi_i(b) = \begin{cases} v_i + t_i(b) = v_i + \sum_{j \neq i} b_j + d_i(b_{-i}) & \text{if } \sum_{i=1}^n b_i \geq 0 \\ d_i(b_{-i}) & \text{otherwise.} \end{cases} \quad (18.7.8)$$

For exactly the same reason as for the mechanism above, one can prove that it is optimal for each agent i to report his true net value. In conclusion, we have the following proposition.

Proposition 18.7.1 *For discrete public good economies under consideration, the truth-telling is a dominant strategy under the Vickrey-Clark-Groves mechanism that implements truthfully the efficient decision rule (the efficient provision of public goods) in dominant strategy.*

If the function $d_i(b_{-i})$ is suitably chosen, the size of the side-payment can be significantly reduced. A nice choice was given by Clarke (1971). He suggested a particular Groves mechanism known as the *Clarke mechanism* (also called *Pivot mechanism*):

The *Pivotal Mechanism* is a special case of the general Groves Mechanism in which $d_i(b_{-i})$ is given by

$$d_i(b_{-i}) = \begin{cases} -\sum_{j \neq i} b_j & \text{if } \sum_{j \neq i} b_j \geq 0 \\ 0 & \text{if } \sum_{j \neq i} b_j < 0. \end{cases}$$

In this case, it gives

$$t_i(b) = \begin{cases} 0 & \text{if } \sum_{i=1}^n b_i \geq 0 \text{ and } \sum_{j \neq i} b_j \geq 0, \\ \sum_{j \neq i} b_j & \text{if } \sum_{i=1}^n b_i \geq 0 \text{ and } \sum_{j \neq i} b_j < 0 \\ -\sum_{j \neq i} b_j & \text{if } \sum_{i=1}^n b_i < 0 \text{ and } \sum_{j \neq i} b_j \geq 0 \\ 0 & \text{if } \sum_{i=1}^n b_i < 0 \text{ and } \sum_{j \neq i} b_j < 0 \end{cases} \quad (18.7.9)$$

i.e.,

$$t_i(b) = \begin{cases} -|\sum_{j \neq i} b_j| & \text{if } (\sum_{i=1}^n b_i)(\sum_{j \neq i} b_i) < 0 \\ -|\sum_{j \neq i} b_j| & \text{if } \sum_{i=1}^n b_i = 0 \text{ and } \sum_{j \neq i} b_j < 0 \\ 0 & \text{otherwise.} \end{cases} \quad (18.7.10)$$

Therefore, the payoff of agent i

$$\phi_i(b) = \begin{cases} v_i & \text{if } \sum_{i=1}^n b_i \geq 0 \quad \text{and } \sum_{j \neq i} b_j \geq 0 \\ v_i + \sum_{j \neq i} b_j & \text{if } \sum_{i=1}^n b_i \geq 0 \quad \text{and } \sum_{j \neq i} b_j < 0 \\ -\sum_{j \neq i} b_j & \text{if } \sum_{i=1}^n b_i < 0 \quad \text{and } \sum_{j \neq i} b_j \geq 0 \\ 0 & \text{if } \sum_{i=1}^n b_i < 0 \quad \text{and } \sum_{j \neq i} b_j < 0. \end{cases} \quad (18.7.11)$$

Remark 18.7.1 Thus, from the transfer given in (18.7.10), adding in the side-payment has the effect of taxing agent i only if he changes the social decision. Such an agent is called the pivotal person. The amount of the tax agent i must pay is the amount by which agent i 's bid damages the other agents. The price that agent i must pay to change the amount of public good is equal to the harm that he imposes on the other agents.

18.7.2 Vickrey-Clark-Groves Mechanisms with Continuous Public Goods

Now we are concerned with the provision of continuous public goods. Consider a public goods economy with n agents, one private good, and K public goods. Denote

x_i : the consumption of the private good by i ;

y : the consumption of the public goods by all individuals;

t_i : transfer payment to i ;

$g_i(y)$: the contribution made by i ;

$c(y)$: the cost function of producing public goods y that satisfies the condition:

$$\sum g_i(y) = c(y).$$

Then, agent i 's budget constraint should satisfy

$$x_i + g_i(y) = w_i + t_i \quad (18.7.12)$$

and his utility functions are given by

$$\bar{u}_i(x_i - w_i, y) = x_i - w_i + u_i(y). \quad (18.7.13)$$

By substitution,

$$\begin{aligned} u_i(t_i, y) &= t_i + (u_i(y) - g_i(y)) \\ &\equiv t_i + v_i(y), \end{aligned}$$

where $v_i(y)$ is called the valuation function of agent i . From the budget constraint,

$$\sum_{i=1}^n \{x_i + g_i(y)\} = \sum_{i=1}^n w_i + \sum_{i=1}^n t_i, \quad (18.7.14)$$

we have

$$\sum_{i=1}^n x_i + c(y) = \sum_{i=1}^n w_i + \sum_{i=1}^n t_i \quad (18.7.15)$$

The feasibility (or balanced) condition then becomes

$$\sum_{i=1}^n t_i = 0. \quad (18.7.16)$$

Recall that Pareto efficient allocations are completely characterized by

$$\begin{aligned} & \max \sum a_i \bar{u}_i(x_i, y) \\ \text{s.t.} \quad & \sum_{i=1}^n x_i + c(y) = \sum_{i=1}^n w_i. \end{aligned}$$

For quasi-linear utility functions it is easily seen that the weights a_i must be the same for all agents since $a_i = \lambda$ for interior Pareto efficient allocations (no income effect), the Lagrangian multiplier, and y is thus uniquely determined for the special case of quasi-linear utility functions $u_i(t_i, y) = t_i + v_i(y)$. Then, the above characterization problem becomes

$$\max_{t_i, y} \left[\sum_{i=1}^n (t_i + v_i(y)) \right], \quad (18.7.17)$$

or equivalently

(1)

$$\max_y \sum_{i=1}^n v_i(y);$$

(2) (feasibility condition): $\sum_{i=1}^n t_i = 0$.

Then, the Lindahl-Samuelson condition is given by:

$$\sum_{i=1}^n \frac{\partial v_i(y)}{\partial y_k} = 0,$$

that is,

$$\sum_{i=1}^n \frac{\partial u_i(y)}{\partial y_k} = \frac{\partial c(y)}{\partial y_k}.$$

Thus, Pareto efficient allocations for quasi-linear utility functions are completely characterized by the Lindahl-Samuelson condition $\sum_{i=1}^n v'_i(y) = 0$ and feasibility condition $\sum_{i=1}^n t_i = 0$.

In a Groves mechanism, it is supposed that each agent is asked to report the valuation function $v_i(y)$. Denote his reported valuation function by $b_i(y)$.

To get the efficient level of public goods, the government may announce that it will provide a level of public goods y^* that maximizes

$$\max_y \sum_{i=1}^n b_i(y).$$

The Groves mechanism has the form:

$$\Gamma = (V, h), \quad (18.7.18)$$

where $V = V_1 \times \dots \times V_n$: is the message space that consists of the set of all possible valuation functions with element $b_i(y) \in V_i$, $h = (t_1(b), t_2(b), \dots, t_n(b), y(b))$ are outcome functions. It is determined by:

- (1) Ask each agent i to report his/her valuation function $b_i(y)$ which may or may not be the true valuation $v_i(y)$;
- (2) Determine y^* : the level of the public goods, $y^* = y(b)$, is determined by

$$\max_y \sum_{i=1}^n b_i(y); \quad (18.7.19)$$

- (3) Determine t : transfer of agent i , t_i is determined by

$$t_i(b) = \sum_{j \neq i} b_j(y^*). \quad (18.7.20)$$

The payoff of agent i is then given by

$$\phi_i(b(y^*)) = v_i(y^*) + t_i(b) = v_i(y^*) + \sum_{j \neq i} b_j(y^*). \quad (18.7.21)$$

The social planner's goal is to have the optimal level y^* that solves the problem:

$$\max_y \sum_{i=1}^n b_i(y).$$

In which case is the individual's interest consistent with social planner's interest? Under the rule of this mechanism, it is optimal for each agent i to truthfully report his true valuation function $b_i(y) = v_i(y)$ since agent i wants to maximize

$$v_i(y) + \sum_{j \neq i} b_j(y).$$

By reporting $b_i(y) = v_i(y)$, agent i ensures that the government will choose y^* which also maximizes his payoff while the government maximizes the social welfare. That is, individual's interest is consistent with the social interest that is determined by the Lindahl-Samuelson condition. Thus, truth-telling, $b_i(y) = v_i(y)$, is a dominant strategy equilibrium.

In general, $\sum_{i=1}^n t_i(b(y)) \neq 0$, which means that a Groves mechanism in general does not result in Pareto efficient outcomes even if it satisfies the Lindahl-Samuelson condition, i.e., it is Pareto efficient to provide the public goods.

As in the discrete case, the total transfer can be very large, just as before, they can be reduced by an appropriate side-payment. The Groves mechanism can be modified to

$$t_i(b) = \sum_{j \neq i} b_j(y) + d_i(b_{-i}).$$

The general form of the Groves Mechanism (Vickrey-Clark-Groves Mechanism) is then $\Gamma = \langle V, t, y(b) \rangle$ such that

- (1) $\sum_{i=1}^n b_i(y(b)) \geq \sum_{i=1}^n b_i(y)$ for $y \in Y$;
- (2) $t_i(b) = \sum_{j \neq i} b_j(y) + d_i(b_{-i})$.

In summary, we have the following proposition.

Proposition 18.7.2 *For continuous public good economies under consideration, the truth-telling is a dominant strategy under the Vickrey-Clark-Groves mechanism that implements truthfully the efficient decision rule $y^*(\cdot)$ (the efficient provision of public goods) in dominant strategy.*

A special case of the Vickrey-Clark-Groves mechanism is independently described by Clark and is called the Clark mechanism (also called the pivotal mechanism) in which $d_i(b_{-i}(y))$ is given by

$$d_i(b_{-i}) = \max_y \sum_{j \neq i} b_j(y). \quad (18.7.22)$$

That is, the pivotal mechanism, $\Gamma = \langle V, t, y(b) \rangle$, is to choose (y^*, t_i^*) such that

- (1) $\sum_{i=1}^n b_i(y^*) \geq \sum_{i=1}^n b_i(y)$ for $y \in Y$;
- (2) $t_i(b) = \sum_{j \neq i} b_j(y^*) - \max_y \sum_{j \neq i} b_j(y)$.

It is interesting to point out that the Clark mechanism contains the well-known Vickery auction mechanism (the second-price auction mechanism) as a special case. Under the Vickery mechanism, the highest bidding person obtains the object, and he pays the second highest bidding price. To see this, let us explore this relationship in the case of a single good auction (Example 9.3.3 in the beginning of this chapter). In this case, the outcome space is

$$Z = \{y \in \{0, 1\}^n : \sum_{i=1}^n y_i = 1\}$$

where $y_i = 1$ implies that agent i gets the object, and $y_i = 0$ means the person does not get the object. Agent i 's valuation function is then given by

$$v_i(y) = v_i y_i.$$

Since we can regard y as a n -dimensional vector of public goods, by the Clark mechanism above, we know that

$$y^* = g(b) = \{y \in Z : \max_{i=1}^n \sum_{i=1}^n b_i y_i\} = \{y \in Z : \max_{i \in N} b_i\},$$

and the truth-telling is a dominate strategy. Thus, if $g_i(v) = 1$, then $t_i(v) = \sum_{j \neq i} v_j y_j^* - \max_y \sum_{j \neq i} v_j y_j = -\max_{j \neq i} v_j$. If $g_i(b) = 0$, then $t_i(v) = 0$. This means that the object is allocated to the individual with the highest valuation and he pays an amount equal to the second highest valuation. No other payments are made. This is exactly the outcome predicted by the Vickery mechanism.

18.7.3 Uniqueness of VCG for Efficient Decision

Do there exist other mechanisms implementing the efficient decision rule y^* ? The answer is no if valuation functions $v(y, \cdot)$ is sufficiently "rich" in a sense. To see this, consider the class of parametric valuation functions $v_i(y, \cdot)$ with continuum type space $\Theta_i = [\underline{\theta}, \bar{\theta}]$.

Proposition 18.7.3 (Laffont and Maskin (1980)) *Suppose $Y = \mathbb{R}$, $\Theta = [\underline{\theta}, \bar{\theta}]$ and $v_i : Y \times \Theta_i \rightarrow \mathbb{R}$, $\forall i \in N$, is differentiable. Then any mechanism implementing an efficient decision rule $y^*(\cdot)$ in dominant strategy is a Vickrey-Clark-Groves mechanism. That is, if a social choice rule $f(\theta) = \{y(\theta), t_1(\theta), \dots, t_n(\theta)\}$ is implemented truthfully in dominant strategy, and*

$$y(\theta) = \arg \max_{y \in Y} \sum_{i=1}^n v_i(y, \theta_i),$$

then we must have $t_i(\theta) = \sum_{j \neq i} v_j(y(\theta), \theta_j) + d_i(\theta_{-i})$, where $d_i(\theta_{-i})$ is independent of θ_i .

Proof. Since

$$y(\theta) = \arg \max_{y \in Y} \sum_{i=1}^n v_i(y, \theta_i),$$

we have

$$\sum_{i=1}^n \frac{\partial v_i}{\partial y}(y(\theta), \theta_i) = 0. \quad (18.7.23)$$

By the requirement of truth implementation in dominant strategy,

$$v_i(y(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i}) \geq v_i(y(\hat{\theta}_i, \theta_{-i}), \theta_i) + t_i(\hat{\theta}_i, \theta_{-i}), \forall \theta_i, \hat{\theta}_i, \theta_{-i}.$$

Then

$$\frac{\partial v_i}{\partial y}(y(\theta), \theta_i) \frac{\partial y}{\partial \theta_i}(\theta) + \frac{\partial t_i}{\partial \theta_i}(\theta_i, \theta_{-i}) = 0, \forall (\theta_i, \theta_{-i}) \in \Theta. \quad (18.7.24)$$

Let $d_i(\theta) \triangleq t_i(\theta) - \sum_{j \neq i} v_j(y(\theta), \theta_j)$. We need to show $d_i(\theta_i, \theta_{-i})$ is independent of θ_i . Indeed, since

$$\begin{aligned} \frac{\partial d_i}{\partial \theta_i} &= \frac{\partial t_i}{\partial \theta_i}(\theta_i, \theta_{-i}) - \sum_{j \neq i} \frac{\partial v_j}{\partial y}(y(\theta), \theta_j) \frac{\partial y}{\partial \theta_i}(\theta_i, \theta_{-i}) \\ &= \frac{\partial t_i}{\partial \theta_i}(\theta_i, \theta_{-i}) + \frac{\partial v_i}{\partial y}(y(\theta), \theta_j) \frac{\partial y}{\partial \theta_i}(\theta_i, \theta_{-i}) - \sum_{j=1}^n \frac{\partial v_j}{\partial y}(y(\theta), \theta_j) \frac{\partial y}{\partial \theta_i}(\theta_i, \theta_{-i}) \\ &= 0, \end{aligned} \quad (18.7.25)$$

we have $d_i(\theta) = d_i(\theta_{-i})$.

The intuition behind the proof is that each agent i only has the incentive to tell the truth when he is made the residual claimant of total surplus, i.e., $U_i(\theta) = S(\theta) + d_i(\theta_{-i})$, which requires a VCG mechanism. This result was also obtained by Green and Laffont (1979), but under much more restrictive assumptions. A particular case of the proposition was also obtained by Green and Laffont (1979). Namely, they allow agents to have all possible valuations over a finite decision set X , which can be described by the Euclidean type space $\Theta_i = \mathbb{R}^{|K|}$ (the valuation of agent i for decision k being represented by θ_{ik}). The proof can be seen in Mas-Colell, Whinston, Green, (1995) without imposing differentiability on v_i .

18.7.4 Balanced VCG Mechanisms

Implementation of a social surplus-maximizing decision rule $y^*(\cdot)$, is a necessary condition for (ex-post) Pareto efficiency, but it is not sufficient. To have Pareto efficiency, we must also guarantee that there is an ex post Balanced Budget (no waste of numeraire):

$$\sum_{i=1}^n t_i(\theta) = 0, \forall \theta \in \Theta. \quad (18.7.26)$$

A trivial case for which we can achieve ex post efficiency is given in the following example.

Example 18.7.1 If there exists some agent i such that $\Theta_i = \{\bar{\theta}_i\}$ a singleton, then we have no incentive problem for agent i ; and we can set

$$t_i(\theta) = - \sum_{j \neq i} t_j(\theta), \forall \theta \in \Theta.$$

This trivially guarantees a balanced budget.

However, in general, there is no such a positive result. When the set of $v(\cdot, \cdot)$ functions is "rich", then there may be no choice rule $f(\cdot) = (y^*(\cdot), t_1(\cdot), \dots, t_n(\cdot))$ where $y^*(\cdot)$ is ex post optimal, which is truthfully implementable in dominant and satisfies (18.7.26), so that it does not result in Pareto efficient outcome.

To have an example where budget balance cannot be achieved, consider a setting with two agents. Under the assumptions of Proposition 18.7.3, any mechanism implementing an efficient decision rule $y^*(\cdot)$ in dominant strategy is a VCG mechanism, hence

$$t_i(\theta) = v_{-i}(y^*(\theta), \theta_{-i}) + d_i(\theta_{-i}).$$

Budget balance requires that

$$0 = t_1(\theta) + t_2(\theta) = v_1(y^*(\theta), \theta_1) + v_2(y^*(\theta), \theta_2) + d_1(\theta_2) + d_2(\theta_1).$$

Letting $S(\theta) = v_1(y^*(\theta), \theta_1) + v_2(y^*(\theta), \theta_2)$ denote the maximum social surplus in state θ , we must therefore have

$$S(\theta) = -d_1(\theta_2) - d_2(\theta_1).$$

Thus, efficiency can only be achieved when maximal total surplus is additively separable in the agents' types, which is unlikely.

To have a robust example where additive separability does not hold, consider the "public good setting" in which $Y = [\underline{y}, \bar{y}] \subset \mathbb{R}$ with $\underline{y} < \bar{y}$, and for each agent i , $\Theta_i = [\underline{\theta}_i, \bar{\theta}_i] \subset \mathbb{R}$, $\underline{\theta}_i < \bar{\theta}_i$ (to rule out the situation described in the above Example), $v_i(y, \theta_i)$ is differentiable in θ_i , $\frac{\partial v_i(y, \theta_i)}{\partial \theta_i}$ is bounded, $v_i(y, \theta_i)$ has the single-crossing property (SCP) in (y, θ_i) and $y^*(\theta_1, \theta_2)$ is in the interior of Y .

Note that by the Envelope Theorem, we have (almost everywhere)

$$\frac{\partial S(\theta)}{\partial \theta_1} = \frac{\partial v_1(y^*(\theta_1, \theta_2), \theta_1)}{\partial \theta_1}.$$

It is clear that if $\frac{\partial v_1(y, \theta_1)}{\partial \theta_1}$ depends on y , and $y^*(\theta_1, \theta_2)$ depends on θ_2 , in general $\frac{\partial S(\theta)}{\partial \theta_1}$ will depend on θ_2 , and therefore $S(\theta)$ will not be additively separable. In fact,

- SCP of $v_1(\cdot)$ implies that $\frac{\partial v_1(y, \theta_1)}{\partial y}$ is strictly increasing in θ_1 .
- SCP of $v_2(\cdot)$ implies that $y^*(\theta_1, \theta_2)$ is strictly increasing in θ_2 .

Consequently, $\frac{\partial S(\theta)}{\partial \theta_1}$ is strictly increasing in θ_2 , thus $S(\theta)$ is not additively separable. In this case, no ex post efficient choice rule is truthfully implementable in dominant strategies.

As such, if the wasted money is subtracted from the social surplus, it is no longer socially efficient to implement the decision rule $y^*(\cdot)$. Instead, we may be interested, e.g., in maximizing the expectation

$$E \left[\sum_i v_i(x(\theta), \theta_i) + \sum_i t_i(\theta) \right]$$

in the class of choice rules $(y(\cdot), t_1(\cdot), \dots, t_I(\cdot))$ satisfying dominant incentive compatibility, given a probability distribution φ over states θ . In this case, we may have interim efficiency as we will see in the section on Bayesian implementation.

Nevertheless, when economic environments are “thin”, we can get some positive results. Groves and Loeb (1975), Tian (1996a), and Liu and Tian (1999) provide such a class of economic environments. An example of such utility functions are: $u_i(t_i, y) = t_i + v_i(y)$ with $v_i(y) = -1/2y^2 + \theta_i y$ for $i = 1, 2, \dots, n$ with $n \geq 3$.

18.8 Nash Implementation

18.8.1 Nash Equilibrium and General Mechanism Design

From Hurwicz’s impossibility theorem, we know that, if one wants to have a mechanism that results in Pareto efficient and individually rational allocations, one must give up the dominant strategy implementation, and then, by Revelation Principle, we must look at more general mechanisms $\langle M, h \rangle$ instead of using a revelation mechanism.

We know the dominant strategy equilibrium is a very strong solution concept. Now, if we adopt the Nash equilibrium as a solution concept to describe individuals’ self-interested behavior, can we design an incentive compatible mechanism which implements Pareto efficient allocations?

For $e \in E$, a mechanism $\langle M, h \rangle$ is said to have a *Nash equilibrium* $m^* \in M$ if

$$h_i(m^*) \succsim_i h_i(m_i, m_{-i}^*) \quad (18.8.27)$$

for all $m_i \in M_i$ and all i . Denote by $NE(e, \Gamma)$ the set of all Nash equilibria of the mechanism Γ for $e \in E$.

It is clear every dominant strategy equilibrium is a Nash equilibrium, but the converse may not be true.

A mechanism $\Gamma = \langle M, h \rangle$ is said to strongly Nash-implement a social choice correspondence F on E if for every $e \in E$

- (a) $NE(e, \Gamma) \neq \emptyset$;
- (b) $h(NE(e, \Gamma)) \subseteq F(e)$.

It fully Nash implements a social choice correspondence F on E if for every $e \in E$

- (a) $NE(e, \Gamma) \neq \emptyset$
 (b) $h(NE(e, \Gamma)) = F(e)$.

The following proposition shows that, if a truth-telling about their characteristics is a Nash equilibrium of the revelation mechanism, it must be a dominant strategy equilibrium of the mechanism.

Proposition 18.8.1 *For a revelation mechanism $\Gamma = \langle E, h \rangle$, a truth-telling e^* is a Nash equilibrium if and only if it is a dominant equilibrium*

$$h(e_i^*, e_{-i}) \succ_i h(e_i, e_{-i}) \quad \forall (e_i, e_{-i}) \in E \& i \in N. \quad (18.8.28)$$

Proof. Since for every $e \in E$ and i , by Nash equilibrium, we have

$$h(e_i, e_{-i}) \succ_i h(e'_i, e_{-i}) \quad \text{for all } e'_i \in E_i.$$

Since this is true for any (e'_i, e_{-i}) , it is a dominant strategy equilibrium. The proof is completed.

Thus, we cannot get any new results if one insists on the choice of revelation mechanisms. To get more satisfactory results, one must give up the revelation mechanism, and look for a more general mechanism with general message spaces.

Notice that, when one adopts the Nash equilibrium as a solution concept, implementation, rather than full or strong implement, in Nash solution may not be a useful requirement. To see this, consider any social choice correspondence F and the following mechanism: each individual's message space consists of the set of economic environments, i.e., it is given by $M_i = E$. The outcome function is defined as $h(m) = a \in F(e)$ when all agents report the same economic environment $m_i = e$, and otherwise it is seriously punished by giving a worse outcome. Then, it is clear the truth-telling is a Nash equilibrium. However, it has a lot of Nash equilibria, in fact infinity number of Nash equilibria. Any false reporting about the economic environment $m_i = e'$ is also a Nash equilibrium. So, when we use Nash equilibrium as a solution concept, we need a social choice rule to be strongly implemented or full implemented in Nash equilibrium.

18.8.2 Characterization of Nash Implementation

Now we discuss what kind of social choice rules can be fully or strongly implemented through Nash incentive compatible mechanism. Maskin in 1977 gave necessary and sufficient conditions for a social choice rule to be fully Nash implementable (This paper was not published till 1999 due to the incorrectness of the original proof. It then appeared in *Review of Economic Studies*, 1999). Maskin's result is fundamental since it not only helps us to understand what kind of social choice correspondence can be fully

Nash implemented, but also gives basic techniques and methods in studying implementability of a social choice rule under other solution concepts.

Maskin's monotonicity condition can be stated in two different ways although they are equivalent.

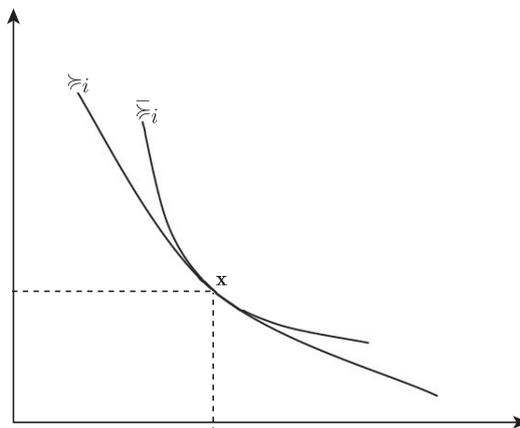


Figure 18.3: An illustration of Maskin's monotonicity.

Definition 18.8.1 (Maskin's Monotonicity) A social choice correspondence $F : E \rightarrow A$ is said to be Maskin's monotonic if for any $e, \bar{e} \in E$, $x \in F(e)$ such that for all i and all $y \in A$, $x \succ_i y$ implies that $x \succ_i y$, then $x \in F(\bar{e})$.

In words, Maskin's monotonicity requires that if an outcome x is socially optimal with respect to economy e , changing economy e to \bar{e} makes all individuals even more like x , then x remains socially optimal with respect to \bar{e} .

Definition 18.8.2 (Another Version of Maskin's Monotonicity) A equivalent condition for a social choice correspondence $F : E \rightarrow A$ to be Maskin's monotonic is that, if for any two economic environments $e, \bar{e} \in E$, $x \in F(e)$ such that $x \notin F(\bar{e})$, there is an agent i and another $y \in A$ such that $x \succ_i y$ and $y \succ_i x$.

Maskin's monotonicity is a reasonable condition, and many well known social choice rules satisfy this condition.

Example 18.8.1 (Weak Pareto Efficiency) The weak Pareto optimal correspondence $P_w : E \rightarrow A$ is Maskin's monotonic.

Proof. If $x \in P_w(e)$, then for all $y \in A$, there exists $i \in N$ such that $x \succ_i y$. Now if for any $j \in N$ such that $x \succ_j y$ implies $x \succ_j y$, then we have $x \succ_i y$ for particular i . Thus, $x \in P_w(\bar{e})$.

Example 18.8.2 (Majority Rule) The majority rule or call the Condorcet correspondence $CON : E \rightarrow A$ for strict preference profile, which is defined by

$$CON(e) = \{x \in A : \#\{i|x \succ_i y\} \geq \#\{i|y \succ_i x\} \text{ for all } y \in A\}$$

is Maskin's monotonic.

Proof. If $x \in CON(e)$, then for all $y \in A$,

$$\#\{i|x \succ_i y\} \geq \#\{i|y \succ_i x\}. \quad (18.8.29)$$

But if \bar{e} is an economy such that, for all i , $x \succ_i y$ implies $x \bar{\succ}_i y$, then the left-hand side of (18.8.29) cannot fall when we replace e by \bar{e} . Furthermore, if the right-hand side rises, then we must have $x \succ_i y$ and $y \bar{\succ}_i x$ for some i , a contradiction of the relation between e and \bar{e} , given the strictness of preferences. So x is still a majority winner with respect to \bar{e} , i.e., $x \in CON(\bar{e})$.

In addition to the above two examples, Walrasian correspondence and Lindahl correspondence with interior allocations are Maskin's monotonic. The class of preferences that satisfy "single-crossing" property and individual's preferences over lotteries satisfy the von Neumann-Morgenstern axioms also automatically satisfy Maskin's monotonicity.

The following theorem shows the Maskin's monotonicity is a necessary condition for full Nash-implementability.

Theorem 18.8.1 *For a social choice correspondence $F : E \rightarrow A$, if it is fully Nash implementable, then it must be Maskin's monotonic.*

Proof. For any two economies $e, \bar{e} \in E$, $x \in F(e)$, then by full Nash implementability of F , there is $m \in M$ such that m is a Nash equilibrium and $x = h(m)$. This means that $h(m) \succ_i h(m'_i, m_{-i})$ for all i and $m'_i \in M_i$. Given $x \succ_i y$ implies $x \bar{\succ}_i y$, $h(m) \bar{\succ}_i h(m'_i, m_{-i})$, which means that m is also a Nash equilibrium at \bar{e} . Thus, by full Nash implementability again, we have $x \in F(\bar{e})$.

Maskin's monotonicity itself can not guarantee a social choice correspondence is fully Nash implementable. However, under the so-called no-veto power, it becomes sufficient.

Definition 18.8.3 (No-Veto Power) A social choice correspondence $F : E \rightarrow A$ is said to satisfy no-veto power if whenever for any i and e such that $x \succ_j y$ for all $y \in A$ and all $j \neq i$, then $x \in F(e)$.

The no-veto power (NVP) condition implies that if $n - 1$ agents regard an outcome is the optimal to them, then it is social optimal. This is a rather weaker condition. NVP is satisfied by virtually all "standard" social choice rules, including weak Pareto efficient and Condorcet correspondences. It

is also often automatically satisfied by any social choice rules when the references are restricted. For example, for private goods economies with at least three agents, if each agent's utility function is strong monotonic, then there is no other allocation such that $n - 1$ agents regard it optimal, so the no-veto power condition holds. The following theorem is given by Maskin (1977, 1999), but a complete proof of the theorem was due to Repullo (1987).

Theorem 18.8.2 *Under no-veto power, if Maskin's monotonicity condition is satisfied, then F is fully Nash implementable.*

Proof. The proof is by construction. For each agent i , his message space is defined by

$$M_i = E \times A \times \mathcal{N}$$

where $\mathcal{N} = \{1, 2, \dots, n\}$. Its elements are denoted by $m_i = (e^i, a^i, v^i)$, which means each agent i announces an economic profile, an outcome, and a real number. Notice that we have used e^i and a^i to denote the economic profile of all individuals' economic characteristics and the outcome announced by individual i , but not just agent i 's economic characteristic and outcome.

The outcome function is constructed in three rules:

Rule(1). If $m_1 = m_2 = \dots = m_n = (e, a, v)$ and $a \in F(e)$, the outcome function is defined by

$$h(m) = a.$$

In words, if players are unanimous in their strategy, and their proposed alternative a is F -optimal, given their proposed profile e , the outcome is a .

Rule (2). For all $j \neq i$, $m_j = (e, a, v)$, $m_i = (e^i, a^i, v^i) \neq (e, a, v)$, and $a \in F(e)$, define:

$$h(m) = \begin{cases} a^i & \text{if } a^i \in L(a, e_i) \\ a & \text{if } a^i \notin L(a, e_i). \end{cases}$$

where $L(a, e_i) = \{b \in A : a R_i b\}$ which is the lower contour set of R_i at a . That is, suppose that all players but one play the same strategy and, given their proposed profile, their proposed alternative a is F -optimal. Then, the outcome is selected from the worse one of a and a_i according to agent i 's preference \succsim_i . As such, no one can gain by deviating from a unanimous strategy.

Rule(3). If neither Rule (1) nor Rule (2) applies, then define

$$h(m) = a^{i^*}$$

where $i^* = \max\{i \in \mathcal{N} : v^i = \max_j v^j\}$. In other words, when neither Rule (1) nor Rule (2) applies, the outcome is the alternative proposed by player with the highest index among those whose proposed number is maximal.

Now we show that the mechanism $\langle M, h \rangle$ defined above fully Nash implements social choice correspondence F , i.e., $h(N(e)) = F(e)$ for all

$e \in E$. We first show that $F(e) \subset h(N(e))$ for all $e \in E$, i.e., we need to show that for all $e \in E$ and $a \in F(e)$, there exists a $m \in M$ such that $a = h(m)$ is a Nash equilibrium outcome. To do so, we only need to show that any m which is given by Rule (1) is a Nash equilibrium. Note that $h(m) = a$ and for any given $m'_i = (e^i, a^i, v^i) \neq m_i$, by Rule (2), we have

$$h(m'_i, m_{-i}) = \begin{cases} a^i & \text{if } a^i \in L(a, e_i) \\ a & \text{if } a^i \notin L(a, e_i). \end{cases}$$

and thus

$$h(m) R_i h(m'_i, m_{-i}) \quad \forall m'_i \in M_i.$$

Hence, m is a Nash equilibrium.

We now show that for each economic environment $e \in E$, if m is a Nash equilibrium, then $h(m) \in F(e)$. First, consider the Nash equilibrium m is given by Rule (1) so that $a \in F(e)$, but the true economic profile is e' , i.e., $m \in NE(e', \Gamma)$ (if it is a Nash equilibrium at e , then it is proved). We need to show $a \in F(e')$. By Rule (1), $h(m) = a$. Let $b \in L(a, e_i)$, so the new message for i is $m'_i = (e, b, v^i)$. Then $h(m'_i, m_{-i}) = b$ by Rule (2). Now, since a is a Nash equilibrium outcome with respect to e' , we have $a = h(m) R'_i h(m'_i, m_{-i}) = b$. Thus, we have shown that for all $i \in N$ and $b \in A$, $a R_i b$ implies $a R'_i b$. Thus, by Maskin's monotonicity condition, we have $a \in F(e')$.

Next, suppose Nash equilibrium m for e' is in the Case (2), i.e, for all $j \neq i$, $m_j = (e, a, v)$, $m_i \neq (e, a, v)$. Let $a' = h(m)$. By Case (3), each $j \neq i$ can induce any outcome $b \in A$ by choosing (R^j, b, v^j) with sufficiently a large v^j (which is greater than $\max_{k \neq j} v_k$), as the outcome at (m'_j, m_{-j}) , i.e., $b = h(m'_j, m_{-j})$. Hence, m is a Nash equilibrium with respect to e' implies that for all $j \neq i$, we have

$$a' R'_j b.$$

Thus, by no-veto power assumption, we have $a' \in F(e')$.

The same argument as the above, if m is a Nash equilibrium for e' is given by Case (3), we have $a' \in F(e')$. The proof is completed.

Although Maskin's monotonicity is very weak, it is violated by some social choice rules such as Solomon's famous judgement. Solomon's solution falls under full Nash implementation, since each woman knows who is the real mother. His solution, which consisted in threatening to cut the baby in two, is not entirely foolproof. What would be have done if the imposer has had the presence of mind to scream like a real mother? Solomon's problem can be formerly described by the language of mechanism design as follows.

Two women: Anne and Bets

Two economies (states): $E = \{\alpha, \beta\}$, where

α : Anne is the real mother

β : Bets is the real mother

Solomon has three alternatives so that the feasible set is given by $A = \{a, b, c\}$, where

a : give the baby to Anne

b : give the baby to Bets

c : cut the baby in half.

Solomon's desirability (social goal) is to give the baby to the real mother,

$f(\alpha) = a$ if α happens

$f(\beta) = b$ if β happens

Preferences of Anne and Bets:

For Anne,

at state α , $a \succ_A^\alpha b \succ_A^\alpha c$

at state β : $a \succ_A^\beta c \succ_A^\beta b$

For Bets,

at state α , $b \succ_B^\beta c \succ_B^\beta a$

at state β : $b \succ_B^\beta a \succ_B^\beta c$

To see Solomon's solution does not work, we only need to show his social choice goal is not Maskin's monotonic. Notice that for Anne, since

$$a \succ_A^\alpha b, c,$$

$$a \succ_A^\beta b, c,$$

and $f(\alpha) = a$, by Maskin's monotonicity, we should have $f(\beta) = a$, but we actually have $f(\beta) = b$. So Solomon's social choice goal is not fully Nash implementable. As such, one needs to adopt a solution concept that is a refinement of Nash equilibrium so that the set of its equilibria is smaller.

18.9 Better Mechanism Design

Maskin's theorem gives necessary and sufficient conditions for a social choice correspondence to be fully Nash implementable. However, due to the general nature of the social choice rules under consideration, the implementing mechanisms in proving characterization theorems turn out to be quite complex. Characterization results show what is possible for the full

or strong implementation of a social choice rule, but not what is realistic. Thus, like most characterization results in the literature, Maskin's mechanism is not natural in the sense that it is not continuous; small variations in an agent's strategy choice may lead to large jumps in the resulting allocations, and further it has a message space of infinite dimension since it includes preference profiles as a component. In this section, we give some mechanisms that have some desired properties.

18.9.1 Groves-Ledyard Mechanism

Groves-Ledyard Mechanism (1977, *Econometrica*) was the first to give a specific mechanism that strongly Nash implements Pareto efficient allocations for public goods economies.

To show the basic structure of the Groves-Ledyard mechanism, consider a simplified Groves-Ledyard mechanism. Public goods economies under consideration have one private good x_i , one public good y , and three agents ($n = 3$). The production function is given by $y = v$.

The mechanism is defined as follows:

$M_i = R_i$, $i = 1, 2, 3$. Its elements, m_i , can be interpreted as the proposed contribution (or tax) that agent i is willing to make.

$t_i(m) = m_i^2 + 2m_j m_k$: the actual contribution t_i determined by the mechanism with the reported m_i .

$y(m) = (m_1 + m_2 + m_3)^2$: the level of public good y .

$x_i(m) = w_i - t_i(m)$: the consumption of the private good.

Then the mechanism is balanced since

$$\begin{aligned} & \sum_{i=1}^3 x_i + \sum_{i=1}^3 t_i(m) \\ &= \sum_{i=1}^3 x_i + (m_1 + m_2 + m_3)^2 \\ &= \sum_{i=1}^3 x_i + y = \sum_{i=1}^3 w_i \end{aligned}$$

The payoff function is given by

$$\begin{aligned} v_i(m) &= u_i(x_i(m), y(m)) \\ &= u_i(w_i - t_i(m), y(m)). \end{aligned}$$

To find a Nash equilibrium, we set

$$\frac{\partial v_i(m)}{\partial m_i} = 0 \quad (18.9.30)$$

Then,

$$\frac{\partial v_i(m)}{\partial m_i} = \frac{\partial u_i}{\partial x_i}(-2m_i) + \frac{\partial u_i}{\partial y} 2(m_1 + m_2 + m_3) = 0 \quad (18.9.31)$$

and thus

$$\frac{\frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x_i}} = \frac{m_i}{m_1 + m_2 + m_3}. \quad (18.9.32)$$

When u_i are quasi-concave, the first order condition will be a sufficient condition for Nash equilibrium.

Making summation, we have at Nash equilibrium

$$\sum_{i=1}^3 \frac{\frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x_i}} = \sum_{i=1}^3 \frac{m_i}{m_1 + m_2 + m_3} = 1 = \frac{1}{f'(v)} \quad (18.9.33)$$

that is,

$$\sum_{i=1}^3 MRS_{yx_i} = MRTS_{yv}.$$

Thus, the Lindahl-Samuelson and balanced conditions hold which means every Nash equilibrium allocation is Pareto efficient.

They claimed that they have solved the free-rider problem in the presence of public goods. However, economic issues are usually complicated. Some agreed that they indeed solved the problem, some did not. There are two weakness of Groves-Ledyard Mechanism: (1) it is not individually rational: the payoff at a Nash equilibrium may be lower than at the initial endowment, and (2) it is not individually feasible: $x_i(m) = w_i - t_i(m)$ may be negative.

How can we design the incentive mechanism to pursue Pareto efficient and individually rational allocations?

18.9.2 Walker's Mechanism

Walker (1981, *Econometrica*) gave such a mechanism. Again, consider public goods economies with n agents, one private good, and one public good, and the production function is given by $y = f(v) = v$. We assume $n \geq 3$.

The mechanism is defined by:

$$M_i = R$$

$$y(m) = \sum_{i=1}^n m_i: \text{the level of public good.}$$

$$q_i(m) = \frac{1}{n} + m_{i+1} - m_{i+2}: \text{personalized price of public good.}$$

$$t_i(m) = q_i(m)y(m): \text{the contribution (tax) made by agent } i.$$

$$x_i(m) = w_i - t_i(m) = w_i - q_i(m)y(m): \text{the private good consumption.}$$

Then, the budget constraint holds:

$$x_i(m) + q_i(m)y(m) = w_i \quad \forall m_i \in M_i \quad (18.9.34)$$

Making summation, we have

$$\sum_{i=1}^n x_i(m) + \sum_{i=1}^n q_i(m)y(m) = \sum_{i=1}^n w_i$$

and thus

$$\sum_{i=1}^n x_i(m) + y(m) = \sum_{i=1}^n w_i$$

which means the mechanism is balanced.

The payoff function is

$$\begin{aligned} v_i(m) &= u_i(x_i, y) \\ &= u_i(w_i - q_i(m)y(m), y(m)) \end{aligned}$$

The first order condition for interior allocations is given by

$$\begin{aligned} \frac{\partial v_i}{\partial m_i} &= -\frac{\partial u_i}{\partial x_i} \left[\frac{\partial q_i}{\partial m_i} y(m) + q_i(m) \frac{\partial y(m)}{\partial m_i} \right] + \frac{\partial u_i}{\partial y} \frac{\partial y(m)}{\partial m_i} \\ &= -\frac{\partial u_i}{\partial x_i} q_i(m) + \frac{\partial u_i}{\partial y} = 0 \\ &\Rightarrow \frac{\frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x_i}} = q_i(m) \quad (\text{FOC for the Lindahl Allocation}) \\ &\Rightarrow N(e) \subseteq L(e) \end{aligned}$$

Thus, if u_i are quasi-concave, it is also a sufficient condition for Lindahl equilibrium. We can also show every Lindahl allocation is a Nash equilibrium allocation, i.e.,

$$L(e) \subseteq N(e) \quad (18.9.35)$$

Indeed, suppose $[(x^*, y^*), q_1^*, \dots, q_n^*]$ is a Lindahl equilibrium. Let m^* be the solution of the following equation

$$\begin{aligned} q_i^* &= \frac{1}{n} + m_{i+1} - m_{i+2}, \\ y^* &= \sum_{i=1}^n m_i \end{aligned}$$

Then we have $x_i(m^*) = x_i^*$, $y(m^*) = y^*$ and $q_i(m^*) = q_i^*$ for all $i \in N$. Thus from $(x(m_i^*, m_{-i}), y(m_i^*, m_{-i})) \in \mathbb{R}_+^2$ and $x_i(m_i^*, m_{-i}) + q_i(m_i^*)y(m_i^*, m_{-i}) = w_i$ for all $i \in N$ and $m_i \in M_i$, we have $(x_i(m^*), y(m^*)) R_i (x_i(m_i^*, m_{-i}), y(m_i^*, m_{-i}))$, which means that m^* is a Nash equilibrium.

Thus, Walker’s mechanism fully implements Lindahl allocations which are Pareto efficient and individually rational.

Walker’s mechanism also has a disadvantage that it is not feasible although it does solve the individual rationality problem. If a person claims large amounts of t_i , consumption of private good may be negative, i.e., $x_i = w_i - t_i < 0$. Tian proposed a mechanism that overcomes Walker’s mechanism’s problem. Tian’s mechanism is individually feasible, balanced, and continuous.

18.9.3 Tian’s Mechanism

In Tian’s mechanism (JET, 1991), everything is the same as Walker’s, except that $y(m)$ is given by

$$y(m) = \begin{cases} a(m) & \text{if } \sum_{i=1}^n m_i > a(m) \\ \sum_{i=1}^n m_i & \text{if } 0 \leq \sum_{i=1}^n m_i \leq a(m) \\ 0 & \text{if } \sum_{i=1}^n m_i < 0 \end{cases} \quad (18.9.36)$$

where $a(m) = \min_{i \in N'(m)} \frac{w_i}{q_i(m)}$ that can be regarded as the feasible upper bound for having a feasible allocation. Here $N'(m) = \{i \in N : q_i(m) > 0\}$.

An interpretation of this formulation is that if the total contributions that the agents are willing to pay were between zero and the feasible upper bound, the level of public good to be produced would be exactly the total taxes; if the total taxes were less than zero, no public good would be produced; if the total taxes exceeded the feasible upper bound, the level of the public good would be equal to the feasible upper bound.

	θ	ϕ	ξ
个体 1	$b \quad a$	$a \quad b$	$b \quad a$
个体 2	$a \quad b$	$b \quad a$	$b \quad a$
$f(\cdot)$	$f(\theta) = \{a, b\}$	$f(\phi) = \{a, b\}$	$f(\xi) = \{b\}$

Figure 18.4: The feasible public good outcome function $Y(m)$.

To show this mechanism has all the nice properties, we need to assume that preferences are strongly monotonically increasing and convex, and further assume that every interior allocation is preferred to any boundary allocations: For all $i \in N, (x_i, y) P_i (x'_i, y')$ for all $(x_i, y) \in \mathbb{R}_{++}^2$ and $(x'_i, y') \in \partial \mathbb{R}_{++}^2$, where $\partial \mathbb{R}_{++}^2$ is the boundary of \mathbb{R}_{++}^2 .

To show the equivalence between Nash allocations and Lindahl allocations. We first prove the following lemmas.

Lemma 18.9.1 *If $(x(m^*), y(m^*)) \in N_{M,h}(e)$, then $(x_i(m^*), y(m^*)) \in \mathbb{R}_{++}^2$ for all $i \in N$.*

Proof: We argue by contradiction. Suppose $(x_i(m^*), y(m^*)) \in \partial \mathbb{R}_+^2$. Then $x_i(m^*) = 0$ for some $i \in N$ or $y(m^*) = 0$. Consider the quadratic equation

$$y = \frac{w^*}{2(y+c)}, \quad (18.9.37)$$

where $w^* = \min_{i \in N} w_i$; $c = b + n \sum_{i=1}^n |m_i^*|$, where $b = 1/n$. The larger root of (18.9.37) is positive and denoted by \tilde{y} . Suppose that player i chooses his/her message $m_i = \tilde{y} - \sum_{j \neq i} m_j^*$. Then $\tilde{y} = m_i + \sum_{j \neq i} m_j^* > 0$ and

$$\begin{aligned} w_j - q_j(m_i^*, m_{-i})\tilde{y} &\geq w_j - [b + (n \sum_{s=1}^n |m_s^*| + \tilde{y})]\tilde{y} \\ &= w_j - (\tilde{y} + b + n \sum_{s=1}^n |m_s^*|)\tilde{y} \\ &= w_j - w^*/2 \geq w_j/2 > 0 \end{aligned} \quad (18.9.38)$$

for all $j \in N$. Thus, $y(m_i^*, m_{-i}) = \tilde{y} > 0$ and $x_j(m_i^*, m_{-i}) = w_j - q_j(m_i^*, m_{-i})y(m_i^*, m_{-i}) = w_j - q_j(m_i^*, m_{-i})\tilde{y} > 0$ for all $j \in N$. Thus $(x_i(m_i^*, m_{-i}), y(m_i^*, m_{-i})) P_i (x_i(m^*), y(m^*))$ by that every interior allocation is preferred to any boundary allocations, which contradicts the hypothesis $(x(m^*), y(m^*)) \in N_{M,h}(e)$. Q.E.D.

Lemma 18.9.2 *If $(x(m^*), y(m^*)) \in N_{M,h}(e)$, then $y(m^*)$ is an interior point of $[0, a(m)]$ and thus $y(m^*) = \sum_{i=1}^n m_i^*$.*

Proof: By Lemma 18.9.1, $y(m^*) > 0$. So we only need to show $y(m^*) < a(m^*)$. Suppose, by way of contradiction, that $y(m^*) = a(m^*)$. Then $x_j(m^*) = w_j - q_j(m^*)y(m^*) = w_j - q_j(m^*)a(m^*) = w_j - w_j = 0$ for at least some $j \in N$. But $x(m^*) > 0$ by Lemma 18.9.1, a contradiction. Q.E.D.

Proposition 18.9.1 *If the mechanism has a Nash equilibrium m^* , then $(x(m^*), y(m^*))$ is a Lindahl allocation with $(q_1(m^*), \dots, q_n(m^*))$ as the Lindahl price vector, i.e., $N_{M,h}(e) \subseteq L(e)$.*

Proof: Let m^* be a Nash equilibrium. Now we prove that $(x(m^*), y(m^*))$ is a Lindahl allocation with $(q_1(m^*), \dots, q_n(m^*))$ as the Lindahl price vector. Since the mechanism is feasible and $\sum_{i=1}^n q_i(m^*) = 1$ as well as $x_i(m^*) + q_i(m^*)y(m^*) = w_i$ for all $i \in N$, we only need to show that each individual is maximizing his/her preference. Suppose, by way of contradiction, that there is some $(x_i, y) \in \mathbb{R}_+^2$ such that $(x_i, y) P_i (X_i(m^*), Y(m^*))$ and $x_i + q_i(m^*)y \leq w_i$. Because of monotonicity of preferences, it will be enough to confine ourselves to the case of $x_i + q_i(m^*)y = w_i$. Let $(x_{i\lambda}, y_\lambda) = (\lambda x_i + (1-\lambda)x_i(m^*), \lambda y + (1-\lambda)y(m^*))$. Then by convexity of preferences we have $(x_{i\lambda}, y_\lambda) P_i (x_i(m^*), y(m^*))$ for any $0 < \lambda < 1$. Also $(x_{i\lambda}, y_\lambda) \in \mathbb{R}_+^2$ and $x_{i\lambda} + q_i(m^*)y_\lambda = w_i$.

Suppose that player i chooses his/her message $m_i = y_\lambda - \sum_{j \neq i} m_j^*$. Since $y(m^*) = \sum_{j=1}^n m_j^*$ by Lemma 18.9.2, $m_i = y_\lambda - y(m^*) + m_i^*$. Thus as $\lambda \rightarrow$

0, $y_\lambda \rightarrow y(m^*)$, and therefore $m_i \rightarrow m_i^*$. Since $x_j(m^*) = w_j - q_j(m^*)y(m^*) > 0$ for all $j \in N$ by Lemma 18.9.1, we have $w_j - q_j(m_i^*, m_{-i})y_\lambda > 0$ for all $j \in N$ as λ is a sufficiently small positive number. Therefore, $y(m_i^*, m_{-i}) = y_\lambda$ and $x_i(m_i^*, m_{-i}) = w_i - q_i(m_i^*)y(m_i^*, m_{-i}) = w_i - q_i(m_i^*)y_\lambda = x_{i\lambda}$. From $(x_{i\lambda}, y_\lambda) P_i(x_i(m^*), y(m^*))$, we have $(x_i(m_i^*, m_{-i}), y(m_i^*, m_{-i})) P_i(x_i(m^*), y(m^*))$. This contradicts $(x(m^*), y(m^*)) \in N_{M,h}(e)$. Q.E.D.

Proposition 18.9.2 *If (x^*, y^*) is a Lindahl allocation with the Lindahl price vector $q^* = (q_1^*, \dots, q_n^*)$, then there is a Nash equilibrium m^* of the mechanism such that $x_i(m^*) = x_i^*$, $q_i(m^*) = q_i^*$, for all $i \in N$, $y(m^*) = y^*$, i.e., $L(e) \subseteq N_{M,h}(e)$.*

Proof: We need to show that there is a message m^* such that (x^*, y^*) is a Nash allocation. Let m^* be the solution of the following linear equations system:

$$\begin{aligned} q_i^* &= \frac{1}{n} + m_{i+1} - m_{i+2}, \\ y^* &= \sum_{i=1}^n m_i \end{aligned}$$

Then we have $x_i(m^*) = x_i^*$, $y(m^*) = y^*$ and $q_i(m^*) = q_i^*$ for all $i \in N$. Thus from $(x(m_i^*, m_{-i}), y(m_i^*, m_{-i})) \in \mathbb{R}_+^2$ and $x_i(m_i^*, m_{-i}) + q_i(m_i^*)y(m_i^*, m_{-i}) = w_i$ for all $i \in N$ and $m_i \in M_i$, we have $(x_i(m_i^*, m_{-i}), y(m_i^*, m_{-i})) R_i(x_i(m^*), y(m^*))$. Q.E.D.

Thus, Tian's mechanism fully Nash implements Lindahl allocations.

18.10 Refining Nash execution, Approximate Nash execution, Nash and strong Nash double execution

Although the sections above give some examples of Nash's enforceable social goals, and the necessary and sufficient conditions, but there are many social objectives that are not monotonous (such as the King Solomon's problem), which is not executable in the sense of Nash equilibrium solution. Two methods are given in the economic literature which can greatly expand the scope of executable social goals. One is by using the Refining Nash execution. Using Nash strategy may lead to multiple Nash equilibrium solutions. But the concept of refining Nash equilibrium solutions gives a method to eliminate those unconvincing Nash equilibriums, which makes the set of Nash equilibrium results greatly smaller and makes it executable under the concept of refining Nash solution. The other is to use the method of approximate Nash execution, which only requires the equilibrium result to be arbitrarily close to the social choice correspondence. In addition, the method of the Nash and strong Nash double execution can make the set of Nash equilibrium results smaller.

18.10.1 Refining Nash execution

What kind of social objectives are executable under the assumption of refining Nash equilibrium solution? When the set $\mathcal{N}(\Gamma, e)$ of Nash equilibrium result of arbitrary mechanisms cannot be a subset of social goal set $F(e)$, the social goal is not Nash executable. However, since the set of refining Nash equilibrium results is much smaller than the set of Nash equilibrium results, the set of refining Nash equilibrium results may be a subset of social goal set $F(e)$. Thus, it may be executable for refining Nash equilibrium solution. This is indeed the case, and this section introduces several concepts of refining Nash executable.

A concept of refining Nash equilibrium solution is a strong Nash equilibrium hypothesis. It means that when a strategy is in a strong equilibrium state, for any alliance formed by a group of people, when given the other people's strategy, no one in the alliance can benefit more from cooperation. Obviously, the strong Nash equilibrium strategy is a stronger equilibrium hypothesis than Nash equilibrium strategy, and every strong Nash equilibrium is Nash equilibrium, whereas the contrary is not true. Therefore, the set of strong Nash equilibrium is a subset of Nash equilibrium. And because the set of this equilibrium is smaller than the set of Nash equilibrium, the set of social selection goals executed by strong Nash equilibrium may be larger. Maskin (1979) proves that: any social choice goal that leads to Pareto optimal allocation and rational allocation for appropriate preference is strongly Nash executable.

The strategic hypothesis of people's egoistic behaviors can also be the strategic hypothesis of subgame perfect Nash equilibrium introduced by Zelteng Selton or other refinement of Nash equilibrium solutions. In addition, there are many kinds of strategic equilibrium solutions to express people's egoistic behaviors, such as the non-dominant Nash equilibrium or other refining Nash equilibrium solutions. Moore-Repullo(1988), Abreu-Sen(1990) and other people proved that almost all social goals are executable under the assumption of the subgame Nash equilibrium solution. Palfrey Srivastava(1991) also proved the same result under the assumption of non-dominant Nash equilibrium.

18.10.2 Approximate Nash execution

People can also expand the range of executable social goals by approximate Nash execute the social goal. Although the Nash equilibrium results set $\mathcal{N}(\Gamma, e)$ cannot be completely contained in the social goals et $F(e)$, but as long as each Nash equilibrium configuration can be arbitrarily close to a particular configuration in $F(e)$ (that means the difference (distance) between the two configurations can be arbitrarily small), we say that this social goal is approachable. Matsushima(1988), Abreu-Sen(1991), Tian(1997)

and other people proved that almost all social goals are approximately approachable.

18.10.3 Nash and strong Nash double execution

Nash equilibrium strategy is a non-cooperative concept of strategic equilibrium, which completely excludes any possibility of cooperation. Although the Nash equilibrium is relatively easy to achieve, it may be unstable: participants tend to respond to the designers through some forms of cooperation, because cooperation may yield greater benefits. In this way, the people's egoistic behavior assumption described by Nash strategic equilibrium may be unrealistic, so it may be more reasonable to adopt the hypothesis of strong Nash strategic equilibrium. Since the set of strong Nash equilibrium is obviously a subset of Nash equilibrium, it is more likely to execute a social goal through strong Nash equilibrium.

Although the strong Nash equilibrium hypothesis is more reasonable, it may not exist or be difficult to solve. In order to be relatively easy to achieve and stable, people naturally require that a social choice can be carried out simultaneously by Nash and strong Nash. This is called double implementation. Suh(1997) gave the necessary and sufficient conditions for a social choice goal to be carried out by Nash and strong Nash double implementation. Because the characteristic result of Suh only considered the sufficient and necessary condition that a social choice goal was carried out by two sides, and does not consider the complexity of the mechanism, Peleg(1996a, 1996b) and the author Tian (1999a, 2000a, 2000b, 2000c, 2000d) presented a set of incentives that implement both Valas and Lindal configurations and other social selection goals, that lead to Pareto optimal and rational allocation.

18.11 Information Efficiency in Mechanism Design

18.11.1 Information Efficiency in Mechanism Design

Consider production economies with L private goods, I consumers and J firms so that total number of agents is $n := I + J$. Consumer i 's characteristic is given by $e_i = (X_i, w_i, R_i)$, where $X_i \subset \mathbb{R}^L$, $w_i \in \mathbb{R}_+^L$, and R_i is convex¹, continuous on X_i , and strictly monotone on the set of interior points of X_i . Producer j 's characteristic is given by $e_j = (\mathcal{Y}_j)$. We assume that, for $j = I + 1, \dots, n$, \mathcal{Y}_j is nonempty, closed, convex, contains 0 (possibility of inaction), and $\mathcal{Y}_j - \mathbb{R}_+^L \subseteq \mathcal{Y}_j$ (free-disposal). We also assume that the economies under consideration have no externalities or public goods.

¹ R_i is convex if for bundles a, b, c with $0 < \lambda \leq 1$ and $c = \lambda a + (1 - \lambda)b$, the relation $a P_i b$ implies $c P_i b$. Note that the term "convex" is defined as in Debreu (1959), not as in some recent textbooks.

An economy is the full vector $e = (e_1, \dots, e_I, e_{I+1}, \dots, e_N)$ and the set of all such production economies is denoted by E and is called neoclassical production economies. E is assumed to be endowed with the product topology.

Let x_i denote the net increment in commodity holdings (net trade) by consumer i and y_j producer j 's (net) output vector. Denoted by $x = (x_1, \dots, x_I)$ and $y = (y_{I+1}, \dots, y_N)$.

An allocation of the economy e is a vector $z := (x, y) \in \mathbb{R}^{NL}$. An allocation $z = (x, y)$ is said to be *individually feasible* if $x_i + w_i \in X_i$ for $i = 1, \dots, I$, and $y_j \in \mathcal{Y}_j$ for $j = I + 1, \dots, n$. An allocation $z = (x, y)$ is said to be *balanced* if $\sum_{i=1}^I x_i = \sum_{j=I+1}^N y_j$. An allocation $z = (x, y)$ is said to be *feasible* if it is balanced and individually feasible for every individual.

An allocation $z = (x, y)$ is said to be *Pareto efficient* if it is feasible and there does not exist another feasible allocation $z' = (x', y')$ such that $(x'_i + w_i)R_i(x_i + w_i)$ for all $i = 1, \dots, I$ and $(x'_i + w_i)P_i(x_i + w_i)$ for some $i = 1, \dots, I$. Denote by $P(e)$ the set of all such allocations.

An important characterization of a Pareto optimal allocation is associated with the following concept. Let $\Delta^{L-1} = \{p \in \mathbb{R}_{++}^L : \sum_{l=1}^L p^l = 1\}$ be the $L - 1$ dimensional unit simplex.

A nonzero vector $p \in \Delta^{L-1}$ is called a vector of *efficiency prices* for a Pareto optimal allocation (x, y) if

- (a) $p \cdot x_i \leq p \cdot x'_i$ for all $i = 1, \dots, I$ and all x'_i such that $x'_i + w_i \in X_i$ and $(x'_i + w_i)R_i(x_i + w_i)$;
- (b) $p \cdot y_j \geq p \cdot y'_j$ for all $y'_j \in \mathcal{Y}_j, j = I + 1, \dots, N$.

In the terminology of Debreu (1959, p. 93), (x, y) is an equilibrium relative to the price system p . It is well known that under certain regularity conditions such as convexity, local non-satiation, etc, every Pareto optimal allocation (x, y) has an efficiency price associated with it as shown in Second Theorem of Welfare Economics in Chapter ??.

We also want a mechanism is individually rational. However, as Hurwicz (1979b) pointed out, it is not quite obvious what the appropriate generalization of the individual rationality concept should be for an economy with production. The following definition of individual rationality of an allocation for an economy with production was introduced by Hurwicz (1979b).

An allocation $z = (x, y)$ is said to be *individually rational* with respect to the fixed share guarantee structure $\gamma_i(e; \theta)$ if $(x_i + w_i)R_i(\gamma_i(e) + w_i)$ for all $i = 1, \dots, I$. Here, $\gamma_i(e; \theta)$ is given by

$$\gamma_i(e; \theta) = \frac{p \cdot \sum_{j=I+1}^N \theta_{ij} y_j}{p \cdot w_i} w_i, \quad i = 1, \dots, I, \quad (18.11.39)$$

where p is an efficiency price vector for e and the θ_{ij} are non-negative fractions such that $\sum_{i=1}^n \theta_{ij} = 1$ for $j = I + 1, \dots, N$, which can be interpreted

as the profit shares of consumer i from producer j . Note that this definition on the individual rationality contains pure exchange as well as constant returns as special cases. Denote by $I_\theta(e)$ the set of all such allocations.

Now we define the competitive equilibria of a private ownership economy in which the i -th consumer owns the share θ_{ij} of the j -th producer, and is, consequently, entitled to the corresponding fraction of its profits. Thus, the ownership structure can be denoted by the matrix $\theta = (\theta_{ij})$. Denoted by Θ the set of all such ownership structures.

An allocation $z = (x, y) = (x_1, x_2, \dots, x_I, y_{I+1}, y_{I+2}, \dots, y_N) \in \mathbb{R}^{IL} \times \mathcal{Y}$ is a θ -Walrasian allocation for an economy e if it is feasible and there is a price vector $p \in \Delta^{L-1}$ such that

- (1) $p \cdot x_i = \sum_{j=I+1}^N \theta_{ij} p \cdot y_j$ for all $i = 1, \dots, I$;
- (2) for all $i = 1, \dots, I$, $(x'_i + w_i) P_i(x_i + w_i)$ implies $p \cdot x'_i > \sum_{j=I+1}^N \theta_{ij} p \cdot y_j$; and
- (3) $p \cdot y_j \geq p \cdot y'_j$ for all $y'_j \in \mathcal{Y}_j$ and $j = I + 1, \dots, N$.

Denoted by $W_\theta(e)$ the set of all such allocations, and by $\mathcal{W}_\theta(e)$ the set of all such price-allocations pair (p, z) .

Let E^c subset E be the subset of production economies on which $W(e) \neq \emptyset$ for all $e \in E^c$ and call such a subset as the Walrasian production economies.

It may be remarked that, every θ -Walrasian allocation is clearly individually rational with respect to the $\gamma_i(e; \theta)$, and also, by the strong monotonicity of preferences, it is Pareto efficient. Thus we have $W_\theta(e) \subset I_\theta(e) \cap P(e)$ for all $e \in E^c$.

We now define the Walrasian process that is a privacy-preserving process and realizes the Walrasian correspondence W_θ , and in which messages consist of prices and trades of all agents. In defining the Walrasian process, it is assumed that the private ownership structure matrix θ is common knowledge for all the agents.

Define the excess demand correspondence of consumer i ($i = 1, \dots, I$) $D_i : \Delta^{L-1} \times \Theta \times \mathbb{R}_+^J \times E_i \rightarrow \mathbb{R}^L$ by

$$D_i(p, \theta, \pi_{I+1}, \dots, \pi_N, e_i) = \{x_i : x_i + w_i \in X_i, p \cdot x_i = \sum_{j=I+1}^N \theta_{ij} \pi_j, (x'_i + w_i) P_i(x_i + w_i) \text{ implies } p \cdot x'_i > \sum_{j=I+1}^N \theta_{ij} \pi_j\} \quad (18.11.40)$$

where π_j is the profit of firm j ($j = I + 1, \dots, N$).

Define the supply correspondence of firm j ($j = I + 1, \dots, N$) $S_j : \Delta^{L-1} \times E_j \rightarrow \mathbb{R}^L$ by

$$S_j(p, e_j) = \{y_j : y_j \in \mathcal{Y}_j, p \cdot y_j \geq p \cdot y'_j \forall y'_j \in \mathcal{Y}_j\} \quad (18.11.41)$$

Note that (p, x, y) is a θ -Walrasian (competitive) equilibrium for economy e with the private ownership structure θ if $p \in \Delta^{L-1}$, $x_i \in D_i(p, \theta, p \cdot y_{I+1}, \dots, p \cdot y_N)$ for $i = 1, \dots, I$, $y_j \in S_j(p, e_j)$ for $j = I+1, \dots, N$, and the allocation (x, y) is balanced.

The Walrasian (competitive) process $\langle M_c, \mu_c, h_c \rangle$ is defined as follows.

Define $M_c = \Delta^{L-1} \times Z$.

Define $\mu_c : E \rightarrow M_c$ by

$$\mu_c(e) = \bigcap_{i=1}^N \mu_{ci}(e_i), \quad (18.11.42)$$

where $\mu_{ci} : E_i \rightarrow M_c$ is defined as follows:

- (1) For $i = 1, \dots, I$, $\mu_{ci}(e_i) = \{(p, x, y) : p \in \Delta^{L-1}, x_i \in D_i(p, \theta, p \cdot y_{I+1}, \dots, p \cdot y_N, e_i) \text{ and } \sum_{i=1}^I x_i = \sum_{j=I+1}^N y_j\}$.
- (2) For $i = I+1, \dots, N$, $\mu_{ci}(e_i) = \{(p, x, y) : p \in \Delta^{L-1}, y_i \in S_i(p, e_i) \text{ and } \sum_{i=1}^I x_i = \sum_{j=I+1}^N y_j\}$.

Thus, we have $\mu_c(e) = W_\theta(e)$ for all $e \in E$.

Finally, the Walrasian outcome function $h_c : M_c \rightarrow Z$ is defined by

$$h_c(p, x, y) = (x, y), \quad (18.11.43)$$

which is an element in $W_\theta(e)$.

The Walrasian process can be viewed as a formalization of resource allocation, simulating the competitive mechanism, which is non-wasteful and individually rational with respect to the fixed share guarantee structure $\gamma_i(e; \theta)$. The competitive message process is privacy-preserving by the construction of the Walrasian process.

Remark 18.11.1 For a given private ownership structure matrix θ , since an element, $m = (p, x_1, \dots, x_I, y_{I+1}, \dots, y_N) \in \mathbb{R}_{++}^L \times \mathbb{R}^{(N)L}$, of the Walrasian message space M_c satisfies the conditions $\sum_{l=1}^L p^l = 1$, $\sum_{i=1}^I x_i = \sum_{j=I+1}^N y_j$, and $p \cdot x_i = \sum_{j=I+1}^N \theta_{ij} p \cdot y_j$ ($i = 1, \dots, I$) and one of these equations is not independent by Walras Law, any Walrasian message is contained within a Euclidean space of dimension $(L+IL+JL) - (1+L+I) + 1 = (L-1)I + LJ$ and thus, an upper bound on the Euclidean dimension of M_c is $(L-1)I + LJ$.

To establish the informational efficiency of the competitive mechanism, we consider a special class of economies, denoted by $E^{cq} = \prod_{i=1}^N E_i^{cq}$, where preference orderings are characterized by Cobb-Douglas utility functions, and efficient production technology are characterized by quadratic functions.

For $i = 1, \dots, I$, consumer i 's admissible economic characteristics in E_i^{cq} are given by the set of all $e_i = (X_i, w_i, R_i)$ such that $X_i = \mathbb{R}_+^L$, $w_i > 0$, and

R_i is represented by a Cobb-Douglas utility function $u(\cdot, a_i)$ with $a_i \in \Delta^{L-1}$ such that $u(x_i + w_i, a_i) = \prod_{l=1}^L (x_i^l + w_i^l)^{a_i^l}$.

For $i = I + 1, \dots, N$, producer i 's admissible economic characteristics are given by the set of all $e_i = \mathcal{Y}_i = \mathcal{Y}(b_i)$ such that

$$\begin{aligned} \mathcal{Y}(b_i) = \{y_i \in \mathbb{R}^L : & b_i^1 y_i^1 + \sum_{l=2}^L (y^l + \frac{b_i^l}{2} (y^l)^2) \leq 0 \\ & -\frac{1}{b_i^l} \leq y_i^l \leq 0 \text{ for all } l \neq 1\}, \end{aligned} \quad (18.11.44)$$

where $b_i = (b_i^1, \dots, b_i^L)$ with $b_i^l > \frac{J}{w_i^l}$. It is clear that any economy in E^{cq} is fully specified by the parameters $a = (a_1, \dots, a_I)$ and $b = (b_{I+1}, \dots, b_N)$. Furthermore, production sets are nonempty, closed, and convex by noting that $0 \in \mathcal{Y}(b_j)$ and their efficient points are represented by quadratic production functions in which (y_i^2, \dots, y_i^L) are inputs and y_i^1 is possibly an output.

Given an initial endowment $\bar{w} \in \mathbb{R}_{++}^{LI}$, define a subset \bar{E}^{cq} of E^{cq} by $\bar{E}^{cq} = \{e \in E^{cd} : w_i = \bar{w}_i \forall i = 1, \dots, I\}$. That is, endowments are constant over \bar{E}^{cq} . Let E^c be the set of all production economies in which Walrasian equilibrium exists.

We then have the following theorem that shows that the Walrasian mechanism is informationally efficient among all smooth resource allocation mechanisms that are informationally decentralized and non-wasteful over the set E^c .

Theorem 18.11.1 (Informational Efficiency Theorem) *Then the Walrasian allocation mechanism $\langle M_c, \mu_c, h_c \rangle$ is informationally efficient among all allocation mechanisms $\langle M, \mu, h \rangle$ defined on E^c that*

- (i) are informationally decentralized;
- (ii) are non-wasteful with respect to \mathcal{P} ;
- (iii) have Hausdorff topological message spaces;
- (iv) satisfy the local threadedness property at some point $e \in \bar{E}^{cq}$.²

That is, $M_c =_F \mathbb{R}^{(L-1)I+LJ} \leq_F M$.

This theorem establishes the informational efficiency of the competitive mechanism within the class of all smooth resource allocation mechanisms which are informationally decentralized and non-wasteful over the class of Walrasian production economies E^c .

²A stationary message correspondence μ is said to be locally threaded at $e \in E$ if there is a neighborhood $N(e) \subset E$ and a continuous function $f : N(e) \rightarrow M$ such that $f(e') \in \mu(e')$ for all $e' \in N(e)$. The stationary message correspondence μ is said to be locally threaded on E if it is locally threaded at every $e \in E$.

The following theorem further shows that the competitive allocation process is the only most informationally efficient decentralized mechanism for production economies among mechanisms that achieve Pareto optimal and individually rational allocations.

Theorem 18.11.2 *Suppose that $\langle M, \mu, h \rangle$ is an allocation mechanism on the class of production economies E^{cq} such that:*

- (i) *it is informationally decentralized;*
- (ii) *it is non-wasteful with respect to \mathcal{P} ;*
- (iii) *it is individually rational with respect to the fixed share guarantee structure $\gamma_i(e; \theta)$;*
- (iv) *M is a $(L - 1)I + LJ$ dimensional manifold;*
- (v) *μ is a continuous function on E^{cq} .*

Then, there is a homeomorphism ϕ on $\mu(E^{cq})$ to M_c such that

- (a) $\mu_c = \phi \cdot \mu$;
- (b) $h_c \cdot \phi = h$.

The conclusion of the theorem is summarized in the following commutative homeomorphism diagram:

$$\begin{array}{ccc}
 E & \xrightarrow{\mu_c} & M_c \\
 \downarrow \mu & \phi \nearrow \swarrow \phi^{-1} & \downarrow h_c \\
 \mu(E) & \xrightarrow{h} & Z
 \end{array}$$

Thus, the above theorem shows that the competitive mechanism is the unique informationally efficient process that realizes Pareto optimal and individually rational allocations over the class of production economies E^c . For a mechanism $\langle M, \mu, h \rangle$, when there does not any homeomorphism ϕ on $\mu(E^{cq})$ to M_c such that (1) $\mu_L = \phi \cdot \mu$ and (2) $h_L \cdot \phi = h$, we may call such a mechanism $\langle M, \mu, h \rangle$ a non-Walrasian allocation mechanism. The Uniqueness Theorem then implies that any non-Walrasian allocation mechanism defined on E^{cq} must use a larger message space.

For public goods economies, Tian (2000e) obtains the similar results, and shows that Lindahl mechanism is the unique informationally efficient process that realizes Pareto optimal and individually rational allocations over the class of production economies that guarantees the existence of Lindahl equilibrium.

18.11.2 Informational Efficiency and Uniqueness of Competitive Market Mechanism

Consider production economies with L private goods, I consumers and J firms so that total number of agents is $n := I + J$. Consumer i 's characteristic is given by $e_i = (X_i, w_i, R_i)$, where $X_i \subset \mathbb{R}^L$, $w_i \in \mathbb{R}_+^L$, and R_i is convex³, continuous on X_i , and strictly monotone on the set of interior points of X_i . Producer j 's characteristic is given by $e_j = (\mathcal{Y}_j)$. We assume that, for $j = I + 1, \dots, n$, \mathcal{Y}_j is nonempty, closed, convex, contains 0 (possibility of inaction), and $\mathcal{Y}_j - \mathbb{R}_+^L \subseteq \mathcal{Y}_j$ (free-disposal). We also assume that the economies under consideration have no externalities or public goods.

An economy is the full vector $e = (e_1, \dots, e_I, e_{I+1}, \dots, e_N)$ and the set of all such production economies is denoted by E and is called neoclassical production economies. E is assumed to be endowed with the product topology.

Let x_i denote the net increment in commodity holdings (net trade) by consumer i and y_j producer j 's (net) output vector. Denoted by $x = (x_1, \dots, x_I)$ and $y = (y_{I+1}, \dots, y_N)$.

An allocation of the economy e is a vector $z := (x, y) \in \mathbb{R}^{NL}$. An allocation $z = (x, y)$ is said to be *individually feasible* if $x_i + w_i \in X_i$ for $i = 1, \dots, I$, and $y_j \in \mathcal{Y}_j$ for $j = I + 1, \dots, n$. An allocation $z = (x, y)$ is said to be *balanced* if $\sum_{i=1}^I x_i = \sum_{j=I+1}^N y_j$. An allocation $z = (x, y)$ is said to be *feasible* if it is balanced and individually feasible for every individual.

An allocation $z = (x, y)$ is said to be *Pareto efficient* if it is feasible and there does not exist another feasible allocation $z' = (x', y')$ such that $(x'_i + w_i) R_i(x_i + w_i)$ for all $i = 1, \dots, I$ and $(x'_i + w_i) P_i(x_i + w_i)$ for some $i = 1, \dots, I$. Denote by $P(e)$ the set of all such allocations.

An important characterization of a Pareto optimal allocation is associated with the following concept. Let $\Delta^{L-1} = \{p \in \mathbb{R}_{++}^L : \sum_{l=1}^L p^l = 1\}$ be the $L - 1$ dimensional unit simplex.

A nonzero vector $p \in \Delta^{L-1}$ is called a vector of *efficiency prices* for a Pareto optimal allocation (x, y) if

- (a) $p \cdot x_i \leq p \cdot x'_i$ for all $i = 1, \dots, I$ and all x'_i such that $x'_i + w_i \in X_i$ and $(x'_i + w_i) R_i(x_i + w_i)$;
- (b) $p \cdot y_j \geq p \cdot y'_j$ for all $y'_j \in \mathcal{Y}_j$, $j = I + 1, \dots, N$.

In the terminology of Debreu (1959, p. 93), (x, y) is an equilibrium relative to the price system p . It is well known that under certain regularity conditions such as convexity, local non-satiation, etc, every Pareto optimal allocation (x, y) has an efficiency price associated with it as shown in Second Theorem of Welfare Economics in Chapter ??.

³ R_i is convex if for bundles a, b, c with $0 < \lambda \leq 1$ and $c = \lambda a + (1 - \lambda)b$, the relation $a P_i b$ implies $c P_i b$. Note that the term "convex" is defined as in Debreu (1959), not as in some recent textbooks.

We also want a mechanism is individually rational. However, as Hurwicz (1979b) pointed out, it is not quite obvious what the appropriate generalization of the individual rationality concept should be for an economy with production. The following definition of individual rationality of an allocation for an economy with production was introduced by Hurwicz (1979b).

An allocation $z = (x, y)$ is said to be *individually rational* with respect to the fixed share guarantee structure $\gamma_i(e; \theta)$ if $(x_i + w_i)R_i(\gamma_i(e) + w_i)$ for all $i = 1, \dots, I$. Here, $\gamma_i(e; \theta)$ is given by

$$\gamma_i(e; \theta) = \frac{p \cdot \sum_{j=I+1}^N \theta_{ij} y_j}{p \cdot w_i} w_i, \quad i = 1, \dots, I, \quad (18.11.45)$$

where p is an efficiency price vector for e and the θ_{ij} are non-negative fractions such that $\sum_{i=1}^n \theta_{ij} = 1$ for $j = I + 1, \dots, N$, which can be interpreted as the profit shares of consumer i from producer j . Note that this definition on the individual rationality contains pure exchange as well as constant returns as special cases. Denote by $I_\theta(e)$ the set of all such allocations.

Now we define the competitive equilibria of a private ownership economy in which the i -th consumer owns the share θ_{ij} of the j -th producer, and is, consequently, entitled to the corresponding fraction of its profits. Thus, the ownership structure can be denoted by the matrix $\theta = (\theta_{ij})$. Denoted by Θ the set of all such ownership structures.

An allocation $z = (x, y) = (x_1, x_2, \dots, x_I, y_{I+1}, y_{I+2}, \dots, y_N) \in \mathbb{R}^{IL} \times \mathcal{Y}$ is a θ -Walrasian allocation for an economy e if it is feasible and there is a price vector $p \in \Delta^{L-1}$ such that

- (1) $p \cdot x_i = \sum_{j=I+1}^N \theta_{ij} p \cdot y_j$ for all $i = 1, \dots, I$;
- (2) for all $i = 1, \dots, I$, $(x'_i + w_i) P_i(x_i + w_i)$ implies $p \cdot x'_i > \sum_{j=I+1}^N \theta_{ij} p \cdot y_j$; and
- (3) $p \cdot y_j \geq p \cdot y'_j$ for all $y'_j \in \mathcal{Y}_j$ and $j = I + 1, \dots, N$.

Denoted by $W_\theta(e)$ the set of all such allocations, and by $\mathcal{W}_\theta(e)$ the set of all such price-allocation pairs (p, z) .

Let E^c subset E be the subset of production economies on which $W(e) \neq \emptyset$ for all $e \in E^c$ and call such a subset as the Walrasian production economies.

It may be remarked that, every θ -Walrasian allocation is clearly individually rational with respect to the $\gamma_i(e; \theta)$, and also, by the strong monotonicity of preferences, it is Pareto efficient. Thus we have $W_\theta(e) \subset I_\theta(e) \cap P(e)$ for all $e \in E^c$.

We now define the Walrasian process that is a privacy-preserving process and realizes the Walrasian correspondence W_θ , and in which messages

consist of prices and trades of all agents. In defining the Walrasian process, it is assumed that the private ownership structure matrix θ is common knowledge for all the agents.

Define the excess demand correspondence of consumer i ($i = 1, \dots, I$) $D_i : \Delta^{L-1} \times \Theta \times \mathbb{R}_+^J \times E_i \rightarrow \mathbb{R}^L$ by

$$D_i(p, \theta, \pi_{I+1}, \dots, \pi_N, e_i) = \{x_i : x_i + w_i \in X_i, p \cdot x_i = \sum_{j=I+1}^N \theta_{ij} \pi_j$$

$$(x_i' + w_i) P_i(x_i + w_i) \text{ implies } p \cdot x_i' > \sum_{j=I+1}^N \theta_{ij} \pi_j\} \quad (18.11.46)$$

where π_j is the profit of firm j ($j = I + 1, \dots, N$).

Define the supply correspondence of firm j ($j = I + 1, \dots, N$) $S_j : \Delta^{L-1} \times E_j \rightarrow \mathbb{R}^L$ by

$$S_j(p, e_j) = \{y_j : y_j \in \mathcal{Y}_j, p \cdot y_j \geq p \cdot y_j' \forall y_j' \in \mathcal{Y}_j\} \quad (18.11.47)$$

Note that (p, x, y) is a θ -Walrasian (competitive) equilibrium for economy e with the private ownership structure θ if $p \in \Delta^{L-1}$, $x_i \in D_i(p, \theta, p \cdot y_{I+1}, \dots, p \cdot y_N)$ for $i = 1, \dots, I$, $y_j \in S_j(p, e_j)$ for $j = I + 1, \dots, N$, and the allocation (x, y) is balanced.

The Walrasian (competitive) process $\langle M_c, \mu_c, h_c \rangle$ is defined as follows.

Define $M_c = \Delta^{L-1} \times Z$.

Define $\mu_c : E \rightarrow M_c$ by

$$\mu_c(e) = \cap_{i=1}^N \mu_{ci}(e_i), \quad (18.11.48)$$

where $\mu_{ci} : E_i \rightarrow M_c$ is defined as follows:

- (1) For $i = 1, \dots, I$, $\mu_{ci}(e_i) = \{(p, x, y) : p \in \Delta^{L-1}, x_i \in D_i(p, \theta, p \cdot y_{I+1}, \dots, p \cdot y_N, e_i) \text{ and } \sum_{i=1}^I x_i = \sum_{j=I+1}^N y_j\}$.
- (2) For $i = I + 1, \dots, N$, $\mu_{ci}(e_i) = \{(p, x, y) : p \in \Delta^{L-1}, y_i \in S_i(p, e_i) \text{ and } \sum_{i=1}^I x_i = \sum_{j=I+1}^N y_j\}$.

Thus, we have $\mu_c(e) = \mathcal{W}_\theta(e)$ for all $e \in E$.

Finally, the Walrasian outcome function $h_c : M_c \rightarrow Z$ is defined by

$$h_c(p, x, y) = (x, y), \quad (18.11.49)$$

which is an element in $W_\theta(e)$.

The Walrasian process can be viewed as a formalization of resource allocation, simulating the competitive mechanism, which is non-wasteful and individually rational with respect to the fixed share guarantee structure $\gamma_i(e; \theta)$. The competitive message process is privacy-preserving by the construction of the Walrasian process.

Remark 18.11.2 For a given private ownership structure matrix θ , since an element, $m = (p, x_1, \dots, x_I, y_{I+1}, \dots, y_N) \in \mathbb{R}_{++}^L \times \mathbb{R}^{(N)L}$, of the Walrasian message space M_c satisfies the conditions $\sum_{l=1}^L p^l = 1$, $\sum_{i=1}^I x_i = \sum_{j=I+1}^N y_j$, and $p \cdot x_i = \sum_{j=I+1}^N \theta_{ij} p \cdot y_j$ ($i = 1, \dots, I$) and one of these equations is not independent by Walras Law, any Walrasian message is contained within a Euclidean space of dimension $(L+IL+JL) - (1+L+I)+1 = (L-1)I + LJ$ and thus, an upper bound on the Euclidean dimension of M_c is $(L-1)I + LJ$.

To establish the informational efficiency of the competitive mechanism, we consider a special class of economies, denoted by $E^{cq} = \prod_{i=1}^N E_i^{cq}$, where preference orderings are characterized by Cobb-Douglas utility functions, and efficient production technology are characterized by quadratic functions.

For $i = 1, \dots, I$, consumer i 's admissible economic characteristics in E_i^{cq} are given by the set of all $e_i = (X_i, w_i, R_i)$ such that $X_i = \mathbb{R}_{++}^L$, $w_i > 0$, and R_i is represented by a Cobb-Douglas utility function $u(\cdot, a_i)$ with $a_i \in \Delta^{L-1}$ such that $u(x_i + w_i, a_i) = \prod_{l=1}^L (x_i^l + w_i^l)^{a_i^l}$.

For $i = I+1, \dots, N$, producer i 's admissible economic characteristics are given by the set of all $e_i = \mathcal{Y}_i = \mathcal{Y}(b_i)$ such that

$$\begin{aligned} \mathcal{Y}(b_i) = \{y_i \in \mathbb{R}^L : & b_i^1 y_i^1 + \sum_{l=2}^L (y_i^l + \frac{b_i^l}{2} (y_i^l)^2) \leq 0 \\ & -\frac{1}{b_i^l} \leq y_i^l \leq 0 \text{ for all } l \neq 1\}, \end{aligned} \quad (18.11.50)$$

where $b_i = (b_i^1, \dots, b_i^L)$ with $b_i^l > \frac{J}{w_i^l}$. It is clear that any economy in E^{cq} is fully specified by the parameters $a = (a_1, \dots, a_I)$ and $b = (b_{I+1}, \dots, b_N)$. Furthermore, production sets are nonempty, closed, and convex by noting that $0 \in \mathcal{Y}(b_j)$ and their efficient points are represented by quadratic production functions in which (y_i^2, \dots, y_i^L) are inputs and y_i^1 is possibly an output.

Given an initial endowment $\bar{w} \in \mathbb{R}_{++}^{LI}$, define a subset \bar{E}^{cq} of E^{cq} by $\bar{E}^{cq} = \{e \in E^{cd} : w_i = \bar{w}_i \forall i = 1, \dots, I\}$. That is, endowments are constant over \bar{E}^{cq} . Let E^c be the set of all production economies in which Walrasian equilibrium exists.

We then have the following theorem that shows that the Walrasian mechanism is informationally efficient among all smooth resource allocation mechanisms that are informationally decentralized and non-wasteful over the set E^c .

Theorem 18.11.3 (Informational Efficiency Theorem) *Then the Walrasian allocation mechanism $\langle M_c, \mu_c, h_c \rangle$ is informationally efficient among all allocation mechanisms $\langle M, \mu, h \rangle$ defined on E^c that*

- (i) are informationally decentralized;
- (ii) are non-wasteful with respect to \mathcal{P} ;
- (iii) have Hausdorff topological message spaces;
- (iv) satisfy the local threadedness property at some point $e \in \bar{E}^{cq}$.⁴

That is, $M_c =_F \mathbb{R}^{(L-1)I+LJ} \leq_F M$.

This theorem establishes the informational efficiency of the competitive mechanism within the class of all smooth resource allocation mechanisms which are informationally decentralized and non-wasteful over the class of Walrasian production economies E^c .

The following theorem further shows that the competitive allocation process is the only most informationally efficient decentralized mechanism for production economies among mechanisms that achieve Pareto optimal and individually rational allocations.

Theorem 18.11.4 *Suppose that $\langle M, \mu, h \rangle$ is an allocation mechanism on the class of production economies E^{cq} such that:*

- (i) it is informationally decentralized;
- (ii) it is non-wasteful with respect to \mathcal{P} ;
- (iii) it is individually rational with respect to the fixed share guarantee structure $\gamma_i(e; \theta)$;
- (iv) M is a $(L - 1)I + LJ$ dimensional manifold;
- (v) μ is a continuous function on E^{cq} .

Then, there is a homeomorphism ϕ on $\mu(E^{cq})$ to M_c such that

- (a) $\mu_c = \phi \cdot \mu$;
- (b) $h_c \cdot \phi = h$.

The conclusion of the theorem is summarized in the following commutative homeomorphism diagram:

$$\begin{array}{ccc}
 E & \xrightarrow{\mu_c} & M_c \\
 \downarrow \mu & \nearrow \phi \searrow \phi^{-1} & \downarrow h_c \\
 \mu(E) & \xrightarrow{h} & Z
 \end{array}$$

⁴A stationary message correspondence μ is said to be locally threaded at $e \in E$ if there is a neighborhood $N(e) \subset E$ and a continuous function $f : N(e) \rightarrow M$ such that $f(e') \in \mu(e')$ for all $e' \in N(e)$. The stationary message correspondence μ is said to be locally threaded on E if it is locally threaded at every $e \in E$.

Thus, the above theorem shows that the competitive mechanism is the unique informationally efficient process that realizes Pareto optimal and individually rational allocations over the class of production economies E^c . For a mechanism $\langle M, \mu, h \rangle$, when there does not any homeomorphism ϕ on $\mu(E^{cq})$ to M_c such that (1) $\mu_L = \phi \cdot \mu$ and (2) $h_L \cdot \phi = h$, we may call such a mechanism $\langle M, \mu, h \rangle$ a non-Walrasian allocation mechanism. The Uniqueness Theorem then implies that any non-Walrasian allocation mechanism defined on E^{cq} must use a larger message space.

For public goods economies, Tian (2000e) obtains the similar results, and shows that Lindahl mechanism is the unique informationally efficient process that realizes Pareto optimal and individually rational allocations over the class of production economies that guarantees the existence of Lindahl equilibrium.

18.12 Biographies

18.12.1 Michael Spence

A. Michael Spence(1943—) was born in New Jersey, United States of America, he received the doctor's degree from Harvard University in 1972. and now he is a professor at New York University and Stanford University, US-A. In 2001, He won the Nobel Prize in Economics, with George Akerlof and Joseph Stiglitz, for his pioneering research on asymmetric information market analysis.

Spence's most important research achievement is how the individuals with information advantage transmit the information "signal" reliably to the individuals with information disadvantage, in order to avoid some problems related to adverse selection in the market. Signals require individuals to take observational and costly measures to convince other individuals of their abilities or, more generally, of the value or quality of their products. Spence's contribution lies in forming and formalizing this idea, and explaining and analyzing its influence at the same time. Spence used education as a signal of productivity in the labour market in his ground breaking study (based on his doctoral thesis) in 1973. The basic point of view was that, the signal would not have a successful effect unless the signal cost was significantly different between the applicant and the job seeker. Employers could not distinguish highly competent job seekers from those with low abilities, unless the latter found that their investment in the education paid off when choosing a lower level of education. If the employer could not distinguish between high and low labor ability, it would lead the labor market to employ the low ability with low wages, forming the phenomenon of "Bad money drives out good" in the labor market. Spence also pointed to the possibility of a different "expected" balance between educa-

tion and wages, where men and whites earn more than women and blacks when productivity is equal.

Spence's subsequent research included extensive applied research that expanded this theory, confirmed the importance of different market signals, and analysed a large number of economic phenomenon, Such as expensive advertising and full guarantees as productivity signals; the active price reduction as a signal of market power; the strategy of delaying wage quotation as a signal of bargaining power; debt financing as a sign of profitability rather than the financing method of issuing new shares; monetary policy at the expense of recession as a signal of strong commitment to lower high inflation.

18.12.2 Joseph E. Stiglize

Joseph E. Stiglize(1943—) was born in Gary, Indiana, a small town famous for producing steel, and gave birth to two great contemporary economists, one Samuelson and the other Stiglitze. In 2001, Stiglitze won the Nobel Prize for Economics for his significant contribution to the creation of information economics. Stiglitze is the most cited economist in the literatures on information economics, and the same is true of broader microeconomics and macroeconomics. Some of his cutting-edge theories, such as adverse selection and moral hazard, have become standard tools for economists and policymakers. Stiglitze is also one of the most famous American economics educators. His book "Economics", which was first published in 1993, was reprinted again and again, and translated into many languages, and was recognized as one of the most classic textbooks on economics. It became another landmark economics introductory textbook in the West after Samuelson's "Economics".

Stiglitze is more concerned about the situation in developing countries, and often addresses problems from the perspective of developing countries. He has sharply accused that the relevant international organizations, which leading the process of economic globalization, ignore the interests of the poor and do nothing to eradicate poverty and promote social justice. On the International Monetary Fund and World Bank's Poverty Eradication Program-Free Trade, Stiglitze's view is that Europeans and Americans are breaking down barriers everywhere in Asia, Africa and Latin America to open up markets. He advocated highlighting the role of governments in macroeconomic regulation and said that the best way to achieve sustained growth and long-term efficiency was to find an appropriate balance between government and the market so as to bring the world economy back to a more equitable and stable growth process, which benefited everyone.

The origins of Stiglitze's ideas may have something to do with his growth. He came from a hardworking family. His father retired as an insurance agent at the age of 95, and his mother, who retired from the post of primary

school teacher at age 67, began teaching people to correct reading until 84. When Dr. Stiglitz was in college, he did very well and was interested in social activities. In 1963, his third year in college, he became president of the student council. And during that time, the American civil rights movement was in full swing, and Dr. Stiglitz took part in a march led by Dr. Martin Luther King in Washington, which culminated in Dr. King's historic speech "I have a Dream". These social activities have great influence on shaping his good-natured and optimistic character, and his efforts to promote fair and just market ideas after becoming famous.

Professor Stiglitz is the author and editor of hundreds of academic papers and works, including undergraduate course material "Public sector Economics"(Norton Corp), "Selection of Modern Economic growth Theory" edited with H.Uzawa (M.I.T. Press, 1969), "Lecture Notes on Public Economics" written with Anthony Barnes Atkinson (McGraw-Hill Books, 1980), "Commodity Price Stability Theory" written with D.M.G.Newbery (Oxford University Press, 1981). In 1987, he founded the Journal of Economic Outlook, which lowered the barriers to specialization created by other major economic journals. In 2008, he proposed several measures to prevent a recurrence of the economic crisis caused by Wall Street's housing bubble in a CNN column. Stiglitz's economics works are extensive, but consistently focus on the role of incomplete information in the process of competition. In several pioneering papers, he proved that: the common assumption that an economic unit has complete information on alternative market opportunities is not as harmless as it seems. These papers are summarized in the paper "Information and Competitive Price Systems", written with Grossman (in the American Economic Review, 1976 II).

18.13 Exercises

Exercise 18.1 Suppose X is an optional set, $R = R_1 \times \cdots \times R_n$ is the social preference set for X . Answer questions in response to the following:

A social selection function $f : R \rightarrow X$ is executable according to the dominant strategy mechanism, if and only if the social selection function is performed by a dominant strategic incentive compatible revelation mechanism.

1. Give the strict definition of the following nouns: (1) Dominant strategic mechanism; (2) The revelation mechanism; (3) Dominant Strategy of incentive compatible revelation Mechanism; (4) Execution
2. State whether the statements above are correct or not, If correct, prove it; otherwise, prove false.

Exercise 18.2 If faced with a restricted domain, the conclusion of the Gibbard-Satterthwaite Impossibility Theorem may not hold true. Please give a concrete structure to prove the above conclusion.

Exercise 18.3 (A simple majority vote with only two candidates) Suppose N voters cast their ballots on two candidates a and b , and each voter stated his or her preferences (just the candidate for whom he or she supports), and then cast his or her vote according to the following simple majority rule:

$$F(e) = \begin{cases} a, & \text{if } \#\{i \in I | a \succ_i^e b\} > \#\{i \in I | b \succ_i^e a\}, \\ b, & \text{if } \#\{i \in I | a \succ_i^e b\} \leq \#\{i \in I | b \succ_i^e a\}. \end{cases}$$

1. Prove: the simple majority rule is anti-manipulation (strategy – proof), that is, every voter will actually declare their preferences, whether or not others are telling the truth, so that $F(\cdot)$ can be implemented truly by dominant strategy.
2. Will this conclusion be true for more than two candidates?

Exercise 18.4 Consider the pure exchange economic environment for two people and two goods. For every $i = 1, 2$, $e^i \in E^i = \{u_c^i, u_l^i\}$, Given

$$\begin{aligned} u_{ci}(x_i^1, x_i^2) &= x_i^1 x_i^2, \\ u_{li}(x_i^1, x_i^2) &= x_i^1 + 2x_i^2. \end{aligned}$$

Therefore, In this environment ($E = E^1 \times E^2$), there are four possible economic forms. For every economy, the endowment is fixed, $\omega^1 = (1, 0)$, $\omega^2 = (1, 2)$.

1. Give the Pareto efficient set, and show the set of personal rationality and Pareto efficiency and their intersection in the graph.
2. Prove: for this economic environment, there is no direct incentive-compatible mechanism to perform both personal rationality and Pareto effectiveness.

Exercise 18.5 Three individuals use a central mechanism to determine whether a public good is provided. It is known that individuals a, b, c report the value of the public goods as 3, 5, -7 .

1. Should the public good be provided?
2. Who is the central human under this mechanism?
3. What should be the transfer payment for each individual?
4. If the true value of the public good to consumer b is 3, will he still keep 5 as report? Why?
5. Is the central mechanism balanced?

Exercise 18.6 Prove: for the public economy in a continuum of private value, under the Victor-Clark-Groves mechanism, all participants speak the truth ($b_1(\mathbf{y}), b_2(\mathbf{y}), \dots, b_n(\mathbf{y}) = (v_1(\mathbf{y}), v_2(\mathbf{y}), \dots, v_n(\mathbf{y}))$) constitute a dominant strategic equilibrium, so this mechanism actually executes the effective social rules according to the dominant equilibrium $y^*(\cdot)$.

Exercise 18.7 Suppose there are two technologies that can reduce pollution, each owned by manufacturer i , $i \in \{1, 2\}$, the technology is convex, and relies on private information θ_i . For simplification, we assume that the cost function is $C_i(q_i) = \frac{1}{2}\theta_i q_i^2$, q_i is the amount of pollution that is reduced by technology i . At the same time, we assume that marginal utility is a linear form, $MU(q) = 1 - 2q$. The total reduction of pollution is $q = q_1 + q_2$.

1. Calculate the effective level for two manufacturers to reduce the amount of pollution, and it is the function of (θ_i, θ_j) .
2. Calculate the effective level of transfer payments in the central mechanism (pivotal).

Exercise 18.8 Consider the availability of public goods. The two towns, each on the banks of a river, now decide whether to build a bridge between the two towns at a cost of $1 > c > 0$, let θ_i be the number of residents in town i who intend to use bridges. Suppose the prior distribution of θ_i is a uniform distribution on $[0, 1]$, and θ_1 is independent with θ_2 . After crossing the bridge to the town on the other side of the river, the residents of town i will have an effect $\gamma\theta_i$ on the residents of town j , which can be positively and negatively affected. Therefore, for town i , the total value of building bridges is $\theta_i + \gamma\theta_j$.

1. For this problem, what is the ordinary VCG mechanism? Find out the hindsight equilibrium associated with it.
2. What are the rules of social choice according to the dominant strategy in this problem?

Exercise 18.9 Consider the following economy composed of individual a, b, c and the public goods y . The utility function of individual i is $u_i(t_i, y) = t_i + v_i(y)$, which $v_i(y) = \bar{\theta}_i \ln y - y$. t_i represents the transfer payment for individual i , and the policy makers do not know the true $\bar{\theta}_i$.

1. Write the Groves mechanism $t_a(\theta), t_b(\theta), t_c(\theta), y(\theta)$, and θ_i is the report value of the true value $\bar{\theta}_i$ for individual i .
2. Write the Groves-Clark mechanism (the Central mechanism).

Exercise 18.10 Suppose $v(\cdot, \theta_i)$ is second order continuous differentiability. And for $\theta_i \in [\underline{\theta}_i, \bar{\theta}_i]$, there is $\partial^2 v_i(y, \theta_i) / \partial y^2 < 0$, $\partial^2 v_i(y, \theta_i) / \partial y \partial \theta_i > 0$. Prove: the continuous differentiable social selection function $f(\cdot) = (y(\cdot), t_1(\cdot), \dots, t_n(\cdot))$ is the real executable of the dominant strategy, if and only if $y(\theta)$ is the nondecreasing function of θ_i and $t_i(\theta_i, \theta_{-i}) = t_i(\underline{\theta}_i, \theta_{-i}) - \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(y(s, \theta_{-i}), s)}{\partial y} \frac{\partial y(s, \theta_{-i})}{\partial s} ds$.

Exercise 18.11 *Probe: if the consumer's preference for public goods is not limited (that is, every utility function $v_i : \mathcal{D} \rightarrow \mathcal{R}$ is likely to appear as θ_i changes), then the mechanism VCG is the only one that can induce agents to speak the truth and realize the effective supply of public goods.*

Exercise 18.12 *Individual 1 and 2 will decide the ownership of an item. Consider the following mechanism: individual 1 and 2 report prices separately, p_1 and p_2 . If $p_1 > p_2$, then individual 1 will get the item and pay p_2 to individual 2; if $p_1 < p_2$, then individual 2 will get the item and pay p_1 to individual 1; if $p_1 = p_2$, then individual 1 and 2 have half the chance to get the item and pay p_1 to the other. Answer the question:*

1. Is this mechanism compatible with dominant strategic incentives?
2. Is the mechanism individual rational?
3. Is the mechanism budget balanced?

Exercise 18.13 *Prove: If the consumer's preference for public goods is not limited (with the change of θ_i , every utility function $v_i : \mathcal{D} \rightarrow \mathcal{R}$ may appear), then mechanism VCG is the only mechanism that can induce agents to speak the truth and realize the effective supply of public goods.*

Exercise 18.14 *Individual 1 and 2 will decide the ownership of an item. Consider the following mechanism: individual 1 and 2 report prices separately, p_1 and p_2 . If $p_1 > p_2$, then individual 1 will get the item and pay p_2 to 2; if $p_1 < p_2$, then individual 2 will get the item and pay p_1 to 1; if $p_1 = p_2$, then individual 1 and 2 have half the chance to get the item and pay p_1 to the other. Answer the question:*

1. Is this mechanism compatible with dominant strategic incentives?
2. Is the mechanism individual rational?
3. Is the mechanism budget balanced?

Exercise 18.15 *Consider the quasilinear economic environment of n participants and a public goods y . Order $y^*(\cdot)$ indicates an ex post facto configuration mechanism, and define $V^*(\theta) = \sum_i v_i(y^*(\theta), \theta_i)$.*

1. Prove that the necessary and sufficient condition for the effective implementation of social choice function can be dominated by the dominant strategy is the function

$$V^*(\cdot) = \sum_i V_i(\theta_{-i}), \text{ for all } i, V_i(\cdot) \text{ is just } \theta_{-i}.$$
2. Using the above conclusions, prove that when $n = 3$, $Y = \mathcal{R}$, $\Theta_i = \mathcal{R}_+$, and for all $i, v_i(y, \theta) = \theta_i y - \frac{1}{2}y^2$, there is an effective social selection function that can be implemented in real terms by dominant strategies. Now

suppose function $v_i(y, \theta_i)$ makes function $V^*(\cdot)$ a continuous and n -degree differentiable function. Prove that a sufficient and necessary condition for the existence of a posteriori effective social selection function is: for any θ ,

$$\frac{\partial^I}{\partial \theta_1 \cdots \partial \theta_n} = 0$$

- Using the conclusion in question (3) to verify that when $n = 2$, there is no post-effective social selection function that can be executed by the dominant strategy.

Exercise 18.16 Consider the economy consisting of $n \geq 3$ individuals and public good y . The utility function of individual i is $u_i(t_i, y) = t_i + v_i(y)$, and $v_i(y) = -1/2y^2 + \theta_i y$.

- Write the Groves mechanism.
- Prove: if $d_i(\theta_{-1}) = \frac{1}{2n} \sum_{j \neq i} \theta_j^2 + \frac{n-1}{2n(n-2)} \sum_{j \neq i} \sum_{k \neq i, j} \theta_j \theta_k$, then the Groves mechanism is balanced.
- Is the Groves mechanism Pareto efficiency?

Exercise 18.17 (Jackson, 2003) Consider two people and the supply of one public good problems. $I = \{1, 2\}$, the type of each participant is distributed as $\theta^i \in \{0, 1\}$, $i = 1, 2$. Public goods are discrete, either providing or not providing, assuming that the supply cost of public goods is $c = \frac{3}{2}$. The utility function of two participants satisfies quasilinearity $u_i(y, t_i, \theta_i) = y\theta_i - t_i$, and $y \in \{0, 1\}$ is the public goods provision decision, t_i is the transfer payment of participant i . The $(y(\mathbf{t}), t_i(\mathbf{t}), i \in I)$ is called balanced budget if satisfying:

$$\sum_{i \in I} t_i(\mathbf{t}) = 0, \forall \mathbf{t}.$$

The $(y(\mathbf{t}), t_i(\mathbf{t}), i \in I)$ is called personal rationality if satisfying:

$$u_i(y(\mathbf{t}), t_i(\mathbf{t})) \geq 0, \forall \mathbf{t}, \forall i \in I.$$

- Prove that the Groves mechanism does not satisfy the balanced budget in this example.
- Prove that the Groves mechanism does not satisfy individual rationality in this example.

Exercise 18.18 (An example of a display principle that results in a non-strong execution) The textbook points out: although the original general mechanism completely or strongly implements the social choice corresponding F , But the

derived display mechanism $\langle E, g \rangle$ could be just execution, not full or strong execution of F . Now consider such counter-example given by Dasgupta, Hammond and Maskin (1979). Assume that the set of socially optional results is $A = \{a, b, c, d, e, p, q, r\}$, individual characteristics are characterized by their preferences. Assume all feature sets of individual 1 and 2 are $\mathcal{R}_1 = \{R_1, R'_1\}$ and $\mathcal{R}_2 = \{R_2, R'_2\}$, these preferences are described as follows:

$$R_1 = \begin{bmatrix} q \\ a - c - e \\ d - b - p \\ r \end{bmatrix}, R'_1 = \begin{bmatrix} c - b - p \\ a - d - e \\ q - r \end{bmatrix}, R_2 = \begin{bmatrix} r \\ d - a - e \\ b - c - p \\ q \end{bmatrix}, R'_2 = \begin{bmatrix} d \\ b - c \\ a \\ e - p - q - r \end{bmatrix}.$$

Consider a social choice rule (social choice rule), and it is defined as following:

$$\begin{aligned} f(R_1, R_2) &= \{a, e\}; \\ f(R'_1, R_2) &= \{c, p, b\}; \\ f(R_1, R'_2) &= \{d\}; \\ f(R'_1, R'_2) &= \{b\}. \end{aligned}$$

1. Prove that Social Choice Rule f satisfies monotonicity and Pareto efficiency.
2. Prove that f can be carried out by the following mechanism g_1 according to dominant equilibrium strong execution, and mechanism g_1 is described as the table, in which the row is the strategic choice of participant 1 and the list is the strategic choice of participant 2.

a	d	e
c	b	p
a	b	e

3. Prove that f cannot be executed by the following direct mechanism g_2 , and mechanism g_2 is described as the table.

	R_2	R'_2
R_1	a	d
R'_1	c	b

Prove that this equilibrium result, which cannot be directly displayed, is not Pareto efficient.

Exercise 18.19 Prove or disprove whether the following social choice correspondence satisfies Maskin monotonicity.

1. Weak Pareto effective correspondence.

2. Condor correspondence.
3. Personal rational configuration set.
4. Valas correspondence.
5. Lindal correspondence.
6. Constrained Valas correspondence.
7. Constrained Lindal correspondence.
8. To the above social choice correspondence which does not satisfy Maskin monotonicity, what kind of restriction to the domain of definition can make it satisfy Maskin monotonicity? Prove your claim.

Exercise 18.20 In the King Solomon case, because social goals do not satisfy Maskin monotonicity, it is inoperable in the sense of Nash equilibrium solution. So can the problem of King Solomon be solved by refining or approximating Nash Implementation? Give your ideas.

Exercise 18.21 If the condition, no negation right, is not established, Maskin monotonicity does not guarantee that the social choice can be fully implemented by Nash. Take an example to prove the above conclusion.

Exercise 18.22 Is the non-veto condition a necessary condition for social choice to be fully implemented by Nash? If so, give proof; if not, give counterexample.

Exercise 18.23 (Borda rule) There are N individuals in a economy, suppose X contains a limited number of options. And for any option x , $B^i(x)$ (Borda count) means the number of options that are not as good as x in X for individual i , Borda rule is:

$$xRy \iff \sum_{i=1}^N B^i(x) \geq \sum_{i=1}^N B^i(y).$$

1. Prove that this correspondence satisfies the condition of Maskin monotonicity.
2. Does such a correspondence satisfy the condition of no negation right?

Exercise 18.24 Assume $N = \{1, 2, 3\}$, $E = \{e, \bar{e}\}$, $A = \{a, b, c\}$. The specific preferences are as follows:

Under the condition of e , $1 : b \succ a \succ c$; $2 : a \succ c \succ b$; $3 : a \succ c \succ b$.

Under the condition of \bar{e} , $1 : b \succ c \succ a$; $2 : c \succ a \succ b$; $3 : c \succ a \succ b$.

For any $\theta \in E$, define the social choice rule F as following:

- (i) $a \in F(\theta)$ if and only if a is better than b by the majority vote rule;
- (ii) $b \in F(\theta)$ if and only if b is better than a by the majority vote rule;
- (iii) $c \in F(\theta)$ if and only if $\{x \in A \mid x \succeq_i^\theta c\} = A$ for any $i \in N$ is true.

Answer the following question:

1. Does F satisfy Maskin monotonicity? Description by definition.
2. Does F satisfy no negation right? Description by definition.

Exercise 18.25 Assume $N = \{1, 2, 3, 4\}$, there are 4 inseparable items a_1, a_2, a_3, a_4 . For any $i \in N$, the initial endowment is $w_i = a_i$, $E = \{e_1, e_2\}$. The specific preferences are as follows:

Under the condition e_1 , 1 : $a_1 \succ a_2 \succ a_3 \succ a_4$; 2 : $a_2 \succ a_1 \succ a_3 \succ a_4$;

3 : $a_3 \sim a_4 \succ a_1 \succ a_2$; 4 : $a_3 \sim a_4 \succ a_2 \succ a_1$.

Under the condition e_2 , 1 : $a_1 \succ a_2 \succ a_3 \succ a_4$; 2 : $a_2 \succ a_1 \succ a_3 \succ a_4$;

3 : $a_1 \succ a_2 \sim a_3 \succ a_4$; 4 : $a_3 \succ a_4 \succ a_2 \succ a_1$.

The social goal is to allocate the 4 items to 4 individuals, each with 1 item. Specifically, the social goal is a response $\sigma : N \rightarrow A$. Let Z represent a collection of all possible configurations. The configuration rule F meets the requirements of individual rationality, which is: for any $e_i \in E$, $F(e_i) = \{\sigma \in Z \mid \sigma_j \succeq_j^{e_i} w_j, \text{ for any } e_i \in E, j \in N\}$. Answer the following questions.

1. Does F satisfy Maskin monotonicity? Description by definition.
2. Does F satisfy no negation right? Description by definition.

Exercise 18.26 Assume $N = \{1, 2, 3\}$, $E = \{e, \bar{e}\}$, $A = \{a, b, c\}$. The specific preferences are as follows:

Under the condition e , 1 : $a \succ b \succ c$; 2 : $c \succ a \succ b$; 3 : $b \succ c \succ a$.

Under the condition \bar{e} , 1 : $a \succ b \succ c$; 2 : $c \succ a \succ b$; 3 : $c \succ b \succ a$.

The social choice rule F is to choose according to the rule of majority voting, and definite result $F(e) = \{a, b, c\}$, $F(\bar{e}) = \{c\}$. Does F Nash executable? Give reasons.

Exercise 18.27 (Moore and Repullo, 1990) Considering the Nash implementation of two participants. $N = \{1, 2\}$, the following condition is called μ_2 .

μ_2 condition: If the μ condition of the 18.8.6 satisfied as defined in the textbook, the following additional condition is met:

for any $(a, e', b, \hat{e}) \in (A \times E \times A \times E)$ satisfies $a = F(e'), b = F(\hat{e})$, there exists a $c = C(a, e', b, \hat{e}) \in C_1(a, e') \cap C_2(b, \hat{e})$, making any $e^* \in E$, satisfying the following conditions:

$$\text{if } c \in C_1(a, e') \subseteq L_1(c, e^*), c \in C_2(b, \hat{e}) \subseteq L_2(c, e^*), \text{ then } c \in F(e^*).$$

Prove: in a two-person environment, F is performed by Nash if and only if μ_2 condition is satisfied.

Exercise 18.28 (Moore and Repullo, 1990) Consider the implementation of Nash in a two-person environment. The social choice rule F satisfies the restrictive veto nature (restricted veto power), if for all $i \in I, e \in E, a \in A$, exist $b \in \text{range}(F) \equiv \{\hat{a} \in A : \hat{a} = F(e), \text{ there is } e \in E\}$, there are:

$$\text{if } A \subseteq \bigcap_{j \neq i} L_j(a, e), \text{ at the same time } a \succeq_i b, \text{ then } a \in F(e).$$

We call the result z is a bad result, if for all $e \in E$ and all $a \in F(e)$, there are $a \succ_i(e)z$, and $\succ_i(e)$ is the preference relationship of participant i under the environment e .

Prove: in the two person environment, if F satisfies the nature of the Makin monotonicity, and satisfies the nature of the restrictive veto, and at the same time, there is a difference result, then F can be Nash execution or condition μ_2 is established.

Exercise 18.29 Try to give a concrete structure to prove Groves-Lydiard mechanism may not be individual rational and individual feasible.

Exercise 18.30 (Concealing endowment mechanism, Tian, 1993) Consider a public economic environment consisting of one private good x_i , K public good y , $n \geq 3$ individuals, and the production function $y = f(v) = v$, written as E . Suppose the preference relationship \succ_i of participant i is concave and strongly monotone with respect to private goods, and for all $i \in N, (x_i, y) P_i(x'_i, y')$, there are $x_i \in \mathcal{R}_{++}, x'_i \in \partial \mathcal{R}_+, \text{ and } y, y' \in \mathcal{R}_+^K$. The $\partial \mathcal{R}_+^m$ is the boundary of \mathcal{R}_+^m .

For $i \in N$, define the information space as

$$M_i = (0, 1] \times (0, \hat{w}_i] \times \mathcal{R}^K \times \mathcal{R}^K,$$

Its general elements can be represented as $(\delta_i, w_i, \phi_i, y_i)$, written as $M = \prod_{i=1}^n M_i$. The personalized price is defined as $q_i(\mathbf{m}) = \frac{1}{n} + m_{i+1} - m_{i+2}$. Define the correspondence $B: M \rightarrow 2^{\mathcal{R}_+^K}$,

$$B(\mathbf{m}) = \{y \in \mathcal{R}_+^K : (1 - \delta(\mathbf{m}))w_i - q_i(\mathbf{m}) \cdot y \geq 0 \forall i \in N\},$$

the $\delta(\mathbf{m}) = \min\{\delta_1, \dots, \delta_n\}$. Define the result function of public goods as $Y: M \rightarrow B$,

$$Y(\mathbf{m}) = \{y : \min_{y \in B(\mathbf{m})} \|y - \tilde{y}\|\},$$

and $\tilde{y} = \sum_{i=1}^n y_i$. For any i , define the tax function $T_i: M \rightarrow \mathcal{R}$,

$$T_i(\mathbf{m}) = q_i(\mathbf{m}) \cdot Y(\mathbf{m}).$$

It can be seen that

$$\sum_{i=1}^n T_i(\mathbf{m}) = q(\mathbf{m}) \cdot Y(\mathbf{m}).$$

The result function of private goods $X(\mathbf{m}): M \rightarrow \mathcal{R}_+$,

$$X_i(\mathbf{m}) = w_i - q_i(\mathbf{m}) \cdot Y(\mathbf{m}).$$

1. Prove that the correspondence $B: M \rightarrow 2^{\mathcal{R}_+^K}$ is a nonempty compact convex continuous correspondence.
2. Prove that $Y(\mathbf{m})$ is a Single-valued continuous function on M .
3. Prove that the above mechanism is a continuous and feasible mechanism.
4. Prove that the above mechanism implements the Lindal correspondence completely on the E .

Exercise 18.31 (Valas corresponding to continuous and feasible complete execution, Tian, 1992) Consider a pure exchange economy $e = (\{X_i, w_i, \succ_i\})$, let E be the set of all the economic classes that make Valas equilibrium exist. Define a continuous mechanism for a fully viable (i.e. individual feasible and balanced configuration) is as follows:

For $i \in N$, define the information space as

$$M_i = \mathcal{R}_{++}^L \times \mathcal{R}^{nL},$$

its general elements can be represented as $m_i = (p_i, x_{i1}, \dots, x_{in})$, written as $M = \prod_{i=1}^n M_i$. Define the price vector function $p: M \rightarrow \mathcal{R}_{++}^L$,

$$p(\mathbf{m}) = \begin{cases} \sum_{i=1}^n \frac{a_i}{a} p_i, & \text{if } a > 0, \\ \sum_{i=1}^n \frac{1}{n} p_i, & \text{if } a = 0, \end{cases}$$

In the function, $a_i = \sum_{j, k \neq i} \|p_j - p_k\|$, $a = \sum_{i=1}^n a_i$ and $\|\cdot\|$ represent the Euclidean module.

Define the correspondence of individual feasibility and balance $B: M \rightarrow \mathcal{R}_+^{nL}$,

$$B(\mathbf{m}) = \{x \in \mathcal{R}_+^{nL} : \sum_{i=1}^n x_i = \sum_{i=1}^n w_i \text{ \& } p(\mathbf{m}) \cdot x_i = p(\mathbf{m}) \cdot w_i, \forall i \in N\}.$$

Let $\tilde{x}_j = \sum_{i=1}^n x_{ij}$, $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$. The result function of private goods is $X: M \rightarrow \mathcal{R}_+^{nL}$ given by the following function

$$X(\mathbf{m}) = \{y \in \mathcal{R}_+^{nL} : \min_{y \in B(\mathbf{m})} \|y - \tilde{x}\|\}.$$

1. Prove the price vector function $p : M \rightarrow \mathcal{R}_{++}^L$.
2. Prove the correspondence $B : M \rightarrow 2^{\mathcal{R}_+^K}$ is a nonempty compact convex continuous correspondence.
3. Prove that $X(\mathbf{m})$ is single-valued continuous function on M .
4. Prove: for every economy $e \in E$, each Nash equilibrium result of this mechanism is a Walras equilibrium configuration.
5. Prove: for every economy $e \in E$, each Walras equilibrium configuration is a result of some Nash equilibrium of this mechanism. In this way, from the problem to the 4 and 5, the above fully feasible mechanism on the full implementation of the Walras correspondence on E .

Exercise 18.32 A monopolist known as a durable commodity whose production cost is 0, and faced with continue consumers $[0, 1]$. Of all consumers, the value of the durable goods to half of the population is v_H , and the value to the other half is v_L , $v_H > v_L$. The monopolist can provide products at time $t = 1, 2$ and in prices p_1, p_2 . Since the products are durable goods, any consumer can choose to buy in the first period, or in the second period, or even choose not to buy in both periods, but they will not buy in both periods. Using δ represents the common discount factor. Therefore, the profit of the Monopoly profit is $\pi_1 + \delta\pi_2$. For the consumer with value v , if he chooses to buy at first period, the surplus is $v - p_1$; if he chooses to buy at second period, the surplus is $\delta(v - p_2)$; if he does not buy anything at two periods, the surplus is 0.

1. Suppose an enterprise can set a single price, that is $p_1 = p_2 = p$. What are the optimal prices and the corresponding profits? item Assume that the monopolist can first fix a price sequence, which is (p_1, p_2) , consumers can decide whether and when to buy according to the price sequence. Give the optimal price sequence and the corresponding profit.
2. In the problem (2), is the optimal price dynamically consistent? That is, when the second phase comes, the monopolist has a motive to change p_2 ? Please give some explanation.
3. Now suppose in the first period, the monopolist sets a price at p_1 , which each consumer decides whether to buy or not. In the second period, the monopolist sets a new price p_2 and consumers then decide whether to buy or not. Solve the subgame perfect equilibrium price, and the corresponding monopoly profits.
4. Finally, assume that before the first phase, the monopolist can produce a certain quantity of the commodity and destroy its capacity. Can monopolists benefit from this strategy? Please explain and give the profit to the monopolist.

Exercise 18.33 (Moore and Repullo, 1988) Consider a economy with two people. Suppose the economy has two condition $E = \{C, L\}$, when $e = C$, the preference of two individuals are Cobb-Douglas utility function, the allocation is $f(C) \in A$, when $e = L$, both individual preferences are fully complementary utility functions (or in the form of a Lyonef function), the allocation is $f(L) \in A$, suppose $A = \{f(C), f(L), x, y\}$, the condition and allocation can be seen in picture 18.6. Consider the following dynamic game g .

First stage: participant 1 provides report r_1 , if $r_1 = L$, the game is over, the result is $f(L)$; if $r_1 = C$, the game goes into the second stage.

Second stage: participant 2 provides report $r_2 = C$, the game is over, the result is $f(C)$, otherwise the game goes into the third stage.

Third stage: participant 1 chooses a result from $\{x, y\}$, the game is over.

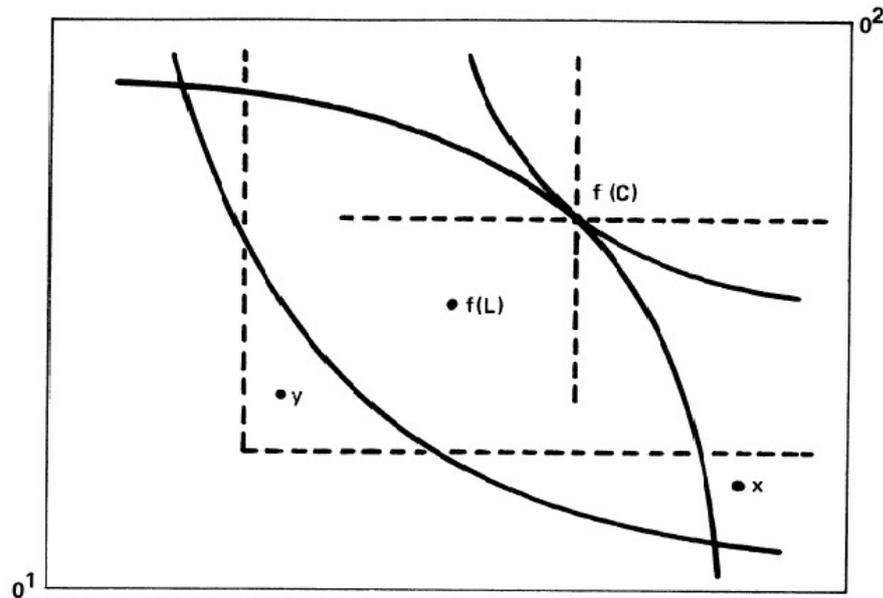


Figure 18.5: Implementation of Subgame Refining equilibrium

Prove: the above social selection rule f can be implemented by dynamic game g with subgame refined equilibrium.

Exercise 18.34 (Jackson, 1992) For environment (N, A, E) , $N = \{1, \dots, n\}$ is the set of participants, a mechanism $\Gamma = (M, g)$ is consist of the information space of participants $M = M^1 \times \dots \times M^n$ and the result function $g : M \rightarrow A$. We say that m^i is **Non-inferior information**, if there does not exist signal m^i weakly dominant of m^i . Written all the non-inferior signal sets of mechanism Γ under the environment e as $U(\Gamma, e)$, all the Nash Equilibriums are written as $N(\Gamma, e)$, all the non-inferior Nash equilibriums are written as $UN(\Gamma, e) =$

$U(\Gamma, e) \cap N(\Gamma, e)$. The social choice rule $F : E \rightarrow A$ can be called as **non-inferior Nash equilibrium execution**, if there exist mechanism (M, g) making $F(e) = g(UN(\Gamma, e)), \forall e \in E$.

Assume $N = \{1, 2\}$, $A = \{a, b\}$, $E = \{(R^1, R^2), (\bar{R}^1, R^2)\}$, $aP^1b, b\bar{P}^1aP^2b$. The social choice rule is $F(R^1, R^2) = b$ and $F(\bar{R}^1, R^2) = a$. The following mechanism (M, g) is described in picture 18.7.

1. Prove that F does not satisfy the Pareto efficiency and the Maskin monotonicity, which means it cannot be carried out by Nash.
2. Prove that F can be executed by mechanism (M, g) as non-inferior Nash.

Exercise 18.35 (Matsushima, 1988; Abreu and Sen, 1991; Diamantaras, 2009) For the environment (N, X', E) , $A' = \{a_1, a_2, \dots, a_K\}$ is a set of socially selectable outcomes. A is the set of all lottery tickets defined on A' , which is $A = \{\mathbf{x} \in [0, 1]^K : \sum_{k=1}^K x_k = 1\}$.

We call the environment not **Completely indifferent** (no total indifference), if for every i and e , there are two results $a, a' \in A'$ in A' , which makes $u_i(a, e) > u_i(a', e)$, and u^i is the utility function of participant i .

We call the ϵ between two social selection rules F and H are close, if for any $e \in E$, there is a one-to-one mapping $\tau : F(E) \rightarrow H(E)$ satisfies $|\mathbf{x} - \tau(\mathbf{x})| = \sqrt{\sum_{k=1}^K (x_k - \tau(x)_k)^2} < \epsilon$, and for any $\mathbf{x} \in F(e)$.

A social choice rule F is called **approximate to Nash execution**, if there is a social rule G which can be executed by Nash and the ϵ between G and F is close.

F is called **ordinal** (ordinal), if for $F(e) \neq F(e')$, there is a participant i and $\mathbf{x}, \mathbf{x}' \in A$ making $u^i(\mathbf{x}, e) \geq u^i(\mathbf{x}', e), u^i(\mathbf{x}', e') > u^i(\mathbf{x}, e')$.

Prove: if the number of participants is over 3, $|N| \geq 3$, The environment is not completely indifferent, at this time arbitrary ordinal social selection rules can be similar to Nash execution.

		M^2									
		m^2									
	m^1	<i>b</i>	<i>a</i>	<i>a</i>	<i>a</i>	...	<i>a</i>	<i>a</i>	<i>a</i>	<i>a</i>	...
		<i>b</i>	<i>a</i>	<i>a</i>	<i>a</i>	...	<i>b</i>	<i>b</i>	<i>b</i>	<i>b</i>	...
		<i>b</i>	<i>b</i>	<i>a</i>	<i>a</i>	...	<i>b</i>	<i>b</i>	<i>b</i>	<i>b</i>	...
		<i>b</i>	<i>b</i>	<i>b</i>	<i>a</i>	...	<i>b</i>	<i>b</i>	<i>b</i>	<i>b</i>	...
M^1		⋮	⋮	⋮	⋮		⋮	⋮	⋮	⋮	
	\bar{m}^1	<i>a</i>	<i>b</i>	<i>b</i>	<i>b</i>	...	<i>b</i>	<i>b</i>	<i>b</i>	<i>b</i>	...
		<i>a</i>	<i>a</i>	<i>a</i>	<i>a</i>	...	<i>a</i>	<i>b</i>	<i>b</i>	<i>b</i>	...
		<i>a</i>	<i>a</i>	<i>a</i>	<i>a</i>	...	<i>a</i>	<i>a</i>	<i>b</i>	<i>b</i>	...
		<i>a</i>	<i>a</i>	<i>a</i>	<i>a</i>	...	<i>a</i>	<i>a</i>	<i>a</i>	<i>b</i>	...
		⋮	⋮	⋮	⋮		⋮	⋮	⋮	⋮	
		⋮	⋮	⋮	⋮		⋮	⋮	⋮	⋮	

Figure 18.6: Non-inferior Nash execution

18.14 Reference

Books and Monographs

Bolton, P. and Dewatripont, M. (2005). *Contract Theory*, MIT Press.

Corchón, L. C. (1996). *Theory of Implementation of Socially Optimal Decisions in Economics*, Macmillan Press, Ltd.

Diamantaras, D. (2009), *A Toolbox for Economic Design*, Palgrave Macmillan.

Green, J. R. and Laffont, J.-J. (1979), *Incentives in public decision making*, North-Holland.

Groves, T and Ledyard, J. (1985). "Incentive Compatibility Since 1972", *Information, Incentive, and Economic Mechanisms*, T. Groves, R. Radner, and S. Reiter, eds.

Hurwicz, L. (1979c). "Balanced Outcome Functions Yielding Walrasian and Lindahl Allocations at Nash Equilibrium Points for Two or More Agents", in *General Equilibrium, Growth, and Trade*, Jerry R. Green and Jose A. Scheinkman, eds., Academic Press, New York.

- Hurwicz, L. (1986a). "Incentive Aspects of Decentralization", in *Handbook of Mathematical Economics*, K. J. Arrow and M. D. Intriligator, eds., Vol. III, North Holland.
- Hurwicz, L. (1986b). "On Informational Decentralization and Efficiency in Resource Allocation Mechanism", in *Studies in Mathematical Economics*, S. Reiter, ed., Mathematical Association of America.
- Hurwicz, L. (1986c). "On the Implementation of Social Choice Rules in Irrational Societies", in *Social Choice and Public Decision Making Essays in Honor of Kenneth J. Arrow*, ed., Vol. I, Cambridge University Press.
- Laffont, J. J. and Maskin, E. (1979). "A Differential Approach to Expected Maximizing Mechanisms", in *Aggregation and Revelation of Preferences*, Laffont, J.J., ed., North Holland.
- Laffont, J. J. and Martimort, D. (2002). *The Theory of Incentives: The Principal-Agent Model*, Princeton University Press.
- Lange, O. (1938). "On the Economic Theory of Socialism", Philadelphia: Lippincott.
- Lange, O. and Taylor, F. M. (1938). *On the Economic Theory of Socialism*, B.E. Lippincott, ed., New York.
- Mas-Colell, A., Whinston, M. D. and Green, J. (1995). *Microeconomic*, Oxford University Press, Chapter 14.
- Moulin, H. (1983), *The Strategy of Social Choice*, North-Holland, Amsterdam.
- Postlewaite, A. (1985). "Implementation in Nash Equilibria in Economic Environments," in *Social Goal and Social Organization Essays in Memory of Elisha Pazner*, Hurwicz, L., Schmeidler, D., and Sonnenschein, H. eds., Cambridge University Press.
- Repullo, R. (1986). "The Revelation Principle under Complete and Incomplete Information", in *Economic Organization as Games*, Binmore, K. and P. Dasgupta, eds., Oxford, Basil Blackwell.
- Thomson, W. (1979). "Maximum Strategies and Elicitation of Preferences", in *Aggregation and Revelation of Preferences*, Laffont, J.J. ed., North-Holland.
- Varian, H. R. (1992). *Microeconomic Analysis, Third Edition*, W.W. Norton and Company.

Wolfstetter, E. (1999). *Topics in Microeconomics—Industrial Organization, Auctions, and Incentives*, Cambridge Press, Chapters 8-10.

Papers

Abreu, R. and Sen, A. (1990). "Subgame Perfect Implementation: A Necessary and Almost Sufficient Condition", *Journal of Economic Theory*, Vol. 50, 285-299.

Abreu, R. and Sen, A. (1991). "Virtual Implementation in Nash Equilibrium", *Econometrica*, Vol. 59, 997-1021.

Bergemann, Dirk and Morris, Stephen (2008). "Ex post implementation", *Games and Economic Behavior*, 63: 527-566

Bochet, Olivier (2007). "Nash implementation with lottery mechanisms", *Social Choice and Welfare*, Vol. 28: 111-125.

Calsarniglia, X. (1977). "Decentralized Resource Allocation and Increasing Returns", *Journal of Economic Theory*, Vol. 14, 263-283.

Danilov, V. (1992). "Implementation via Nash Equilibrium", *Econometrica*, Vol. 60, 43-56.

Dasgupta, P., Hammond, P. and Maskin, E. (1979). "The Implementation of Social Choice Rules: Some General Results on Incentive Compatibility", *Review of Economic Studies*, Vol. 46, 185-216.

Duggan, J. (1993). "Virtual Implementation in Bayesian Equilibrium with Infinite Types", *Parts I and II*, Mimeo.

Duggan, J. (1997), "Virtual Bayesian implementation", *Econometrica*, 67:1175-1199.

Dutta, B., and Sen, Arunava (1991), "Implementation under strong equilibria: a complete characterization", *Journal of Mathematical Economics*, 20:49-67.

Green, Jerry and Laffont, Jean-Jacques (1977). "Characterization of Satisfactory Mechanisms for the Revelation of Preferences for Public Goods.", *Econometrica*, 45: 427-438.

Gibbard, A. (1973), "Manipulation of voting schemes: a general result", *Econometrica*, 41: 587-601.

Groves, T. and Loeb, M. (1975). "Incentives and Public Inputs", *Journal of Public Economics*, Vol. 4, 211-226.

- Groves, T. and Ledyard, J. (1977). "Optimal Allocation of Public Goods: A Solution to the Free Rider Problem", *Econometrica*, Vol. 45, No. 4, 783-811.
- Hayek, and von, F. A. (1935). "The Present State of the Debate", in *Collectivist Economic Planning*, F. A. von Hayek, ed., London, 201-243.
- Hayek, and von, F. A. (1945). "The Use of Knowledge in Society", *American Economic Review*, Vol. 35, 519-530.
- Holmstrom, B. (1979), "Groves Scheme on Restricted Domain," *Econometrica*, Vol. 47, 1137-1147.
- Hurwicz, L. (1972). "On Informational Decentralized Systems", in *Decision and Organization*, Radner, R. and C. B. McGuire, eds., in Honor of J. Marschak, North-Holland, 297-336.
- Hurwicz, L. (1979a). "Outcome Function Yielding Walrasian and Lindahl Allocations at Nash Equilibrium Point", *Review of Economic Studies*, Vol. 2, 397-419.
- Hurwicz, L. (1979b). "On Allocations Attainable Through Nash Equilibria", *Journal of Economic Theory*, Vol. 21, No. 1, 140-165.
- Hurwicz, L. (1979d). "Socialism and Incentives: Developing a Framework", *Journal of Comparative Economics*, Vol. 3, 207-216.
- Hurwicz, L. (1999). "Revisiting Externalities", *Journal of Public Economic Theory*, Vol. 1, 225-246.
- Hurwicz, L., Maskin, E. and Postlewaite, A. (1984). "Feasible Implementation of Social Choice Correspondences by Nash Equilibria", Mimeo.
- Hurwicz, L., and D. Schmeidler (1978), "Construction of outcome functions guaranteeing existence and Pareto-optimality of Nash equilibria", *Econometrica*, 46: 1447-1474.
- Hurwicz, L., and Walker, M. (1990), "On the Generic Nonoptimality of Dominant-Strategy Allocation Mechanism: A General Theorem That Includes Pure Exchange Economies", *Econometrica* Vol. 58, 683-704.
- Jackson, M.O. (1992). "Implementation in Undominated Strategies: A Look at Bounded Mechanisms", *Review of Economic Studies*, Vol.59, No. 4, 757-775.
- Jackson, M.O. (2001) "A Crash Course in Implementation Theory", *Social Choice and Welfare*, Vol. 18: 655-708.
- Jackson, M.O. (2003) "Mechanism Theory", Working Paper

- Jordan, J. S. (1982). "The Competitive Allocation Process in Informationally Efficient Uniquely", *Journal of Economic Theory*, Vol. 28, 1-18.
- Laffont, J.J. and Maskin, E. (1980). "A differential approach to dominant strategy mechanisms", *Econometrica*, Vol. 48, No. 6, 1507-1520.
- Lange, O. (1936-1937). "On the Economic Theory of Socialism", *Review of Economic Studies*, Vol. 4.
- Lange, O. (1942). "The Foundations of Welfare Economics", *Econometrica*, Vol. 10, 215-228.
- Ledyard, J., and J. Roberts (1974), "On the incentive problem with public goods", Discussion Paper 116.
- Lerner, A. P. (1944). *Economics of Control*, New York.
- Li, Q., Nakmura, S. and Tian, G. (1995). "Nash Implementation of the Lindahl Correspondence with Decreasing Returns to Scale Technology", *International Economic Review*, Vol. 36, 34-50.
- Liu, L., and Tian, G. (1999). "A Characterization of Optimal Dominant Strategy Mechanisms", *Review of Economic Design*, Vol. 4, 205-218.
- Maskin, E. (1999). "Nash Equilibrium and Welfare Optimality", *Review of Economic Studies*, Vol. 66, 23-38.
- Matsushima, H. (1988). "A New Approach to the Implementation Problem", *Journal of Economic Theory*, Vol. 45, 128-144.
- McKelvey, R. (1989). "Game Forms for Nash Implementation of General Social Correspondence", *Social Choice and Welfare*, Vol. 6, 139-156.
- Mookherjee, D., and Reichelstein, S. (1990). "Implementation via Augmented Revelation Mechanisms", *Review of Economic Studies*, Vol. 57, 453-475.
- Moore, J., and Repullo, R. (1988). "Subgame Perfect Implementation", *Econometrica*, Vol. 56, 1191-1220.
- Moore, J., and Repullo, R. (1990). "Nash Implementation: A Full Characterization", *Econometrica*, Vol. 56, 1083-1099.
- Mount, K., and Reiter, S. (1974). "Informational Size of Message Spaces", *Journal of Economic Theory*, Vol. 8, 161-191.
- Palfrey, T. and Srivastava, S. (1991). "Nash Implementation Using Undominated Strategy", *Econometrica*, Vol. 59, 479-502.

- Peleg, B., (1996a). "A continuous double implementation of the constrained Walrasian equilibrium". *Economic Design* 2, 89-97.
- Peleg, B., (1996b). "Double implementation of the Lindahl equilibrium by a continuous mechanism". *Economic Design*, 2, 311-324
- Postlewaite, A., and Wettstein, D. (1989). "Continuous and Feasible Implementation", *Review of Economic Studies*, Vol. 56, 603-611.
- Reichelstein, S., and S. Reiter (1988), "Game forms with minimal message spaces", *Econometrica*, 56: 661-692.
- Repullo, R. (1987). "A Simple Proof of Maskin's Theorem on Nash Implementation", *Social Choice and Welfare*, Vol. 4, 39-42.
- Saijo, T. (1987), "On constant Maskin monotonic social choice functions", *Journal of Economic Theory*, 42: 382-386.
- Saijo, T. (1988), "Strategy space reduction in Maskin's theorem", *Econometrica*, 56: 693-700.
- Samuelson, PA (1954), "The pure theory of public expenditure", *The review of economics and statistics*, Vol. 36, No. 4, 387-389.
- Samuelson, PA (1955), "Diagrammatic Exposition of a Theory of Public Expenditure", *The review of economics and statistics*, 37, 350-356.
- Satterthwaite, M.A. (1975), "Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions", *Journal of Economic Theory*, 10: 187-217.
- Schmeidler, D. (1980). "Walrasian Analysis via Strategic Outcome Functions", *Econometrica*, Vol. 48, 1585-1593.
- Shao, R. and Zhou, L. (2016a). "Voting and optimal provision of a public good," *Journal of Public Economics*, 134, 35-41
- Shao, R. and Zhou, L. (2016b). "Optimal Allocation of an Indivisible Good," *Games and Economic Behavior*, forthcoming.
- Suh, S.C. (1997), "Double implementation in Nash and strong Nash equilibria", *Social Choice and Welfare* 14: 439-447.
- Tian, G. (1988). "On the Constrained Walrasian and Lindahl Correspondences," *Economics Letters*, 26, 299-303.
- Tian, G. (1989). "Implementation of the Lindahl Correspondence by a Single-Valued, Feasible, and Continuous Mechanism", *Review of Economic Studies*, Vol. 56, 613-621.

- Tian, G. (1990). "Completely Feasible and Continuous Nash-Implementation of the Lindahl Correspondence with a Message Space of Minimal Dimension", *Journal of Economic Theory*, Vol. 51, 443-452.
- Tian, G. (1991). "Implementation of Lindahl Allocations with Nontotal-Nontransitive Preferences", *Journal of Public Economics*, Vol. 46, 247-259.
- Tian, G. (1992). "Implementation of the Walrasian Correspondence without Continuous, Convex, and Ordered Preferences", *Social Choice and Welfare*, Vol. 9, 117-130.
- Tian, G. (1993). "Implementing Lindahl Allocations by a Withholding Mechanism", *Journal of Mathematical Economics*, Vol. 22, 169-179.
- Tian, G. (1994). "Implementation of Linear Cost Share Equilibrium Allocations", *Journal of Economic Theory*, Vol. 64, 568-584.
- Tian, G. (1994a). "On Informational Efficiency and Incentive Aspects of Generalized Ratio Equilibria", *Journal of Mathematical Economics*, Vol. 23, 323-337.
- Tian, G. (1994b). "Implementation of Linear Cost Share Equilibrium Allocations", *Journal of Economic Theory*, Vol. 64, 568-584.
- Tian, G. (1995a). "Nash Implementation of the Lindahl Correspondence with Decreasing Returns to Scale Technology," (with Qi Li and S. Nakamura), *International Economic Review*, 36, 37-52.
- Tian, G. (1995b). "On Nash-Implementation in the Presence of Withholding," (with Qi Li), *Games and Economic Behavior*, 9, 222-233.
- Tian, G. (1995c). "Ratio-Lindahl Equilibria and an Informationally Efficient and Implementable Mixed-Ownership System," (with Qi Li), *Journal of Economic Behavior and Organization*, 26, 391-411
- Tian, G. (1996a). "On the Existence of Optimal Truth-Dominant Mechanisms", *Economics Letters*, Vol. 53, 17-24.
- Tian, G. (1996b). "Continuous and Feasible Implementation of Rational Expectation Lindahl Allocations", *Games and Economic Behavior*, Vol. 16, 135-151.
- Tian, G. (1997). "Virtual Implementation in Incomplete Information Environments with General Sets of Alternatives and Types", *Journal of Mathematical Economics*, Vol. 28, 313-339.

- Tian, G. (1999a). "Double Implementation in Economies with Production Technologies Unknown to the Designer", *Economic Theory*, Vol. 13, 1999, 689-707.
- Tian, G. (1999b). "Bayesian Implementation in Exchange Economies with State Dependent Preferences and Feasible Sets", *Social Choice and Welfare*, Vol. 16, 99-119.
- Tian, G. (2000a). "Double Implementation of Lindahl Allocations by a Continuous and Feasible Mechanism", *Social Choice and Welfare*, Vol. 17, 125-141.
- Tian, G. (2000b). "Incentive Mechanism Design for Production Economies with Both Private and Public Ownership", *Games and Economic Behavior*, Vol. 33, 294-320.
- Tian, G. (2000c). "Implementation of Balanced Linear Cost Share Equilibrium Solution in Nash and Strong Nash Equilibria", *Journal of Public Economics*, Vol. 76, 239-261.
- Tian, G. (2000d). "Double Implementation of Linear Cost Share Equilibrium Allocations", *Mathematical Social Sciences*, Vol. 40, 175-189.
- Tian, G. (2000e). "A Unique Informationally Efficient Allocation Mechanism in Economies with Public Goods", Mimeo.
- Tian, G. (2003). "A Solution to the Problem of Consumption Externalities", *Journal of Mathematical Economics*, Vol.39, 831-847.
- Tian, G. (2004). "A Unique Informationally Efficient Allocation Mechanism in Economies with Consumption Externalities", *International Economic Review*, Vol. 45, 79-111.
- Tian, G. (2005). "Implementation in Production Economies with Increasing Returns", *Mathematical Social Sciences*, Vol. 49, 309-325.
- Tian, G. (2006). "The Unique Informational Efficiency of the Competitive Mechanism in Economies with Production", *Social Choice and Welfare*, Vol. 26, 155-182.
- Tian, G. (2009a). "Implementation in Economies with Non-Convex Production Technologies Unknown to the Designer", *Games and Economic Behavior*, Vol. 66, 526-545.
- Tian, G. (2009b). "Implementation of Pareto Efficient Allocations", *Journal of Mathematical Economics*, Vol. 45, 113-123.

- Tian, G. (2010). "Implementation of Marginal Cost Pricing Equilibrium Allocations with Transfers in Economies with Increasing Returns to Scale", *Review of Economic Design*, Vol. 14, 163-184.
- Tian, G., and Li, Q. (1991). "Completely Feasible and Continuous Implementation of the Lindahl Correspondence with Any Number of Goods", *Mathematical Social Sciences*, Vol. 21, 67-79.
- Tian, G., and Li, Q. (1994). "An Implementable and Informational Efficient State-Ownership System with General Variable Returns", *Journal of Economic Theory*, Vol. 64, 268-297.
- Tian, G., and Li, Q. (1995a). "On Nash-Implementation in the Presence of Withholding", *Games and Economic Behavior*, Vol. 9, 222-233.
- Tian, G., and Li, Q. (1995b). "Ratio-Lindahl Equilibria and an Informationally Efficient and Implementable Mixed-Ownership System", *Journal of Economic Behavior and Organization*, Vol. 26, 391-411.
- Walker, M. (1978). "A Note on the Characterization of Mechanisms for the Revelation of Preferences", *Econometrica*, Vol. 46, 147-152.
- Walker, M. (1980). "On the Nonexistence of a Dominant Strategy Mechanism for Making Optimal Public Decisions", *Econometrica*, Vol. 48, 1521-1540.
- Walker, M. (1981). "A Simple Incentive Compatible Scheme for Attaining Lindahl Allocations", *Econometrica*, Vol. 49, 65-71.

Chapter 19

Incomplete Information and Bayesian-Nash Implementation

19.1 Introduction

This chapter tends to analyze the implementation problem with incomplete information. The so-called incomplete information refers to that agents do not know each other's preferences, technology, endowments and other information.

We consider the implementation of social choices under the concept of two more specific solutions. One is the concept of dominant equilibrium solution and the other is Nash equilibrium solution and its refinements. Dominant equilibrium is the strongest concept about the self-interest behavior of individuals. This concept requires the least amount of information for both designers and agents. Besides own information, it does not need to know any other participants' information.

Although the mechanism that dominant equilibrium used as a solution is able to make people truly show their preferences, in which the VCG mechanism can even effectively provide public goods or inseparable items, the truth-telling mechanism cannot implement Pareto optimal allocation. Nash strategic equilibrium and its refinements are all very weak concepts about self-interest behavior solution, which have a strict requirement to information. Each economic person not only requires to know their own economic characteristics, but also requires to know the economics characteristics of all other participants. However, although the Pareto optimal allocation can be implemented under the mechanism that Nash equilibrium used as a solution, it is necessary to abandon the requirement that people speak the truth.

In this way, under the concept of these two solutions, whether Hwowitz's impossibility theorem, VCG mechanism, or Nash implementable mechanism all show that the Pareto optimal allocation of resources and truth-

telling preferences is generally impossible to achieve at the same time. Then, is there a concept of solution that, in a weaker sense, can the Pareto optimal allocation of resources and truth-telling preferences can be achieved at the same time? The answer is YES. As long as the Bayesian incentive compatibility constraint is established (when others tell the truth, I tell the truth is the optimal choice), the expected external mechanism shows that telling the truth and Pareto optimal allocation can be achieved at the same time.

We will use Bayesian-Nash equilibrium solution concept that introduced by Harsanyi to describe self-interested behavior of participants. Nash, Selten and Harsanyi received the 1994 Nobel Prize in economics for their invention of Nash equilibrium, sub-game refinement Nash equilibrium and Bayesian-Nash equilibrium respectively. Bayesian solution assumes that although each agent does not know economic characteristics of the others, he/she knows its probability distribution. The corresponding implementation is Bayesian-Nash implementation. People can still design many Bayesian incentive compatibility mechanisms and fully implementability that depicts general social selection rules.

In the following part, we will first give its basic analysis framework, introducing basic model with incomplete information, Bayesian-Nash equilibrium and the basic concepts and definitions of Bayesian implementation in various senses. For simplicity, until the end of this chapter, we are all considering Bayesian incentive compatibility and Bayesian implementation problems of social selection functions (rather than the set of social selection functions). In the last two sections of this chapter, we will consider the Bayesian implementation problem and ex post implementable problem of the social selection set.

19.2 Basic Analytical Framework

19.2.1 Model

Throughout we follow the notation introduced in the previous chapter. Let Z denote the set of outcomes, $A \subseteq Z$ the set of feasible outcomes, and Θ_i agent i 's space of types. Unless indicated explicitly, for simplicity, we assume each individual's preferences are given by parametric utility functions with private values, denoted by $u_i(x, \theta_i)$, where $x \in Z$ and $\theta_i \in \Theta_i$. Assume that all agents and the designer know that the vector of types, $\theta = (\theta_1, \dots, \theta_n)$ is distributed according to $\varphi(\theta)$ a priori on a set Θ .

Each agent knows his own type θ_i , and therefore computes the conditional distribution of the types of the other agents:

$$\varphi(\theta_{-i}|\theta_i) = \frac{\varphi(\theta_i, \theta_{-i})}{\int_{\Theta_{-i}} \varphi(\theta_i, \theta_{-i}) d\theta_{-i}}.$$

As usual, a mechanism is a pair, $\Gamma = \langle M, h \rangle$. Given m with $m_i : \Theta_i \rightarrow M_i$, agent i 's expected utility at θ_i is given by

$$\begin{aligned} \Pi_{\langle M, h \rangle}^i(m(\theta); \theta_i) &\equiv E_{\theta_{-i}}[u_i(h(m_i(\theta_i), m_{-i}(\theta_{-i})), \theta_i) \mid \theta_i] \\ &= \int_{\Theta_{-i}} u_i(h(m(\theta)), \theta_i) \varphi(\theta_{-i} \mid \theta_i) d\theta_{-i}. \end{aligned} \quad (19.2.1)$$

A strategy $m^*(\cdot)$ is a *Bayesian-Nash equilibrium (BNE)* of $\langle M, h \rangle$ if for all $\theta_i \in \Theta_i$,

$$\Pi^i(m^*(\theta_i); \theta_i) \geq \Pi^i(m_i, m_{-i}^*(\theta_{-i}); \theta_i) \quad \forall m_i \in M_i.$$

That is, if player i believes that other players are using strategies $m_{-i}^*(\cdot)$ then he maximizes his expected utility by using strategy $m_i^*(\cdot)$. Denote by $B(\theta, \Gamma)$ the set of all Bayesian-Nash equilibria of the mechanism.

Remark 19.2.1 In the present private value incomplete information setting, a message $m_i : \Theta_i \rightarrow M_i$ is a dominant strategy strategy for agent i in mechanism $\Gamma = \langle M, h \rangle$ if for all $\theta_i \in \Theta_i$ and all possible strategies $m_{-i}(\theta_{-i})$,

$$E_{\theta_{-i}}[u_i(h(m_i(\theta_i), m_{-i}(\theta_{-i})), \theta_i) \mid \theta_i] \geq E_{\theta_{-i}}[u_i(h(m'_i, m_{-i}(\theta_{-i})), \theta_i) \mid \theta_i]. \quad (19.2.2)$$

for all $m'_i \in M_i$.

Since condition (19.2.2) holding for all θ_i and all $m_{-i}(\theta_{-i})$ is equivalent to the condition that, for all $\theta_i \in \Theta_i$,

$$u_i(h(m_i(\theta_i), m_{-i}), \theta_i) \geq u_i(h(m'_i, m_{-i}), \theta_i) \quad (19.2.3)$$

for all $m'_i \in M_i$ and all $m_{-i} \in M_{-i}$,¹ this leads the following definition of dominant equilibrium.

Thus, for such a private value model, a message m^* is a dominant strategy equilibrium of a mechanism $\Gamma = \langle M, h \rangle$ if for all i and all $\theta_i \in \Theta_i$,

$$u_i(h(m_i^*(\theta_i), m_{-i}), \theta_i) \geq u_i(h(m'_i(\theta_i), m_{-i}), \theta_i)$$

for all $m'_i \in M_i$ and all $m_{-i} \in M_{-i}$, which is the same as that in the case of complete information.

Remark 19.2.2 It is clear every dominant strategy equilibrium is a Bayesian-Nash equilibrium, but the converse may not be true. Bayesian-Nash equilibrium requires more sophistication from the agents than dominant strategy equilibrium. Each agent, in order to find his optimal strategy, must have a correct prior $\varphi(\cdot)$ over states, and must correctly predict the equilibrium strategies used by other agents.

¹(19.2.2) implies (19.2.3) by noting $m_{-i}(\theta_{-i}) = m_{-i}$, and (19.2.3) implies (19.2.2) because $E_{\theta_{-i}}[u_i(h(m_i, m_{-i}(\theta_{-i})), \theta_i) \mid \theta_i] = u_i(h(m_i, m_{-i}), \theta_i)$ for all $m_i \in M_i$.

19.2.2 Bayesian Incentive-Compatibility and Bayesian Implementation

To see specifically what is implementable in Bayesian-Nash equilibrium (BNE), till the last two sections, the social choice goal is given by a single-valued social choice function rather than the set of social choice functions. We focus on the issues of Bayesian Incentive-Compatibility and Bayesian Implementation of a social choice function. Furthermore, by Revelation Principle below, without loss of generality, we can focus on direct revelation mechanisms.

Let $f : \Theta \rightarrow A$ be a social choice function. Since at Bayesian-Nash equilibrium, every agent reaches his interim utility maximization, Bayesian implementation sometimes is called interim implementation. Like implementation in dominant strategy and Nash strategy, we similarly have the strong Bayesian-Nash implementation and Bayesian-Nash implementation.

Definition 19.2.1 A mechanism $\Gamma = \langle M, h \rangle$ is said to *strongly Bayesian implement* a social choice function f on Θ if for all Bayesian-Nash equilibrium m^* , we have $h(m^*(\theta)) = f(\theta)$, $\forall \theta \in \Theta$. If such a mechanism exists, we call f is *strongly Bayesian implementable*.

When there are multiple Bayesian-Nash equilibria, it may result in some undesirable equilibrium outcomes, i.e., it is not equal to f (we will provide such an example). Then, we have the following weaker concept of Bayesian implementability.

Definition 19.2.2 A mechanism $\Gamma = \langle M, h \rangle$ is said to *Bayesian implement* a social choice function f on Θ if there exists a Bayesian-Nash equilibrium m^* , such that $h(m^*(\theta)) = f(\theta)$, $\forall \theta \in \Theta$. If such a mechanism exists, we call f is *Bayesian implementable*.

Like dominant implementation, we will see that a social choice function is Bayesian implementable if and only if it is truthfully Bayesian implementable.

Definition 19.2.3 We call a social choice function f is *truthfully Bayesian implementable* or *Bayesian incentive compatible*, if truth-telling: $m^*(\theta) = \theta$, $\forall \theta \in \Theta$ is a Bayesian-Nash equilibrium of revelation mechanism $\Gamma = (\Theta, f)$, i.e.,

$$E_{\theta_{-i}}[u_i(f(\theta_i, \theta_{-i})), \theta_i] \geq E_{\theta_{-i}}[u_i(f(\theta'_i, \theta_{-i})), \theta_i], \quad \forall i, \forall \theta_i, \theta'_i \in \Theta_i.$$

Bayesian incentive-compatibility means that for social choice rule f , every agent will report his type truthfully provided *all* other agents are employing their truthful strategies and thus every truthful strategy profile is a Bayesian equilibrium of the direct mechanism $\langle \Theta, f \rangle$. Notice that Bayesian incentive compatibility does not say what is the best response of an agent

when other agents are not using truthful strategies. When a mechanism has multiple equilibria, non-truth-telling may be a Bayesian-Nash equilibrium, resulting in an undesirable outcome that is not equal to f . When Bayesian incentive compatibility is just a basic requirement, it in general only ensures a social choice rule is truthfully Bayesian implementable but not strongly Bayesian implementable, i.e., it may also contain some undesirable equilibrium outcomes when a mechanism has multiple equilibria.

Similarly, we have the following revelation principle.

Proposition 19.2.1 (Revelation Principle) *A social choice rule $f(\cdot)$ is implementable in Bayesian-Nash equilibrium (in short, BNE) if and only if it is truthfully implementable in BNE.*

Proof. The proof is the same as before: Suppose that there exists a mechanism $\Gamma = (M_1, \dots, M_n, g(\cdot))$ and an equilibrium strategy profile $m^*(\cdot) = (m_1^*(\cdot), \dots, m_n^*(\cdot))$ such that $g(m^*(\cdot)) = f(\cdot)$ and $\forall i, \forall \theta_i \in \Theta_i$. We then have

$$E_{\theta_{-i}}[u_i(g(m_i^*(\theta_i), m_{-i}^*(\theta_{-i})), \theta_i) \mid \theta_i] \geq E_{\theta_{-i}}[u_i(g(m_i', m_{-i}^*(\theta_{-i})), \theta_i) \mid \theta_i], \forall m_i' \in M_i.$$

One way to deviate for agent i is by pretending that his type is $\hat{\theta}_i$ rather than θ_i , i.e., sending message $m_i' = m_i^*(\hat{\theta}_i)$. This gives

$$E_{\theta_{-i}}[u_i(g(m_i^*(\theta_i), m_{-i}^*(\theta_{-i})), \theta_i) \mid \theta_i] \geq E_{\theta_{-i}}[u_i(g(m_i^*(\hat{\theta}_i), m_{-i}^*(\theta_{-i})), \theta_i) \mid \theta_i], \forall \hat{\theta}_i \in \Theta_i.$$

But since $g(m^*(\theta)) = f(\theta)$, $\forall \theta \in \Theta$, we must have $\forall i, \forall \theta_i \in \Theta_i$, there is

$$E_{\theta_{-i}}[u_i(f(\theta_i, \theta_{-i}), \theta_i) \mid \theta_i] \geq E_{\theta_{-i}}[u_i(f(\hat{\theta}_i, \theta_{-i}), \theta_i) \mid \theta_i], \forall \hat{\theta}_i \in \Theta_i.$$

Although Bayesian incentive-compatibility is a necessary and sufficient for f to be Bayesian implementable, it is not a sufficient condition for f to be strongly Bayesian implementable. Strong Bayesian implementability requires all BNE outcomes are equal to $f(\theta)$. The goal for designing a mechanism is to reach a desirable equilibrium outcome, but it may also result in an undesirable equilibrium outcome. Thus, while the incentive compatibility requirement is central, it may not be sufficient for a mechanism to give all of desirable outcomes. The severity of this multiple equilibrium problem has been exemplified by Demski and Sappington (JET, 1984), Postlewaite and Schmeidler (JET, 1986), Repullo (Econometrica, 1988), and others. The implementation problem involves designing mechanisms to ensure that all equilibria result in desirable outcomes.

Under what conditions, does there exist a mechanism that has a unique Bayesian incentive-compatible equilibrium outcome? If such a mechanism exists, by definition, we know a social choice function f is strongly Bayesian implementable. One such condition is the nonexistence of undominated Bayesian equilibrium.

Definition 19.2.4 A message $m \in M$ is weakly dominated, if there is an agent i , θ_i , and another Bayesian message $m'_i(\theta_i) \neq m_i$, such that for all $m_{-i} \in M_{-i}$,

$$\Pi^i(m_i(\theta_i), m_{-i}; \theta_i) \leq \Pi^i(m'_i(\theta_i), m_{-i}; \theta_i)$$

with strict inequality for some $m_{-i} \in M_{-i}$.

Definition 19.2.5 $m \in M$ is a *undominated Bayesian equilibrium* of $\Gamma = \langle M, h \rangle$, if it is a Bayesian-Nash equilibrium that is not weakly dominated.

Palfrey and Srivastava(1989, JPE))prove the following proposition.

Proposition 19.2.2 *For the class of private value economic environments, if there exists a mechanism, such that all agents do not have weakly dominated strategies, then any Bayesian incentive-compatible social choice function is strongly Bayesian implementable.*

In the remainder of the chapter, we mainly consider truthful Bayesian implementability of a social choice rule. The last two sections of the chapter will discuss the necessary and sufficient conditions for full Bayesian implementability of a general social choice rule—the set of social choice functions under more general interdependent value model.

19.3 Truthful Implementation of Pareto Efficient Outcomes

Once again, let us return to the quasilinear setting. From the section on dominant strategy implementation in the last chapter, we know that there is, in general, no ex-post Pareto efficient implementation if the dominant equilibrium solution behavior is assumed. However, in quasilinear environments, relaxation of the equilibrium concept from DS to BNE allows us to implement ex post Pareto efficient choice rule $f(\cdot) = (y^*(\cdot), t_1(\cdot), \dots, t_i(\cdot))$, where $\forall \theta \in \Theta$,

$$y(\theta) \in \operatorname{argmax}_{y \in Y} \sum_{i=1}^n v_i(x, \theta_i), \quad (19.3.4)$$

and there is a balanced budget:

$$\sum_{i=1}^n t_i(\theta) = 0. \quad (19.3.5)$$

The Expected Externality Mechanism described below was suggested independently by D' Aspermont and Gerard-Varet (1979) and Arrow (1979) and are also called AGV mechanisms in the literature. This mechanism enables us to have ex-post Pareto efficient implementation under the following additional assumption.

Assumption 19.3.1 *Types are distributed independently: $\phi(\theta) = \prod_i \phi_i(\theta_i), \forall \theta \in \Theta$.*

To see this, take the VCG transfer for agent i and instead of using other agents' announced types, take the expectation over their possible types

$$t_i(\hat{\theta}) = E_{\theta_{-i}} \left[\sum_{j \neq i} v_j(y^*(\hat{\theta}_i, \theta_{-i}), \theta_j) \right] + d_i(\hat{\theta}_{-i})$$

(by the above assumption the expectation over θ_{-i} does not have to be taken conditionally on θ_i). Note that unlike in VCG mechanisms, the first term only depends on agent i 's announcement $\hat{\theta}_i$, and not on other agents' announcements. This is because it sums the expected utilities of agents $j \neq i$ assuming that they tell the truth and given that i announced $\hat{\theta}_i$, and does not depend on the actual announcements of agents $j \neq i$. This means that $t_i(\cdot)$ is less "variable", but on average it will cause i 's incentives to be lined up with the social welfare.

To see that i 's incentive compatibility is satisfied given that agent $j \neq i$ announce truthfully, observe that agent i solves

$$\begin{aligned} \max_{\hat{\theta}_i \in \Theta_i} E_{\theta_{-i}} [v_i(y(\hat{\theta}_i, \theta_{-i}), \theta_i) + t_i(\hat{\theta}_i, \theta_{-i})] \\ = \max_{\hat{\theta}_i \in \Theta_i} E_{\theta_{-i}} \left[\sum_{j=1}^n v_j(y(\hat{\theta}_i, \theta_{-i}), \theta_j) \right] + E_{\theta_{-i}} d_i(\theta_{-i}). \end{aligned} \quad (19.3.6)$$

Again, agent i 's announcement only matters through the decision $y^*(\hat{\theta}_i, \theta_{-i})$. Furthermore, by the definition of the efficient decision rule $y^*(\cdot)$, the designer always chooses $y^*(\cdot)$ to max social surplus (19.3.4) for each realization of $\theta_{-i} \in \Theta_{-i}$. Then, when agent i announces truthfully: $\hat{\theta}_i = \theta_i$, maximization of (19.3.6) is consistent with the social goal (19.3.4). As such, when the social surplus (19.3.4) is maximized, agent i 's expected utility (19.3.6) is maximized by choosing the decision $y^*(\theta_i, \theta_{-i})$, which can be achieved by announcing truthfully: $\hat{\theta}_i = \theta_i$. Therefore, truthful announcement maximizes the agent i 's expected utility as well. Thus, BIC is satisfied.

Remark 19.3.1 The argument relies on the assumption that other agents announce truthfully. Thus, it is in general not a dominant strategy for agent i to announce the truth. Indeed, if agent i expects the other agents to lie, i.e., announce $\hat{\theta}_{-i}(\theta_{-i}) \neq \theta_{-i}$, then agent i 's expected utility is

$$E_{\theta_{-i}} [v_i(y^*(\hat{\theta}_i, \hat{\theta}_{-i}(\theta_{-i})), \theta_i) + \sum_{j \neq i} v_j(y^*(\hat{\theta}_i, \theta_{-i}), \theta_j)] + E_{\theta_{-i}} d_i(\theta_{-i}),$$

which may not be maximized by truthfully announcement.

Furthermore, we can now choose functions $d_i(\cdot)$ so that the budget is balanced. To see this, let

$$\xi_i(\theta_i) = E_{\theta_{-i}}[\sum_{j \neq i} v_j(y^*(\theta_i, \theta_{-i}), \theta_j)],$$

so that the transfers in the Expected Externality mechanism are $t_i(\theta) = \xi_i(\theta_i) + d_i(\theta_{-i})$. We will show that we can use the $d(\cdot)$ functions to "finance" the $\xi(\cdot)$ functions in the following way:

Let

$$d_j(\theta_{-j}) = - \sum_{i \neq j} \frac{1}{n-1} \xi_i(\theta_i).$$

Then

$$\begin{aligned} \sum_{j=1}^n d_j(\theta_{-j}) &= -\frac{1}{n-1} \sum_{j=1}^n \sum_{i \neq j} \xi_i(\theta_i) = -\frac{1}{n-1} \sum_{j \neq i} \sum_{i=1}^n \xi_i(\theta_i) \\ &= -\frac{1}{n-1} \sum_{i=1}^n (n-1) \xi_i(\theta_i) = - \sum_{i=1}^n \xi_i(\theta_i), \end{aligned}$$

and therefore

$$\sum_{i=1}^n t_i(\theta) = \sum_{i=1}^n \xi_i(\theta_i) + \sum_{i=1}^n d_i(\theta_{-i}) = 0.$$

Thus, we have shown that when agents's Bernoulli utility functions are quasilinear and agents' types are statistically independent, there is an ex-post efficient social choice function that is implementable in Bayesian-Nash equilibrium.

19.4 Characterization of DIC and BIC under Linear Mod- e

Consider the quasi-linear environment with the decision set $Y \subset \mathbb{R}^n$, and the type spaces $\Theta_i = [\underline{\theta}_i; \bar{\theta}_i] \subset \mathbb{R}$ for all i . Each agent i 's utility takes the form

$$\theta_i y_i + t_i.$$

Note that these payoffs satisfy SCP.

The decision set Y depends on the application.

Two examples:

1. Allocating a private good: $Z = \{y \in \{0, 1\}^n : \sum_i y_i = 1\}$.
2. Provision of a nonexcludable public good: $Y = \{(q, \dots, q) \in \mathbb{R}^n : q \in \{0, 1\}\}$.

We can fully characterize both dominant incentive compatibility (DIC) and Bayesian incentive compatibility (BIC) social choice rules in this environment.

Let $U_i(\theta) = \theta_i y_i(\theta) + t_i(\theta)$. We first have the following proposition.

Proposition 19.4.1 (DIC Characterization Theorem) *In the linear model, social choice rule $(y(\cdot), t_1(\cdot), \dots, t_I(\cdot))$ is Dominant Incentive Compatible if and only if for all $i \in N$,*

- (1) $y_i(\theta_i, \theta_{-i})$ is nondecreasing in θ_i , (DM);
- (2) $U_i(\theta_i, \theta_{-i}) = U_i(\underline{\theta}_i, \theta_{-i}) + \int_{\underline{\theta}_i}^{\theta_i} y_i(\tau, \theta_{-i}) d\tau, \forall \theta \in \Theta$. (DIC-FOC)

PROOF. Necessity: Dominant strategy incentive compatibility implies that for each $\theta'_i > \theta_i$, we have

$$U_i(\theta_i, \theta_{-i}) \geq \theta_i y_i(\theta'_i, \theta_{-i}) + t_i(\theta'_i, \theta_{-i}) = U_i(\theta'_i, \theta_{-i}) + (\theta_i - \theta'_i) y_i(\theta'_i, \theta_{-i})$$

and

$$U_i(\theta'_i, \theta_{-i}) \geq \theta'_i y_i(\theta_i, \theta_{-i}) + t_i(\theta_i, \theta_{-i}) = U_i(\theta_i, \theta_{-i}) + (\theta'_i - \theta_i) y_i(\theta_i, \theta_{-i}).$$

Thus

$$y_i(\theta'_i, \theta_{-i}) \geq \frac{U_i(\theta'_i, \theta_{-i}) - U_i(\theta_i, \theta_{-i})}{\theta'_i - \theta_i} \geq y_i(\theta_i, \theta_{-i}). \quad (19.4.7)$$

Expression (19.4.7) implies that $y_i(\theta_i, \theta_{-i})$ must be nondecreasing in θ_i by noting that $\theta'_i > \theta_i$. In addition, letting $\theta'_i \rightarrow \theta_i$ in (19.4.7) implies that for all θ_i , we have

$$\frac{\partial U_i(\theta)}{\partial \theta_i} = y_i(\theta),$$

and thus

$$U_i(\theta_i, \theta_{-i}) = U_i(\underline{\theta}_i, \theta_{-i}) + \int_{\underline{\theta}_i}^{\theta_i} y_i(\tau, \theta_{-i}) d\tau, \forall \theta \in \Theta.$$

Sufficiency. Consider any two types θ'_i and θ_i . Without loss of generality, suppose $\theta_i > \theta'_i$. If DM and DICFOC hold, then

$$\begin{aligned} U_i(\theta_i, \theta_{-i}) - U_i(\theta'_i, \theta_{-i}) &= \int_{\theta'_i}^{\theta_i} y_i(\tau, \theta_{-i}) d\tau \\ &\geq \int_{\theta'_i}^{\theta_i} y_i(\theta'_i, \theta_{-i}) d\tau \\ &= (\theta_i - \theta'_i) y_i(\theta'_i, \theta_{-i}). \end{aligned}$$

Thus, we have

$$U_i(\theta_i, \theta_{-i}) \geq U_i(\theta'_i, \theta_{-i}) + (\theta_i - \theta'_i)y_i(\theta'_i, \theta_{-i}) = \theta_i y_i(\theta'_i, \theta_{-i}) + t_i(\theta'_i, \theta_{-i}).$$

Similarly, we have derive that

$$U_i(\theta'_i, \theta_{-i}) \geq U_i(\theta_i, \theta_{-i}) + (\theta'_i - \theta_i)y_i(\theta_i, \theta_{-i}) = \theta'_i y_i(\theta_i, \theta_{-i}) + t_i(\theta_i, \theta_{-i}).$$

Hence, the decision rule $(y(\cdot), t_1(\cdot), \dots, t_n(\cdot))$ is dominant strategy incentive compatible. \square

When agent i announces $\hat{\theta}$ and the others announce θ_{-i} truthfully, we have the interim expected consumption $\bar{y}_i(\hat{\theta}_i) \equiv E_{\theta_{-i}} y_i(\hat{\theta}_i, \theta_{-i})$ and transfer $\bar{t}_i(\hat{\theta}_i) \equiv E_{\theta_{-i}} t_i(\hat{\theta}_i, \theta_{-i})$. Thus agent i 's interim expected utility $\bar{U}_i(\theta_i, \hat{\theta}_{-i}) = \theta_i \bar{y}_i(\hat{\theta}_i) + \bar{t}_i(\hat{\theta}_i)$ depends only on the interim expected consumption $E_{\theta_{-i}} y_i(\hat{\theta}_i, \theta_{-i})$ and interim transfer $E_{\theta_{-i}} t_i(\hat{\theta}_i, \theta_{-i})$.

Define

$$\bar{U}_i(\theta_i) \equiv \bar{U}_i(\theta_i, \theta_{-i}) = E_{\theta_{-i}} U_i(\theta) = \theta_i \bar{y}_i(\theta_i) + \bar{t}_i(\theta_i),$$

which is agent i 's interim expected utility when he announces θ_{-i} truthfully. Then, BIC means that truth-telling is optimal at the interim stage. By the same method in the proof of the above proposition for $\bar{U}_i(\theta_i)$, we can similarly prove the following result on BIC.

Proposition 19.4.2 (BIC Characterization Theorem) *In the linear model, social choice rule $(y(\cdot), t_1(\cdot), \dots, t_I(\cdot))$ is Bayesian Incentive Compatible if and only if for all $i \in I$,*

- (1) $E_{\theta_{-i}} y_i(\theta_i, \theta_{-i}) = \bar{y}_i(\theta_i)$ is nondecreasing in θ_i (BM);
- (2) $E_{\theta_{-i}} U_i(\theta_i, \theta_{-i}) = E_{\theta_{-i}} U_i(\underline{\theta}_i, \theta_{-i}) + \int_{\underline{\theta}_i}^{\theta_i} E_{\theta_{-i}} y_i(\tau, \theta_{-i}) d\tau, \forall \theta_i \in \Theta_i$. (BICFOC)

Note that (BICFOC) is a pooling of (DICFOC), and similarly (BM) is a pooling of (DM). The latter means that BIC allows to implement some decision rules that are not DS implementable.

The proposition implies that for any two BIC mechanisms that implement the same decision rule $y(\cdot)$, the interim expected utilities $E_{\theta_{-i}} U_i(\theta)$ and thus the transfers $E_{\theta_{-i}} t_i(\theta)$ coincide up to a constant.

Corollary 19.4.1 *In the linear model, for any BIC mechanism that implements the ex post Pareto efficient decision rule $y(\cdot)$, there exists a VCG mechanism with the same interim expected transfers and utilities.*

Proof. If $(y^*(\cdot), t(\cdot))$ is a BIC mechanism and $(y^*(\cdot), \tilde{t}(\cdot))$ is a VCG mechanism, then by BIC Characterization Theorem (Proposition 19.4.2), $E_{\theta_{-i}} t_i(\theta) =$

$E_{\theta_{-i}} \tilde{t}_i(\theta) + c_i, \forall \theta_i \in \Theta_i$. Then letting $\bar{t}_i(\theta) = \tilde{t}_i(\theta) + c_i$, $(y^*(\cdot), \bar{t}(\cdot))$ is also a VCG mechanism, and $E_{\theta_{-i}} \bar{t}_i(\theta) = E_{\theta_{-i}} t_i(\theta)$.

For example, the expected externality mechanism is interim-equivalent to a VCG mechanism. Its only advantage is that it allows to balance the budget ex post, i.e., in each state of the world. More generally, if a decision rule $y(\cdot)$ is DS implementable, the only reason to implement it in a BIC mechanism that is not DIC is we care about ex post transfers/utilities rather than just their interim or ex ante expectations.

19.5 Impossibility of BIC Pareto Efficient Optimal Contract

19.5.1 Participation Constraints

In general mechanism design considered in this chapter so far, we have not been yet thinking of the mechanism as a contract or principal-agent issue. Since if a mechanism is a contract, it must be voluntary, i.e. it should satisfy participation constraint. Thus, if we would put the principal-agent theory into the framework of general mechanism design, a mechanism should be individually rational. As such, when agents' types are not observed, a social choice function that can be successfully implemented should satisfy not only the incentive compatibility either in a dominant strategy or Bayesian Nash dominant strategy, depending on the equilibrium concept used, but also participation constraints that are relevant in the environment under study.

When using dominant strategy as a solution concept, Hurwicz impossibility theorem tells us that, there is no any truth-telling mechanism that results in Pareto efficient allocations. As for Bayesian incentive compatibility, such a mechanism exists such as the Expected Externality mechanism. Then, a question is, if imposing the individual rationality constraint, so that we can think of a mechanism as a contract, is such a mechanism still exist? The answer is negative.

If thinking of the mechanism as a contract, the following issues are raised:

- Will the agents accept it voluntarily, i.e., are their participation constraints satisfied?
- If one of the agents designs the contract, he will try to maximize his own payoff subject to the other agents' participation constraints. What will the optimal contract look like?

To analyze these questions, we need to impose additional restrictions on the social choice rule in the form of participation constraints. These constraints depend on when agents can withdraw from the mechanism, and

what they get when they do. Let $\hat{u}_i(\theta_i)$ be the utility of agent i if he withdraws from the mechanism. (This assumes that when an agent withdraws from the mechanism he does not care what the mechanism does with other agents.)

Definition 19.5.1 The social choice rule $f(\cdot)$ is

1. Ex Post Individually Rational if for all i ,

$$U_i(\theta) \equiv u_i(f(\theta), \theta) \geq \hat{u}_i(\theta_i), \forall \theta \in \Theta.$$

2. Interim Individually Rational if for all i ,

$$E_{\theta_{-i}}[U_i(\theta_i, \theta_{-i})] \geq \hat{u}_i(\theta_i), \forall \theta \in \Theta.$$

3. Ex ante Individually Rational if for all i ,

$$E_{\theta}[U_i(\theta)] \geq E_{\theta_i}[\hat{u}_i(\theta_i)].$$

Note that ex post IRs imply interim IR, which in turn imply ex ante IRs, but the reverse may not be true. Then, the constraints imposed by voluntary participation are most severe when agents can withdraw at the ex post stage.

The ex post IRs arise when the agent can withdraw in any state after the announcement of the outcome. For example, they are satisfied by any decentralized bargaining procedure. These are the hardest constraints to satisfy.

The Interim IRs arise when the agent can withdraw after learning his type θ_i ; but before learning anything about other agents' types. Once the agent decides to participate, the outcome can be imposed on him. These constraints are easier to satisfy than ex post IR.

With the ex-ante participation constraint the agent can commit to participating even before his type is realized. These are the easiest constraints to satisfy. For example, in a quasi-linear environment, whenever the mechanism generates a positive expected total surplus, i.e.,

$$E_{\theta}[\sum_i U_i(\theta)] \geq E_{\theta}[\sum_i \hat{u}_i(\theta_i)],$$

all agents' ex ante IR can be satisfied by reallocating expected total surplus among agents through lump-sum transfers, which will not disturb agents' incentive constraints or budget balance.

For this reason, we will focus mainly on interim IR. In the following, we will illustrate further the limitations on the set of implementable social choice functions that may be caused by participation constraints by the important theorem given by Myerson-Satterthwaite (1983).

19.5.2 Myerson-Satterthwaite Impossibility Theorem

Even though Pareto efficient mechanisms do exist (e.g. the expected externality mechanism), it remains unclear whether such a mechanism may result from private contracting among the parties. We have already seen that private contracting need not yield efficiency. In the Principal-Agent model, the Principal offers an efficient contract to extract the agent's information rent. However, this leaves open the question of whether there is some contracting/bargaining procedure that would yield efficiency. For example, in the P-A model, if the agent makes an offer to the principal, who has no private information on his own, the agent would extract all the surplus and implement efficient as a result. Therefore, we focus on a bilateral situation in which both parties have private information. In this situation, it turns out that generally there does not exist an efficient mechanism that satisfies both agents' participation constraints.

Consider the setting of allocating an indivisible object with two agents - a seller and a buyer: $I = \{S, B\}$. Each agent's type is $\theta_i \in \Theta_i = [\underline{\theta}_i, \bar{\theta}_i] \subset \mathbb{R}$, where $\theta_i \sim \varphi_i(\cdot)$ are independent, and $\varphi_i(\cdot) > 0$ for all $\theta_i \in \Theta_i$. Let $y \in \{0, 1\}$ indicate whether B receives the good. A social choice rule is then $f(\theta) = (y(\theta), t_1(\theta), t_2(\theta))$. The agents' utilities can then be written as

$$\begin{aligned} u_B(y, \theta_B) &= \theta_B y + t_B, \\ u_S(y, \theta_S) &= -\theta_S y + t_S. \end{aligned}$$

It is easy to see that an efficient decision rule $y^*(\theta)$ must have²

$$y^*(\theta_B, \theta_S) = \begin{cases} 1 & \text{if } \theta_B > \theta_S, \\ 0 & \text{if } \theta_B < \theta_S. \end{cases}$$

We could use an expected externality mechanism to implement an efficient decision rule in BNE with ex post budget balance. However suppose that we have to satisfy Interim IR:

$$\begin{aligned} E_{\theta_S}[\theta_B y(\theta_B, \theta_S) + t_B(\theta_B, \theta_S)] &\geq 0, \\ E_{\theta_B}[-\theta_S y(\theta_B, \theta_S) + t_S(\theta_B, \theta_S)] &\geq 0. \end{aligned}$$

As for budget balance, let us relax this requirement by requiring only ex ante Budget Balance:

$$E_{\theta}[t_B(\theta_B, \theta_S) + t_S(\theta_B, \theta_S)] \leq 0.$$

Unlike ex post budget balance considered before, ex ante budget balance allows us to borrow money as long as we break even on average. It

²Given our full support assumption on the distributions, ex ante efficiency dictates that the decision rule coincide with $y^*(\cdot)$ almost everywhere.

also allows us to have a surplus of funds. The only constraint is that we cannot have an expected shortage of funds.

We can then formulate a negative result:

Theorem 19.5.1 (Myerson-Satterthwaite Theorem, JET 1983) *In the two-party trade setting above, suppose each agent's type is $\theta_i \in \Theta_i = [\underline{\theta}_i, \bar{\theta}_i] \subset \mathbb{R}$, where $\theta_i \sim \varphi_i(\cdot)$ are independent and $(\underline{\theta}_B, \bar{\theta}_B) \cap (\underline{\theta}_S, \bar{\theta}_S) \neq \emptyset$ (gains from trade are possible but not certain). Then there is no BIC social choice rule that has the efficient decision rule and satisfies ex ante Budget Balance and interim IR.*

Proof. Consider first the case where $[\underline{\theta}_B, \bar{\theta}_B] = [\underline{\theta}_S, \bar{\theta}_S] = [\underline{\theta}, \bar{\theta}]$.

By Corollary 19.4.1 above, we know that any BIC mechanism in the linear environment that implements the ex post Pareto efficient decision rule $y(\cdot)$, there exists a VCG mechanism with the same interim expected transfers and utilities. As such, we can restrict attention to VCG mechanisms while preserving ex ante budget balance and interim IR. Such mechanisms take the form

$$\begin{aligned} t_B(\theta_B, \theta_S) &= -\theta_S y^*(\theta_B, \theta_S) + d_B(\theta_S), \\ t_S(\theta_B, \theta_S) &= \theta_B y^*(\theta_B, \theta_S) + d_S(\theta_B). \end{aligned}$$

By interim IR of B's type $\underline{\theta}$, $E_{\theta_S}[\underline{\theta}_B y(\underline{\theta}_B, \theta_S) + t_B(\underline{\theta}_B, \theta_S)] \geq 0$, using the fact that $y^*(\underline{\theta}, \theta_S) = 0$ with probability 1, we must have $E_{\theta_S} d_B(\theta_S) \geq 0$. Similarly, by interim IR of S's type $\bar{\theta}$, $E_{\theta_B}[-\bar{\theta}_S y(\theta_B, \bar{\theta}_S) + t_S(\theta_B, \bar{\theta}_S)] \geq 0$, using the fact that $y^*(\theta_B, \bar{\theta}) = 0$ with probability 1, we must have $E_{\theta_B} d_S(\theta_B) \geq 0$. Thus, adding the transfers, we have

$$\begin{aligned} E_{\theta}[t_B(\theta_B, \theta_S) + t_S(\theta_B, \theta_S)] &= E_{\theta}[(\theta_B - \theta_S)y^*(\theta_B, \theta_S)] + E_{\theta_S}[d_B(\theta_S)] + E_{\theta_B}[d_S(\theta_B)] \\ &\geq E_{\theta}[\max\{\theta_B - \theta_S\}] > 0 \end{aligned}$$

since $\Pr(\theta_B > \theta_S) > 0$. Therefore, ex ante budget balance cannot be satisfied.

Now, in the general case, let $(\underline{\theta}, \bar{\theta}) = (\underline{\theta}_B, \bar{\theta}_B) \cap (\underline{\theta}_S, \bar{\theta}_S)$, and observe that any type of either agent above $\bar{\theta}$ [or below $\underline{\theta}$] has the same decision rule, and therefore must have the same transfer, as this agent's type $\bar{\theta}$ [resp. $\underline{\theta}$]. Therefore, the payments are the same as if both agents having valuations distributed on $[\underline{\theta}, \bar{\theta}]$; with possible atoms on $\underline{\theta}$ and $\bar{\theta}$. The argument thus still applies.

The intuition for the proof is simple: In a VCG mechanism, in order to induce truthful revelation, each agent must become the residual claimant for the total surplus (since it is given by the sum of individuals' values plus transfers). This means that in case trade is implemented, the buyer pays the seller's cost for the object, and the seller receives the buyer's valuation for the object. Any additional payments to the agents must be nonnegative, in order to satisfy interim IRs of the lowest-valuation buyer and the highest-cost seller. Thus, each agent's utility must be at least equal to the total

surplus. In BNE implementation, agents receive the same expected utilities as in the VGC mechanism, thus again each agent's expected utility must equal at least to the total expected surplus. This cannot be done without having an expected infusion of funds equal to the expected surplus.

Myerson-Satterthwaite Impossibility Theorem is based on the assumption that agents' types are independent. If agents' types are dependent, Cremer-McLean Full Surplus Extraction Theorem to be discussed later tells us that the conclusion may not be true.

Interpretation of the Theorem: Who is designing the mechanism to maximize expected total surplus subject to agents' interim IRs?

- Agents themselves at the ex ante stage? They would face ex ante rather than interim IRs, which would be easy to satisfy. For example, in a quasi-linear environment, whenever the mechanism such as AVG mechanism generates a positive expected total surplus $E_\theta[\sum_i U_i(\theta)] \geq E_\theta[\sum_i \hat{u}_i(\theta_i)]$, all agents' ex ante IR can be satisfied by reallocating expected surplus among agents through lump-sum transfers, which will not disturb agents' incentive constraints or budget balance.
- An agent at the interim stage? He would be interested not in efficiency but in maximizing his own payoff. As such, it cannot form incentive compatibility.
- A benevolent mediator at the interim stage? But where would such a mediator come from?

A better interpretation of the result is as an upper bound on the efficiency of decentralized bargaining procedures. Indeed, any such procedure can be thought of as a mechanism, and must satisfy interim IR (indeed, even ex post IR), and ex ante budget balance (indeed, even ex post budget balance). The Theorem says that decentralized bargaining in this case cannot be efficient. In the terminology of the Coase Theorem, private information creates a "transaction cost".

Cramton-Gibbons-Klemperer (1987) show that when the object is divisible, and is jointly owned initially, efficiency can be attained if initial ownership is sufficiently evenly allocated. For example, suppose that the buyer initially owns $\hat{y} \in [0, 1]$ of the object. Efficiency can be achieved when \hat{y} is sufficiently close to 1/2. Thus, when a partnership is formed, the partners can choose initial shares so as to eliminate inefficiencies in dissolution. While this result can be applied to study optimal initial allocations of property rights, it does not explain why property rights are good at all. That is, interim IRs stemming from property rights can only hurt the parties in the model. For example, the parties could achieve full efficiency by writing an ex ante contract specifying the AGV mechanism and not allowing withdrawal at the interim stage. One would have to appeal to a difficulty of

specify in complicated mechanisms such as AGV to explain why the parties would look for optimal property rights that facilitate efficient renegotiation rather than specifying an efficient mechanism ex ante without allow in subsequent withdrawal or renegotiation.

19.6 The Revenue Equivalence Theorem in Auctions

Let us consider again the setting of allocating an indivisible object among n risk-neutral buyers: $Y = \{y \in \{0, 1\}^n : \sum_i y_i = 1\}$, and the payoff of buyer i is

$$\theta_i y_i + t_i.$$

The buyers' valuations are independently drawn from $[\underline{\theta}_i, \bar{\theta}_i]$ with $\underline{\theta}_i < \bar{\theta}_i$ according to a strictly positive density $\varphi_i(\cdot)$, and c.d.f. denoted by $\Phi_i(\cdot)$.

Suppose that the object initially belongs to a seller (auctioneer), who is the residual sink of payments. The auctioneer's expected revenue in a social choice rule $(y(\cdot), t_1, \dots, t_n)$ can be written as

$$-\sum_i E_{\theta} t_i(\theta) = \sum_i E_{\theta} [\theta_i y_i(\theta) - U_i(\theta)] = \sum_i E_{\theta} [\theta_i y_i(\theta)] - \sum_i E_{\theta_i} E_{\theta_{-i}} [U_i(\theta_i, \theta_{-i})].$$

The first term is the agents' expected total surplus, while the second term is the sum of the agents' expected utilities. By BIC Characterization Theorem (Proposition 19.4.2), the second term is fully determined by the decision (object allocation) rule $y(\cdot)$ together with the lowest types' interim expected utilities $E_{\theta_{-i}} [U_i(\underline{\theta}_i, \theta_{-i})]$. Since the total surplus is pinned down as well, we have the following result.

Theorem 19.6.1 (The Revenue Equivalence Theorem) *Suppose that two different auction mechanisms have Bayesian-Nash Equilibria in which (i) the same decision (object allocation) rule $y(\cdot)$ is implemented, and (ii) each buyer i has the same interim expected utility when his valuation is $\underline{\theta}_i$. Then the two equilibria of the two auction mechanisms generate the same expected revenue for the seller.*

Note that even when the decision rule and lowest agents' utilities are fixed, the seller still has significant freedom in designing the auction procedure, since there are many ways to achieve given interim expected transfers $\bar{t}_i(\theta_i)$ with different ex post transfer $t_i(\theta)$.

For example, suppose that the buyers are symmetric, and that the seller wants to implement an efficient decision rule $y^*(\cdot)$, and make sure that the lowest-valuation buyers receive zero expected utility. We already know that this can be done in dominant strategies - using the Vickrey (second-price sealed-bid) auction. More generally, consider a k^{th} price sealed-bid auction, with $1 \leq k \leq n$. Here the winner is the highest bidder, but he pays the k^{th} highest bid. Suppose that buyers' valuations are i.i.d. Then it

can be shown that the auction has a unique equilibrium, which is symmetric, and in which an agent's bid is an increasing function of his valuation, $b(\theta_i)$. (See, e.g., Fudenberg- Tirole "Game Theory," pp. 223-225.) Agent i receives the object when he submits the maximum bid, which happens when he has the maximum valuation. Therefore, the auction implements an efficient decision rule. Also, a buyer with valuation $\underline{\theta}$ wins with zero probability, hence he has zero expected utility. Hence, the Revenue Equivalence Theorem establishes that k^{th} price sealed-bid auctions generate the same expected revenue for the seller for all k .

How can this be true, if for any given bid profile, the seller receives a higher revenue when k is lower? The answer is that the bidders will submit lower bids when k is lower, which exactly offsets the first effect. For example, we know that in the second-price auction, buyer will bid their true valuation. In the first price auction, on the other hand, buyers will obviously bid less than their true valuation, since bidding their true valuation would give them zero expected utility. The revenue equivalence theorem establishes that the expected revenue ends up being the same. Remarkably, we know this without solving for the actual equilibrium of the auction. With $k > 2$, the revenue equivalence theorem implies that buyers will bid more than their true valuation.

Now we can also solve for the seller's optimal auction. Suppose the seller also has the option of keeping the object to herself. The seller will be called "agent zero", her valuation for the object denoted by θ_0 , and whether she keeps the object will be denoted by $y_0 \in \{0, 1\}$. Thus, we must have $\sum_{i=0}^n y_i = 1$.

The seller's expected payoff can be written as

$$\theta_0 E_{\theta} y_0(\theta) + \text{Expected Revenue} = \theta_0 E_{\theta} y_0(\theta) + \sum_{i=1}^n E_{\theta} [\theta_i y_i(\theta)] - \sum_{i=1}^n E_{\theta_i} E_{\theta_{-i}} [U_i(\theta_i, \theta_{-i})].$$

Thus, the seller's expected payoff is the difference between total surplus and the agents' expected information rents.

By BIC Characterization Theorem (Proposition 19.4.2), the buyer's expected interim utilities must satisfy

$$E_{\theta_{-i}} [U_i(\theta_i, \theta_{-i})] = E_{\theta_{-i}} [U_i(\underline{\theta}_i, \theta_{-i})] + \int_{\underline{\theta}_i}^{\theta_i} E_{\theta_{-i}} y_i(\tau, \theta_{-i}) d\tau, \forall \theta_i \in \Theta_i. \text{ (BICFOC)}$$

By the buyers' interim IR, $E_{\theta_{-i}} [U_i(\underline{\theta}_i, \theta_{-i})] \geq 0, \forall i$, and for a given decision rule y , the seller will optimally choose transfers to set $E_{\theta_{-i}} [U_i(\underline{\theta}_i, \theta_{-i})] = 0, \forall i$. Furthermore, just as in the one-agent case, upon integration by parts, buyer i 's expected information rent can be written as

$$E_{\theta} E_{\theta_{-i}} \left[\frac{1}{h_i(\theta_i)} y_i(\theta) \right],$$

where $h_i(\theta_i) = \frac{\varphi_i(\theta_i)}{1 - \Phi_i(\theta_i)}$ is the hazard rate of agent i . Substituting in the seller's payoff, we can write it as the expected "virtual surplus":

$$E_\theta \left[\theta_0 y_0(\theta) + \sum_{i=1}^n \left(\theta_i - \frac{1}{h_i(\theta_i)} \right) y_i(\theta) \right].$$

Finally, for $i \geq 1$, let $\nu_i(\theta_i) = \theta_i - 1/h_i(\theta_i)$, which we will call the "virtual valuation" of agent i . Let also $\nu_0(\theta_0) = \theta_0$. Then the seller's program can be written as

$$\begin{aligned} \max_{x(\cdot)} E_\theta & \quad [\sum_{i=0}^n \nu_i(\theta_i) y_i(\theta_i)] \quad s.t. \quad \sum_{i=0}^n y_i(\theta) = 1, \\ & \quad E_{\theta_{-i}} y_i(\theta_i, \theta_{-i}) \text{ is nondecreasing in } \theta_i, \forall i \geq 1. \quad (BM) \end{aligned}$$

If we ignore Bayesian Monotonicity (BM) above, we can maximize the above expectation for each state θ independently. The solution then is to give the object to the agent who has the highest virtual valuation in the particular state: we have $y_i(\theta) = 1$ when $\nu_i(\theta_i) > \nu_j(\theta_j)$ for all agents $j \neq i$. The decision rule achieving this can be written as $y(\theta) = y^*(\nu_0(\theta_0), \nu_1(\theta_1), \dots, \nu_n(\theta_n))$, where $y^*(\cdot)$ is the first-best efficient decision rule. Intuitively, the principal uses agents' virtual valuation rather than their true valuations because she cannot extract all of the agents' rents. An agent's virtual valuation is lower than his true valuation because it accounts for the agent's information rents which cannot be extracted by the principal. The profit-maximizing mechanism allocates the object to the agent with the highest virtual valuation.

Under what conditions can we ignore the monotonicity constraint? Note that when the virtual valuation function $\nu_i(\theta_i) = \theta_i - 1/h_i(\theta_i)$ is an increasing function³, then an increase in an agent's valuation makes him more likely to receive the object in the solution to the relaxed problem. Therefore, $y_i(\theta_i, \theta_{-i})$ is nondecreasing in θ_i for all θ_{-i} , and so by DIC Characterization Theorem (Proposition 19.4.1), the optimal allocation rule is implementable not just in BNE, but also in DS. The DIC transfers could be constructed by integration using DICFOC; the resulting transfers take a simple form:

$$t_i(\theta) = p_i(\theta_{-i}) y_i(\theta), \text{ where } p_i(\theta_{-i}) = \inf \{ \hat{\theta}_i \in [\underline{\theta}_i, \bar{\theta}_i] : y_i(\hat{\theta}_i, \theta_{-i}) = 1 \}.$$

Thus, for each agent i there is a "pricing rule" $p_i(\theta_{-i})$ that is a function of others' announcements, so that the agent wins if his announced valuation exceeds the price, and he pays the price whenever he wins. This makes truth-telling a dominant strategy by the same logic as in the Vickrey auction: lying cannot affect your own price but only whether you win or lose, and you want to win exactly when your valuation is above the price you face.

³A sufficient condition for this is that the hazard rate $h_i(\theta_i)$ is a nondecreasing function. This is the same argument as in the one-agent case (note that $\nu_{y\theta\theta} = 0$ for the linear utility function $\nu(y, \theta) = \theta y$).

19.7 Cremer-McLean Full Surplus Extraction Theorem

A central theme of mechanism design under incomplete information is about surplus extraction ability of agents. Whatever the realistic observation or economic intuition tells us that the existence of private information makes the principal has to give up some positive information rent. However, the Myerson-Satterthwaite Theorem is based not only on the assumption of private value but also on the assumption that agents' types are independent. If agents' types are dependent, can the Myerson-Satterthwaite Impossibility Theorem that there is no BIC social choice rule that has the efficient decision rule and satisfies ex ante Budget Balance and interim IR be still true?

The answer is no, and we will have a positive result. The above observation and economic intuition are no longer true. This is the basic ideal of the full surplus extraction theorem in Cremer and McLean (1988). This theorem proves that, when private types are correlated, the principal of the mechanism can extract all information rents of agents through verifying information reported, and thus obtains Pareto efficient outcome. Using the ideas of Cremer and McLean (1988), we can see that with correlated types, "almost anything is implementable". Such a conclusion is called Cremer-McLean Theorem, which has a wide application. In auction theory, seller can design a Pareto efficient design mechanism. Cremer-McLean's full surplus extraction theorem holds for any kind of correlated types.

Consider an economy with n agents. Each agent's type is discrete, $\theta_i \in \Theta_i \equiv \{\theta_i^1, \theta_i^2, \dots, \theta_i^{k_i}\}$, $\forall i \in N = \{1, \dots, n\}$, and his utility function is quasilinear with private values, i.e., $U_i(\mathbf{y}, \mathbf{t}, \theta_i) = v_i(\mathbf{y}, \theta_i) + t_i$. Assume agents are correlated and have density function $\pi(\theta)$. When agent i 's type is θ_i , his beliefs about other agents types are given by

$$\pi_i(\theta_{-i}|\theta_i) = \frac{\pi(\theta)}{\sum_{\theta'_{-i} \in \Theta_{-i}} \pi(\theta'_{-i}, \theta_i)}.$$

For each agent i , form matrix Π_i with element $\pi_i(\theta_{-i}|\theta_i)$. As such, matrix Π_i has k_i rows and $\sum_{j \neq i} k_j$ columns. Each row represents agent i 's beliefs distribution about other agents' type when his type is θ_i . If agents' types are independent, all rows are the same, and thus the ranks of the matrix Π_i is 1. If types are correlated, then different rows represent different beliefs. We assume Π_i is row full rank that is k_i , which implies an agent's beliefs distribution about others' beliefs is different.

The following well-known Cremer-McLean Full Surplus Extraction Theorem proves that, even under requirement of dominant incentive-compatibility, the principal can extract all surplus. The ideal of the proof is pretty simpler. We first take any efficient and dominant incentive-compatible mechanism such as VCG mechanism, and then form a corresponding efficient and

dominant incentive-compatible mechanism with equality individual rational constraint so that information rent is zero. As a result, we obtain a Pareto efficient dominant incentive-compatible mechanism. Formally, we have the following Cremer-McLean Full Surplus Extraction Theorem.

Theorem 19.7.1 (Cremer-McLean Full Surplus Extraction Theorem) *Suppose agents's types under private value model are correlated, and the information matrix Π_i has row full rank. Then, for any efficient and dominant incentive-compatible social choice rule $(y(\cdot), t_1(\cdot), t_2(\cdot))$, there exists another dominant incentive-compatible and efficient social choice rule $(y(\cdot), \tilde{t}_1(\cdot), \tilde{t}_2(\cdot))$ with the same decision rule $y(\cdot)$ in which interim individual rationality constraints are binding, and thus the first-best is implementable.*

Proof. For any dominant incentive-compatible social choice rule $(y(\cdot), t_1(\cdot), t_2(\cdot))$, since agents have private quasi-linear utility functions, agent i 's interim expected utility with θ_i at equilibrium can be written as

$$\bar{U}_i(\theta_i) = \sum_{\theta_{-i}} \pi_i(\theta_{-i}|\theta_i) [u_i(\mathbf{y}(\theta), \theta_i) + t_i(\theta)].$$

Let $\bar{u}_i^* = [\bar{U}(\theta_i^1), \bar{U}(\theta_i^2), \dots, \bar{U}(\theta_i^{m_i})]'$. Since Π_i is row full rank, there is a vector $c_i = (c_i(\theta_{-i}))_{\theta_{-i} \in \Theta_{-i}}$ with $\sum_{j \neq i} m_j$ columns, such that

$$\Pi_i c_i = -\bar{u}_i^*,$$

that is, for $\forall \theta_i$, we have

$$\bar{U}_i(\theta_i) + \sum_{\theta_{-i}} \pi_i(\theta_{-i}|\theta_i) c_i(\theta_{-i}) = 0.$$

To guarantee i 's interim rationality is binding, let

$$\tilde{t}_i(\theta) = t_i(\theta) + c_i(\theta_{-i}).$$

Construct a new mechanism, called Cremer-McLean mechanism.

$$(\mathbf{y}(\theta), \tilde{t}_i(\cdot)) = (\mathbf{y}(\theta), t_i(\theta) + c_i(\theta_{-i})), \forall i \in N.$$

Since $c_i(\theta_{-i})$ is uncorrelated with θ_i , Cremer-McLean mechanism is still dominant incentive-compatible and efficient. But, under Cremer-McLean mechanism, agent i 's interim expected utility at equilibrium becomes

$$\begin{aligned} \bar{V}_i(\theta_i) &= \sum_{\theta_{-i}} \pi_i(\theta_{-i}|\theta_i) [u_i(\mathbf{y}(\theta), \theta_i) + \tilde{t}_i(\theta)] \\ &= \sum_{\theta_{-i}} \pi_i(\theta_{-i}|\theta_i) [u_i(\mathbf{y}(\theta), \theta_i) + t_i(\theta) + c_i(\theta_{-i})] \\ &= \bar{U}_i(\theta_i) + \sum_{\theta_{-i}} \pi_i(\theta_{-i}|\theta_i) c_i(\theta_{-i}) \\ &= 0. \end{aligned}$$

Thus, all agents' interim individual rationality constraints are binding, and the designer extracts all the surplus, especially, the designer can extract agents' surplus through VCG mechanism to reach the first-best outcome.

The above Cremer-McLean Theorem is based on private value model. In fact, Cremer-McLean Theorem still holds for interdependent model. Although VCG mechanism with interdependent values is not dominant incentive-compatible, the generalized VCG mechanism (by modifying VCG mechanism) is dominant incentive-compatible so that Cremer-McLean Theorem still holds for interdependent model. Such a mechanism is discussed in auction theory with interdependent values.

It may be pointed out, besides Cremer-McLean Full Surplus Extraction Theorem rely on the assumptions of quasi-linear utility function and unlimited liability, it also relies on an implicit assumption that there is no collusion, or it does not hold. As such, a major criticism towards CM's result comes from its vulnerability to collusion among agents. In the FSE mechanism, payments to and from agents depend on the reports of other agents. Therefore, it is highly susceptible to collusion among the agents, especially in nearly independent environments where these payments are very large.

The pioneering work that studies collusion in principal-multiagent setting is due to Laffont and Mortimort (1997, 2000). In procurement/public good settings with two agents, they show that if the types are correlated, preventing collusion entails strict cost to the principal (LM, 2000). Che and Kim (2006) shows that the results obtained by Laffont and Mortimort (1997, 2000) depends the two-agent assumption. For $n > 2$, that show that agents' collusion, including both reporting manipulation and arbitrage, is harmless to the principal in a broad class of circumstances.

It is still unknown what outcome could be implemented for two-agent nonlinear pricing environment when types are correlated and arbitrage is allowed. Meng and Tian (2015) recently show that it makes a big difference if types are positively or negatively related. Under negative correlation, the principal can exploit the conflict of interest inside the coalition to prevent collusion at no cost, i.e., CM's result is still true. However, under positive correlation, however, the threat collusion forces the principal to distort the allocation away from the first-best level obtained without collusion.

19.8 Characterization of Bayesian Implementability

This section investigates the necessary and sufficient conditions for full Bayesian implementability of any set of social choice functions in general economic environments (say, allowing interdependent types).

Assume agent i 's parametric utility function depends on type θ , denoted by $u_i(x, \theta)$, where $x \in Z$ and $\theta_i \in \Theta_i$. The designer and agents know

$\theta = (\theta_1, \dots, \theta_n)$ has density function $\varphi(\theta)$ on $\Theta = \prod_{i \in N} \Theta_i$.

Let $X = \{x : \Theta \rightarrow A\}$ be the set of all feasible outcomes, and $\hat{F} \subseteq X$ a social choice rule that is the set of social choice functions $\hat{F} = \{f_1, f_2, \dots\}$. When a social choice rule contains only one social choice function $\hat{F} = \{f\}$, it is called social choice function, denoted by f . We will see below that a set of social choice functions \hat{F} in general differs from social choice correspondence, unless every status $\theta \in \Theta$ is common knowledge event and the set satisfies closureness.

Given a mechanism $\langle M, h \rangle$, like Nash implementation, Bayesian implementation also involves the relationship between \hat{F} and $\mathcal{B}(\Gamma)$.

Definition 19.8.1 A mechanism $\Gamma = \langle M, h \rangle$ is said to *fully Bayesian implements* \hat{F} , if

- (i) for every $f \in \hat{F}$, there is a Bayesian-Nash equilibrium \mathbf{m}^* , such that $h(\mathbf{m}^*) = f(\cdot)$;
- (ii) If \mathbf{m}^* is Bayesian-Nash equilibrium, then $h(\mathbf{m}^*) \in \hat{F}$.

If such a mechanism exists, we call \hat{F} is *fully Bayesian implementable*. If only condition (ii) is satisfied, we call \hat{F} is *strongly Bayesian implementable*. When \hat{F} contains only one social choice function, the above definition reduces to the definition for a social choice function is strongly Bayesian implementable.

Like implementation issue under other solution concepts, full Bayesian implementability needs to deal with two issues: Condition (i) requires to seek a mechanism such that any outcome determined by social choice rule is a Bayesian-Nash equilibrium outcome of the mechanism, while Condition (ii) requires all Bayesian-Nash equilibrium outcomes of the mechanism are the outcomes under social choice rules. As for Bayesian-Nash implementation, while the incentive compatibility requirement is central, it may not be sufficient for a mechanism to give all of desirable outcomes because of multiple equilibria. The severity of this multiple equilibrium problem is a critical issue to be solved in full or strong Bayesian implementation. Then, we need find conditions such that there exists a mechanism in which all Bayesian-Nash equilibrium outcomes are the outcomes under social choice rule.

We first consider an example given by Palfrey and Srivastava(1989), which shows that even for social choice function, how the issue of multiple Bayesian-Nash equilibria affects full Bayesian implementability.

Example 19.8.1 Consider an exchange economy with two agents and two goods. Agent 1 has two preferences (types) θ_1 and θ'_1 , and agent 2 only has one preference θ_2 . Agent 1's preference is private information. Under preference profile $\theta = (\theta_1, \theta_2)$, Pareto efficient allocation is $x(\theta)$, under

$\theta' = (\theta'_1, \theta_2)$, Pareto efficient allocation is $x(\theta')$, see Figure 19.1. In the above two allocations $x(\theta)$ and $x(\theta')$, agent 1 with type θ_1 likes $x(\theta)$ more, while agent 1 with θ'_1 feels two allocations are indifferent, but θ_2 likes $x(\theta')$ more. Consider social choice rule $f(\theta) = x(\theta)$ and $f(\theta') = x(\theta')$, it is Pareto efficient.

In the above exchange economy, consider the following mechanism $\Gamma = (\Theta, h(\cdot), m(\cdot))$: Each agent (mainly agent 1) reports his types $M_1 = \Theta_1 = \{\theta_1, \theta'_1\}$, $M_2 = \{\theta_2\}$. If reported message is $m = (\theta_1, \theta_2)$, then $h(\theta_1, \theta_2) = x(\theta)$; if reported message is $m = (\theta'_1, \theta_2)$, then $h(\theta'_1, \theta_2) = x(\theta')$. This mechanism has Bayesian-Nash equilibria: One is $m_1(\theta_1) = \theta_1, \forall \theta_1 \in \Theta_1; m_2(\theta_2) = \theta_2$, another is $m_1(\tilde{\theta}_1) = \theta'_1, \forall \tilde{\theta}_1 \in \Theta_1; m_2(\theta_2) = \theta_2$. However, these two Bayesian-Nash equilibria are equally desirable from Pareto optimality criterion. When the report is $\theta' = (\theta'_1, \theta_2)$, the second Bayesian-Nash equilibrium outcome is not Pareto efficient. This example shows that we have multiple equilibria problem in Bayesian implementation.

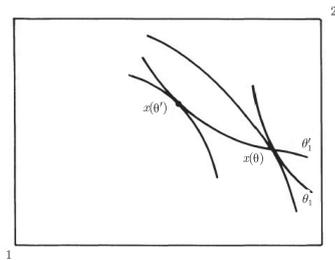


Figure 19.1: Bayesian implementation under exchange economies with two agents and two goods.

To eliminate undesirable Bayesian-Nash equilibria, we consider the following mechanism, see Table 19.1.

Table 19.1 Full Bayesian implementable social choice rule

		Agent 2	
		θ_2	ρ
Agent 1	θ_1	$x(\theta)$	$x(\theta')$
	θ'_1	$x(\theta')$	$x(\theta)$

In this mechanism, agent 2 has two different reports θ_2 and ρ . If agent reports ρ , then $h(\theta_1, \rho) = x(\theta')$ and $h(\theta'_1, \rho) = x(\theta)$. Under this mechanism, $m_1(\theta_1) = \theta_1$ is not agent 1's Bayesian-Nash equilibrium, or agent 2 would report ρ , resulting outcome $x(\theta')$. This is because $x(\theta')$ is strictly preferred to $x(\theta)$ to agent 2. When agent 2 chooses N , the optimal report of agent 1 with θ_1 will be $m_1(\theta_1) = \theta'_1$.

One can easily see the mechanism has two Bayesian-Nash equilibria:

Equilibrium 1: $m(\theta_1) = \theta_1, m(\theta'_1) = \theta'_1, m_2(\theta_2) = \theta_2$, its equilibrium outcome is $h(\theta_1, \theta_2) = x(\theta)$ and $h(\theta'_1, \theta_2) = x(\theta')$;

Equilibrium 2: $m(\theta_1) = \theta'_1, m(\theta'_1) = \theta_1, m_2(\theta_2) = \rho$, resulting in equilibrium outcome $h(\theta_1, \rho) = x(\theta)$ and $h(\theta'_1, \rho) = x(\theta')$.

Thus, although this mechanism has two Bayesian-Nash equilibria, the outcome at these two equilibria is the same as outcome under social choice rule, i.e., this mechanism fully Bayesian implements f .

For a general social choice set \hat{F} , Pastlewaite–Schmeidler (JET, 1986), Palfrey–Srivastava (RES, 1989), Mookherjee–Reichelstein (RES, 1990), Jackson (Econometrica, 1991), Dutta–Sen (Econometrica, 1994), Tian (Social Choices and Welfare, 1996; Journal of Math. Econ., 1999) provided necessary and sufficient conditions for a social choice set \hat{F} to be fully Bayesian implementable. Under some technical conditions, they showed that a social choice set \hat{F} is fully Bayesian implementable if and only if \hat{F} is Bayesian incentive compatible (it deals with the first problem that any outcome determined by social choice rule is a Bayesian-Nash equilibrium outcome of the mechanism), and it is Bayesian monotonic (it solves the second problem: all Bayesian-Nash equilibrium outcomes of the mechanism are the outcomes under social choice rules).

We first discuss these conditions. It will be seen that, Bayesian monotonicity is similar to Maskin monotonicity, using expected utility function replaces Bernoulli utility function.

Definition 19.8.2 A mechanism $\Gamma = \langle M, h \rangle$ Bayesian implements social choice set \hat{F} , if there exists a Bayesian-Nash equilibrium \mathbf{m}^* , such that $h(\mathbf{m}^*) \in \hat{F}$. If such a mechanism exists, we call that \hat{F} is Bayesian Implementable..

The following important concept of Bayesian incentive-compatibility solves the first problem of full Bayesian implementability.

Definition 19.8.3 A social choice set \hat{F} is said to Bayesian incentive-compatibility or truthfully Bayesian implementable, if for all $f \in \hat{F}$, truthtelling, i.e., $\mathbf{m}^*(\theta) = \theta, \forall \theta \in \Theta$, is Bayesian-Nash equilibrium of a revelation mechanism $\Gamma = (\Theta, f)$, i.e. for any $f \in \hat{F}$, we have

$$E_{\theta_{-i}}[u_i(f(\theta_i, \theta_{-i}), \theta) \mid \theta_i] \geq E_{\theta_{-i}}[u_i(f(\theta'_i, \theta_{-i}), \theta) \mid \theta_i], \quad \forall i, \forall \theta_i, \theta'_i \in \Theta_i.$$

We then have the first necessary condition for full Bayesian implementability of a social choice set \hat{F} .

Proposition 19.8.1 If social choice set \hat{F} is Bayesian implementable, then it satisfies Bayesian incentive-compatibility

Proof. Suppose mechanism $\Gamma = \langle M, h \rangle$ Bayesian implements social choice set \hat{F} . If some social choice function $f \in \hat{F}$ is not Bayesian incentive-compatible, then there is i and $\theta_i, \theta'_i \in \Theta_i$, such that

$$E_{\theta_{-i}}[u_i(f(\theta_i, \theta_{-i}), \theta) | \theta_i] < E_{\theta_{-i}}[u_i(f(\theta'_i, \theta_{-i}), \theta) | \theta_i]. \quad (19.8.8)$$

Let $\mathbf{m} \in \mathcal{B}(\Gamma)$ such that $h \circ \mathbf{m} = f$. When agent i 's type is θ_i , if he chooses $m_i(\theta_i)$, then his expected utility is given by

$$E_{\theta_{-i}}[u_i(h(m_i(\theta_i), \mathbf{m}_{-i}(\theta_{-i})), \theta) | \theta_i] = E_{\theta_{-i}}[u_i(f(\theta_i, \theta_{-i}), \theta) | \theta_i].$$

If he chooses a message $m'_i = m_i(\theta'_i)$, his expected utility is

$$E_{\theta_{-i}}[u_i(h(m_i(\theta'_i), \mathbf{m}_{-i}(\theta_{-i})), \theta) | \theta_i] = E_{\theta_{-i}}[u_i(f(\theta'_i, \theta_{-i}), \theta) | \theta_i].$$

By (19.8.8), we know the agent has incentive to deviate from $m_i(\theta_i)$, contradicting to $\mathbf{m} \in \mathcal{B}(\Gamma)$. \square

We now discuss the second necessary condition for a social choice set to fully Bayesian implementable, i.e., Bayesian monotonicity condition. Consider a revelation mechanism, an agent i , and a strategy $\alpha_i : \Theta_i \rightarrow \Theta_i$. If agent i tell the truth, it implies $\alpha_i(\theta_i) = \theta_i, \forall \theta_i \in \Theta_i$, otherwise α_i is called a deception strategy of agent i . We call $\alpha(\theta) = (\alpha_1(\theta_1), \dots, \alpha_n(\theta_n))$ is a deception, if at least one agent has a deception strategy.

Let

$$\alpha_{-i}(\theta_{-i}) = (\alpha_1(\theta_1), \dots, \alpha_{i-1}(\theta_{i-1}), \alpha_{i+1}(\theta_{i+1}), \dots, \alpha_n(\theta_n)).$$

For a social choice function f and deception α , $f \circ \alpha$ denotes the social choice under deception. When social status is θ , social choice outcome is $f \circ \alpha(\theta) = f(\alpha(\theta))$. For every $\theta' \in \Theta$, define $f_{\alpha_i(\theta_i)}(\theta') \equiv f(\alpha_i(\theta_i), \theta'_{-i})$, i.e. it is the outcome when only agent i chooses deception strategy.

Similar to Maskin monotonicity, we have the following Bayesian monotonicity condition.

Definition 19.8.4 (Bayesian monotonicity) A social choice set \hat{F} is said to satisfy *Bayesian monotonicity*, if for any $f \in \hat{F}$ and deception α that results in $f \circ \alpha \notin \hat{F}$, there exists a agent i and a function $y : \Theta_{-i} \rightarrow A$, such that

$$E[u_i(f(\theta_i, \theta_{-i}), \theta) | \theta_i] \geq E[u_i(y(\theta_{-i}), \theta) | \theta_i] \quad (19.8.9)$$

for all $\theta_i \in \Theta_i$, and for some θ'_i , we have

$$E[u_i(f(\theta'_i, \theta_{-i}), \theta') | \theta'_i] < E[u_i(y(\theta_{-i}), \theta') | \theta'_i]. \quad (19.8.10)$$

Bayesian monotonicity is an variation of Maskin monotonicity under incomplete information. Its role is to avoid undesirable outcomes become Bayesian-Nash equilibria. Consider a mechanism $\Gamma = \langle M, h \rangle$ that fully

Bayesian implements a social choice set \hat{F} , and a social choice function $f \in \hat{F}$ that can be Bayesian implemented by Bayesian-Nash equilibrium \mathbf{m} , i.e., for any $\theta \in \Theta$, $h(\mathbf{m}(\theta)) = f(\theta)$. Suppose agent i adopts a deception strategy α . Then strategy $\mathbf{m} \circ \alpha$ results in an outcome given by $f \circ \alpha$. If $f \circ \alpha \notin \hat{F}$, then $\mathbf{m} \circ \alpha$ is not Bayesian-Nash equilibrium. The inequality in Bayesian monotonicity condition (19.8.10) avoids the possibility that $\mathbf{m} \circ \alpha$ becomes a Bayesian-Nash equilibrium, while another inequality (19.8.9) ensures no one has incentive to cheat.

We then have another necessary condition for full Bayesian implementability, i.e., Bayesian monotonicity.

Proposition 19.8.2 *If a social choice set \hat{F} is fully Bayesian implementable, then \hat{F} must satisfy Bayesian monotonicity.*

Proof. Suppose $\Gamma = \langle M, h \rangle$ fully Bayesian implements a social choice set \hat{F} . Then, for any social choice function $f \in \hat{F}$, there is a Bayesian-Nash equilibrium \mathbf{m}^* such that $f(\theta) = h(\mathbf{m}^*(\theta))$, $\forall \theta \in \Theta$. Let α is a deception such that $f \circ \alpha \notin \hat{F}$. Consider strategy $\mathbf{m}^* \circ \alpha$. Under this strategy, for any $\theta \in \Theta$, agents's strategy profile is $\mathbf{m}^* \circ \alpha(\theta) = \mathbf{m}^*(\alpha(\theta))$. Since $f \circ \alpha \notin \hat{F}$, Full Bayesian implementability implies that $\mathbf{m}^* \circ \alpha$ is not a Bayesian-Nash equilibrium, which implies that there is $\theta'_i \in \Theta_i$ such that the agent has incentive to choose some $m'_i \neq m_i^*(\alpha_i(\theta'_i))$ such that

$$E[u_i(h(\mathbf{m}^* \circ \alpha(\theta'_i, \theta_{-i})), (\theta'_i, \theta_{-i}) | \theta'_i)] < E[u_i(h(m'_i, \mathbf{m}^*_{-i}(\alpha_{-i}(\theta_{-i}))), (\theta'_i, \theta_{-i}) | \theta'_i)]. \quad (19.8.11)$$

Define $y : \Theta_{-i} \rightarrow A$: $y(\theta_{-i}) = h(m'_i, \mathbf{m}^*_{-i}(\theta_{-i}))$. We have

$$y(\alpha_{-i}(\theta_{-i})) = h(\mathbf{m}^*_{-i}(m'_i, \alpha_{-i}(\theta_{-i}))).$$

Thus, from the above inequality (19.8.11), we can obtained (19.8.10), while (19.8.9) comes from the fact that f can be Bayesian implemented by $h \circ \mathbf{m}^*$. In this case, for any θ_i , all agents have incentive to tell the truth. \square

Addition to Bayesian incentive-compatibility and Bayesian monotonicity, to enable them also to be sufficient conditions, some technic conditions are needed. For full implementable social choice set \hat{F} , first they need satisfy closureness condition. We call a subset of a type space $\Theta' \subseteq \Theta$ a *common knowledge event*, if for any $\theta' = (\theta'_i, \theta'_{-i}) \in \Theta'$, $\theta = (\theta_i, \theta_{-i}) \notin \Theta'$, we have $\varphi(\theta'_{-i} | \theta_i) = 0$, $\forall i$.

If an agent does not know the true status, for all possible statuses, the agent need predict what messages the other agents will report. All such possible states consist of common knowledge, which are bases for reporting messages.

Definition 19.8.5(Closureness of Social Choice Set) Let Θ_1 and Θ_2 be a partition of Θ , i.e., $\Theta_1 \cap \Theta_2 = \emptyset$ and $\Theta_1 \cup \Theta_2 = \Theta$. A social choice set \hat{F} is said to have *closureness*, if for any $f_1, f_2 \in \hat{F}$, there is $f \in \hat{F}$, satisfying $f(\theta) = f_1(\theta)$, $\forall \theta \in \Theta_1$; $f(\theta) = f_2(\theta)$, $\forall \theta \in \Theta_2$.

If every state $\theta \in \Theta$ is common knowledge, it reduces to economic environment with complete information discussed in last chapter, and thus a social choice set that satisfies closure becomes social choice correspondence. If a social choice set \hat{F} does not satisfy closure, say, $\Theta = \{\theta, \theta'\}$, and every state is common knowledge, $\hat{F} = \{f_1, f_2\}$ satisfies $f_1(\theta) = f_2(\theta') = a; f_1(\theta') = f_2(\theta) = b, a \neq b$, then the social choice set \hat{F} is not fully Bayesian implementable. This is because, if \hat{F} is fully Bayesian implementable, we need they are Bayesian-Nash equilibrium under two states a and b , and then by the definitions of f_1 and f_2 , no way ensures implementable outcomes are different under two different states. We should notice that social choice set does not equal to social choice correspondence, i.e., $\hat{F} \neq F$, where $F(\theta) = F(\theta') = \{a, b\}$.

Jackson(1991) characterizes the sufficient condition for full Bayesian implementation of a social choice set in incomplete information environments. We first give the definition of economic environment in incomplete information.

Definition 19.8.6 An environment is called a *economic environment*, if for any state $\theta \in \Theta$ and any outcome $y \in Y$, there are two agents i and j , and two outcomes y_i and y_j , we have

$$u_i(y_i, \theta) > u_i(y, \theta)$$

and

$$u_j(y_j, \theta) > u_j(y, \theta).$$

Intuitively, for incomplete information economic environments, for any social choice function and state, there are at least two agents who hope to change social choice outcome under the state, which implies that a social choice outcome cannot make all agents reach their satiated points. If every utility functions are monotonic in private goods economies, this condition is satisfied. Such a condition is also satisfied for public good economies and economies with externalities.

We now state the following theorem given by Jackson(1991)without proof.

Proposition 19.8.3(Necessity and Sufficiency for Full Bayesian Implementability) For economic environments with $N \geq 3$ agents, suppose that a social choice set \hat{F} satisfies closure condition. Then, \hat{F} is fully Bayesian implementable if and only if it satisfies Bayesian incentive-compatibility and Bayesian monotonicity.

Under more general environments, Jackson(1991) strengthens Bayesian monotonicity by introducing monotonicity-no-veto-power condition and shows that in economic environments with $N \geq 3$ agents, if a social choice function Bayesian incentive-compatibility and monotonicity-no-veto-power

condition, it is fully Bayesian implementable. For environments with two agents, Dutta and Sen (1994) provide sufficient conditions for a social choice function to be fully Bayesian implementable.

One can similarly investigate refinement of Bayesian implementation and virtual Bayesian implementation such as in Palfrey–Srivastava (JPE, 1989) as we discuss in the earlier section of this chapter.

19.9 Characterization of Ex-post Implementability

Earlier, we discussed the Bayesian implementable mechanism design problem with Bayesian equilibrium as the solution. One of the basic assumptions discussed in this issue is that people in the game have common knowledge about the distribution of participants' types. However, applying a common knowledge of the type distribution is a big limitation on the application. This is due to the fact that, the common knowledge is very difficult to achieve under the incomplete information including the mechanism designers, if not completely unachievable. Wilson (1987) summed up the validity and limitations of game theory on the analysis of real problems. There was a popular quote: "Game theory has a great advantage in analyzing the consequences of trading rules that presumably really are common knowledge; it is deficient to the extent it assumes other features to be common knowledge, such as one agent's probability assessment about another's preferences or information. I foresee that the progress of game theory as depending on successive reductions in the base of common knowledge required to conduct useful analyses of practical problems. Only by repeated weakening of common knowledge assumptions will the theory approximate reality."

In the development of mechanism design theory, especially in the practice of auctions design in recent years, many researchers have focused on a more stable equilibrium concept, ex post equilibrium. Simply speaking (there is a strict definition later), the so-called ex post equilibrium means that the participant's interim equilibrium strategy will not have the motivation to change or repent afterwards (that is, after knowing the true type of other people). The dominant equilibrium under private value we discussed earlier is a typical post-event equilibrium. Because under the dominant strategy, participants do not have to consider the probability distribution of other participant types, i.e. strategy is independent of belief. If the participant's utility function depends on the signals of other participants, then the concept of post-equilibrium needs to be introduced. In this section, we mainly discuss the issue of the social choice set (function or correspondence) that can be ex post implementation.

Similar to the Nash implementation and Bayesian implementation discussed earlier, ex post implementation also has its own relevant incentive

compatibility conditions and monotonicity conditions. This section discusses the necessary and sufficient conditions for ex post implementation. The discussion is mainly based on Bergemann and Morris (2005, 2008). Assume the set of participants is $N = \{1, \dots, n\}$, the type participant i is $\theta_i \in \Theta_i$, the type combination for all participants is $\theta \in \Theta$. Assume the set of feasible result is A , the utility function of participant i is $u_i : A \times \Theta \rightarrow \mathbb{R}$. The value of participants depends on the type of other people, that is, the related value. A social choice function is denoted as f . The set of all social selection functions is denoted as F , and a set of social choices is denoted as $\hat{F} \subseteq F$. $\Gamma = \langle M_1, \dots, M_n, h(\cdot) \rangle$ is a mechanism, the participant's signal pure strategy is $m_i : \Theta_i \rightarrow M_i$, after which the ex post equilibrium definition is introduced.

Definition 19.9.1(Ex post equilibrium) Strategy combination $\mathbf{m}^* = (m_1^*, \dots, m_n^*)$ is a **Ex post equilibrium** of mechanism $\Gamma = \langle M_1, \dots, M_n, h(\cdot) \rangle$, if for any $i \in N, \theta \in \Theta, m_i \in M_i$, there is :

$$u_i(h(m_i^*, \mathbf{m}_{-i}^*), \theta) \geq u_i(h(m_i, \mathbf{m}_{-i}^*), \theta).$$

Obviously, if the effectiveness of the participants depends only on their own type, that is $u_i(h(\mathbf{m}(\theta)), \theta) = u_i(h(\mathbf{m}(\theta)), \theta_i), \forall i \in N$, the ex post mentioned before is a dominant strategy. In this way, as a corollary, the necessary and sufficient conditions for the ex post full implementation of the set of social choices in this section, are a sufficient and necessary condition for the social choice to be fully dominating complete implementable under the private value. In the game of incomplete information, ex post equilibrium has the characteristic of ex post no regret, that is, even if the participant knows the type information of other participants, he will not change his signal strategy.

Definition 19.9.2(ex post full implementation) Social choice set \hat{F} is (pure-strategy)**ex post complete implementable**, if there is a mechanism $\Gamma = \langle M_1, \dots, M_n, h(\cdot) \rangle$ such that:

(i) For any $f \in \hat{F}$, there is an ex post equilibria, \mathbf{m}^* satisfies that:

$$h(\mathbf{m}^*(\theta)) = f(\theta), \quad \forall \theta \in \Theta;$$

(ii) For any ex post equilibria \mathbf{m}^* , there is a social choice function, $f \in \hat{F}$, satisfies that:

$$h(\mathbf{m}^*(\theta)) = f(\theta), \quad \forall \theta \in \Theta.$$

The above implementation requires that the equilibrium of the mechanism be exactly the same as the result of the social selection set, that is, the full implementation. Similar to previous Nash implementation with complete

information and Bayesian implementation with incomplete information, a social choice usually needs to satisfy the conditions for incentive compatibility and monotonicity to be ex post implementable. Here we introduce ex post incentives compatibility and ex post monotonicity.

Definition 19.9.3(ex post incentives compatibility) Social choice set \hat{F} is **ex post incentives compatible**, if for any $f \in \hat{F}$, $i \in N$, $\theta \in \Theta$, and $\theta'_i \in \Theta_i$, there is:

$$u_i(f(\theta), \theta) \geq u_i(f(\theta'_i, \theta_{-i}), \theta).$$

Social choice set \hat{F} is **Ex post strictly compatible**, if the above inequality is strictly established, that is, for all the $i \in N$, $\theta \in \Theta$, $\theta'_i \neq \theta_i$, there is:

$$u_i(f(\theta), \theta) > u_i(f(\theta'_i, \theta_{-i}), \theta).$$

In a direct mechanism, consider the participant's i manipulation information disclosure. Let $\alpha_i : \Theta_i \rightarrow \Theta_i$ is a fraud of participants i , noted as $\alpha_i(\theta_i) \neq \theta_i$, $\alpha(\theta)$ is a fraud combination of the participant. Under the social selection function f , this deception will result in a social choice of $f(\alpha(\theta)) \neq f(\theta)$. The ex post monotonicity condition make sure that for each fraud, there exists an alert provided by one of the participants, and the provision of the alert satisfies incentive compatibility.

Definition 19.9.4(ex post monotonicity) Social choice set \hat{F} has the nature of **ex post monotonicity**, if for any $f \in \hat{F}$ and $f \circ \alpha \notin \hat{F}$'s fraud α , there is a participant $i \in N$ and an outcome y , such that:

$$u_i(y, \theta) > u_i(f(\alpha(\theta)), \theta), \quad (19.9.12)$$

and

$$u_i(f(\theta'_i, \alpha_{-i}(\theta_{-i}))) \geq u_i(y, (\theta'_i, \alpha_{-i}(\theta_{-i}))), \quad \forall \theta'_i \in \Theta_i. \quad (19.9.13)$$

The inequality above (19.9.12) means that when a fraud α appears, the participant i has incentive to provide the alert y ; but the inequality (19.9.13) means that the alert will not appear in the absence of deception. For convenience, we define a set that given other participants' types are θ_{-i} , participant i doesn't have incentive to provide alert in all the types $\theta_i \in \Theta_i$:

$$Y_i^f(\theta_{-i}) \equiv \{y | u_i(y, (\theta'_i, \theta_{-i})) \leq u_i(f(\theta'_i, \theta_{-i}), (\theta'_i, \theta_{-i})), \forall \theta'_i \in \Theta_i\}. \quad (19.9.14)$$

The inequality above (19.9.13) means $y \in Y_i^f(\theta_{-i})$. $Y_i^f(\theta_{-i})$ is called reward set, that is to reward participant i disclose information truthfully, which depend on social choice set f . In the meantime, we define a successful reward set: $Y_i^{f*}(\theta_{-i}) = \{y : u_i(y, \theta) > u_i(f(\alpha(\theta)), \theta)\} \cap Y_i^f(\theta_{-i})$. In this set, the participant i only sends out an alert on fraud, thus avoiding some

bad choices. Here we discuss the necessary conditions and sufficient conditions for full implementation of the ex post full implementation. Note: Compare ex post monotonicity and Manski monotonicity, maybe you will feel that ex post monotonicity is stronger than Manski monotonicity. Because in the definition of ex post monotonicity, the truth-telling limitation is satisfied for all $\theta'_i \in \Theta_i$ in $(\theta'_i, \alpha_{-i}(\theta_{-i}))$, but Manski monotonicity only require it holds on $\alpha(\theta)$. Actually, this difference has no effect on the ex post implementation and Nash implementation. Indeed, as proved by Bergemann and Morris (2008), in general, they do not have contain relations. Ex post monotonicity does not mean that Maskin monotonicity, nor is Manski monotonicity. For example, the weak Pareto effective allocation satisfies the Manski monotonicity, but it does not meet the ex post monotonicity. Single-item mechanism under the relevant value satisfies ex post monotonicity, but does not meet the Manski monotonicity. However, for environments that meet the single-crossing nature, it can be shown that the ex post monotonicity and the Manski monotonicity are equivalent.

Definition 19.9.5(Necessary conditions for ex post implementation) *If social choice set $\cap F$ is ex post complete implementable, then it must satisfy ex post incentive compatibility and ex post monotonicity.*

PROOF. Let $\Gamma = (M_1, \dots, M_n; h(\cdot))$ is a mechanism of an ex post implementable social choice set $\cap F$. For any $f \in \cap F$, according to implementable conditions, there is an ex post equilibria \mathbf{m}^* , such that $f = h \circ \mathbf{m}^*$. Because \mathbf{m}^* is an ex post equilibria, then for any $i \in N, \theta'_i \in \Theta_i, \theta \in \Theta$, there will be:

$$u_i(h(\mathbf{m}^*(\theta)), \theta) \geq u_i(h(m_i^*(\theta'_i), \mathbf{m}_{-i}^*(\theta_{-i})), \theta).$$

Because $f(\theta) = h(\mathbf{m}^*(\theta)), f(\theta'_i, \theta_{-i}) = h(m_i^*(\theta'_i), \mathbf{m}_{-i}^*(\theta_{-i}))$, the social choice set we have $\cap F$ satisfies ex post incentive compatibility. Consider any fraud α that satisfies $f \circ \alpha \notin \cap F$, then $\mathbf{m}^* \circ \alpha$ must not be an equilibria in some θ . That means, there is a group of $i \in N, m_i \in M_i$, such that:

$$u_i(h(m_i, \alpha_{-i}(\theta_{-i})), \theta) > u_i(h(\mathbf{m}^*(\alpha(\theta))), \theta).$$

Let $y \equiv h(m_i, \alpha_{-i}(\theta_{-i}))$, then we have

$$u_i(y, \theta) > u_i(h(\mathbf{m}^*(\alpha(\theta))), \theta).$$

Because \mathbf{m}^* is an ex post equilibria and $f = h \circ \mathbf{m}^*$, then we have:

$$\begin{aligned} u_i(f(\theta'_i, \alpha_{-i}(\theta_{-i})), (\theta'_i, \alpha_{-i}(\theta_{-i}))) &= u_i(h(\mathbf{m}^*(\theta'_i, \alpha_{-i}(\theta_{-i}))), (\theta'_i, \alpha_{-i}(\theta_{-i}))) \\ &\geq u_i(h(m_i, \mathbf{m}_{-i}^*(\alpha_{-i}(\theta_{-i}))), (\theta'_i, \alpha_{-i}(\theta_{-i}))) \\ &= u_i(y, (\theta'_i, \alpha_{-i}(\theta_{-i}))). \end{aligned}$$

Therefore, the alert y satisfies the incentive compatibility of participant i or $y \in Y_i^f(\alpha_{-i}(\theta_{-i}))$. We have the ex post monotonicity of \hat{F} . $\square \square$ Bergemann and Morris (2008), under the economic environment (see definition 19.8.6), also obtained similar sufficient conditions for Bayesian implementation.

Definition 19.9.6(Sufficient conditions of ex post implementation) *In a society with more than three members of the economic environment, if the social choice set \hat{F} satisfies for ex post incentive compatibility and ex post monotonicity, then it is ex post implementable.*

For social choice set \hat{F} , Bergemann and Morris (2008) constructed the mechanism below to implement (ex post).

First of all, the signal space of each participant is not just its own type, $m_i = (\theta_i, f_i, z_i, y_i)$, in which θ_i is the type signal of participant i himself; f_i is social choice function constructed by participant i ; z_i is similar to the positive integer in the Manski implementation mechanism, but the boundary that limit z_i here is $N = \{1, \dots, n\}$; y_i is the reward(outcome) that participant i proposed; the signal space of i is $M_i = \Theta_i \times \hat{F} \times N \times Y$, in which Y is the set of all the possible outcome. The implementation mechanism is limited by the following rules:

Rule 1: If $\forall i, f_i = f$, then $h(\mathbf{m}) = f(\boldsymbol{\theta})$. Rule 2: If there is a participant j and a social choice rule $f \in \hat{F}$, satisfy $\forall i \neq j$ we have $f_i = f, f_j \neq f$, if $y_j \in Y_j^f(\boldsymbol{\theta}_{-j})$ then choose y_j ; otherwise $f(\boldsymbol{\theta})$. Rule 3: Expect the two conditions above, when $j(z) = \sum_{i=1}^n z_i \pmod{n}$, then choose outcome $y_{j(z)}$.

Now, let us prove the theorem of sufficient conditions for ex post implementation in an economic environment. **PROOF.** First, we rewrite the signal of participants: $m_i(\theta_i) = (m_i^1(\theta_i), m_i^2(\theta_i), m_i^3(\theta_i), m_i^4(\theta_i)) \in \Theta_i \times \hat{F} \times N \times Y$. Here are three steps to prove. Step one: For any given $f \in \hat{F}$, there is an ex post equilibria \mathbf{m}^* , such that, $h(\mathbf{m}^*(\boldsymbol{\theta})) = f(\boldsymbol{\theta}), \forall \boldsymbol{\theta} \in \Theta$.

Consider the strategy combination $(m_i^*(\theta_i) = (\theta_i, f, \dots), i \in N)$. According to rule one, $h(\mathbf{m}^*(\boldsymbol{\theta})) = f(\mathbf{m}^*(\boldsymbol{\theta}))$. We prove this combination is an ex post equilibria. Given other participants' strategy $(m_j^*(\theta_j), j \neq i)$, if participant i deviate from signal strategy $m_i^*(\theta_i)$, choosing to send a signal $m_i(\theta_i) = (\theta'_i, f_i, \dots)$. When $y_i \notin Y_i^f(\boldsymbol{\theta}_{-i})$, then according to rule two, society will choose $f(\theta'_i, \boldsymbol{\theta}_{-i})$, then for participant i , the profit of deviation is:

$$u_i(f(\theta'_i, \boldsymbol{\theta}_{-i}), (\theta_i, \boldsymbol{\theta}_{-i})) - u_i(f(\theta_i, \boldsymbol{\theta}_{-i}), (\theta_i, \boldsymbol{\theta}_{-i})) \leq 0.$$

The above inequality is derived by the ex post incentives compatibility and the participants do not have incentive to deviate. When $f_i \neq f, y_i \in Y_i^f(\boldsymbol{\theta}_{-i})$, according to rule two, society will choose y_i , for participant i , the profit of deviation:

$$u_i(y, (\theta_i, \boldsymbol{\theta}_{-i})) - u_i(f(\theta_i, \boldsymbol{\theta}_{-i}), (\theta_i, \boldsymbol{\theta}_{-i})) \leq 0.$$

The inequality above is derived by the definition(19.9.14) of $Y_i^f(\boldsymbol{\theta}_{-i})$ or the ex post monotonicity condition(19.9.13). When $f_i = f$, according to rule one, the other signal choices of participant will not change the choice of whole society.

Step two: for any ex post equilibrium \mathbf{m}^* , there is a social choice function $f \in \hat{F}$, such that $m_i^{*2}(\theta_i) = f, \forall i \in N, \theta \in \Theta$, then according to rule one, $h \circ \mathbf{m}^* = f$.

Proof by contradiction: suppose with one ex post equilibria \mathbf{m}^* , for any $f \in \cap F$, there is a i and θ_i , such that $m_i^{*2}(\theta_i) \neq f$. Then, there is a state θ , such that rule one is not applicable.

Let's assume that rule 2 can be applied to the state θ , which means that there is a j and f such that $f_i = f, i \neq j$. Now, for any participant $i \neq j$ that status is θ_i , sending a signal $m_i(\cdot, f_i, z_i, y_i)$ when envisioning the other participant's status as θ_{-i} , in which $f_i \neq f, i = \sum_{k=1}^n z_k \pmod n$. Therefore, society will choose y_i under rule 3 and the utility is $u_i(y_i, \theta)$. Hence, if \mathbf{m}^* is an ex post equilibria, for any $y \in Y, i \neq j$, it must satisfy $u_i(h(\mathbf{m}^*(\theta)), \theta) \geq u_i(y, \theta)$, which contradicts with the condition of economics environment.

Now assume rule 3 can be applied in state θ , which means for any participant with state θ_i , sending the signal $m_i(\cdot, f_i, z_i, y_i)$ when envisioning that the other participant's status is θ_{-i} , in which $i = \sum_{k=1}^n z_k \pmod n$. Then by rule 3, the society will choose y_i and the utility level is $u_i(y_i, \theta)$. Therefore, if \mathbf{m}^* is an post equilibria, for any $y \in Y, i \neq j$, there must be $u_i(h(\mathbf{m}^*(\theta)), \theta) \geq u_i(y, \theta)$. It contradicts with the condition of economics environment.

Step three: for any social choice function $f \in \cap F$ and any ex post equilibria \mathbf{m}^* that satisfies $m_i^{*2}(\theta_i) = f, \forall i, \theta_i$, we have $f \circ m^{*1} \in \cap F$.

Proof by contradiction: Assume $f \circ m^{*1} \notin \cap F$. According to ex post monotonicity, there is an $i, \theta, y \in Y_i^f(m_{-i}^{*1}(\theta_{-i}))$, satisfies:

$$u_i(y, \theta) > u_i(f(m^{*1}(\theta)), \theta).$$

Consider a participant i with status θ_i , in the condition of believing other participants' status are θ_{-i} , he/she will send signal $m_i = (\cdot, f_i, \cdot, y)$ that satisfy $f_i \neq f$. Given other participants choose their equilibrium strategy, by rule 2, the society choice is $h(m_i, \mathbf{m}_{-i}^*(\theta_{-i})) = y$. The participant i 's expect utility under m_i is:

$$u_i(h(m_i, \mathbf{m}^*(\theta_{-i})), \theta) = u_i(y, \theta) > u_i(f(\mathbf{m}^{*1}(\theta)), \theta) = u_i(h(\mathbf{m}^*(\theta)), \theta).$$

It contradicts with that \mathbf{m}^* is an ex post equilibria.

From the above three steps, we have proved that under the economic environment, the ex post incentive compatibility and ex post monotonicity are sufficient condition for ex post implementation. \square \square For the non-economic environment, Bergemann and Morris (2008), under the condition of adding "no veto power", have proved similar sufficient conditions for ex post full implementation. In this article, for the economic environment that satisfies the single-crossing nature, they also proved that any set of social choices with strict ex post incentive compatibility and consisted of internal

points satisfies ex post monotonicity conditions and is therefore fully implementable. Besides, when the number of participants is greater than 2, the next chapter will introduce the generalized direct VCG mechanism under the relevant value environment also satisfies the ex post monotonicity and has the unique ex post equilibria. In addition, Ohashi (2012) gave similar sufficient conditions and necessary conditions for ex post implementation in the two-participant environments. Additionally, in the mechanism design, using the concept of ex post equilibrium, Bergemann and Morris (2005) established a robust mechanism design analysis framework.

19.10 Biographies

19.10.1 Jean Jacques Laffont

Jean-Jacques Laffont (1947-2004), founder of Toulouse's Industrial Economics Institute (Institut D'Economie Industrielle, IDEI), one of the founders of the new regulatory economics, is also one of the founders of information economics and motivation theory. Jean-Jacques Laffont born in Toulouse, southeastern France, was a post-war generation in France. This generation grew up under the influence of General de Gaulle. They have a strong patriotism and are committed to the revitalization of the French nation. In 1968, Laffont graduated from the University of Toulouse, which has a profound mathematics education tradition. He obtained a master's degree in mathematics. Later, he went to Paris University to pursue further studies and received a doctorate in applied mathematics in 1972. In the fall of 1973, the young Laffont came to Harvard University in the United States under the supervision of Kenneth Arrow. Together with Maskin and Elhanan Helpman, he became the legend of the economics community - so called the three Musketeers at Harvard Economics Department. Laffont received a Ph.D. in Economics (1975) from Harvard University in just one and a half years, which is very rare in the history of Harvard. In 1978, Laffont gave up a faculty position in the United States and returned to the University of Toulouse after alternative military service. In a very tough environment, he spread economics and unremittingly created a new area of economics. Along the research direction of doctoral thesis, he quickly made important contributions in the field of public economics and mechanism design. In the 1970s, the general equilibrium theory was still in the dominant position in economics, but Laffont has been deeply aware of the important position of incentive mechanism design theory in economics in the future. The focus on economics incentives makes Laffont choose incentive theory as his main research field. At the end of the 1970s, information economics, an important branch of modern economics, was emerging. The main subject of research in information economics is incentives problem under incomplete

information and asymmetric information. Integrated with the game theory methodologically, information economics has successfully explained many problems that cannot be solved under the framework of general equilibrium. This shows the strong vitality of the theory, which greatly promotes the development of enterprise theories and industrial organization theories. Laffont took information economics as the basic framework for his research on incentives and began to explore the integration of incentive theory. In 1979, Professor Laffont's book **Incentives in Public Decision Making** (co-authored with Jerry Green), established his authoritative position in the field of public economics. Laffont and Tirole created a general framework for incentive regulation, which led to the birth of new regulatory economics. The new regulatory economics combines the basic ideas of public economics and industrial organization theory, the basic methods of information economics and mechanism design theory. The basic ideas and methods of incentive regulation proposed in the new regulatory economics successfully solve the regulation problem under asymmetric information. Laffont and Tirole published a book **A Theory of Incentives in Procurement and Regulation** in 1993, which completed the construction of a theoretical framework for new regulatory economics, thus laying their academic leadership position in this field. Professor Laffont is an extremely diligent and productive scholar. In his short 57-year-old life, he published 12 books and more than 300 high-level academic papers. His academic contributions have earned him a high reputation in the economics world. Professor Laffont does not just wait to win the Nobel Prize as many other well-known economists. He still continued to spread and develop new areas of motivation theory. Even in the fight against cancer, he still insisted on completing the new book *Regulation and Development* (December 2003). Unexpectedly, this book has become his legacy. As a distinguished economist, Professor Jean-Jacques Laffont has been recognized by the economics community for his outstanding contributions and achievements in the fields of mechanism design theory, public economics, incentive theory, and new regulatory economics. He was elected Chair of the Econometric Society (1992), Chairman of the European Economic Association (1998), Honorary Member of the American Economic Association (1991), Foreign Honorary Member of the American Academy of Arts and Sciences (1993), and he was awarded Yrjo-Jahnsson Prize from European Economic Society in 1993 (the prize is commensurate with the Clarke Prize of the American Economic Association). If he was alive, he would at least win the Nobel Prize in Economics together with Tirole. Perhaps this is the greatest last wish of his life.

19.10.2 Roger B. Myerson

Roger Bruce Myerson (born 1951) is professor at the University of Chicago. He was awarded Nobel Prize in economics in 2007 for his contribution

to the creation and development of mechanism design theory and auction theory, including the revelation principle, The optimal mechanism design, revenue-equivalence theorem and etc. Myerson was born in Boston, USA. In 1976, he received a Ph.D. in applied mathematics from Harvard University. His doctoral thesis is "A Theory of Cooperative Games". He has taught at the University of Chicago since 2007 and his research expertise includes the game system in the field of economics and the voting system in political science. Myerson's "Optimal Auction Design" published in 1981 was the cornerstone of the optimal mechanism design. The optimal auction theory seeks to solve the problem of how to design a system to maximize incentives for participants in economic activities given the distribution of information, that is, the design of optimal contracts. He also published books **Game theory: analysis of conflict and probability models for economic decisions**. After Vickrey, a large number of scholars began to pay attention to auction theory. Among them, Myerson deserves special mention. He used the newly developed mechanism design theory to revisit the auction theory and promoted Vickrey's theory on this basis. Myerson, came to the conclusion through rigorous mathematical analysis: In a series of assumptions (the bidder's evaluate the items independently, the bidders only care about their own expected returns and etc.), all possible auction mechanisms will bring the same expected benefits to the auctioneer. Obviously, this conclusion surpassed previous research ideas of Vickrey, who research the specific auction forms. He was able to study all possible auctions, which greatly advanced the auction theory. Myerson is an economist who can make abstract economic theory practical. He has solved many economic problems for the, such as California's power crisis, and is well known in the economics community. In the 1980s, the power reform in California in the United States was to break the drawbacks of power monopoly, but it was impossible for the power industry to implement complete competition. The best way is to use oligopoly. Myerson applied the theory of mechanism design and game theory to design a plan for California's power reform. This plan has been implementing so far and has achieved good results. In addition, he also solved the problem of recruiting students at the American Medical School. American doctors are high-income groups, but medical schools are mostly private. Without controlling the number of students in medical schools, the quality and income of doctors cannot be guaranteed. The U.S. government introduced Myerson's mechanism design principles into relevant laws to limit the number of medical college admissions. Myerson's contribution to the real economy has impressed economists in the United States: It is therefore possible to create an "Economic Engineering" and to make economics as practical as engineering, to design economic phenomena as well.

19.11 Exercises

Exercise 19.1 Consider an economics environment with one seller and one buyer. The value of buyer is v_b and the value of seller is v_s . We know that v_b and v_s both are uniform distribution on $[0, 1]$, and the two are independent. The given trading mechanism is as follows: the buyer and the seller simultaneously make offers at the purchase price p_b and the selling price p_s respectively. If $p_b \geq p_s$, the transaction is conducted with the price $p = (p_b + p_s)/2$, and the profit of buyers and sellers are:

$$\begin{aligned} v_b - p, \\ p - v_s. \end{aligned}$$

If $p_b < p_s$, there will be no transaction and the profit of buyers and sellers are 0. It is known that under this mechanism, there are multiple equilibrium bidding strategies. For example, there is a linear equilibrium bidding strategy: $p_i(v_i) = A_i + c_i v_i$, $i \in b, s$. Please give a linear equilibrium strategy for buyers and sellers and prove that whether it can be ex post effective implemented.

Exercise 19.2 (Arrow, 1979; d'Aspremont and Gerard-Varet, 1979; Diamantaras, 2009) Assume the participants' set is N , $|N| = n$, the type set of participants i is Θ_i . The utility function is $u_i(d, \theta_i, t_i) = v_i(d, \theta_i) + t_i$. AVG mechanism $(d(\theta), t_i(\theta))$ satisfies:

$$d(\theta) \in \operatorname{argmax}_d \sum_{i \in N} v_i(d, \theta_i);$$

$$t_i(\theta) = E_{\theta_{-i}} \left[\sum_{j \neq i} v_j(d(\theta), \theta_j) | \theta_i \right] - \frac{1}{n} \sum_{k \neq i} E_{\theta_{-i}} \left[\sum_{j \neq k} v_j(d(\theta), \theta_j) | \theta_k \right].$$

Prove: if each participant's type distribution is independent, the above AVG mechanism satisfies Bayesian incentive compatibility and interim budget balance.

Exercise 19.3 (Diamantaras, 2009) Consider an economic environment with t -two participant and one public good. The type of each participant is $\Theta_i = \{0, 1\}$, $i = 1, 2$. The two types have equal probability, and the types of two participants are independently distributed. Participant i 's evaluation to the public good is θ_i . Assume the cost of public good is $c = \frac{3}{4}$.

Prove that in this environment there is no mechanism that satisfies the following features at the same time: (i) Interim participation constraints; (ii) Bayesian incentive compatibility; (iii) The public goods decision-making mechanism is effective; the (iv) mechanism is feasible, i.e. it does not require external extra resources.

Exercise 19.4 (Palfrey and Srivastava, 1989) Consider an economy that the set of participants is $I = \{1, 2, 3\}$, the social feasible set is $A = \{a, b\}$ and each participant's type is $T^i = \{t_a, t_b\}$. Assume that the types of participants are all independently and identically distributed, $\text{Prob}(t^i = t_b) = q > 0.5^{1/2}$, each participant's utility function is the same, satisfying:

$$u^i(a, t_a) = 1 > 0 = u^i(b, t_a);$$

$$u^i(b, t_b) = 1 > 0 = u^i(a, t_b).$$

Assume that the social selection function f is described as follows:

$$f(t^1, t^2, t^3) = \begin{cases} t_a, & \text{If } t^1 = t^2 = t_a, \text{ or } t^1 = t^3 = t_a, \text{ or } t^2 = t^3 = t_a; \\ t_b, & \text{if } t^1 = t^2 = t_b, \text{ or } t^1 = t^3 = t_b, \text{ or } t^2 = t^3 = t_b. \end{cases}$$

1. Prove that $f(\cdot)$ is the only allocation rule that satisfies the following five characteristics: (i) It is incentive compatible; (ii) It is valid ex ante, interim, and ex post; (iii) $f(\mathbf{t})$ is the majority voting decision rule; (iv) it maximizes the Arrow social welfare function; (v) It can be dominated implemented in a direct display mechanism.
2. Proves that $f(\cdot)$ is not strong Bayesian implementable (hint: proves that it does not meet the Bayesian monotonicity).

Exercise 19.5 (Palfrey and Srivastava, 1989) Assume the participant's set is $I = \{1, 2, 3\}$, the set of social feasible outcome is $A = \{a, b\}$, each participant's type is $T_i = \{t^a, t^b\}$. Assuming each participant's type is independent and identically distributed. $\text{prob}(t_i = t^b) = q > 0.5^{1/2}$. Assuming that each participant's utility function is the same, depending on the type of other participants, they satisfy:

$$u_i(a, \mathbf{t}) = \begin{cases} 1, & \text{If at least two participants are of the type } t^a, \\ 0, & \text{other.} \end{cases}$$

$$u_i(b, \mathbf{t}) = \begin{cases} 1, & \text{If at least two participants are of the type } t^b, \\ 0, & \text{other.} \end{cases}$$

Assume that the social selection function f is described as follows:

$$f(t_1, t_2, t_3) = \begin{cases} t^a, & \text{if } t_1 = t_2 = t^a, \text{ or } t_1 = t_3 = t^a, \text{ or } t_2 = t_3 = t^a, \\ t^b, & \text{if } t_1 = t_2 = t^b, \text{ or } t_1 = t_3 = t^b, \text{ or } t_2 = t_3 = t^b. \end{cases}$$

Prove that this selection function can not be implemented by the non-inferior Bayesian equilibrium. The so-called non-inferior means that for everyone, there is no other strategy weaker dominated it.

Exercise 19.6 (Bayesian Incentive Compatible Characterization) In the linear model of independent distribution of private value, social choice rule $(\mathbf{y}(\cdot), t_1(\cdot), \dots, t_n(\cdot))$ is Bayesian compatible if and only if for all $i \in N$.

1. Prove $\bar{y}_i(\theta_i)$ is non-decreasing about θ_i .
2. Prove $E_{\theta_{-i}} U_i(\theta_i) = E_{\theta_{-i}} [U_i(\theta_i, \theta_{-i})] + \int_{\theta_i}^{\theta_i} E_{\theta_{-i}} y_i(\tau, \theta_{-i}) d\tau, \forall \theta_i \in \Theta_i$.

Exercise 19.7 A seller and a buyer are bargaining for an inseparable item. The value of the seller is $\theta_b = 10$. There are two possible values for the buyer, $\theta_s \in \{0, 9\}$. Let t be the period of the transaction ($t = 1, 2, \dots$). P represents the agreed price. The discount factor for both the buyer and the seller is δ .

1. In the currently set economic environment, what is the set of feasible options?
2. Imagine that under a Bayesian Nash equilibrium of the bargaining process, when the value of the seller is 0, the transaction will happen immediately; and when the value of the seller is θ_s , the agreed transaction price is $(10 + \theta_s)/2$. When the value of the seller is 9, which is the earliest possible time for the transaction?

Exercise 19.8 Consider Myerson-Satterthwaite bilateral trade model under discrete condition. Given the value of buyer v and the cost of seller c are uniform distribution over 1, 2, 3, 4.

1. Prove that effective trade is incentive-compatible.
2. What does this example tell us about the Myerson-Satterthwaite effectiveness conclusion?

Exercise 19.9 Consider a direct incentive compatible mechanism (q, \mathbf{t}) , \mathbf{t} is ex ante budget balanced, that is $E_{\theta} \sum_{i=1}^N t_i(\theta) = 0$. Assume the type is independent distributed.

1. Prove: there is another Bayesian incentive incompatible mechanism (q, \mathbf{t}') , \mathbf{t}' is ex post budget balanced, $\sum_{i=1}^N t_i(\theta) = 0$.
2. Prove: $E_{\theta_{-i}} t_i(\theta) = E_{\theta_{-i}} t'_i(\theta)$.
3. Prove: if there is an individual rationality VCG mechanism (q^*, \mathbf{t}) , such that $E_{\theta} \sum_{i=1}^N t_i(\theta) \geq 0$, then there is a Bayesian incentive computable, budget balanced, individual rational and effect mechanism (q^*, \mathbf{t}') .

Exercise 19.10 Consider a bilateral trading economy where two participants initially own a unit of a commodity. Each person's value for every unit of product consumed is θ_i ($i = 1, 2$). Assume that the distribution of θ_i is a uniform distribution over $[0, 1]$ and is independent of each other.

1. Describe the trading rules in an ex post effective social selection function.
2. Consider the following mechanism: Each participant gives a bid, and the highest bidder obtains the unit of goods from another person and pay the offer. Give a symmetric Bayesian Nash equilibrium for this mechanism.
3. What is the social selection function that can be implemented by this mechanism? Verify that it is incentive compatible. Is it effective? Does it satisfy individual rationality? Intuitively, why does it differ from the conclusion of Myerson-Satterthwaite theorem?

Exercise 19.11 Consider the problem of bilateral transactions between a supplier and a consumer. The supplier's cost per unit is determined by its cost type, and the cost type is the supplier's private information. When the supplier's type is θ , if the consumer's total payment is x , the former's profit is $x - \theta q$. At the same time, for q units of goods, the value of the consumer depends on the provider's type θ , which is represented by $\pi(q|\theta) = (4 + 2\theta)q^{1/2}$. Therefore, the consumer obtains q unit goods from the supplier whose type is θ by paying x , and the net surplus is $(4 + 2\theta)\sqrt{q} - x$. Any participant can refuse to participate in the transaction, then $q = 0$ and $x = 0$.

1. When the supplier's type θ is common knowledge, find the expression that maximizes the quantity of goods remaining on both supply and demand side, $q^*(\theta)$.
2. Assume that the supplier's type may be $\theta_L = 2$ or $\theta_H = 3$. The probability of being a low-cost θ_L type is p_L , and the probability of being a high-cost θ_H type is $p_H = 1 - p_L$. For the buyer's expected net return maximization problem that takes into account incentive constraints and participation constraints, find a trading plan and list the constrained optimization problems. (Hint: $\pi(q|\theta_L) = 8\sqrt{q}$ and $\pi(q|\theta_H) = 10\sqrt{q}$.)
3. Give a formula for calculating the buyer's optimal incentive compatible trading plan.
4. Consider the following case: the buyer believes that the supplier's cost type is subject to a uniform distribution over the $[2, 3]$ interval. Give a formula for calculating the buyer's optimal incentive compatible trading plan. This formula maximizes the buyer's expected surplus, satisfying the participation constraint and information incentive constraint.
5. Consider another case: the supplier's cost type is $\theta_L = 2$ or $\theta_H = 3$. In all incentive-compatible plans where the buyer's surplus is non-negative for any type of supplier, find the optimal deal plan for both types of suppliers.
6. For the solution to the problem (5), prove: if p_L is close enough to 0 and p_H is close enough to 1, then there is a pooling strategy that can give the buyer

a non-negative expected surplus, which is better than the (5) trading plan in question.

Exercise 19.12 There are two construction companies bid for construction of a government building. Each construction company is one of two types: a *H*-type enterprise can use the cost C_H to construct a high-quality project that values at v_H ; and a *L* type enterprise can use the cost C_L to construct a high-quality project that values at v_L . Here, $C_H > C_L$, $v_H > v_L$. Assume that $v_H - C_H > v_L - C_L > 0$. It is known that the types of companies are independent of each other and it is private information. Each company has a probability of v to become a high quality company. Here we consider the direct display mechanism, when the company declares its type as θ and another company declares its type as θ' , we use $q(\theta, \theta')$, $\theta, \theta' \in \{H, L\}$ to represent the probability that the previous company won the bid; similarly, we use $T(\theta, \theta')$ to represent the government's payment. Given that both the government and the two companies are risk-neutral, the government's goal is to maximize its expected surplus of $E(v - \omega)$.

1. Write the government's optimal bidding design problem that satisfies participation constraints and incentive compatibility constraints.
2. Find the government's optimal bidding mechanism.

Exercise 19.13 (Dana and Spier, 1994) Assume that there are two companies, $j = 1, 2$, to obtain production rights for a given market through competition. A social planner designs an optimal production rights auction mechanism to maximize the expectations of social surplus functions. The social surplus function is defined as:

$$W = \sum_j \pi_j + S + (\lambda - 1) \sum_j t_j,$$

Among them, π_j is the total (pre-transfer) enterprise j profit, S is the consumer's surplus, $\lambda > 1$ is the shadow cost of public funds, and t_j indicates the transfer from the company j to the planner. The auction decides each company's transfer payments and the structure of a market, that is, either two companies do not obtain production rights, or one company obtains production rights, or both have the right to production. Every firm j observes its fixed production cost of θ_j alone. The fixed costs θ_1 and θ_2 are the independent identical distribution over $[\theta^1, \theta^2]$. The density function $\phi(\cdot)$ and the distribution function $\Phi(\cdot)$ are all continuously differentiable. Suppose $\frac{\Phi(\cdot)}{\phi(\cdot)}$ is increasing about θ . Companies have a common marginal cost $c < 1$ and they produce homogenous products. The inverse market demand is $p(x) = 1 - x$. If both companies have obtained production rights, they will become Cournot competitors.

1. Write the problem of the optimal auction mechanism for production rights.
2. Describe its optimal auction mechanism.

Exercise 19.14 There is a seller ($i = 0$) owns two identical inseparable goods. At the same time, there are two buyers ($i = 1, 2$). For a unit of goods, buyers and sellers each have a willingness to pay θ_i and they are private information. The three participants' willingness to pay is subject to a uniform distribution over $[0, 1]$ and is independent of each other. This is a common knowledge. For the portion of the product that exceeds one unit, everyone's willingness to pay is 0.

1. Describe the effective allocation function $f(\theta_0, \theta_1, \theta_2)$ for these two units of goods.
2. For an incentive compatibility mechanism that Bayesian implemented effectively allocation over this two unit of goods, what is the interim expected utility of each participant?
3. What is the participation constraint for each participant?
4. When there is no external transfer, is there a mechanism that satisfies the participation constraint in question (3), as well as incentive compatibility constraints, and can effectively allocate these two units of goods?

Exercise 19.15 There are two types of students looking for work in the market: L and H . There are at least two potential employers. The profit increase for hiring i students is π_i , and $\pi_H > \pi_L$. Recruitment is conducted as follows: (1) student observes his own type (not observed by the employer); (2) student selects an unproductive education level e , which is costly and can be observed by the employer; (3) The employer offer a salary to future workers (current students), that is w , based on observed level of education (Burtland competition); (4) student chooses employer and become a worker. The utility function for i type workers who accept w wages is $U(w, e) = w - c_i(e)$. The corresponding employer's profit paid to the i type worker with w salary is $\pi_i - w$. Assume $C'_H(e) > C'_L(e)$ for all $e > 0$. Only considers pure strategic equilibrium in the whole process

1. Does this model satisfy Spence-Mirrlees single-crossing property? Explain your answer.
2. What is the level of education attained by a student who is of type L in separating equilibrium? Derive the expression of the maximum and minimum education level of workers of type H in the separating equilibrium.
3. Derive the expression of the maximum and minimum education level of all the students in the pooling equilibrium.
4. Which equilibrium outcome is more dominant? Explain your answer.
5. Assume that the government taxes t for education level (that is, a person who receives e year of education will pay a tax of te), and the government holds all the tax revenue. How does this tax change the separating and pooling equilibrium conditions you derived in question (2) and (3)?

6. Only consider effective separating equilibrium. Based on the (5) question, we derived the partial derivative of H and L workers' utility functions respect to tax t . Write the sign of the partial derivative. How does this tax affect the welfare of workers?

Exercise 19.16 Answers the following questions:

1. Under conditions of complete information, can single-crossing derive Manski monotonicity? If yes, prove it; if not, give a counterexample.
2. Under conditions of complete information, can single-crossing derive Bayesian monotonicity? If yes, prove it; if not, give a counterexample.

Exercise 19.17 Give an example to illustrate the independence between the Mankin monotonic property and the ex post monotonic property .

Exercise 19.18 (The non-Nash monotonicity of generalized VCG mechanism, Bergemann and Morris, 2008) Consider revised Vickrey mechanism in relative value conditions. We call $(\Theta, \mathbf{y}^*, \mathbf{t}^*)$ is generalised VCG mechanism. If the highest bidder gets the good and the payment price is $\max_{j \neq i} u_j(v_i(\theta_{-i}), \theta_{-i})$ (not directly dependent in θ_i).

1. Prove generalised VCG mechanism satisfies ex post monotonicity.
2. Prove generalised VCG mechanism satisfies ex post incentive compatibility.
3. However, generalised VCG mechanism doesn't satisfies Manski monotonicity, so that it is not Nash implementable. Give a proof.

Exercise 19.19 Prove: For an economic environment with a participation size of $n = 2$, if the social selection function f satisfies ex post incentive compatibility and post monotony, then f is ex post implementable.

Exercise 19.20 For an economic environment with a participation size of $N = 2$, if the social selection set F contains more than 1 elements. try to explain whether ex post incentive compatibility and ex post monotonicity are the sufficient condition for F to be ex post implementable.

Exercise 19.21 Bergemann and Morris (2008) derive: for general case that participants' size $N \geq 3$, if social choice function f satisfies ex post incentive compatibility and ex post monotonicity-no-veto-power condition, then f is ex post implementable. Try to explain whether this conclusion can be generalized to the case that participants' size $N = 2$.

Exercise 19.22 (Bergemann and Morris, 2005) Consider two participants 1 and 2. Their type sets are $\Theta_1 = \{\theta_1, \theta'_1\}$ and $\Theta_2 = \{\theta_2, \theta'_2\}$. The social feasible outcome set is $A = \{a, b, c\}$. With different outcomes and types of participants, their utility can be described by the following table: (where each of the two numbers in each box respectively represents the utility of participant 1 and 2):

a	θ_2	θ'_2
θ_1	$(1,0)$	$(-1,2)$
θ'_1	$(0,0)$	$(0,0)$

b	θ_2	θ'_2
θ_1	$(-1,2)$	$(1,0)$
θ'_1	$(0,0)$	$(0,0)$

c	θ_2	θ'_2
θ_1	$(0,0)$	$(0,0)$
θ'_1	$(1,1)$	$(1,1)$

Social planners consider the following social selection rule, F , to maximize the sum of the utility of two participants, described by the following table:

F	θ_2	θ'_2
θ_1	$\{a,b\}$	$\{a,b\}$
θ'_1	$\{c\}$	$\{c\}$

Prove: social choice rule F can not be ex post implemented.

Exercise 19.23 (Bergemann and Morris, 2005) Consider two participants 1 and 2. Their type sets are $\Theta_1 = \{\theta_1, \theta'_1, \theta''_1\}$ and $\Theta_2 = \{\theta_2, \theta'_2\}$. Assume the social feasible outcome set is $A = \{a, b, c, d\}$. With different outcomes and types of participants, their utility can be described by the following table: (where each of the two numbers in each box respectively represents the utility of participant 1 and 2):

a	θ_2	θ'_2	b	θ_2	θ'_2
θ_1	$(0,2)$	$(0,2)$	θ_1	$(0,0)$	$(0,0)$
θ'_1	$(-4,0)$	$(1,0)$	θ'_1	$(0,2)$	$(0,0)$
θ''_1	$(-4,0)$	$(-4,0)$	θ''_1	$(-4,0)$	$(0,0)$

c	θ_2	θ'_2	d	θ_2	θ'_2
θ_1	$(0,0)$	$(-4,0)$	θ_1	$(-4,0)$	$(-4,0)$
θ'_1	$(0,0)$	$(0,2)$	θ'_1	$(1,0)$	$(-4,0)$
θ''_1	$(0,0)$	$(0,0)$	θ''_1	$(0,2)$	$(0,2)$

Social planners consider the following social selection rule, F , to maximize the sum of the utility of two participants, described by the following table:

F	θ_2	θ'_2
θ_1	a	a
θ'_1	b	c
θ''_1	d	d

Assume that planners can transfer payments to participants. Each participant has a quasi-linear utility function.

Prove: When satisfying ex post budget balance, social choice rule F can not be ex post implemented.

19.12 Reference

Books and Monographs

- Bolton, P. and Dewatripont, M. (2005), *Contract Theory*, MIT Press.
- Corchón, L. C. (1996). *Theory of Implementation of Socially Optimal Decisions in Economics*, Macmillan Press, Ltd.
- Diamantaras, D. (2009). *A Toolbox for Economic Design*, Palgrave Macmillan.
- Laffont, J. J., and Martimort, D. (2002). *The Theory of Incentives: The Principal-Agent Model*, Princeton University Press.
- Mas-Colell, A., Whinston, M. D. and Green, J. (1995). *Microeconomic*, Oxford University Press, Chapter 14.
- Segal, I. (2007). *Lecture Notes in Contract Theory*, Stanford University.

Papers

- Abreu, D., and Matsushima H. (1990). "Virtual Implementation in Iteratively Undominated Strategies: Incomplete Information", Mimeo.
- Arrow, K. (1979). "The Property Rights Doctrine and Demand Revelation Under Incomplete Information", in *Economics and Human Welfare*, ed. by M. Boskin. New York: Academic Press.
- Anil Arya , Glover Jonathan and Rajan Uday (2000)"Implementation in Principal Agent Models of Adverse Selection", *Journal of Economic Theory*, 93: 87-109
- Bergemann, Dirk and Morris Stephen (2005). "Robust Mechanism Design", *Econometrica*, Vol. 73, No. 6, 527-566
- Bergemann, Dirk and Morris Stephen (2008). "Ex Post Implementation", *Games and Economic Behavior*, Vol. 63, 527-566
- Che, Y., and Kim, J. (2006). "Robustly Collusion-Proof Implementation", *Econometrica*, Vol. 74, 1063-1107.
- Cramton Peter , Gibbons, Robert and Klemperer Paul (1987). "Dissolving a Partnership Efficiently", *Econometrica*, Vol. 55, No. 3, 615-632.

- Cremer, J. and RP Mclean (1988). "Full Extraction of Surplus in Bayesian and Dominant Strategy Auctions", *Econometrica*, Vol. 56, No. 6, 1247-57.
- d'Aspremont, C., and Gerard-Varet, L. (1979). "Incentives and Incomplete Information", *Journal of Public Economics*, Vol. 11, 25-45.
- Dana, JD and Spier, KE (1994). "Designing a Private Industry: Government Auctions with Endogenous Market Structure", *Journal of Public Economics*, Vol. 53, No. 1, 127-147.
- Demski, JS and D Sappington (1984). "Optimal Incentive Contracts with Multiple Agents", *Journal of Economic Theory*, Vol. 33, No. 1, 152-171.
- Duggan, J. (1993). "Virtual Implementation in Bayesian Equilibrium with Infinite Types", Parts I and II, Mimeo.
- Dutta, B., and Arunava Sen (1994), "Bayesian Implementation: The Necessity of Infinite mechanisms", *Journal of Economic Theory*, Vol. 64, 130-141.
- Groves, T and Ledyard, J. (1985). "Incentive Compatibility Since 1972", *Chapter 2: Information, Incentive, and Economic Mechanisms*, T. Groves, R. Radner, and S. Reiter, eds.
- Jackson, M. . O. (1991). "Bayesian Implementation", *Econometrica*, Vol. 59, 461-477.
- Laffont, J., and Martimort, D. (1997). "Collusion Under Asymmetric Information", *Econometrica*, Vol. 65, 875-911.
- Laffont, J., and Martimort, D. (2000). "Mechanism Design with Collusion and Correlation", *Econometrica*, Vol. 68, 309-342.
- Meng, D., and Tian, G. (2014). "Two-Agent Collusion-Proof Mechanism with Correlation and Arbitrage", working paper.
- Matsushima, H. (1993). "Bayesian Monotonicity with Side Payments", *Journal of Economic Theory*, Vol. 59, 107-121.
- Myerson, R. B., and Satterthwaite, M. A. (1983). "Efficient Mechanisms for Bilateral Trading", *Journal of Economic Theory*, Vol. 29, 265-281.
- Ohashi, Yoshihiro (2012), "Two-person Ex Post Implementation", *Games and Economic Behavior*, Vol. 75, 435-440.
- Palfrey, T., and Srivastava, S. (1987). "On Bayesian Implementable Allocations", *Review of Economic Studies*, Vol. 54, 193-208.

- Palfrey, T., and Srivastava, S. (1989). "Mechanism Design with Incomplete Information: A Solution to the Implementation Problem", *Journal of Political Economy*, Vol. 97, 668-691.
- Palfrey, T., and Srivastava, S. (1989). "Implementation with Incomplete Information in Exchange Economies", *Econometrica*, Vol. 57, 115-134.
- Palfrey, T., and Srivastava, S. (1991). "Nash Implementation Using Undominated Strategy", *Econometrica*, Vol. 59, 479-502.
- Postlewaite, A., and Schmeidler, D. (1986). "Implementation in Differential Information Economies", *Journal of Economic Theory*, Vol. 39, 14-33.
- Repullo, R. (1986). "The Revelation Principle under Complete and Incomplete Information", in *Economic Organization as Games*, Binmore, K. and P. Dasgupta, eds., Oxford, Basil Blackwell.
- Tian, G. (1996b). "Continuous and Feasible Implementation of Rational Expectation Lindahl Allocations", *Games and Economic Behavior*, Vol. 16, 135-151.
- Tian, G. (1997). "Virtual Implementation in Incomplete Information Environments with General Sets of Alternatives and Types", *Journal of Mathematical Economics*, Vol. 28, 313-339.
- Tian, G. (1999b). "Bayesian Implementation in Exchange Economies with State Dependent Preferences and Feasible Sets", *Social Choice and Welfare*, Vol. 16, 99-119.
- Wilson, R. (1987). "Game-Theoretic Analyses of Trading Processes". In: Bewley, T. (Ed.), *Advances in Economic Theory: Fifth World Congress*. Cambridge University Press, Cambridge, 33-70.
- Yoshihiro, Ohashi (2012). "Two-person Ex Post Implementation", *Games and Economic Behavior*, Vol. 75, 435-440

Chapter 20

Dynamic Mechanism Design

20.1 Introduction

In Chapters 16-19, we discussed the optimal contract design and the general mechanism design, which have a common feature. Contracts, mechanisms and institutions are to a large extent synonymous. They all mean rules of the game, which describe what actions the parties can undertake, and what outcomes these actions would entail. Indeed, almost all games observed in daily life are not given by nature, but designed by someone or organization: in chess, basketball, football. Constitution is another typical example of mechanism design. The rules are designed to achieve desired outcomes: given an institutional environment and certain constraints faced by the designer, what rules are feasible? What mechanisms are optimal among those that are feasible?

While mechanism design theory considers designing rules of more general games such as institutional design, contract theory proved useful for more manageable smaller questions, concerning specific contracting practices. In addition, a contract is characterized by the following features: (1) a contract is designed by one of the parties themselves; (ii) participation is voluntary; (iii) parties may be able to renegotiate the contract later on

In practical environments, people often repeatedly interact each other, which is different from one time interaction. In this chapter, we discuss the dynamics of incentive contracts and dynamic mechanism design in different backgrounds.

20.2 Dynamic Contracts under Full Commitment

In this section we discuss a simple dynamic contracting problem, where the principal has perfect commitment capacity. We derive second-best dynamic contracts in three cases: (1) constant types, (2) independent types, and (3) correlated types over time.

20.2.1 Constant Types

For simplicity, consider a situation of monopoly selling in two periods, where the type of the agent (or consumer) is constant over time. We discuss the second-best contracting of the principal (or monopolist). Let us first start with the simplest case where the agent's type, $\theta \in \{\theta_H, \theta_L\}$, is constant over time. As in Chapter 16, we assume that the probability of type θ_L is β .

The timing of contracting is described in figure (20.1). At time 0, the agent learns his type θ ; at time 0.25, the principal offers a long term contract $(q_1, T_1(q_1); q_2, T(q_1, q_2))$ to the agent for both periods; at time 0.5, the agent decides whether to accept the contract or not, the interaction continues if accepting and is over otherwise; in period 1, the agent buys q_1 units, and pays for $T_1(q_1)$; in period 2, the agent buys q_2 units, and pays for $T(q_1, q_2)$ to the principal, where the transfer is dependent on the current and the past activity.

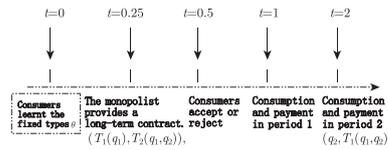


Figure 20.1: timing of contracting with constant types

The agent's utility function is:

$$u(\theta, q_1, q_2, T_1, T_2) = \theta v(q_1) - T_1 + \delta[\theta v(q_2) - T_2(q_1, q_2)],$$

where δ is the discount rate, and his status quo utility level is 0.

The principal's utility function is:

$$T_1 - cq_1 + \delta[T_2 - cq_2].$$

As the previous discussion of static adverse selection, if there is only one period, we know the second-best contract is described as $(q_H^{SB}, T_H^{SB}; q_L^{SB}, T_L^{SB})$, where $q_H^{SB} = q_H^{FB}$ and q_L^{SB} satisfy

$$\begin{aligned} \theta_H v'(q_H^{SB}) &= c, \\ \theta_L v'(q_L^{SB}) &= \frac{c}{1 - \left(\frac{1-\beta}{\beta} \frac{\theta_H - \theta_L}{\theta_L}\right)}. \end{aligned}$$

T_H^{SB} and T_L^{SB} satisfy

$$\begin{aligned} \theta_H v(q_H^{SB}) - T_H^{SB} &= U_H = \Delta\theta v(q_L^{SB}), \\ \theta_L v(q_L^{SB}) - T_L^{SB} &= U_L = 0. \end{aligned}$$

This is the result we discussed in Chapter 16 on nonlinear pricing that was first studied by Maskin and Riley (Rand J. of Economics, 1984).

If the principal has full commitment power, the second-best contract for two periods is twice the repetition of the static second-best contract as shown below. Since the principal can commit intertemporally, she can omit the information learned in period one, and strictly enforce the static second-best contract. In this case, the revelation principle remains valid. In the following, we discuss this long-term second-best contract, denoted by (q_1, q_2, T) , where T is the two-period discounted transfer, i.e., $T = T_1 + \delta T_2$. The complete form of the contract is $(q_{1H}, q_{2H}, T_H; q_{1L}, q_{2L}, T_L)$. If the contract is incentive feasible, then the below four conditions are satisfied:

$$U_H \equiv \theta_H(v(q_{1H}) + \delta v(q_{2H})) - T_H \geq U_L + \Delta\theta(v(q_{1L}) + \delta v(q_{2L})), \quad (20.2.1)$$

$$U_L \equiv \theta_L(v(q_{1L}) + \delta v(q_{2L})) - T_L \geq U_H - \Delta\theta(v(q_{1H}) + \delta v(q_{2H})); \quad (20.2.2)$$

$$U_H \geq 0, \quad (20.2.3)$$

$$U_L \geq 0. \quad (20.2.4)$$

The above can be described the following constrained optimal problem:

$$\begin{aligned} \max_{(q_{1H}, q_{2H}, U_H; q_{1L}, q_{2L}, U_L)} & (1 - \beta)[\theta_H(v(q_{1H}) + \delta v(q_{2H})) - U_H - c(q_{1H} + \delta q_{2H})] \\ & + \beta[\theta_L(v(q_{1L}) + \delta v(q_{2L})) - U_L - c(q_{1L} + \delta q_{2L})] \end{aligned} \quad (20.2.5)$$

and the same time satisfying the conditions of (20.2.1), (20.2.2), (20.2.3), and (20.2.4).

As the same precious logic, only the conditions of (20.2.1) and (20.2.4) are binding, so we have: $U_L = 0$, $U_H = \Delta\theta(v(q_{1L}) + \delta v(q_{2L}))$. Substituting them into the equation (20.2.5), we get the first order conditions for q_{1H} , q_{2H} ; q_{1L} , q_{2L} respectively:

$$\begin{aligned} \theta_H v'(q_{1H}^{SB}) &= \theta_H v'(q_{2H}^{SB}) = c, \\ \theta_L v'(q_{1L}^{SB}) &= \theta_L v'(q_{2L}^{SB}) = \frac{c}{1 - (\frac{1-\beta}{\beta} \frac{\theta_H - \theta_L}{\theta_L})}. \end{aligned}$$

Hence, the second-best long-term contract with full commitment for two periods is twice the repetition of the static second-best contract.

20.2.2 Independent Dynamic Types

Now we discuss the other extreme case of dynamic contracting, i.e., independent type over time.

The timing of contracting with agent's independent type is described in figure (20.2). Comparing the figure (20.1), at time 0, the agent only learns his type in period 1, and only at the time 1.5, he can learn his type in period 2. The type distribution of the consumer is i.i.d, the probability of θ_L is β , and the other setting is the same to the case of constant type.

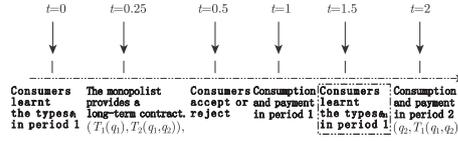


Figure 20.2: timing of contracting with independent types

We start with the interaction in the period 2. When θ_1 is the agent's first-period announcement on his type, then the contract $(q_{2H}(\theta_1), T_{2H}(\theta_1); q_{2L}(\theta_1), T_{2L}(\theta_1))$ in the second period depends on θ_1 , and the following constraints have to be specified: contract in period 2 should satisfy incentive compatibility

$$U_{2H}(\theta_1) \equiv \theta_H(\theta_1)v(q_{2H}(\theta_1)) - T_{2H}(\theta_1) \geq U_{2L}(\theta_1) + \Delta\theta v(q_{2L}(\theta_1)), \quad (20.2.6)$$

$$U_{2L}(\theta_1) \equiv \theta_L v(q_{2L}(\theta_1)) - T_{2L}(\theta_1) \geq U_{2H}(\theta_1) - \Delta\theta v(q_{2H}(\theta_1)), \quad (20.2.7)$$

where $U_{2H}(\theta_1)$ and $U_{2L}(\theta_1)$ are the second-period equilibrium utility for θ_H and θ_L respectively.

The participation constraint at second period is not binding since in two periods interaction under full commitment, if the agent's total discounted utility is not below the status quo utility, he should accept the two-period contracting. As such, $(q_{1H}, T_{1H}, q_{2H}(\theta_H), T_{2H}(\theta_H))$ and $(q_{1L}, T_{1L}, q_{2L}(\theta_L), T_{2L}(\theta_L))$ constitutes a two-period incentive compatible contract.

Let us also denote the first-period rents by $U_{1H} = \theta_H v(q_{1H}) - T_{1H}$ and $U_{1L} = \theta_L v(q_{1L}) - T_{1L}$. At date 1, beside the conditions of (20.2.6) and (20.2.7), the feasible incentive constraints at the first-period should include the following conditions:

$$U_{1H} + \delta[(1 - \beta)U_{2H}(\theta_H) + \beta U_{2L}(\theta_H)] \geq U_{1L} + \Delta\theta v(q_{1L}) + \delta[(1 - \beta)U_{2H}(\theta_L) + \beta U_{2L}(\theta_L)]; \quad (20.2.8)$$

$$U_{1L} + \delta[(1 - \beta)U_{2H}(\theta_L) + \beta U_{2L}(\theta_L)] \geq U_{1H} - \Delta\theta v(q_{1H}) + \delta[(1 - \beta)U_{2H}(\theta_H) + \beta U_{2L}(\theta_H)], \quad (20.2.9)$$

$$U_{1H} + \delta[(1 - \beta)U_{2H}(\theta_H) + \beta U_{2L}(\theta_H)] \geq 0, \quad (20.2.10)$$

$$U_{1L} + \delta[(1 - \beta)U_{2H}(\theta_L) + \beta U_{2L}(\theta_L)] \geq 0, \quad (20.2.11)$$

where the inequality functions (20.2.8) and (20.2.9) are the incentive compatibility constraints, and (20.2.10) and (20.2.11) the participation constraints.

Obviously, only the constraints of (20.2.8) and (20.2.11) are binding at the optimum of the principal's problem, and thus we have:

$$\delta[(1 - \beta)U_{2H}(\theta_L) + \beta U_{2L}(\theta_L)] = -U_{1L}, \quad (20.2.12)$$

$$\delta[(1 - \beta)U_{2H}(\theta_H) + \beta U_{2L}(\theta_H)] = -U_{1H} + \Delta\theta v(q_{1L}). \quad (20.2.13)$$

Thus, the incentive feasibility of second-period contract means the ex ante participation constraints, i.e. (20.2.12) and (20.2.13) should be binding.

Then, following the same logic of adverse selection under ex ante participation constraints discussed in Chapter 16, there is no allocative distortion, i.e., $q_{2H}(\theta_H) = q_{2H}(\theta_L) = q_H^*$ and $q_{2L}(\theta_H) = q_{2L}(\theta_L) = q_L^*$. At the same time, the second period rents are given by

$$U_{2H}(\theta_L) = \frac{-U_{1L}}{\delta} + \beta\Delta\theta v(q_H^*), \quad (20.2.14)$$

$$U_{2L}(\theta_L) = \frac{-U_{1L}}{\delta} - (1 - \beta)\Delta\theta v(q_H^*), \quad (20.2.15)$$

$$U_{2H}(\theta_H) = \frac{-U_{1H} + \Delta\theta v(q_L^{SB})}{\delta} + \beta\Delta\theta v(q_H^*), \quad (20.2.16)$$

$$U_{2L}(\theta_H) = \frac{-U_{1H} + \Delta\theta v(q_L^{SB})}{\delta} - (1 - \beta)\Delta\theta v(q_H^*). \quad (20.2.17)$$

$$(20.2.18)$$

Hence, the optimal consumptions corresponding to the inefficient draws of types in both periods are such that $q_{1L} = q_L^{SB}$ and $q_{2L} = q_L^*$, respectively. The agent gets a positive rent only when his type is θ_H at date 1, and his expected intertemporal informational rent over both periods is $\Delta\theta v(q_L^{SB})$, which is positive.

We then have the following proposition.

Proposition 20.2.1 *With independent types and a risk-neutral agent, the optimal long-term contract for two periods with full commitment combines the optimal static contract written interim for period 1 and the optimal static contract written ex ante for periods 2. In particular, the expected rent of the agent only equals the expectation of the agent's rent when he is efficient in period one and is worth $\Delta\theta v(q_L^{SB})$, which is positive.*

The same result would also be obtained if the risk-neutral agent can drop the contract at the period 2 when he does not get a positive rent in the second period. In this case, the participation constraint is needed, i.e.,

$$(1 - \beta)U_{2H}(\theta) + \beta U_{2L}(\theta) \geq 0, \forall \theta \in \{\theta_H, \theta_L\}.$$

Thus, it is enough to $U_{1L} = 0$ and $U_{1H} = \Delta\theta v(q_L^{SB})$ so that the right-hand sides of (20.2.12) and (20.2.13) equal to zero.

20.2.3 Correlated Types

Let us generalize the previous information structures and turn now to the more general case where the agent's type are imperfectly correlated over time. In this case, there are new features of second-best dynamic contracting. The problem firstly were studied by Baron and Besanko (1984), in which they derived the second-best contracts with correlated over time and full commitment of the principal.

In this subsection, we remain the hypothesis of full commitment, and study the long-term contract in the background of intertemporal price discrimination. In this case, the agent learned his first-period type $\theta_1 \in \{\theta_H, \theta_L\}$, and the principal only know its distribution. The agent's second-period type is imperfectly correlated to the first-period one, we assume their correlation is:

$$\beta_i = \text{prob}(\theta_2 = \theta_H | \theta_i) \text{ for } i = H, L.$$

We also suppose that $\beta_H \geq \beta_L$, which means the intertemporal correlation is positive, i.e., when type at period 1 is high-type, the probability that the type is a high-type is also higher than the probability that the type is a low-type at period 2. When $\beta_H = 1 > \beta_L = 0$, it is equivalent to the case of constant types; when $\beta_H = \beta_L = \beta$, it is equivalent to the case of independent types.

The timing of the contract is the same as in figure (20.2). In this framework, a direct revelation mechanism requires that the agent report the new information in each period he has learned on his current type. Typically, a direct revelation mechanism is a four-tuple $\{(T_1(\tilde{\theta}_1), q_1(\tilde{\theta}_1)), (T_2(\tilde{\theta}_1, \tilde{\theta}_2), q_2(\tilde{\theta}_1, \tilde{\theta}_2))\}$ for all pair $(\tilde{\theta}_1, \tilde{\theta}_2)$, where $\tilde{\theta}_1$ (resp. $\tilde{\theta}_2$) is the date 1 (resp. date 2) announcement on his first-period (resp. second-period) type.

The important point to note here is that the first-period report can now be used by the principal to update his beliefs on the agent's second period type. This report can be viewed as an informative signal that is useful for improving second-period contracting. The difference is that now the signal used by the principal to improve second-period contracting is not exogenously given by nature but comes from the first-period report $\tilde{\theta}_1$ of the agent on his type θ_1 . Hence, this signal can be strategically manipulated by the agent in the first period in order to improve his second-period rent.

If the agent's type θ_t in period t accepts the contract (T_t, q_t) , his utility in period t is $U_{it} = \theta_i v(q_t) - T_t$. We assume that $v(q_t)$ is concave, and further the agent is infinitely averse. Assume that the common discount rate is δ . In deriving the second-best long-term contract, the principal's objective is to maximize her total discount utility while the contract is subject to some intertemporal incentive feasible constraints.

We first discuss the second-period constraints, given the agent's first-period report $\tilde{\theta}_1$. Because the agent is infinitely risk averse below zero wealth, his ex post participation constraint in period 2 is written as:

$$U_{H2}(\tilde{\theta}_1) = \theta_H v(q_{H2}(\tilde{\theta}_1)) - T_{H2}(\tilde{\theta}_1) \geq 0, \quad (20.2.19)$$

$$U_{L2}(\tilde{\theta}_1) = \theta_L v(q_{L2}(\tilde{\theta}_1)) - T_{L2}(\tilde{\theta}_1) \geq 0. \quad (20.2.20)$$

Moreover, inducing information revelation by the agent in period 2 requires to satisfy the following incentive constraints:

$$U_{H2}(\tilde{\theta}_1) = \theta_H v(q_{H2}(\tilde{\theta}_1)) - T_{H2}(\tilde{\theta}_1) \geq U_{L2}(\tilde{\theta}_1) + \Delta\theta v(q_{L2}(\tilde{\theta}_1)) \quad (20.2.21)$$

$$U_{L2}(\tilde{\theta}_1) = \theta_L v(q_{L2}(\tilde{\theta}_1)) - T_{L2}(\tilde{\theta}_1) \geq U_{H2}(\tilde{\theta}_1) - \Delta\theta v(q_{H2}(\tilde{\theta}_1)) \quad (20.2.22)$$

From the equations of (20.2.21) and (20.2.22), we get $q_{H2}(\tilde{\theta}_1) \geq q_{L2}(\tilde{\theta}_1)$, and thus the monotonic condition remains valid.

Given the agent's first-period announcement $\tilde{\theta}_1$, in deriving the second-best contract, the principal is to solve the following problem:

$$\begin{aligned} \pi_2(\tilde{\theta}_1, q_{H2}(\tilde{\theta}_1), q_{L2}(\tilde{\theta}_1)) &= \max \beta(\tilde{\theta}_1)(T_{H2}(\tilde{\theta}_1) - cq_{H2}(\tilde{\theta}_1)) \\ &\quad + (1 - \beta(\tilde{\theta}_1))(T_{L2}(\tilde{\theta}_1) - cq_{L2}(\tilde{\theta}_1)) \end{aligned} \quad (20.2.23)$$

subject to the constraints of (20.2.21), (20.2.22), (20.2.19), and (20.2.20).

Again, only the constraints of (20.2.21) and (20.2.20) are binding, and then the second-period monopolist's profit is:

$$\begin{aligned} \pi_2(\tilde{\theta}_1, q_{H2}(\tilde{\theta}_1), q_{L2}(\tilde{\theta}_1)) &= \beta(\tilde{\theta}_1)(\theta_H v(q_{H2}(\tilde{\theta}_1)) - cq_{H2}(\tilde{\theta}_1)) + \\ &\quad (1 - \beta(\tilde{\theta}_1))(\theta_L v(q_{L2}(\tilde{\theta}_1)) - cq_{L2}(\tilde{\theta}_1)) - \\ &\quad \beta(\tilde{\theta}_1)\Delta\theta v(q_{L2}(\tilde{\theta}_1)). \end{aligned}$$

Let us now consider period 1. The type θ_i -agent's rent in period 1 is:

$$\begin{aligned} U_{H1} &= \theta_H v(q_{H1}) - T_{H1}, \\ U_{L1} &= \theta_L v(q_{L1}) - T_{L1}, \end{aligned}$$

where $q_{i1} = q_1(\theta_i)$, $T_{i1} = T_1\theta_i$, $i \in \{H, L\}$.

The type θ_i agent's second-period rent is:

$$\begin{aligned} EU_2(\theta_H) &= \beta_H \Delta\theta v(q_{L2}(\theta_H)), \\ EU_2(\theta_L) &= \beta_L \Delta\theta v(q_{L2}(\theta_L)). \end{aligned}$$

Thus, the incentive compatibility constraints can be written as:

$$\begin{aligned} U_{H1} + \delta\beta_H \Delta\theta v(q_{L2}(\theta_H)) &\geq U_{L1} + \Delta\theta v(q_{L1}) + \delta\beta_H \Delta\theta v(q_{L2}(\theta_H)), \\ U_{L1} + \delta\beta_L \Delta\theta v(q_{L2}(\theta_L)) &\geq U_{H1} - \Delta\theta v(q_{H1}) + \delta\beta_L \Delta\theta v(q_{L2}(\theta_L)). \end{aligned} \quad (20.2.24)$$

From the inequality functions of (20.2.24) and (20.2.25), we get:

$$\delta\Delta\theta(v(q_{L2}) - v(q_{H2}))(\theta_H)(\beta_H - \beta_L) + \Delta\theta(v(q_{H1}) - v(q_{L1})) \geq 0.$$

From the above, we have $q_{H2} \geq q_{L2}$, and thus: $q_{H1} \geq q_{L1}$. Hence, at the both periods, the incentive compatible consumption plans both satisfy the monotonic condition.

Due to the agent's infinite risk aversion, his participation constraint in period one is written as

$$U_{H1} \geq 0, \quad (20.2.26)$$

$$U_{L1} \geq 0. \quad (20.2.27)$$

Then, in designing whole long-term second-best contract, the principle is to solve the following problem:

$$\begin{aligned} \pi = \max & \beta(T_{H1} - cq_{H1}) + (1 - \beta)(T_{L1} - cq_{L1}) + \\ & \delta[\beta\pi_2(\theta_H, q_{H2}(\theta_H), q_{L2}(\theta_H)) + \\ & (1 - \beta)\pi_2(\theta_L, q_{H2}(\theta_L), q_{L2}(\theta_L))] \end{aligned}$$

subject to the constraints of (20.2.24) and (20.2.25), (20.2.26), and (20.2.27).

From the same logic, only the constraints of (20.2.24) and (20.2.27) are binding. The above optimal problem can be rewritten as:

$$\begin{aligned} \pi = \max_{\{(q_{L1}, q_{L2}(\theta_L), q_{H2}(\theta_L)), (q_{H1}, q_{L2}(\theta_H), q_{H2}(\theta_H))\}} & \beta(\theta_H v(q_{H1}) - cq_{H1} - U_{H1}) + \\ & (1 - \beta)(\theta_L v(q_{L1}) - cq_{L1}) + \\ & \delta[\beta\pi_2(\theta_H, q_{H2}(\theta_H), q_{L2}(\theta_H)) + \\ & (1 - \beta)\pi_2(\theta_L, q_{H2}(\theta_L), q_{L2}(\theta_L))], \end{aligned}$$

where

$$U_{H1} = \Delta\theta v(q_{L1}) + \delta\beta_H \Delta\theta(v(q_{L2}(\theta_L)) - v(q_{L2}(\theta_H))),$$

and $\pi_2(\theta_H, q_{H2}(\theta_H), q_{L2}(\theta_H)), \pi_2(\theta_L, q_{H2}(\theta_L), q_{L2}(\theta_L))$ is derived from the inequality (20.2.24).

The first order condition is the following:

$$\theta_H v'(q_{H1}) = c. \quad (20.2.28)$$

Hence, $q_{H1} = q_H^*$, which means that for the type θ_H -agent, his first-period consumption is efficient (first-best). From the following equation

$$\left(1 - \frac{\Delta\theta}{\theta_L} \frac{\beta}{1 - \beta}\right) \theta_L v'(q_{H1}) = c, \quad (20.2.29)$$

we have $q_{L1} = q_H^{SB}$, and thus, for the type θ_L agent, his first-period is not efficient, and downward distortion exists.

At the same time, because

$$\theta_H v'(q_{H2}(\theta_H)) = c, \quad (20.2.30)$$

$$\theta_H v'(q_{H2}(\theta_L)) = c, \quad (20.2.31)$$

we have $q_{H2}(\theta_H) = q_{H2}(\theta_L) = q_H^*$, and then for the second-period type θ_H agent, his consumption is also efficient.

Also, by

$$\theta_L v'(q_{L2}(\theta_H)) = c, \quad (20.2.32)$$

we get $q_{L2}(\theta_H) = q_L^*$. Thus, for both the first-period type θ_H and the second-period type θ_L , their consumption are efficient too.

Since

$$\left(1 - \frac{\Delta\theta}{\theta_L} \frac{(1-\beta)\theta_L + \beta\beta_H}{(1-\theta_L)(1-\theta)}\right) \theta_L v'(q_{L2}(\theta_L)) = c, \quad (20.2.33)$$

at both periods, for the type θ_L agent, his consumption $q_{L2}(\theta_L)$ are equivalent to the case of constant types, i.e., $\beta_H = 1, \beta_L = 0$, and then we have

$$\left(1 - \frac{\Delta\theta}{\theta_L} \frac{\beta}{1-\beta}\right) \theta_L v'(q_{L2}(\theta_L)) = c,$$

we get the second-best consumption, $q_{L2}(\theta_L) = q_{L1} = q_H^{SB}$.

As the case of independent types, i.e., $\beta_H = \beta_L = \beta$, we have:

$$\left(1 - \frac{\Delta\theta}{\theta_L} \frac{\beta}{(1-\beta)^2}\right) \theta_L v'(q_{L2}(\theta_L)) = c.$$

Hence, we have $q_{L2}(\theta_L) < q_{L1} = q_H^{SB}$.

20.3 Dynamic Contracts under Different Commitment Power

Now we discuss how the different degree of commitment affects the agent's incentives. If the principal has no full commitment power, the agent may have no incentives to reveal his type in early time, since the principal can use her information to reduce the agent's rent in later time. In the regulation background, the regulated price is dependent on the cost reported by a regulated monopolist. If the cost information is used in the future regulation, there is phenomena of ratchet effect, which destroys the revelation incentives of true information by the regulated monopolist. This problem is deeply analyzed by Freixas, Guesnerie and Tirole (1985).

In this section, we first assume that the principal (designer) can commit to a contract forever, and then consider what happens when she cannot commit against modifying the contract as new information arrives. We discuss the dynamic contracting of monopoly selling. Hart and Tirole (1988) analyzed this problem in detail, and we adopt the simplified version from segal (2010).

For this purpose, we consider a principal-agent relationship that is repeated over time, over a finite or infinite horizon. First, as a benchmark, we suppose that the agent's type θ is realized ex ante and is then constant over time. For simplicity, we focus on a 2-period P-A model with a constant type and stationary payoffs. That is, suppose the payoffs are stationary. The common discount rate is δ .

The agent's type θ is realized before the relationship starts and is the same in both periods. Suppose the type space is $\{\theta_H, \theta_L\}$, $\theta_H > \theta_L > 0$, and

the probability of θ_H is β . We will allow δ to be smaller or greater than one, the latter can be interpreted as capturing situations in which the second period is very long.

The outcome of contracting in this model depends on the degree of the principal's ability to commit not to modify the contract after the rest period. If the principal cannot commit, she may modify the optimal commitment contract using the information revealed by the agent in the rest period. To what extent can the principal commit not to modify the contract and avoid the ratchet and renegotiation problems? The literature has considered three degrees of commitment:

1. Full Commitment: The principal can commit to any contract ex-ante. We have already considered this case. The Revelation Principle works, and we obtain a simple replication of the static model.
2. Long Term Renegotiable Contracts: The principal cannot modify the contract unilaterally, but the contract can be renegotiated if both the principal and the agent agree to do it. Thus, the principal cannot make any modifications that make the agent worse off, but can make modifications that make the agent better off. Thus, the ratchet problem does not arise in this setting (the high type would not accept a modification making him worse off), but the renegotiation problem does arise.
3. Short-Term Contracts: The principal can modify the contract unilaterally after the rest period. This means that the ex ante contract has no effect in the second period, and after the rest period the parties contract on the second-period outcome. This setting gives rise to both the ratchet problem and the renegotiation problem.

A key feature of the cases without commitment is that when δ is sufficiently high, the principal will no longer want the agent to reveal his type fully in the rest period, she prefers to commit herself against contract modification by having less information at the modification stage. For intermediate levels of δ , the principal will prefer to have partial revelation of information in the rest period, which allows her to commit herself against modification. This partial revelation will be achieved by the agent using a mixed strategy, revealing his type with some probability and pooling with some probability.

For simplicity, we assume the consumer (agent) has unit demand of the good. Denote $x_{it} \in \{0, 1\}$ as the purchase decision of type i agent. Suppose the production cost is normalized to 0, p_t is the price in time t .

The total discounted utility of the type i agent is:

$$U(x_{i1}, x_{i2}) = \sum_{t=1}^2 \delta^{t-1} (\theta_i x_{it} - p_t).$$

The total discounted profit of the principal is

$$\pi(p_1, p_2) = \sum_{t=1}^2 \delta^{t-1} p_t [\beta x_{H,t} + (1 - \beta) x_{L,t}],$$

where p_t is the price of the good at time t .

20.3.1 Contracting with Full Commitment

As a benchmark, we first discuss the simplest case, i.e. the one-period optimal contract.

Note that if $p \leq \theta_L$, $x_H = x_L = 1$, and the profit is p ; if $\theta_L < p \leq \theta_H$, $x_H = 1, x_L = 0$, and the profit is βp ; if $p > \theta_H$, $x_H = x_L = 0$, and the profit is 0. Let $\bar{\beta} = \frac{\theta_L}{\theta_H}$. Then the principal's optimal price then is:

$$p^*(\beta) = \begin{cases} \theta_H & \text{if } \beta > \bar{\beta}, \\ \theta_L & \text{if } \beta < \bar{\beta}, \\ \theta_H \text{ or } \theta_L & \text{if } \beta = \bar{\beta}. \end{cases}$$

and the agent's choice is:

$$(x_H^*, x_L^*) = \begin{cases} (1, 0) & \text{if } \beta > \bar{\beta}, \\ (1, 1) & \text{if } \beta < \bar{\beta}, \\ (1, 0) \text{ or } (1, 1) & \text{if } \beta = \bar{\beta}. \end{cases}$$

The principal's one-period profit is:

$$\pi_1^*(\beta) = \max\{\beta\theta_H, \theta_L\}.$$

If $\beta < \bar{\beta}$, the utility of type θ_H is $U(\theta_H) = \theta_H - \theta_L$, which is also the rent of the type θ_H .

Suppose now that there are two periods. We already know that any two-period contract can be replaced with a repetition of the same static contract. Therefore, the optimal commitment contract sets the same price, i.e., $p_t^* = p^*$, $t = 1, 2$. The principal's profit is $\pi^*(\beta) = (1 + \delta)\pi_1^*(\beta)$, which is the most profit as she can earn.

20.3.2 Dynamic Contracting with No Commitment

If the principal lacks full commitment, will she want to modify this contract after the rest period? This depends on $p^*(\beta)$. If $\beta \leq \bar{\beta}$, and therefore $p_1^*(\beta) = \theta_L$, then the two types pool in the first period, and the principal receives no information. As such, she has no incentives to modify the same optimal static contract in the second period.

However, when $\beta > \bar{\beta}$, the monopolist may have incentives to modify the contract unilaterally. Indeed, if the consumer does not purchase the

good, i.e., $x_1 = 0$, the monopolist know the consumer is θ_L , and then she will modify the contract so that $p_2^*(\beta) = \theta_H$. Once the principal knows following a purchase that she deals with a high type and following no purchase that she deals with a low type, she wants to restore efficiency for the low type by the price to θ_L in the second period. Expecting this price reduction following no purchase, the high type will not buy in the rest period, and then the contract is not implementable. Thus, when the principal deals optimally with the renegotiation problem, the ratcheting problem may arise and further hurt the principal. As result, there are only short-term contracts in this situation.

In the following, we suppose that $\beta > \bar{\beta}$. Let the parties play the following two-period game: (1) Principal offers price p_1 ; (2) Agent chooses $x_1 \in \{0, 1\}$; (3) Principal offers price p_2 ; (4) Agent chooses $x_2 \in \{0, 1\}$.

We solve for the principal's preferred weak Perfect Bayesian Equilibrium (PBE) in this extensive-form game of incomplete information. We do this by considering possible continuation PBEs following different rest-period price choices p_1 . Then the principal's optimal PBE can be constructed by choosing the optimal continuation PBE following any price choice p_1 , and finding the price p_1^* that gives rise to the optimal continuation PBE for the principal.

The principal's strategy in the continuation game is a function $p_2(x_1)$, which sets the second-period price following the agent's first-period choice x_1 . Let $\hat{\beta}(x_1) = \text{prob}(\theta) = \text{prob}(\theta_H|x_1)$ be the principal's posterior belief that the agent is a high type after x_1 is chosen. In the following, we analyze three possible equilibrium types: "full revelation" equilibrium, "no revelation" equilibrium and "partial revelation" equilibria.

"Full Revelation" Equilibrium:

Under the first-period price p_1 , there is only one possible "full revelation" (or separating) equilibrium: $x_H = 1$ and $x_L = 0$. The principal's posterior belief is given by

$$\hat{\beta}(x_1) = \begin{cases} 1 & \text{if } x_1 = 1, \\ 0 & \text{if } x_1 = 0. \end{cases}$$

The principal's second-period price is:

$$p_2(x_1) = \begin{cases} \theta_H & \text{if } x_1 = 1, \\ \theta_L & \text{if } x_1 = 0. \end{cases}$$

Therefore, if such equilibrium exists, the incentive compatibility constraints for the types $\{\theta_H, \theta_L\}$ should hold:

$$\theta_H - p_1 \geq \delta(\theta_H - \theta_L), \quad \text{if type is } \theta_H, \quad (20.3.34)$$

$$\theta_L - p_1 \leq 0, \quad \text{if type is } \theta_L. \quad (20.3.35)$$

In the incentive compatibility inequality function (20.3.34) for the type θ_H , the left hand is his utility of first-period consumption, and the right hand is his utility of second-period consumption (with second-period price $p_2 = \theta_L$), the inequality (20.3.34) can be rewritten as:

$$p_1 \leq (1 - \delta)\theta_H + \delta\theta_L. \quad (20.3.36)$$

In the incentive compatibility inequality function (20.3.35) for the type θ_L , the left hand is his utility of first-period consumption, and the right hand is his utility of second-period consumption (with second-period price $p_2 = \theta_H$), the inequality (20.3.35) can be rewritten as

$$p_1 \geq \theta_L. \quad (20.3.37)$$

From the inequality functions (20.3.36) and (20.3.37), we have $\delta \leq 1$. Thus *the necessary condition for existence of "full revelation" equilibrium is that $\delta \leq 1$* . If $\delta > 1$, then the incentive compatibility inequality function (20.3.36) for type θ_H means $p_1 \leq \theta_L$, then both types of θ_H and θ_L choose to buy at the first-period, and for the type θ_L , he will not buy at the second-period (since $p_2 = \theta_H$), belonging to "take the money and run".

Therefore, when $\delta \leq 1$ holds, "full revelation" equilibrium exists, and $p_1^R = (1 - \delta)\theta_H + \delta\theta_L$, the principal's profit is

$$\pi^R = \beta[(1 - \delta)\theta_H + \delta\theta_L] + \delta[\beta\theta_H + (1 - \beta)\theta_L] = \beta\theta_H + \delta\theta_L,$$

and the rent for type θ_H is $\delta(\theta_H - \theta_L)$.

"No Revelation" Equilibrium

In the "no revelation" (or pooling) equilibrium, $x_{H1} = x_{L1}$. Then $\hat{\beta}(x_1) = \beta > \bar{\beta}$, and $p_2 = \theta_H$. Although there are two possible "no revelation" equilibria, i.e., $x_{H1} = x_{L1} = 0$, and $x_{H1} = x_{L1} = 1$, for the principal, the optimal (profit maximizing) "no revelation" equilibrium is $x_{H1} = x_{L1} = 1$, therefore *the necessary condition for such equilibrium is that the participation constraint should hold, and the first-period price is $p_1 = \theta_L$* .

The principal's profit is

$$\pi^p = \theta_L + \delta\beta\theta_H,$$

and the rent for type θ_H is $\theta_H - \theta_L$.

Now we compare π^R and π^p . For $\delta \leq 1$, by

$$\pi^R - \pi^p = (\beta\theta_H - \theta_L)(1 - \delta),$$

we have $\pi^R \geq \pi^p$.

"Partial Revelation" Equilibrium

Let ρ_i be the probability of the purchase for type θ_i under the first-period price p_1 . Since type θ_H has more incentives to purchase than the type θ_L , so it must be true that $\rho_H > \rho_L$, and then the ex post belief is

$$\hat{\beta}(x_1 = 1) = \frac{\beta\rho_H}{\beta\rho_H + (1-\beta)\rho_L} \geq \beta > \bar{\beta},$$

Therefore, $p_2(x_1 = 1) = \theta_H$. There are two possibilities for the second-period price:

(1) If $\hat{\beta}(x_1 = 0) \leq \bar{\beta}$, then $p_2(1) = \theta_H$ and $p_2(0) = \theta_L$.

By $\rho_H > \rho_L \geq 0$, we have

$$\theta_H - p_1 \geq \delta(\theta_H - \theta_L),$$

which means that $p_1 \leq (1-\delta)\theta_H + \delta\theta_L$. If $\delta > 1$, both types have incentives to buy in period 1, then $\rho_H = \rho_L = 1$, it is not "partial revelation". If $\delta \leq 1$, the principle has two options:

(a) $p_1 = (1-\delta)\theta_H + \delta\theta_L$. Then $\rho_H \in (0, 1]$, $\rho_L = 0$, and her profit is

$$\pi^S = \rho_H\beta[(1-\delta)\theta_H + \delta\theta_L] + \delta[\beta\theta_H + (1-\beta)\theta_L],$$

which is equal to that of the "full revelation" case, i.e., $\pi^S = \pi^R$,

(b) $p_1 = \theta_L$. Then $\rho_H = 1$, $\rho_L \in (0, 1)$, and her profit is $\pi^S = \pi^p$. Since $\pi^R > \pi^p$, the principal chooses $p_1 = (1-\delta)\theta_H + \delta\theta_L$, which is the same as the "full revelation" case.

(2) If $\hat{\beta}(x_1 = 0) \geq \bar{\beta}$, then $p_2(x_1 = 1) = p_2(x_1 = 0) = \theta_H$.

Since $\hat{\beta}(x_1 = 0) = \frac{\beta(1-\rho_H)}{\beta(1-\rho_H) + (1-\beta)(1-\rho_L)} > 0$, then $\rho_H < 1$, and thus it must be that $p_1 \geq \theta_H$. Indeed, if $p_1 < \theta_H$, then $\rho_H = 1$, and $\rho_L = 0$. The first-period price must be $p_1 = \theta_H$. Due to

$$\hat{\beta}(x_1 = 0) = \frac{\beta(1-\rho_H)}{\beta(1-\rho_H) + (1-\beta)} \geq \bar{\beta},$$

we have

$$\rho_H \leq \hat{\rho}_H = \frac{\beta\theta_H - \theta_L}{\beta(\theta_H - \theta_L)},$$

and thus the principal's profit is

$$\pi^S = \beta\theta_H[\hat{\rho}_H + \delta].$$

20.3. DYNAMIC CONTRACTS UNDER DIFFERENT COMMITMENT POWER 507

If $\delta \leq 1$, then $\pi^R \geq \pi^p$. We compare π^R and π^S .

$$\begin{aligned}
 \pi^S - \pi^R &= \beta\theta_H\hat{\rho}_H + \beta\theta_H\delta - \beta\theta_H - \delta\theta_L \\
 &= \beta\theta_H\hat{\rho}_H + \delta(\beta\theta_H - \theta_L) - \beta\theta_H \\
 &= \theta_H \frac{\beta\theta_H - \theta_L}{\theta_H - \theta_L} + \delta(\beta\theta_H - \theta_L) - \beta\theta_H \\
 &= (\beta\theta_H - \theta_L) \left[\frac{\theta_H}{\theta_H - \theta_L} + \delta \right] - \beta\theta_H \\
 &= \beta\theta_H \left[\frac{\theta_H}{\theta_H - \theta_L} + \delta - 1 \right] - \theta_L \left[\frac{\theta_H}{\theta_H - \theta_L} + \delta \right].
 \end{aligned}$$

Therefore, $\pi^S \geq \pi^R$ is equivalent to

$$\beta \geq \beta^{RS} \equiv \frac{\theta_L \theta_H + \delta(\theta_H - \theta_L)}{\theta_H \theta_L + \delta(\theta_H - \theta_L)}.$$

In the following, we discuss the principal's contract choice under $\delta > 1$. Since there is no "full revelation" equilibrium, we need to compare π^p and π^S .

$$\begin{aligned}
 \pi^S - \pi^p &= \beta\theta_H\hat{\rho}_H + \beta\theta_H\delta - \theta_L - \delta\beta\theta_H \\
 &= \beta\theta_H\hat{\rho}_H - \theta_L \\
 &= \frac{\beta\theta_H - \theta_L}{\theta_H - \theta_L} \theta_H - \theta_L \\
 &= \frac{\theta_H}{\theta_H - \theta_L} \left[\beta - \frac{\theta_L}{\theta_H} \left(2 - \frac{\theta_L}{\theta_H} \right) \right].
 \end{aligned}$$

Thus $\pi^S \geq \pi^p$ is equivalent to

$$\beta \geq \beta^{PS} \equiv \frac{\theta_L}{\theta_H} \left(2 - \frac{\theta_L}{\theta_H} \right).$$

From the above discussion, we have the following result.

Proposition 20.3.1 *In the lack of commitment, the optimal contract for the principal entail:*

If $\beta \leq \bar{\beta}$, the principal chooses the same contract as the full commitment case and it can be implemented;

If $\delta \leq 1$ and $\beta \in (\bar{\beta}, \beta^{RS}]$, the optimal contract for the principal is $p_1 = (1 - \delta)\theta_H + \delta\theta_L$, $p_2(x_1 = 1) = \theta_H$, and $p_2(x_1 = 0) = \theta_L$;

If $\delta \leq 1$ and $\beta > \beta^{RS}$, the principal chooses the "full revelation" equilibrium, the optimal contract is $p_1 = \theta_H$, and $p_2(x_1 = 1) = p_2(x_1 = 0) = \theta_H$;

If $\delta > 1$ and $\beta \in (\bar{\beta}, \beta^{PS}]$, the principal chooses the “no revelation” equilibrium, the optimal contract is $p_1 = \theta_L$, and $p_2(x_1 = 1) = p_2(x_1 = 0) = \theta_H$;

If $\delta > 1$ and $\beta > \beta^{PS}$, the principal chooses the “partial revelation” equilibrium, the optimal contract is $p_1 = \theta_H$, and $p_2(x_1 = 1) = p_2(x_1 = 0) = \theta_H$.

20.3.3 Dynamic Contracts with Partial Commitment

Next, we turn to the case of partial commitment, in this case, the principal can revise contract only with the consent of the agent. The timing of this dynamic contracting is the following: (1) principal offers a contract $(p_1; p_2(x_1))$; (2) agent chooses x_1 ; (3) principal offers a new contract $p'_2(x_1)$; (4) agent accepts $p'_2(x_1)$ or rejects and sticks to the original contract; (4) agent chooses x_1 .

We will look for the principal’s preferred PBE of this contract. Note that a renegotiation offer $p'_2(x_1)$ following the agent’s choice x_1 will be accepted by the agent if and only if $p'_2(x_1) < p_2(x_1)$. Therefore, a long-term renegotiable contract commits the principal against raising the price, but does not commit her against lowering the price. Analysis is simplified by observing that the principal can, without loss of generality, offer a contract that is not renegotiated in equilibrium. We first define renegotiation-proof contract.

Definition 20.3.1 A Renegotiation-Proof (RNP) contract is one that is not renegotiated in the continuation equilibrium.

The following principle can simplify analysis of the above problem.

Proposition 20.3.2 (Renegotiation-Proof Principle) *For any PBE outcome of the contract, there exists another PBE that implements the same outcome and in which the principal offers a RNP contract.*

PROOF. If the principal offers $(p_1; p_2(1), p_2(0))$ and in equilibrium renegotiates to $p'_2(x_1) < p_2(x_1)$ after the agent’s choice of $x_1 \in \{0, 1\}$, then the contract $(p_1; p'_2(1), p'_2(0))$ is RNP. \square

Dewatripont (1989) discussed the Renegotiation-Proof Principle in detail. The RNP Principle is similar to the Revelation Principle, i.e., for any equilibrium, one can always find a new equilibrium that has the same equilibrium outcome. By the RNP Principle, the optimal dynamic contract can be implemented by a RNP contract. As such, we can restrict our attention to RNP mechanisms.

When $\beta \leq \bar{\beta}$, with full commitment power, all consumers will purchase at date 1, and thus the consumer’s action will not modify the principal’s belief. As such, a contract with full commitment can be always implementable even though the principal may modify the contract without the

consent of the agent, and of course it can be implementable with the consent of the agent. As such, we only need to consider the case of $\beta > \bar{\beta}$.

First, the contract with full commitment is not a RNP contract. This is because, under such a contract, the principal offers $p_1 = p_2(1) = p_2(0) = \theta_H$, and the agent chooses $x_1(\theta_H) = x_2(\theta_H) = 1, x_1(\theta_L) = x_2(\theta_L) = 0$. However, with $x_1 = 0$ observed, the consumer can be identified as θ_L type, then $p_2(0) = \theta_L$ will be accepted by the consumer with θ_L and thus this will increase the principal's profit. As such, she has incentives to modify the contract. But, she cannot do so without the consent of the agent.

Secondly, any short-term contract with no commitment can be implementable with the consent of the agent. This is because, if p_1^N and $(p_2(1)^N, p_2(0)^N)$ are two short-term contracts, it is clear that with x_1 observed, the contract $(p_1^N; p_2(1)^N, p_2(0)^N)$ in the second period can be implemented. If the principal provides a new contract $(p'_2(1), p'_2(0))$ that can be accepted by the agent, we then must have $p'_2(1) < p_1^N(1)$ or $p'_2(0) < p_1^N(0)$. But, for the profit maximizing dynamic contract, given x_1 and the principal's belief, $p_1^N(x_1)$ is an optimal choice. As such, the principal does not have incentives to modify the original contract.

Thirdly, do long-term renegotiable contracts offer an advantage relative to short-term contracts? This is true when the ability to commit to a low price is useful. Consider the following contract $(p_1 = \theta_H + \delta\theta_L; p_2(1) = 0, p_2(0) = \theta_L)$. This contract provides a separating equilibrium under the consent of the agent. Indeed, θ_H -consumer's utility with purchase in the first period is

$$\theta_H - (\theta_H + \delta\theta_L) + \delta\theta_H = \delta(\theta_H - \theta_L) > 0,$$

and his utility without purchase in the first period is

$$\delta(\theta_H - \theta_L).$$

Thus, the incentive compatibility and individual rationality constraints are satisfied for θ_H .

As for θ_L , his utility with purchase in the first period is

$$\theta_L - (\theta_H + \delta\theta_L) + \delta\theta_L = -(\theta_H - \theta_L) < 0,$$

and his utility without purchase in the first period is

$$\delta(\theta_L - \theta_L = 0).$$

Thus, the incentive compatibility and individual rationality constraints are satisfied for θ_L .

In addition, the principal will not provide a new the contract $(p'_2(1), p'_2(0))$ for the second period. This is because, if the agent agrees, then

$p'_2(1) < 0$ or $p'_2(0) < \theta_L$, which makes the principal worse off. Hence, the contract with consent of the agent is RNP.

However, the contract without the consent of the agent is not implementable. Indeed, once x_1 is observed, the consumer must be θ_H -type, and then the principal will choose $p_2(1) = \theta_H$. As such, the contract without commitment power is not implementable.

20.4 Sequential Screening

From the above discussion, we learn that in dynamic mechanism design, the agent has information advantage with different degree over time. In dynamic contracting environments, these information advantages will turn into the information rent for agent. In fact, the agent (consumer) may need some time to learn his future consumption type, and has some private information in different degree over time, the principal (monopolist) use some mechanism to elicit the private information over time, such as dynamic contracting to screen agents over time. Courty and Li (2000) introduce sequential screening to analyze the above dynamic principal-agent problems.

20.4.1 An Example of Sequential Screening

Consider the demand for airplane tickets. Travellers typically do not know their valuations for tickets until just before departure, but they know in advance their likelihood to have high and low valuations. A monopolist can wait until the travellers learn their valuations and charge the monopoly price, and more consumer surplus can be extracted by requiring them to reveal their private information sequentially. An illustration of such monopoly practice is a menu of refund contracts, each consisting of an advance payment and a refund amount in case the traveller decides not to use the ticket. By selecting a refund contract from the menu, travellers reveal their private information about the distribution of their valuations, and by deciding later whether they want the ticket or the specified refund, they reveal what they have learned about their actual valuation.

Suppose that at time $t = 0$, one-third of all potential buyers are leisure travellers (type L) whose valuation is uniformly distributed on $\theta_L \in [1, 2]$, and two-thirds are business travellers (type B) whose valuation is uniformly distributed on $\theta_B \in [0, 1] \cup [2, 3]$. Intuitively, business travellers face greater valuation uncertainty than leisure travellers. Suppose that cost of flying an additional traveller is 1. Suppose monopolist and travellers are all risk neutral. At time $t = 1$, all travellers know their true value. The monopolist design contract in time $t = 0$.

We first discuss the monopolist's pricing in $t = 1$. If the seller waits until travellers have privately learned their valuations, she faces a valuation distribution that is uniform on $[0, 3]$, i.e., type distribution function being $F(x) = \frac{x}{3}$, the problem of monopolist is:

$$\max_p (p - 1)(1 - F(p)),$$

the monopoly price is $p^m = 2$, and her profit is $\pi^m = \frac{1}{3}$.

Now we discuss the monopolist's contract in $t = 0$. Travellers only know their general types, i.e., leisure type or business type. Suppose instead that the seller offers two contracts before the travellers learn their valuation, one with an advance payment of 1.5 and no refund (unchangeable), and the other with an advance payment of 1.75 and a partial refund of 1 (changeable, but with .75 cancellation fee). Contracts: price being $p_l = 1.5$ without refund. Leisure travellers strictly prefer the contract with no refund, (p_l, r_l) . Business travellers are indifferent between the two contracts so we assume that they choose the contract with refund, (p_b, r_b) .

Since for the leisure type, if choosing the contract (p_l, r_l) , his expected utility is:

$$\int_1^2 (\theta_L - 1.5) d\theta_L = 0 > \int_1^2 (\theta_L - 1.75) d\theta_L.$$

For the business type, if choosing the contract (p_b, r_b) , his expected utility is:

$$\begin{aligned} \frac{1}{2} \int_0^1 (1 - 1.75) d\theta_B + \frac{1}{2} \int_2^3 (\theta_B - 1.75) d\theta_B &= 0 = \\ \frac{1}{2} \int_0^1 (\theta_B - 1.5) d\theta_B + \frac{1}{2} \int_2^3 (\theta_B - 1.5) d\theta_B. \end{aligned}$$

Therefore, such screening contracts satisfy the incentive compatibility and participation constraint, and the expected rent for both types are 0. In such contracts, the monopolist's profit is:

$$1/3(1.5 - 1) + 2/3[\int_0^1 (1.75 - 1) d\theta_B + \int_2^3 (1.75 - 1) d\theta_B] = \frac{2}{3} > \pi^m.$$

In time $t = 1$, for business type, if $\theta_B \in [0, 1]$, he will choose refund, and if $\theta_B \in [2, 3]$, he will travel, and the monopolist can get profit 0.75 from each business traveller.

The above example reveals the basic idea of sequential screening: On one hand, with time moving, the agent has more advantage on information. To screen such information, the principal need to give up more information rent, the earlier screening is, more reduction of agent's information rent would be. On the other hand, with time moving, more information makes increase in efficiency of outcome, earlier screening results in the loss

of efficiency. Therefore, in sequential screening, there is a trade off between allocative efficiency and information rent in time.

There are many other example of sequential mechanisms that take different forms such as hotel reservations (cancellation fees), car rentals (free mileage vs. fixed allowance), telephone pricing (calling plans), public transportation (day pass), and utility pricing (optional tariffs). Sequential price discrimination can also play a role in contracting problems such as taxation and procurement where the agent's private information is revealed sequentially.

20.4.2 Sequential Screening under Incomplete Information

In this subsection, we consider the problem of designing the optimal menu of refund contracts for two ex ante types of potential buyers.

Consider a monopoly seller of airplane tickets facing two types of travellers, B and L , with proportion β_B and β_L respectively. We can think of type B as the "business traveller" and type L as the "leisure traveller". There are two periods. In the beginning of period one, the traveller privately learns his type. The seller and the traveller contract at the end of period one. In the beginning of period two, the traveller privately learns his actual valuation $v \in [\underline{v}, \bar{v}]$ for the ticket, and then decides whether to travel. Each ticket costs the seller c . The seller and the traveller are risk-neutral, and do not discount. The reservation utility of each type of traveller is normalized to zero. The business type may value the ticket more in the sense of first-order stochastic dominance (FSD): type B 's distribution of valuation G_B first-order stochastically dominates the leisure type's distribution G_L if $G_B(v) \leq G_L(v)$ for all v in the range of valuations $[\underline{v}, \bar{v}]$. Alternatively, the business type may face greater valuation uncertainty in the sense of mean-preserving-spread (MPS): if G_B dominates G_L by MPS, then for G_B and G_L , and for their random variables v_B and v_L , there exists v_ϵ so that the related variable respectively, and $v_B = v_L + v_\epsilon$ with v_ϵ independent of v_L , or equivalently, for all $v \in [\underline{v}, \bar{v}]$, $\int_{\underline{v}}^v (G_B(u) - G_L(u)) du \geq 0$.

Example 20.4.1 Let us consider again the example from the last subsection. G_B and G_L are described as followed:

$$G_B(v) = \begin{cases} \frac{v}{2}, & \text{if } v \in [0, 1], \\ \frac{1}{2}, & \text{if } v \in [1, 2], \\ \frac{v-1}{2}, & \text{if } v \in [2, 3]. \end{cases}$$

and

$$G_L(v) = \begin{cases} 0, & \text{if } v \in [0, 1], \\ v - 1, & \text{if } v \in [1, 2], \\ 1, & \text{if } v \in [2, 3]. \end{cases}$$

The distribution function of v_e is:

$$G_e(v) = \frac{v+2}{4}, v \in [-2, 2].$$

Thus,

$$\int_v^v (G_B(u) - G_L(u))du = \begin{cases} \frac{v^2}{4} \geq 0, & \text{if } v \in [0, 1], \\ \frac{1}{4} - \frac{(v-1)(v-2)}{2} \geq 0, & \text{if } v \in [1, 2], \\ \frac{(v-3)^2}{4} \geq 0, & \text{if } v \in [2, 3]. \end{cases}$$

Thus, G_B dominates G_L by MPS.

A refund contract consists of an advance payment a at the end of period one and a refund k that can be claimed at the end of period two after the traveller learns his valuation. Clearly, regardless of the payment a , the traveler use it only if he values the ticket more than k . The seller offers two refund contracts (a_B, k_B, a_L, k_L) . The profit maximization problem can be written as:

$$\begin{aligned} \max (a_B, k_B; a_L, k_L) & \beta_B [a_B - k_B G_B(k_B) - c(1 - G_B(k_B))] \\ & + \beta_L [a_L - k_L G_L(k_L) - c(1 - G_L(k_L))] \end{aligned} \quad (20.4.38)$$

subject to

$$\begin{aligned} -a_B + k_B G_B(k_B) + \int_v^{\bar{v}} \max\{v, k_B\} dG_B(v) \\ \geq -a_L + k_L G_B(k_L) + \int_v^{\bar{v}} \max\{v, k_L\} dG_B(v); \end{aligned} \quad (20.4.39)$$

$$\begin{aligned} -a_L + k_L G_L(k_L) + \int_v^{\bar{v}} \max\{v, k_L\} dG_L(v) \\ \geq -a_B + k_B G_L(k_B) + \int_v^{\bar{v}} \max\{v, k_B\} dG_L(v); \end{aligned} \quad (20.4.40)$$

$$-a_B + k_B G_B(k_B) + \int_v^{\bar{v}} \max\{v, k_B\} dG_B(v) \geq 0; \quad (20.4.41)$$

$$-a_L + k_L G_L(k_L) + \int_v^{\bar{v}} \max\{v, k_L\} dG_L(v) \geq 0. \quad (20.4.42)$$

The constraints of (20.4.41) and (20.4.42) are the participation constraints in period one, and (20.4.39) and (20.4.40) are the incentive compatibility constraints in period one.

We can verify that in second-best contracts, only the constraints of (20.4.39) and (20.4.42) are binding. Since $\max\{v, k_L\}$ is concave function of v , by the properties of MPS, we have:

$$\int_v^{\bar{v}} \max\{v, k_L\} dG_B(v) \geq \int_v^{\bar{v}} \max\{v, k_L\} dG_L(v),$$

and the constraint of (20.4.39) is binding, i.e.,

$$-a_B + k_B G_B(k_B) + \int_{\underline{v}}^{\bar{v}} \max\{v, k_B\} dG_B(v) \geq -a_L + k_L G_B(k_L) + \int_{\underline{v}}^{\bar{v}} \max\{v, k_L\} dG_B(v),$$

so that we have

$$-a_B + k_B G_B(k_B) + \int_{\underline{v}}^{\bar{v}} \max\{v, k_B\} dG_B(v) \geq -a_L + k_L G_B(k_L) + \int_{\underline{v}}^{\bar{v}} \max\{v, k_L\} dG_L(v) \geq 0,$$

Thus we get the participation constraint (20.4.41) for type B .

Let us firstly omit the incentive compatible constraint (20.4.40) for type L (below we show it is loosely satisfied). Substituting the equations of (20.4.39) and (20.4.42) into (20.4.38), the profit maximization problem can be rewritten as:

$$\max(k_B, k_L) \int_{k_B}^{\bar{v}} \beta_B(v - c) dv + \int_{k_L}^{\bar{v}} \beta_L(v - c) g_L(v) - \beta_B(G_L(v) - G_B(v)) dv \quad (20.4.43)$$

In the objective function (20.4.43), define $S_t(k_t) = \int_{k_t}^{\bar{v}} (v - c) g_L(v) dv$ as the consumers' surplus for type $t \in \{L, B\}$, and $R_B(k_L) = \int_{k_L}^{\bar{v}} (G_L(v) - G_B(v)) dv$ as the information rent for type B .

The solution for the problem is satisfied:

$$k_B = c, \quad (20.4.44)$$

$$k_L = \arg \max_k f_L S(k) - f_B Rk. \quad (20.4.45)$$

The second-best solution is the tradeoff between allocation efficiency and information rent, the same logic with principal-agent problem.

Next we show that under the constraints (20.4.44) and (20.4.45), the incentive compatibility constraint (20.4.40) for L is satisfied.

From the binding of (20.4.39), it means that $a_L - a_B = \int_{k_B}^{k_L} G_B(v) dv$, and thus

$$\begin{aligned} -a_L + k_L G(k_L) + \int_{k_L}^{\bar{v}} v dG_L(v) &= -a_B + k_B G(k_L) + \int_{k_B}^{\bar{v}} v dG_L(v) \\ &\quad - \int_{k_B}^{k_L} (G_B(v) - G_L(v)) dv. \end{aligned}$$

Therefore (20.4.40) is equivalent to $\int_{k_B}^{k_L} (G_B(v) - G_L(v)) dv \leq 0$. So we need only to verify that $\int_{k_B}^{k_L} (G_B(v) - G_L(v)) dv \leq 0$. When $k_L = c$, it is obviously true. When $k_L \neq c$, suppose by way of contradiction that $\int_c^{k_L} (G_B(v) - G_L(v)) dv > 0$, and consider a new contract that $k'_B = k'_L = c$. We have

$S_L(k'_L) = S_L(c) > S_L(k_L)$, and the information rent is:

$$\begin{aligned} R_B(k'_L) &= \int_c^{\bar{v}} (G_L(v) - G_B(v))dv \\ &= \int_{k_L}^{\bar{v}} (G_L(v) - G_B(v))dv + \int_c^{k_L} (G_L(v) - G_B(v))dv \\ &\leq \int_{k_L}^{\bar{v}} (G_L(v) - G_B(v))dv = R_B(k_L), \end{aligned}$$

which is a contradiction to the second-best contract (k_B, k_L) , so it must be that $\int_c^{k_L} (G_B(v) - G_L(v))dv \leq 0$. Thus, the incentive compatible constraint (20.4.40) holds under the binding constraints of (20.4.39) and (20.4.42).

In the following, we discuss k_L when $k_L \neq c$. When $k_L > c$, it means that the type L is rationed, and when $k_L < c$, it means the type L is subsidized.

For the buyer's surplus for type L , $S_L(k_L) = \int_{k_L}^{\bar{v}} (v - c)g_L(v)dv$, $\forall k_L \in [\underline{v}, \bar{v}]$, $S_L(c) \geq S_L(k_L)$, if $k_L = c$, its surplus is biggest. However for the information rent for type H , $R_B(k_L) = \int_{k_L}^{\bar{v}} (G_L(v) - G_B(v))dv = \int_{\underline{v}}^{k_L} (G_B(v) - G_L(v))dv \geq 0$, when $k_L = \underline{v}$ or $k_L = \bar{v}$, $R_B(k_L) = 0$, we know in the interior of $[\underline{v}, \bar{v}]$, the rent R_B has an extreme point.

Consider the following special case where $R_B(k_L)$ is a single-peaked function, which means there exists z such that $\frac{dR_B(k_L)}{dk_L} > 0 \forall k_L < z$ and $\frac{dR_B(k_L)}{dk_L} < 0 \forall k_L > z$. If the density functions g_B and g_L are symmetric at point z , we must have $k_L < c$ (or $k_L > c$) if $c < z$ (or $c > z$). To see this, we only discuss the case of $c < z$ (the case of $c > z$ is similar). Let k_L^{SB} be second-best solution.

(1) $k_L^{SB} \notin (c, z]$. If $k_L \in (c, z]$, then $\frac{dS_L(k_L)}{dk_L} < 0$ and $\frac{dR_B(k_L)}{dk_L} > 0$. Thus, if k_L decreases, then $S_L(k_L) - R_B(k_L)$ increases.

(2) $k_L^{SB} \notin (z, 2z - c]$. If $k_L \in (z, 2z - c]$, consider a new refund $\tilde{k}_L = 2z - k_L$ so that $R_B(\tilde{k}_L) = R_B(k_L)$. Thus, for $c \leq \tilde{k}_L < k_L$, we have $S_L(\tilde{k}_L) > S_L(k_L)$.

(3) $k_L^{SB} \notin [2z - c, \infty)$. If $k_L > 2z - c$, consider a new refund $\tilde{k}_L = 2z - k_L$ so that $R_B(\tilde{k}_L) = R_B(k_L)$. Since $k > 2z - c$, for $\tilde{k} = 2z - k$, we get:

$$-\frac{dS_L(k)}{dk} = (k - c)g_L(k) = (k - c)g_L(\tilde{k}) > (c - \tilde{k})g_L(\tilde{k}) = \frac{dS_L\tilde{k}}{d\tilde{k}},$$

in which the second equality is from the symmetric hypothesis of g_L at z and the third equality is from $k + \tilde{k} = 2z > 2c$.

We then have

$$\begin{aligned} S_L(c) - S_L(k_L) &= \int_c^{k_L} -\frac{dS_L(k)}{dk} dk \\ &> \int_{\tilde{k}_L}^c \frac{dS_L(\tilde{k})}{dk} dk \\ &= S_L(c) - S_L(\tilde{k}_L), \end{aligned}$$

Thus, $S_L(\tilde{k}_L) > S_L(k_L)$, which is contradictory to the second-best solution of k_L .

From the above discussion, we obtain $k_L^{SB} < c$.

Example 20.4.2 Return to the example from the last subsection and let $c = 1$. We show the contract given in this example is in fact the second-best screening contract to the principal. Indeed, $S_L(k_L)$ and $R_B(k_L)$ are

$$S_L(k_L) = \int_{k_L}^{\bar{v}} (v - c)g_L(v)dv \begin{cases} \frac{1}{2}, & \text{if } k_L \in [0, 1], \\ k_L - \frac{k_L^2}{2}, & \text{if } k_L \in [1, 2], \\ 0, & \text{if } k_L \in [2, 3], \end{cases}$$

and

$$R_B(k_L) = \int_{\underline{v}}^{k_L} (G_B(u) - G_L(u))du = \begin{cases} \frac{k_L^2}{4} \geq 0, & \text{if } k_L \in [0, 1], \\ \frac{1}{4} - \frac{(k_L-1)(k_L-2)}{2} \geq 0, & \text{if } k_L \in [1, 2], \\ \frac{(k_L-3)^2}{4} \geq 0, & \text{if } k_L \in [2, 3]. \end{cases}$$

So, $z = 1.5$, $k_L^{SB} = \arg \max_k \frac{1}{3}S_L(k) - \frac{2}{3}R_B(k)$, and therefore $k_L^{SB} = 0$. Its graphic illustration is shown the figure (20.3).

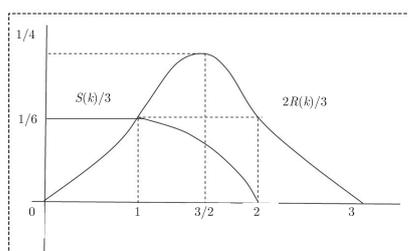


Figure 20.3: Sequential Screening for two types

Since the participation constraint (20.4.42) is binding for type L , if $k_L = 0$, $a_L = 1.5$; and since $k_B = c = 1$, for type B , its participation constraint (20.4.39) is binding too. Therefore, we get $a_B = 1.75$. Thus, $(a_L = 1.5, k_L = 0; a_B = 1.75, k_B = 1)$ is the second-best screening contract to the principal.

In Courty and Li (2000), they further discuss the continuous type case, interested reader can find more detail analysis in their paper.

20.5 Efficient Budget-Balanced Dynamic Mechanism

In this section we turn to the dynamic efficient mechanism in the framework of general mechanism design with more than one agent. Athey and Segal (Econometrica, 2013) provide an analysis framework of dynamic mechanism design, and construct an efficient dynamic mechanism.

The Vickery-Clark-Groves (VCG) mechanism established the existence of an incentive-compatible and efficient mechanism for a general class of static mechanism design problems. The VCG mechanism provides incentives for truthful reporting of private information under the assumption of private values (other agents' private information does not directly affect an agent's payoff) and that preferences are quasilinear so that incentives can be provided using monetary transfers. One shortcoming of VCG mechanism is not budget-balanced ex post. Subsequently, a pair of classic papers, Arrow (1979) and d'Aspremont and Gerard-Varet (1979) (AGV), constructed an efficient and incentive-compatible mechanism, called the expected externality mechanism, in which the transfers were budget-balanced, and thus resulting in Pareto efficient outcomes, using the solution concept of Bayesian Nash equilibrium, under the additional assumption that private information is independent across agents.

In this section, we discuss the efficient dynamic mechanism design, which is also budget-balanced, by omitting the requirement of ex post participation constraints. In a static setting, the AGV mechanism gives every agent an incentive to report truthfully given his beliefs about opponents' types and truth-telling, by giving him a transfer equal to the "expected externality" his report imposed on the other agents. Thus, an agent's current beliefs about opponents' types play an important role in determining his transfer. However, in a dynamic setting, these beliefs evolve over time as a function of opponent reports and the decisions those reports induce. *If the transfers are constructed using the agents' prior beliefs at the beginning of the game, the transfers will no longer induce truthful reporting after agents have gleaned some information about every other's types.*

If, instead, the transfers are constructed using beliefs that are updated using earlier reports, this will undermine the incentives for truthful reporting at the earlier stages. Athey and Segal (2013) construct a mechanism, called balanced team mechanism, that achieves budget-balance property. Such mechanism sustains an equilibrium in truthful strategies by giving each agent in each period an incentive payment equal to the change in the expected present value (EPV) of the other agents' utilities that is induced by his current report. On the one hand, these incentive payments cause each agent to internalize the expected externality imposed on the other agents by his reports. On the other hand, *the expected incentive payment to an agent is zero when he reports truthfully no matter what reporting strategies the other agents use.* The latter property makes the budget balanced by letting the

incentive payment to a given agent being paid by the other agents without affecting those agents' reporting incentives.

Before we construct such a mechanism, let us first analyze the static case, discuss the difficulties in dynamic mechanism, and then we introduce the dynamic efficient mechanism proposed by Athey and Segal.

Consider a seller (agent 1) and a buyer (agent 2) who engage in a two-period relationship. In each period $t = 1, 2$, they can trade a contractible quantity $x_t \in R_+$.

Before the first period, the seller privately observes a random type $\tilde{\theta}_1 \in [1, 2]$, whose realization θ_1 determines his cost. The cost function is given by

$$C(\theta_1, x_t) = \frac{1}{2}\theta_1 x_t^2, \quad t = 1, 2.$$

where x_t is output at time t .

The buyer's value per unit of the good in period 1 is equal to 1, and in period 2 it is given by a random type $\tilde{\theta}_2 \in [0, 1]$ whose realization she privately observes between the periods.

When information is complete, the optimal problem at period 1 is:

$$\max_{x_1} x_1 - \frac{1}{2}\theta_1 x_1^2,$$

giving us $x_1(\theta_1) = \frac{1}{\theta_1}$, and the optimal problem at period 2 is:

$$\max_{x_2} \theta_2 x_2 - \frac{1}{2}\theta_1 x_2^2,$$

giving us $x_2(\theta_1, \theta_2) = \frac{\theta_2}{\theta_1}$. Thus, an efficient (surplus-maximizing) mechanism must have trading decisions x_1 and x_2 determined by the decision rules: $x_1(\theta_1) = \frac{1}{\theta_1}$ and $x_2(\theta_1, \theta_2) = \frac{\theta_2}{\theta_1}$.

When information is incomplete, but the agents can observe their own types, although there are trades for two periods, it is a static interaction. At this situation, the problem of designing an efficient mechanism comes down to designing transfers to each agent as a function of their reports. In this simple setting, each agent makes only one report. Let us first consider the AGV mechanism for this problem, where we assume that the seller and buyer observe their type at the same time. Taking this as a static benchmark, we discuss the incentive change under the asynchronous observations and reports.

20.5.1 Efficient Budget-Balanced Static Mechanism

Since the AGV mechanism can make agent to internalize the externality induced by his action, and in the efficient decision rule, agents have incentives to reveal their information. In the following, we show how AGV

mechanism can implement efficient rule, i.e., $x_1(\theta_1) = \frac{1}{\theta_1}$ and $x_2(\theta_1, \theta_2) = \frac{\theta_2}{\theta_1}$.

Let $\gamma_i(\theta_i)$ be the payoff to agent i , to encourage the buyer to reveal his type, the transfer to him must be the expected externality to the seller:

$$\begin{aligned}\gamma_2(\theta_2) &= -E_{\tilde{\theta}_1}[\frac{1}{2}\tilde{\theta}_1(x_1(\tilde{\theta}_1))^2 + \frac{1}{2}\tilde{\theta}_1(x_2(\tilde{\theta}_1, \theta_2))^2] \\ &= -\frac{1}{2}E_{\tilde{\theta}_1}[\frac{1}{\tilde{\theta}_1}](1 + (\theta_2)^2).\end{aligned}$$

We verify that under such transfer, the buyer has the incentive to reveal true type. Let the buyer's report type as $\hat{\theta}_2$, his expected utility is:

$$\gamma_2(\hat{\theta}_2) + E_{\tilde{\theta}_1}[x_1\tilde{\theta}_1 + \theta_2x_2(\tilde{\theta}_1, \hat{\theta}_2)] = -\frac{1}{2}E_{\tilde{\theta}_1}[\frac{1}{\tilde{\theta}_1}](1 + (\hat{\theta}_2)^2) + E_{\tilde{\theta}_1}[\frac{1}{\tilde{\theta}_1}](1 + \theta_2\hat{\theta}_2),$$

and then the first condition for $\hat{\theta}_2$ is:

$$-E_{\tilde{\theta}_1}[\frac{1}{\tilde{\theta}_1}](\hat{\theta}_2) + E_{\tilde{\theta}_1}[\frac{1}{\tilde{\theta}_1}]\theta_2 = 0,$$

which gives us

$$\hat{\theta}_2 = \theta_2.$$

In the similar logic, the transfer to seller is equal to the externality to the buyer:

$$\begin{aligned}\gamma_1(\theta_1) &= E_{\tilde{\theta}_2}[x_1(\theta_1) + \tilde{\theta}_2x_2(\theta_1, \tilde{\theta}_2)] \\ &= \frac{1}{\theta_1}[1 + E_{\tilde{\theta}_2}(\tilde{\theta}_2)^2].\end{aligned}$$

We can easily check that under such transfer, the seller has incentives to reveal his type.

The above transfer is not budget-balanced. However, due to the independence between $\gamma_i(\theta_i)$ and $\theta_j, j \neq i$, when the transfer to agent θ_i is

$$\psi_i(\theta_i, \theta_j) = \gamma_i(\theta_i) - \gamma_j(\theta_j), i \neq j, i, j \in \{1, 2\},$$

it is a budget-balanced AGV mechanism. So in the static mechanism, the efficient rule is Bayesian implementable and budget-balanced. Actually, we know this result in Chapter 19.

20.5.2 Incentive Problem in Dynamic Environments

Now we turn to our dynamic model, where the buyer (agent 2) observes his type $\tilde{\theta}_2$ between the time 1 and time 2, and the seller (agent 1) reported her type at time 1. The above AGV mechanism cannot implement the

efficient rule $x_1(\theta_1) = \frac{1}{\theta_1}$ and $x_2(\theta_1, \theta_2) = \frac{\theta_2}{\theta_1}$. This is because, under the dynamic setting, when the seller reports his true type θ_1 , the buyer can learn the seller's type θ_1 from the first-period trade $x_1(\theta_1)$, and then the transfer $\gamma_2(\theta_2)$ cannot induce the true revelation for the buyer. Indeed, if the buyer announces his type being $\hat{\theta}_2$, his expected utility is

$$\gamma_2(\hat{\theta}_2) + \theta_1[x_1\theta_1 + \theta_2x_2(\theta_1, \hat{\theta}_2)] = -\frac{1}{2}E_{\hat{\theta}_1}\left[\frac{1}{\theta_1}\right](1 + (\hat{\theta}_2)^2) + \frac{1}{\theta_1}(1 + \theta_2\hat{\theta}_2). \quad (20.5.46)$$

In this dynamic setting, the buyer can learn the seller's private information. For the buyer's expected utility (20.5.46), the first-order condition for $\hat{\theta}_2$ is

$$-E_{\hat{\theta}_1}\left[\frac{1}{\theta_1}\right](\hat{\theta}_2) + \frac{\theta_2}{\theta_1} = 0,$$

and so

$$\hat{\theta}_2 = \frac{1}{\theta_1} (E_{\hat{\theta}_1}\left[\frac{1}{\theta_1}\right])^{-1}.$$

Therefore, when $\frac{1}{\theta_1} > E_{\hat{\theta}_1}\left[\frac{1}{\theta_1}\right]$, the buyer has incentives to over-report his value; otherwise has incentives to underreport his value. The reason that the buyer has incentives to distort his type, is that it cannot fully internalize the externality induced by his action.

However, if we let buyer bear externality so that it is given by:

$$\tilde{\gamma}_2(\theta_1, \theta_2) = -\frac{1}{2}\theta_1(1 + (\theta_2)^2),$$

then under this transfer, the buyer's incentive can be restored. In this case, we have

$$\tilde{\gamma}_2(\theta_1, \hat{\theta}_2) + \theta_1[x_1\theta_1 + \theta_2x_2(\theta_1, \hat{\theta}_2)] = -\frac{1}{2}\theta_1\left[\frac{1}{\theta_1}\right](1 + (\hat{\theta}_2)^2) + \frac{1}{\theta_1}(1 + \theta_2\hat{\theta}_2). \quad (20.5.47)$$

The first condition for $\hat{\theta}_2$ then is:

$$-\frac{1}{\theta_1}(\hat{\theta}_2) + \frac{\theta_2}{\theta_1} = 0,$$

and thus

$$\hat{\theta}_2 = \theta_2.$$

Although $\gamma_1(\theta_1)$ and $\tilde{\gamma}_2(\theta_1, \theta_2)$ are incentive compatible for truthful revelation, they are not budget-balanced. In order us to restore budget-balance property, the transfers to the seller and buyer consumer must be:

$$\psi_1(\theta_1, \theta_2) = \gamma_1(\theta_1) - \tilde{\gamma}_2(\theta_1, \theta_2),$$

$$\psi_2(\theta_1, \theta_2) = \tilde{\gamma}_2(\theta_1, \theta_2) - \gamma_1(\theta_1).$$

However $\tilde{\gamma}_2(\theta_1, \theta_2)$ depends on θ_1 , such transfer $\psi_1(\theta_1, \theta_2)$ is not incentive compatible for θ_1 .

20.5.3 Efficient Budget-Balanced Dynamic Mechanism

The above difficulty is called the problem of contingent deviation. Athey and Segal (2013) proposed a mechanism that overcomes this difficulty. Similar to the AGV mechanism, their construction proceeds in two steps: (i) construct incentive compatible transfers $\gamma_1(\theta)$ and $\gamma_2(\theta)$ to make each agent report truthfully if he expected the other to do so, where $\theta = (\theta_1, \theta_2)$; (2) charge each agent's incentive compatible transfer to the other agent, making the total transfer to agent i equal to $\psi_i(\theta) = \gamma_i(\theta) - \gamma_j(\theta)$. However, in contrast to AGV transfers, the incentive transfer $\gamma_i(\theta)$ to agent i will now depend not just on agent i 's announcements θ_i , but also on those of the other agents. How do we then ensure that step (ii) does not destroy incentives? For this purpose, we ensure that even though agent i can affect the other's incentive payment $\gamma_i(\theta_i, \theta_{-i})$, he cannot manipulate the expectation of the payment given that agent i reports truthfully. We achieve this by letting $\gamma_i(\theta_i, \theta_{-i})$ be the change in the expectation of agent i 's utility, conditional on all the previous announcements, that is brought about by the report of agent i . (In the general model in which an agent reports in many periods, these incentive transfers would be calculated in each period for the latest report.) No matter what reporting strategy agent i adopts, if he believes agent i reports truthfully, his expectation of the change in his expected utility due to agent i 's future announcements is zero by the law of iterated expectations. Hence agent i can be charged $\gamma_i(\theta_i, \theta_{-i})$ without affecting his incentives.

For the above situation, our construction entails giving the buyer an incentive transfer of

$$\gamma_2(\theta_1, \theta_2) = -\frac{1}{2\theta_1}[(\theta_2)^2 - E_{\hat{\theta}_2}(\tilde{\theta}_2)^2].$$

Such transfer is incentive compatible for the buyer. This is because, upon the announcement $\hat{\theta}_2$, the buyer's utility is

$$\gamma_2(\theta_1, \hat{\theta}_2) + \theta_1[x_1\theta_1 + \theta_2x_2(\theta_1, \hat{\theta}_2)] = -\frac{1}{2\theta_1}[(\hat{\theta}_2)^2 - E_{\hat{\theta}_2}(\tilde{\theta}_2)^2] + \frac{1}{\theta_1}(1 + \theta_2(\hat{\theta}_2)).$$

The first condition to $\hat{\theta}_2$ is:

$$-\frac{1}{\theta_1}(\hat{\theta}_2) + \left[\frac{1}{\theta_1}\right]\theta_2 = 0,$$

giving us $\hat{\theta}_2 = \theta_2$, which means the incentive compatibility.

Although $\gamma_2(\theta_1, \hat{\theta}_2)$ depends on θ_1 ,

$$E_{\hat{\theta}_2}[\gamma_2(\theta_1, \tilde{\theta}_2)] = 0, \forall \theta_1.$$

Therefore, the seller's report does not affect the expected transfer to the buyer. As we know, $\gamma_1(\theta_1)$ is the incentive transfer for the seller, so the total

transfer for the agents are:

$$\begin{aligned}\psi_1(\theta_1, \theta_2) &= \gamma_1(\theta_1) - \gamma_2(\theta_1, \theta_2), \\ \psi_2(\theta_1, \theta_2) &= -\psi_1(\theta_1, \theta_2).\end{aligned}$$

Hence this transfer scheme are budget-balanced and incentive compatible for the seller, because the seller reports earlier than the buyer, and the transfer to the seller is:

$$E_{\tilde{\theta}_2} \psi_1(\theta_1, \tilde{\theta}_2) = \gamma_1(\theta_1).$$

In this mechanism proposed by Athey and Segal (2013), the transfer for each agent is equal to the change of other agents' expected present value, and such mechanism achieves the incentive compatibility and budget balancedness simultaneously.

Athey and Segal (2013) generalize the idea of using incentive payments that give an agent the change in the expected present value of opponent utilities induced by his report to design an efficient mechanism for a general dynamic model, interested readers can find more detail in their paper.

Dynamic mechanism design is a hot theoretical topic in recent years. Bergemann and Valimaki (Econometrica, 2010) proposed an alternative efficient dynamic mechanism, but with new properties, such as satisfying the ex post participation constraints and "efficient exit" conditions. Pavan, Segal and Toikka (Econometrica, 2014) developed a general allocation model and derived the optimal dynamic revenue-maximizing mechanism. For interested readers, the last chapter of the monograph by Borgers (2015) is also a good reference on dynamic mechanism design.

20.6 Biographies

20.6.1 Jean Tirole

Jean Tirole(1953-), the world-renowned master of economics, the economic's genius, is known as the modern "genius economist", and his contribution is enough to make all economists wonder: more than 300 high-level papers, 11 monographs, covering all the important fields of economics-from macroeconomics to industrial organization theory, from the game theory to the incentive theory, to the international finance, to the cross research of economics and psychology, and Jean Tirole has made a disruptive contribution to rewrite these areas. In 2000, as a summary of the regulation theory and policy research of monopoly industry in the past ten years, he and Lafont co-authored book *Telecom Competition* provided the most authoritative theoretical basis for the analysis and policy formulation of the competition and regulation of telecom and network industries. Now Tirole is a director of the institute of industrial economics at the university of Toulouse in

France and a visiting professor at the Massachusetts institute of technology. He was the President of the world econometric society and President of the European economic association and was awarded the Nobel Prize in economics in 2014 for his “analysis of market forces and regulation”.

Tirole was born in 1953 in Troyes, on the Seine river, whose father was a gynecologist and whose mother was a Greek and French teacher. Tirole’s mother attached great importance to family education. When Tirole was very young, she taught him the theory of the importance of knowledge. In 1978, after getting a PhD in applied mathematics at the ninth university in Paris, Tirole, who developed an interest in economics, went on to study at the prestigious Massachusetts institute of technology and received his doctorate in economics in 1981.

Tirole’s startling intuition about economics is something that ordinary economists cannot have. Having an extraordinary ability to generalize and synthesize, he is always able to put the most essential law and the most important achievement in any field of economics into the most simple economics model and language expression, and collate them into a theoretical framework of the system. Tirole, who inherited the tradition of French scholars’ emphasis on the humanities, coupled with a deep mathematical foundation, soon showed a remarkable talent for studying economics. He mainly studied macroeconomics and finance at the time, and in 1982 and 1985, he published two classic papers in *Econometrica*, “The Possibility of Rational Expectations of Speculation” and “Asset Bubbles and Generation Overlapping Models”, which established his authority in the field.

What was the most successful on European continent since the economic revival of the 1980s was the institute of industrial economics (IDEI) at the university of Toulouse in France. In 1988, Tirole returned from the United States to France, founded the world-renowned IDEI coupled with the famous economist Professor Jean-Jacques Laffont, and as a research director, Tirole eventually resigned as a tenured professor at the Massachusetts institute of technology. He has made remarkable contributions to the revitalization of French and European economics. Today, IDEI has become the world’s most recognized center for industrial economics and an academic center of economics in Europe.

20.6.2 Thomas Schelling

Thomas C. Schelling (1921-) is an American scholar, economist, an expert on foreign affairs, national security, nuclear strategy and arms control, and one of the founders of limited war theory. He was born in California in April 1921 and received his doctorate in economics from Harvard University in 1948. In 1977 he won Frank E. Seidman award for outstanding contribution to political economy. In 2005, he as well as Robert Aumann got the Nobel Prize in economics.

Unlike game theory, which has traditionally used mathematics extensively, Schelling's main research field is called "non-mathematical game". Schelling and Aumann further developed the non-cooperative game theory and began to deal with some major problems in the field of sociology. They came from different angles respectively—Aumann from the view of mathematics, Schelling from the Angle of economics, and thought that it was possible to reconstruct the analytical paradigm of human interaction from the game theory. What was the most important, Schelling pointed out, was that many social interactions that people are familiar with can be understood from the perspective of non-cooperative games; Aumann also found that some long-term social interactions can be analyzed deeply with formal non-cooperative game theory.

Schelling's game theory based on the breakthrough on the basis of neo-classical economic theory analysis method, different from the mainstream game theory in the research method and the focus, thereby improves, enriches and develops the modern game theory. In his classic book *The Strategy of Conflict*, Schelling first defined and clarified concepts such as deterrence, coercive threats and commitments, strategic mobility, and so on, and began to study social science issues as a unified analytical framework for the insights of game theory, and make a very detailed analysis of the bargaining and conflict management theory. Bargaining theory is the main contribution of Schelling's early period. Although he did not deliberately set out to establish a formal model, many of his views were later shaped by the new development of game theory. The concepts that he defined are also the most basic concepts in game theory, for example, the incredible threat from the concept of perfect equilibrium comes from the Schelling's concept of feasible equilibrium.

His fruitful work contributed to the new development of game theory and accelerated the application of this theory in the field of social science. In particular, his research on strategic commitments explains many phenomena (such as the company's competitive strategy, the mandate of political decision-making). In 1988, when the American economic society rated him as an "outstanding senior member", his comments were: "Schelling's theory of social relations and his application to the theory are derived from his fruitful integration of theory with practice. He has an unusual talen-

t, which made him can actually involve with the nature of the social and economic conditions of the participants with the same or different interests, and can specific and vividly describe the nature of this out." The Nobel committee evaluated him: "Schelling, the self-proclaimed 'itinerant economist', proved to be a very distinguished and pioneering explorer."

20.7 exercises

Exercise 20.1 Consider the principal-agent model of the two phases with full commitment. The enterprise as the principal to produce a certain product, the value of q unit products to the client is $S(q)$, satisfying $S' > 0$, $S'' < 0$, $S(0) = 0$. The enterprise intends to entrust the production of q_1 and q_2 to the agent in the two periods, and provide salaries of $t(q_1)$ and $t(q_1, q_2)$. The agent has two types: $\underline{\theta}$ and $\bar{\theta}$, where $\underline{\theta}$ means the marginal cost of production is low, whereas $\bar{\theta}$ means the marginal cost of production is high, and the agent knows his type information. However, the client knows the probability information, that is, the probability of each type of the agent is $\beta(\bar{\theta}) = \bar{\beta}$ and $\beta(\underline{\theta}) = \underline{\beta}$ respectively, where $\bar{\beta} + \underline{\beta} = 1$. The goal of an enterprise is to maximize $S(q_1) - t(q_1) + S(q_2) - t(q_1, q_2)$. Meanwhile, it satisfies the agent's incentive compatibility constraint and participation constraint.

1. Under the assumption of fixed dynamic type, solve the principal-agent problem.
2. Under the assumption of independent dynamic type, solve the model and compare the result with the above.

Exercise 20.2 Under the model of the previous problem, we add the type correlation of the two phases. At the first phase, the agent observed himself type of the first phase $\theta_1 \in \{\underline{\theta}, \bar{\theta}\}$, and the client just knows that the probability of $\theta_1 = \underline{\theta}$ is β . For the information of the second phase, the agent can't observe it, and there is only one correlation between the two periods, namely, the probability of the marginal cost of the second period is

$$\underline{\beta} = \text{prob}(\underline{\theta}|\underline{\theta});$$

$$\bar{\beta} = \text{prob}(\underline{\theta}|\bar{\theta}).$$

When $\underline{\beta} > \bar{\beta}$, it is positive correlation; when $\underline{\beta} = 1 > \bar{\beta} = 0$, it is the fixed dynamic type. Using the ways of analyzing consumers and monopolists in the text of the textbook, analyze the principal-agent problem of positive correlation dynamic type.

Exercise 20.3 (Laffont and Tirole, 1993) For dynamic contracts with no commitment capability, consider the following model: for each phase of production activity, the enterprise must complete a project at a cost

$$C_\tau = \beta - e_\tau, \tau = 1, 2, \quad (20.7.48)$$

Where β is the parameter not varying with time, but only the enterprise knows the value of this parameter; e_τ reflects the cost reduction of τ , Or the level of effort that business managers put in. At τ , social welfare is

$$W_\tau = S - (1 + \lambda)(C_\tau + t_\tau) + U_\tau, \quad (20.7.49)$$

Where $U_\tau - \psi(e_\tau)$ is the utility of manager, S is the social value of the project itself, λ is the shadow cost of capital, total cost is $(1 + \lambda)(C_\tau + t_\tau)$. Both the social planner and the enterprise manager's discount factor are δ .

Consider a $[\underline{\beta}, \bar{\beta}]$ continuum situation, whose Prior probability distribution is $F_1(\cdot)$, density function $f_1(\cdot)$ is strictly positive in $[\underline{\beta}, \bar{\beta}]$, and $\frac{d(F_1(\beta)/f_1(\beta))}{d\beta} > 0$. Under this assumption, the optimal static mechanism is completely separable.

1. Proof. For any scheme $t_1(\cdot)$ of the first period, there is a subsequent equilibrium of non-complete separation.
2. consider any incentive scheme $t_1(\cdot)$ of the first period, assuming ψ' 's lower bound is some positive. Proof: for any ϵ , there is $\beta_\epsilon < \bar{\beta}$ such that when $\beta_n \geq \beta_\epsilon$, for any n , there is not a subsequent equilibrium making social planners obtain more benefits than a fully mixed contract.

Exercise 20.4 Consider the bilateral economy between buyer and seller. When ex ante the buyer meets the seller, the buyer only has the private signal of the value to commodity θ , but he does not know fully the real value of product until accepting the mechanism that the seller designed. Ex ante signal τ is discrete: $\tau \in \{\tau_L, \tau_H\}$. Each ex ante type is equally likely, $\theta \in [0, 1]$. The distributions of the two types respectively are $F(\theta|\tau_H) = \theta$ and $F(\theta|\tau_L) = \sqrt{\theta}$. Direct revelation mechanism (q, t) is also represented as a quaternion group $(q_L, q_H, t_L, t_H) : [0, 1] \rightarrow [0, 1]^2 \times \mathcal{R}^2$ in this case.

1. for $i \in \{L, H\}$, define random variable $\gamma_i = F(\theta|\tau_i)$. Proof: γ_i is randomly independent on τ and uniformly distributed in $[0, 1]$.
2. Let $(\tilde{q}^*, \tilde{t}^*)$ represent the optimal one in the mechanism that Incentives are compatible with observable values. To prove

$$\tilde{q}_H^*(\gamma) = 1, \quad \forall \gamma \in [0, 1];$$

$$\tilde{q}_L^*(\gamma) = \begin{cases} 0, & \text{if } \gamma < 1/2, \\ 1, & \text{others.} \end{cases}$$

3. To prove that the optimal mechanism of observable γ brings expected utility $1/12$ to ex ante type τ_H buyers.

4. For the next question, assume that only the buyer secretly observed γ . Assume that the arrangement mechanism $(\tilde{q}_L^*, \tilde{t}_L)$ leads to that ex ante type τ_L buyer reports γ truthfully after reporting τ_L . proof: in this case, there is $\bar{\tau}_L \in R$ satisfying

$$\tilde{t}_L(\gamma) = \begin{cases} \bar{t}_L, & \text{if } \gamma < 1/2, \\ \bar{t}_L + 1/4, & \text{others.} \end{cases}$$

5. Proof. If $(\tilde{q}_L^*, \tilde{t}_L)$ leads to that ex ante type τ_L reports γ truly, for any $\gamma \in (1/4, 1/2)$, ex ante type τ_H will not report truly γ , and will report γ exceeds $1/2$ inversely, after reporting τ_L .
6. Proof. If $(\tilde{q}_L^*, \tilde{t}_L)$ leads to that ex ante type τ_L reports γ truly, the expected utility of ex ante type τ_H exceeds the expected utility of ex ante type τ_L by $11/96 > 1/12$.
7. Explaining the problems (3) and (4) means that compared with γ fully disclosed and observed, the sellers, in order to realize $(\tilde{q}^*, \tilde{t}^*)$, have to provide higher information rents to the buyers of ex ante type τ_H , when only the buyers observe γ privately.

Exercise 20.5 (Borgers, 2015) Considering the economic environment of the sequential screening, there is a seller and a buyer whose ex ante and ex post type are discrete. Before the buyer met the seller, he had one discrete signal $\tau \in \{1, 3\}$. The seller gives a mechanism. The buyer then accepts the mechanism and receives a second discrete signal $\sigma \in \{1, 3\}$. The value of the buyer is $\theta = \theta_{\tau\sigma} = \tau + \sigma$. Each signal value is equally likely. The opportunity cost for the seller to sell the product is $c = 1/2$. For each $(\tau, \sigma) \in \{1, 3\}^2$, a direct revelation mechanism (q, t) specifies the probability of sale $q_{\tau,\sigma} \in [0, 1]$ and payment $t_{\tau,\sigma} \in R$.

1. Proof. Any one direct revelation mechanism will result in the following lies of unbalance path. The buyer of ex ante type $\tau = 3$ has best reported $\sigma = 3$ after reporting $\tau = 1$, but the buyer of ex ante type $\tau = 1$ has best reported $\sigma = 1$ after reporting $\tau = 3$.
2. Proof. Optimal mechanism (q, t) satisfies $q_{11} = 1, q_{13} = q_{31} = q_{33} = 1$.
3. Assume the buyer could not obtain post ante signal σ . Thus, the expected value of the buyer of ex ante type $\tau = 1$ to product is $\theta_1 = 3$, but the expected value of the buyer of ex ante type $\tau = 3$ to product is $\theta_3 = 5$. To prove that $q_1 = q_3 = 1$ is optimal in a (static) direct revelation mechanism $(q_\tau, t_\tau)_{\tau=1,3}$.
4. Proof. A buyer's situation being receiving an ex post private signal is worse than the ones not being able to receive an ex post signal.

Exercise 20.6 Consider the problem that a client intends to entrust an agent to produce a q unit product. The value of q unit product to the client is $S(q)$, where $S' > 0$, $S'' < 0$, $S(0) = 0$. The client can't observe the cost of production of the agent, the marginal cost subordinates to the collection $\theta \in \{\underline{\theta}, \bar{\theta}\}$, and the probability of the agent being the high efficiency or low efficiency type is respectively v and $1 - v$. The cost function is $C(q, \underline{\theta}) = \underline{\theta}q$, whose probability is v , or the cost function is $C(q, \bar{\theta}) = \bar{\theta}q$, whose probability is $1 - v$, and the agent gets transfer payment t .

1. Assuming it's static, let's solve for the optimal contract.
2. Assuming that there are two periods, the target function of the client is $V = S(q_1) - t_1 + \delta(S(q_2) - t_2)$. The risk neutral agent also has the same discount factor, and the objective function is $U = t_1 - \theta q_1 + \delta(t_2 - \theta q_2)$. Try to solve the fixed dynamic type problem.

Exercise 20.7 (independent type and risk neutral) Now suppose that the distributions of the marginal cost of the agent's two periods are independent, $\theta \in \{\underline{\theta}, \bar{\theta}\}$, and the probabilities of the agent being the high efficiency or low efficiency type in two periods are the same, respectively being v and $1 - v$. The utility function of a risk-neutral agent is $U = t_1 - \theta_1 q_1 + \delta(t_2 - \theta_2 q_2)$, then solve the optimal contract for independent dynamic type under complete commitment.

Exercise 20.8 (the moral hazard problem of repeating two periods) In the moral hazard model of chapter 17, we now assume that the relationship between the principal and the agent is repeated for two periods, and the utility function of the agent of risk aversion is $U = u(t_1) - \psi(e_1) + \delta(u(t_2) - \psi(e_2))$, $e_i \in \{0, 1\}$, and $\psi(1) = \psi$, $\psi(0) = 0$. The effort level of the agent at each period will generate a random return q_i with the probability $PI(e_i)$, where $\pi_0 = \pi(0)$, $\pi_1 = \pi(1)$, the principal is risk neutral, and the utility function is $V = S(q_1) - t_1 + \delta(S(q_2) - t_2)$. In this two-stage problem, solve the optimal contract.

Exercise 20.9 (prevent re-negotiate) Assuming that the relationship between the principal and the agent is repeated twice, and the utility function of the agent of risk aversion is $U = u(t_1) - \psi(e_1) + \delta(u(t_2) - \psi(e_2))$, $e_i \in \{0, 1\}$, and $\psi(1) = \psi$, $\psi(0) = 0$. The effort level of the agent at each period will generate a random return q_i with the probability $PI(e_i)$, where $\pi_0 = \pi(0)$, $\pi_1 = \pi(1)$, the principal is risk neutral, and the utility is $V = S(q_1) - t_1 + \delta(S(q_2) - t_2)$. All conditions are consistent with the previous one, given that $u_2(q_1)$ is the utility that the agent can obtain in the second period, guaranteed by the principal if the output in the first period is q_1 . The condition for preventing renegotiation is $u_2(q_1) \geq 0$.

1. To solve the optimal contract problem when the condition of renegotiation is met.
2. What happens to the optimal contract problem when the agent is allowed to borrow?

Exercise 20.10 (the moral hazard problem of repeating infinite periods) Now extend the two-phase problem in the previous one to the infinite periods, assuming that the relationship between the principal and the agent is repeated infinitely, and the utility function of the risk-averse agent is $U = u(t_1) - \psi(e_1) + \delta(u(t_2) - \psi(e_2))$, $e_i \in \{0, 1\}$, and $\psi(1) = \psi$, $\psi(0) = 0$. The effort level of the agent at each period will generate a random return q_i with the probability $PI(e_i)$, where $\pi_0 = \pi(0)$, $\pi_1 = \pi(1)$, the principal is risk neutral, and the utility is $V = S(q_1) - t_1 + \delta(S(q_2) - t_2)$.

1. The optimal contract problem is depicted with recursive structure and the state variable of this dynamic programming problem is found.
2. Solve the optimal contract and prove that it has markov property.

Exercise 20.11 (Bolton, 2005) Consider a two-phase durable-goods monopoly. When a seller faces a buyer, the buyer's reservation utility to the durable goods is $v \in \{v_L, v_H\}$, $v_H > v_L > 0$. The initial belief of the seller to the reservation utility of the buyer is $\Pr(v = v_H) = 0.5$. The cost of producing goods by the seller takes two values for the same probability: $c \in \{c_L, c_H\}$ and $c_H > c_L \geq 0$. The seller's cost and the value of the buyer's reservation utility are independent distribution and private information. Suppose $v_L - c_L \geq \frac{v_H - c_L}{2}$, and $v_L - c_H \geq \frac{v_H - c_H}{2}$. The common discount factor is $\delta > 0$.

1. What are the conditions for a mixed equilibrium in the following situations?
 - (1) The sellers of both types set such a price: $p_1 = v_H - \delta(v_H - v_L)/2$.
 - (2) The buyer of the type v_H accepts the price for the probability $\gamma = \frac{v_H + c_H - 2v_L}{v_H - v_L}$, but the buyer of the type v_L rejects the price for the probability 1.
 - (3) After the price is rejected in the first period, the seller of the type c_L will set the price $p_2^L = v_L$ in the second period, but the seller of the type c_H will set the price $p_2^H = v_H$ in the second phase.
2. Explain why the seller of the type c_L can benefit from the cost of private information, while the type c_H is not.

Exercise 20.12 (two-stage control model) Consider a two-stage control model where the type of enterprise is endogenous. Firstly, the controller proposes an income function $R_1(q)$, which provides the payment for an output level q . Next, the enterprise inputs sunk cost I , which is only visible to the enterprise himself. The production cost of each phase is $c(q, I)$, where $c(0, I) = c_q(0, I) = 0$, $c_{qq}(q, I) > 0$, $c_I(q, I) < 0$. The enterprise selects the output of q_1 , thus generating a first-phase return of $R_1(q_1) - c(q_1, I) - I$. If the enterprise gives up, it gets $-I$ and the game ends. If the enterprise chooses to produce q , the first-phase return of the regulators is $q_1 - R_1(q_1)$. In the second phase, the regulator observes q_1 then raises $R_2(q)$, and the enterprise chooses or gives up a production q_2 . Accordingly, the second-phase returns of the two sides are respectively $R_2(q_2) - c(q_2, I)$ and $q_2 - R_2(q_2)$.

1. What is the manager's total commitment strategy?

2. Proof: if there is no commitment, when $I > 0$, there is no pure strategic bayesian refining Nash equilibrium.

Exercise 20.13 (repeated moral hazard, Rogerson, 1985) Consider a two-stage moral hazard problem. At the time 1, the agent selects the effort level of a , thereby gaining an independent identically distributed profit in each phase: $q_1, q_2 \in \{q_L, q_H\}$. The probability of the occurrence of q_H is $p(a)$, which is strictly increasing with a , and the probability of the occurrence of q_L is $1 - p(a)$. For all $a \in [0, \bar{a}]$, $a < \infty$, $1 > p(a) > 0$. The utility function of the agent is $u(w) - a$, $u' > 0$, $u'' < 0$. The agent could not borrow, so his income would be consumed entirely. The principal is risk neutral and can deposit and loan at zero interest rate.

1. The payments of the agent depending on output in the first stage and the second stage are expressed by $\{w_L, w_H, w_{LL}, w_{LH}, w_{HL}, w_{HH}\}$. Prove that the optimal contract satisfies:

$$\frac{1}{u'(w_i)} = \frac{p(a)}{u(w_{iH})} + \frac{1 - p(a)}{u'(w_{iL})}, i = H, L.$$

2. Proof: under the optimal contract, when $1/u'$ is concave function, $w_i \leq p(a)w_{iH} + [1 - p(a)]w_{iL}$, $i = H, L$.
3. Let's assume that under the above optimal contract, the first phase allows the agent to deposit a loan after the q_1 is realized and explain why he is willing to deposit it.

Exercise 20.14 (Bolton, 2005) Consider investment insurance under private information. A risk-averse agent invests $p/2$ on a project, and there is a random shock to incomes in two phases $t = 1, 2$, where w_i is equal to 1 with a probability of p , equal to 0 with a probability of $1 - p$, $\Pr(w_2 = 1|w_1 = 1) = \gamma \leq p$, $\Pr(w_2 = 1|w_1 = 0) = \mu \geq p$, and $\gamma > 0.5$, $p < 1$. The utility function of the agent is time-separable, $U(c_1, c_2) = u(c_1) + u(c_2)$. $u(c)$ is the following piecewise linear function: when $c \geq 1/2$, $u(c) = c/2 + 1/4$; when $c < 1/2$, $u(c) = c$. At the beginning of each period, the agent can obtain the insurance of the insurance income impact with the insurance fair rate.

1. The optimal consumption configuration is depicted in the assumption that the agent cannot make private deposits.
2. Assuming that the income shock is private information, prove that the agent can't get any insurance coverage when only the spot contract is available.
3. Suppose the agent can deposit and borrow money at zero interest rate from a bank. Describe the optimal lending margins of the agent.
4. When is the insurance of the form of deposit and loan the optimal contract?

Exercise 20.15 (international lending, Atkeson, 1991) *There are two types of subjects in an indefinite economy, one of which is the risk-averse agent (borrower), the other of which is risk-neutral principal (lender). After the agent borrows money, he can invest it and generate a random return of investment. The distribution of returns depends on the number of investments. The agent has an initial endowment of $Y_0 - d_0$ at 0 phase. At the beginning of phase t , the agent borrows b_t from the principal; At the end of period t , the agent returns d_{t+1} to the principal, and the utility of the agent depends on the consumption level of each phase. Set $(b_t, d_{t+1}(Y_{t+1}))$ as a loan contract, and make $Q_t = Y_t - d_t$.*

1. Define the feasible configuration of the agent.
2. Write the participation constraint of the principal and the agent of the dynamic problem.
3. Write the incentive compatibility constraint of the agent of the dynamic problem, that is, the agent does select the optimal level of investment.
4. Assuming that once the agent defaults, the principal will terminate the loan agreement, that is, the agent will return to the autarky. Write an incentive constraint to prevent the agent from defaulting.
5. Try to solve the optimal contract problem. When the output level is low, the agent has the phenomenon of capital outflow.

20.8 References

Textbooks and Monographs

- Borgers, T. (2015). *An Introduction to the Theory of Mechanism Design*, Oxford University Press.
- Bolton, P., and Dewatripont, M. (2005). *Contract Theory*, MIT Press.
- Laffont, J. J. and Martimort, D. (2002). *The Theory of Incentives: The Principal-Agent Model*, Princeton University Press.
- Laffont, J. J. and Tirole, J. (1993). *A theory of incentives in regulation and procurement*, MIT Press.
- Mas-Colell, A., Whinston, M. D. and Green, J. (1995). *Microeconomic*, Oxford University Press, Chapter 14.
- Segal, I. (2010). *Lecture Notes in Contract Theory*, Stanford University.

Papers:

- Arrow, K. (1979). "The Property Rights Doctrine and Demand Revelation Under Incomplete Information", in *Economics and Human Welfare*, ed. by M. Boskin. New York: Academic Press.
- Athey, Susan and Ilya Segal (2013). "An Efficient Dynamic Mechanism", *Econometrica*, Vol. 81, No.6, 2463-2485.
- Atkeson A. (1991) "International lending with moral hazard and risk of repudiation". *Econometrica*, Vol. 59, No. 4, 1069-1089.
- Baron, D., and Besanko, D.(1984). "Regulation and Information in a Continuing Relationship", *Information Economics and Policy*, Vol.1, 447-470.
- Bergemann, D., and Valimaki, J. (2010). "The Dynamic Pivot Mechanism", *Econometrica*, Vol.78, No.2, 771-789.
- Courty, P., and Li, H. (2000). "Sequential Screening", *Review of Economic Studies*, Vol.67, 697-717.
- d'Aspremont, C., and Gerard-Varet, L. (1979). "Incentives and Incomplete Information", *Journal of Public Economics*, Vol.11, 25-45.
- Dewatripont, M. (1989). "Renegotiation and Information Revelation over Time: The Case of Optimal Labor Contracts", *Quarterly Journal of Economics*, Vol.104, 589-619.
- Eso, P., and Szentes, B. (2007). "Optimal Information Disclosure in Auctions and the Handicap Auction", *Review of Economic Studies*, Vol.74, No.3, 705-731.
- Freixas, X., Guesnerie, R. and Tirole, J. (1985). "Planning under Incomplete Information and the Ratchet Effect", *Review of Economic Studies*, Vol.52, 173-91.
- Hart, O., and Tirole, J. (1988). "Contract Renegotiation and Coasian Dynamics", *Review of Economic Studies*, Vol.55, 509-40.
- Myerson, R. B., and Satterthwaite, M. A. (1983). "Efficient Mechanisms for Bilateral Trading", *Journal of Economic Theory*, Vol.29, 265-281.
- Pavan, A., Segal, I., and Toikka, J. (2014). "Dynamic Mechanism Design: A Myersonian Approach", *Econometrica*, Vol. 82, No. 2, 601-653.
- Rothschild, M. and Stiglitz, J. E. (1970), "Increasing Risk: I. A. Definition", *Journal of Economic Theory*, 2, 225-243.
- Rogerson W P. (1985). "Repeated moral hazard", *Econometrica*, Vol. 53, No. 1, 69-76.

- Spear S E, Srivastava S. (1987). "On repeated moral hazard with discounting", *The Review of Economic Studies*, 1987, 54(4): 599-617.